The classification of two-dimensional extended conformal field theories

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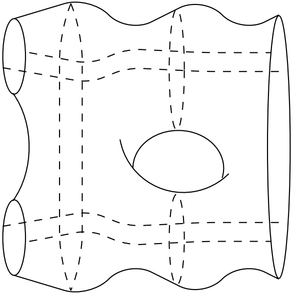
These slides: `https://dmitripavlov.org/wichita.pdf`

arXiv:2011.01208, arXiv:2111.01095 (joint with Daniel Grady)



Main theorem 1: conformal field theory

Theorem

The following smooth ∞-categories are equivalent:

- *extended conformal field theories;*
- *Serre-twisted homotopy coherent representations of the Lie group* $\mathbf{R}^2 \rtimes \widetilde{\mathrm{Conf}}(2)$ *on a 2-dualizable* object.*

Notation:

- $\widetilde{\mathrm{Conf}}(2)$*: the universal covering of* $\mathrm{Conf}(2)$*.*
- $\mathrm{Conf}(2)$*:* $z \mapsto \sum_{k\geq 1} a_k z^k$, $a_1 \neq 0$, *group operation: composition.*
- *Serre-twisted: restricting to* $\mathbf{Z} \subset \widetilde{\mathrm{Conf}}(2) \subset \mathbf{R}^2 \rtimes \widetilde{\mathrm{Conf}}(2)$ *yields Serre automorphisms.*
- *Example: if Serre automorphisms are trivial, get representations of* $\mathbf{R}^2 \rtimes \mathrm{Conf}(2)$*.*

Main theorem 2: $2|1$-Euclidean field theory

Theorem

The following smooth ∞-categories are equivalent:

- *extended* $2|1$*-Euclidean field theories;*
- *Serre-twisted homotopy coherent representations of the Lie supergroup* $\widetilde{\mathrm{Euc}}(2|1)$ *on a 2-dualizable object.*

Notation:

- $\widetilde{\mathrm{Euc}}(2|1)$*: the universal covering of* $\mathrm{Euc}(2|1) = \mathbf{R}^{2|1} \rtimes \mathrm{Spin}(2)$.
- *Serre-twisted: restricting to* $\mathbf{Z} \subset \widetilde{\mathrm{Euc}}(2|1)$ *yields Serre automorphisms.*
- *Serre automorphisms trivial* $\Longrightarrow$ *representations of* $\mathrm{Euc}(2|1)$.

Origins of functorial field theory

- 1948 (Feynman): path integral formulation of quantum mechanics
- 1949 (Feynman–Kac): the Feynman–Kac formula
- Later: path integral used in QFT, no longer rigorous
- 1980s (Witten): properties of path integrals for (conformal) field theory
- 1980s (Segal): mathematical formulation of conformal field theory

Further developments

- late 1980s (Atiyah, Kontsevich, ...): topological theories: easier to construct and study, but less relevant for physics
- 1992 (Freed, Lawrence): extended field theories (correspond to locality in physics)
- 1995 (Baez–Dolan): the topological cobordism and tangle hypotheses
- 2002 (Stolz–Teichner): modern formulation of nontopological field theories (including supersymmetry); the Stolz–Teichner program on $2|1$-EFTs and TMF
- 2004 (Costello): the $(\infty, 2)$-category of topological 2-dimensional bordisms
- 2006 (Hopkins–Lurie); 2015 (Calaque–Scheimbauer): the (∞, d)-category of topological bordisms

Previous results on the topological cobordism hypothesis

- 2008 (Lurie): outline of a proof of the topological cobordism hypothesis
- 2017 (Ayala–Francis): a different approach, conditional on a conjecture
- 2004 (Costello), 2009 (Schommer-Pries):
  the 2-dimensional topological cobordism hypothesis
- 2006 (Galatius–Madsen–Tillmann–Weiss);
  2011 (Bökstedt–Madsen); 2017 (Schommer-Pries):
  the invertible case

Low-dimensional nontopological field theories

Examples of 2-dimensional nonextended nontopological field theories:

- 2007 (Pickrell): Riemannian 2-dimensional field theory
- 2018 (Runkel–Szegedy): volume-dependent 2-dimensional field theory

Classifications of holonomy maps, transport functors, and 1-dimensional nontopological field theories:

- 1990 (Barrett), 1994 (Caetano–Picken), 2007 (Schreiber–Waldorf): parallel transport for bundles
- 2000 (Mackaay–Picken), 2002 (Bunke–Willerton–Turner), 2008 (Schreiber–Waldorf): parallel transport for gerbes
- 2015 (Berwick-Evans–P.), 2020 (Ludewig–Stoffel): 1-dimensional field theories

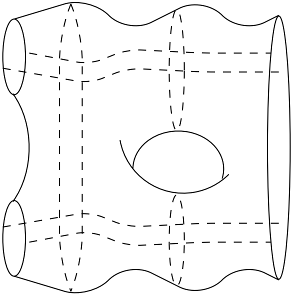
A brief overview of functorial field theory

- Formalizes the Schrödinger picture of quantum field theory
- $(d-1)$-manifold M (e.g., space) $\mapsto$ a Hilbert space of states $F(M)$;
- Bordism $B\colon M_1 \to M_2$ (e.g., spacetime) $\mapsto$ quantum propagator $F(B)\colon F(M_1) \to F(M_2)$;
- F is meant to axiomatize the path integral (functional integral) construction of QFT:
- $F(M)$: the infinite-dimensional vector space of functions on the space of fields on M;
- $\langle F(B)(s_1), s_2 \rangle = \int_{s\colon s_1 \to s_2} \exp(\frac{i}{\hbar} S(s))$;
- Here $s\colon s_1 \to s_2$ is a field on B restricting to s_i on M_i.

Features of the geometric bordism category

- Locality: k-bordisms with corners of all codimensions (up to d) with compositions in d directions
  $\Longrightarrow$ symmetric monoidal d-category of bordisms
- Isotopy: chain complexes to encode BV-BRST
  $\Longrightarrow$ must encode (higher) diffeomorphisms between bordisms
  $\Longrightarrow$ symmetric monoidal (∞, d)-categories
- Geometric (nontopological) structures on bordisms:
  Riemannian/Lorentzian metrics,
  complex/conformal/symplectic/contact structures,
  principal G-bundles with connection and isos,
  higher gauge fields (Kalb–Ramond, Ramond–Ramond)
  $\Longrightarrow$ an $(\infty, 1)$-sheaf of geometric structures
- Smoothness: values of field theories depend smoothly on bordisms
  $\Longrightarrow$ $(\infty, 1)$-sheaf of (∞, d)-categories of bordisms

How to compose bordisms



Geometric structures

Definition

Given $d \geq 0$, the site FEmb_d has

- Objects: submersions $T \to U$ with d-dimensional fibers, where $U \cong \mathbf{R}^n$ is a cartesian manifold;
- Morphisms: commutative squares with $T \to T'$ a fiberwise open embedding over a smooth map $U \to U'$;
- Covering families: open covers on total spaces T.

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Definition

Given $d \geq 0$, a d-dimensional geometric structure is a simplicial presheaf $\mathcal{S}\colon \mathsf{FEmb}_d^{\mathrm{op}} \to \mathsf{sSet}$.

Example:

- $T \to U \mapsto$ the set of fiberwise Riemannian metrics on $T \to U$;
- $(T \to T', U \to U') \mapsto$ the restriction map from T' to T.

Examples of geometric structures

- fiberwise Riemannian, Lorentzian, pseudo-Riemannian metrics; positive/negative sectional/Ricci curvature;
- fiberwise conformal, complex, symplectic, contact, Kähler structures;
- fiberwise foliations, possibly with transversal metrics;
- smooth map to a target manifold M (traditional σ-model);
- smooth map to an orbifold or ∞-sheaf on manifolds;
- fiberwise etale map or an open embedding into a target manifold N;
- fiberwise topological structures: orientation, framing, etc.
- fiberwise differential n-forms (possibly closed).

Examples of geometric structures: gauge transformations

Definition

- Send a d-manifold M to (the nerve of) the groupoid $\mathrm{B}_\nabla G(M)$:
  - Objects: principal G-bundles on T with a fiberwise connection on $T \to U$ (gauge fields);
  - Morphisms: connection-preserving isomorphisms (gauge transformations).

Examples of geometric structures: (higher) gauge transformations

- Principal G-bundles with connection on M (gauge fields, e.g., the electromagnetic field);
- Bundle gerbe with connection on M (B-field, Kalb–Ramond field).
- Bundle 2-gerbe with connection on M (supergravity C-field).
- Bundle $(d-1)$-gerbes with connection on M (Deligne cohomology, Cheeger–Simons characters, ordinary differential cohomology, circle d-bundles).
- Geometric tangential structures: geometric Spin^c-structure, String (Waldorf), Fivebrane (Sati–Schreiber–Stasheff), Ninebrane (Sati). (Vanishing of anomaly.)
- differential K-theory (Ramond–Ramond field). Requires ∞-groupoids.

The geometric cobordism hypothesis

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}\colon \mathsf{FEmb}_d^{\mathsf{op}} \to \mathsf{sSet}$.

Constructions:

- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.
- A d-dimensional functorial field theory valued in $\mathcal{V}$ with geometric structure $\mathcal{S}$ is a smooth symmetric monoidal (∞, d)-functor $\mathfrak{Bord}_d^{\mathcal{S}} \to \mathcal{V}$.
- The simplicial set of d-dimensional functorial field theories valued in $\mathcal{V}$ with geometric structure $\mathcal{S}$ is the derived mapping simplicial set

$$\mathsf{FFT}_{d,\mathcal{V}}(\mathcal{S}) = \mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V}).$$

Can be refined to a derived internal hom.

The geometric cobordism hypothesis

Conjectures (for topological field theories):

- Freed, Lawrence (1992): $\mathsf{FFT}_{d,\mathcal{V}}$ is an ∞-sheaf.
- Baez–Dolan (1995), Hopkins–Lurie (2008):

$$\mathsf{FFT}_{d,\mathcal{V}}(\mathcal{S}) \simeq \mathbf{R}\,\mathrm{Map}(\mathcal{S}, \mathcal{V}^\times).$$

$\mathcal{V}^\times$: fully dualizable objects and invertible morphisms.

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Theorem (Grady–P., The geometric cobordism hypothesis)

Part I (Locality): $\mathfrak{Bord}_d$ is a left adjoint functor:

$$\mathbf{R}\operatorname{Map}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V}) \simeq \mathbf{R}\operatorname{Map}(\mathcal{S}, \mathcal{V}_d^\times),$$

where $\mathcal{V}_d^\times = \mathrm{FFT}_{d,\mathcal{V}}$, i.e., $\mathcal{V}_d^\times(T \to U) = \mathrm{FFT}_{d,\mathcal{V}}(T \to U)$.

Part II (Framed GCH): The evaluation-at-points map

$$\mathcal{V}_d^\times(\mathbf{R}^d \times U \to U) = \mathrm{FFT}_{d,\mathcal{V}}(\mathbf{R}^d \times U \to U) \to \mathcal{V}^\times(U)$$

is a weak equivalence of simplicial sets functorial in U.

Computing with GCH

- How to compute $\mathcal{V}_d^\times$?
- How to compute $\mathbf{R}\,\mathrm{Map}(\mathcal{S}, \mathcal{V}_d^\times)$?

Computing $\mathcal{V}_d^\times$

- Already know $\mathcal{V}_d^\times(\mathbf{R}^d \times U \to U) \simeq \mathcal{V}^\times(U)$, functorial in $U \in \mathsf{Cart}$.
- What are the structure maps for functoriality in FEmb_d?
- Step 1: Guess a map $\mathcal{W} \to \mathcal{V}_d^\times$.
- Step 2: For every U, prove $\mathcal{W}(\mathbf{R}^d \times U \to U) \to \mathcal{V}_d^\times(\mathbf{R}^d \times U \to U) \to \mathcal{V}^\times(U)$ is a weak equivalence.

Example ($\mathcal{V} = \mathrm{B}^d\mathrm{U}(1)$; prequantum FFTs)

- Step 1a: $\mathcal{W}(\mathbf{R}^d \times U \to U) = U\Gamma(\Omega_U^d(\mathbf{R}^d \times U) \leftarrow \cdots \leftarrow \Omega_U^1(\mathbf{R}^d \times U) \leftarrow \mathrm{C}^\infty(\mathbf{R}^d \times U, \mathrm{U}(1)))$.
- Step 1b: $\mathcal{W} \to \mathcal{V}_d^\times$: $\omega \mapsto (B \mapsto \exp(\frac{i}{\hbar}\int_B \omega))$.
- Step 2: Poincaré lemma: $\mathcal{W}(\mathbf{R}^d \times U \to U) \xrightarrow{\sim} \mathrm{B}^d\mathrm{C}^\infty(U, \mathrm{U}(1))$

How to compute $\mathbf{R}\operatorname{Map}(\mathcal{S}, \mathcal{W})$?

Two main options:

- Use the theory of natural operations, working on the site FEmb_d.
  Examples: differential characteristic classes yield prequantum field theories.
- Use an adjunction to switch to a different category: $\operatorname{Fun}(\mathsf{Cart}^{\mathsf{op}}, \mathsf{sSet}^{\mathrm{O}(d)})$.
  Examples: classification of conformal or Euclidean field theories.

Categories of geometric structures

Proposition

The functors q^* and ι^* are right Quillen equivalences.

$$
\begin{array}{ccccc}
\mathcal{S}h(\mathrm{FEmb}_d) & \xleftarrow{\rho^*} & \mathcal{S}h(\mathfrak{FEmb}_d) & \xrightarrow{\iota^*} & \mathcal{S}h(\mathrm{Cart})^{\mathrm{O}(d)} \\
\downarrow q^* & & \downarrow q^* & \nearrow \iota^* & \\
\mathcal{S}h(\mathrm{FEmbCart}_d) & \xleftarrow{\rho^*} & \mathcal{S}h(\mathfrak{FEmbCart}_d). & &
\end{array}
$$

- $\mathcal{S}h(C)$: simplicial presheaves on C, Čech-local model structure
- $\mathfrak{FEmb}_d$: like FEmb_d, but enriched in spaces
- $\mathrm{FEmbCart}_d$: full subcategory of FEmb_d on $\mathrm{D}_U := (\mathbf{R}^d \times U \to U)$
- $\mathfrak{FEmbCart}_d$: equivalent to $\mathrm{Cart} \times \mathrm{BO}(d)$ by C^∞ Kister–Mazur

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\mathcal{S}h(\mathrm{FEmbCart}_d) & \xleftarrow{\ \rho^*\ } & \mathcal{S}h(\mathfrak{FEmbCart}_d). & &
\end{array}$$

The functor $\rho_!$ adds "d-thin homotopies" to a geometric structure.
d-dimensional holonomy is invariant under d-thin homotopies.
$d = 1$: Kobayashi, Barrett, Caetano–Picken
$d > 1$: Bunke–Turner–Willerton, Picken, Mackaay–Picken

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Recipe to compute $\mathbf{R}\,\mathrm{Map}(\mathcal{S}, \rho^*\mathcal{V}_d^\times)$.

- Use q^* to move to $\mathsf{FEmbCart}_d$ / $\mathfrak{FEmbCart}_d$. (Suppressed from the notation.)
- $\mathbf{R}\,\mathrm{Map}(\mathcal{S}, \rho^*\mathcal{V}_d^\times) \simeq \mathbf{R}\,\mathrm{Map}(\rho_!\mathcal{S}, \mathcal{V}_d^\times)$.
- Compute $\rho_!\mathcal{S}$.
- $\mathbf{R}\,\mathrm{Map}(\rho_!\mathcal{S}, \mathcal{V}_d^\times) \simeq \mathbf{R}\,\mathrm{Map}(\iota^*\rho_!\mathcal{S}, \iota^*\mathcal{V}_d^\times)$. ($\mathrm{C}^\infty$ Kister–Mazur)

How to compute $\rho_! \mathcal{S}$?

Notation:

- $\mathsf{FEmbCart}_d$: Objects $\mathsf{D}_U = (\mathbf{R}^d \times U \to U)$, morphisms: fiberwise open embeddings.
- $\mathfrak{FEmbCart}_d$: Objects $\mathfrak{D}_U$, space of morphisms.
- $\rho\colon \mathsf{FEmbCart}_d \to \mathfrak{FEmbCart}_d$: inclusion.
- $\rho_!\colon \mathcal{S}h(\mathsf{FEmbCart}_d) \to \mathcal{S}h(\mathfrak{FEmbCart}_d)$: left Kan extension.

Computation:

- $\rho_! \mathcal{S} = \rho_! \operatorname{hocolim}_{\mathsf{D}_U \to \mathcal{S}} Y(\mathsf{D}_U) = \operatorname{hocolim}_{\mathsf{D}_U \to \mathcal{S}} Y(\mathfrak{D}_U)$.
- Evaluate on $\mathfrak{D}_W$:

$$(\rho_! \mathcal{S})(\mathfrak{D}_W) = \operatorname*{hocolim}_{\mathsf{D}_U \to \mathcal{S}} \mathfrak{FEmbCart}_d(\mathfrak{D}_W, \mathfrak{D}_U).$$

- $\mathfrak{FEmbCart}_d(\mathfrak{D}_W, \mathfrak{D}_U)$ is 1-truncated. Ob: $\varphi\colon \mathsf{D}_W \to \mathsf{D}_U$. Mor $\gamma\colon \varphi \to \varphi'$: isotopy classes of isotopies from φ to φ' (form a $\mathbf{Z}$-torsor).

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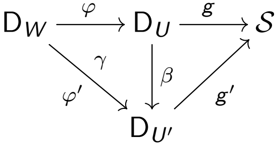
- Thomason's theorem: hocolim computed as the Grothendieck construction F. Ob: $\mathsf{D}_W \xrightarrow{\varphi} \mathsf{D}_U \xrightarrow{g} \mathcal{S}$. Mor $(\varphi, g) \to (\varphi', g')$: $\beta\colon \mathsf{D}_U \to \mathsf{D}_{U'}$: $g = g'\beta$, $\gamma\colon \beta\varphi \to \varphi'$.



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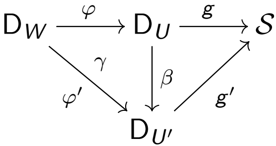

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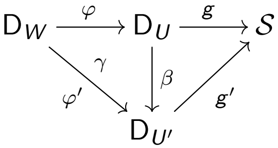

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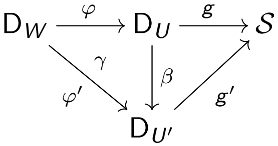
- Grothendieck construction F:



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  - $W \to \mathbf{R}^2$: the displacement of the origin.
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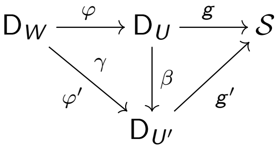

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- Quillen's Theorem A: $*/\pi$ is a directed poset $\Longrightarrow$ weakly contractible nerve
- Theorem: $(\rho_! \mathcal{S})(\mathfrak{D}_W) \simeq \mathrm{BC}^\infty(W, \mathbf{R}^2 \rtimes \widetilde{\mathrm{Conf}}(2))$.

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- Theorem: $(\rho_! \mathcal{S})(\mathfrak{D}_W) \simeq \mathrm{BC}^\infty(W, \mathbf{R}^2 \rtimes \widetilde{\mathsf{Conf}}(2))$.
- Theorem: $\mathbf{R}\,\mathrm{Map}(\mathcal{S}, \mathcal{V}_d^\times) \simeq \mathbf{R}\,\mathrm{Map}(\mathrm{B}(\mathbf{R}^2 \rtimes \widetilde{\mathsf{Conf}}(2)), \iota^* \mathcal{V}_d^\times)$.

Applications (current)

- Consequence of the GCH: smooth invertible FFTs are classified by the smooth Madsen–Tillmann spectrum. (Previous work: Galatius–Madsen–Tillmann–Weiss, Bökstedt–Madsen, Schommer-Pries.)
- The Stolz–Teichner conjecture: concordance classes of extended FFTs have a classifying space. (Proof: Locality + the smooth Oka principle (Berwick-Evans–Boavida de Brito–P.).
- Construction of power operations on the level of FFTs (extending Barthel–Berwick-Evans–Stapleton).
- (Grady) The Freed–Hopkins conjecture (Conjecture 8.37 in *Reflection positivity and invertible topological phases*)

Applications (ongoing)

- Construction of prequantum FFTs from geometric/topological data. Differential characteristic classes as FFTs. (cf. Berthomieu 2008; Bunke–Schick 2010; Bunke 2010).
- Atiyah–Singer index invariants (index, η-invariant, determinant line, index gerbe) as a fully extended FFT (cf. Bunke 2002; Hopkins–Singer 2002; Bunke–Schick 2007).
- Quantization of functorial field theories. Examples: 2d Yang–Mills.

Example: the prequantum Chern–Simons theory (1)

Input data:

- G: a Lie group;
- $\mathcal{S} = \mathsf{B}_\nabla G$ (fiberwise principal G-bundles with connection);
- $\mathcal{V} = \mathsf{B}^3\mathrm{U}(1)$ (a single k-morphism for $k < 3$; 3-morphisms are $\mathrm{U}(1)$ as a Lie group).

Output data: a fully extended 3-dimensional G-gauged FFT:

$$\mathfrak{Bord}_3^{\mathsf{B}_\nabla G} \to \mathsf{B}^3\mathrm{U}(1).$$

- Closed 3-manifold $M \mapsto$ the Chern–Simons action of M;
- Closed 2-manifold $B \mapsto$ the prequantum line bundle of B;
- Closed 1-manifold $C \mapsto$ the Wess–Zumino–Witten gerbe (B-field) of C (Carey–Johnson–Murray–Stevenson–Wang);
- Point $\mapsto$ the Chern–Simons 2-gerbe (Waldorf).

Example: the prequantum Chern–Simons theory (2)

Step 1 Compute $\mathcal{V}_3^\times = (\mathrm{B}^3\mathrm{U}(1))_3^\times$.

Step 1a W is the fiberwise Deligne complex of $T \to U$:

$$W(T \to U) = \Omega^3 \leftarrow \Omega^2 \leftarrow \Omega^1 \leftarrow \mathrm{C}^\infty(T, \mathrm{U}(1)).$$

Step 1b $W \to \mathcal{V}_3^\times$: a fiberwise 3-form ω on $T \to U$
$\mapsto$ framed FFT: 3-bordism $B \mapsto \exp(\int_B \omega)$.

Step 1c The composition

$$W(T \to U) \to \mathcal{V}_3^\times(T \to U) \to \mathcal{V}^\times(U) = \mathrm{B}^3\mathrm{C}^\infty_{\mathsf{fconst}}(T, \mathrm{U}(1))$$

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Step 2 Construct a point in

$$\begin{aligned} &\mathbf{R}\,\mathrm{Map}(\mathrm{B}_\nabla G, W) \\ &= \mathbf{R}\,\mathrm{Map}(\Omega^1(-, \mathfrak{g})//\mathrm{C}^\infty(-, G), \mathrm{B}^3\mathrm{C}^\infty_{\mathsf{fconst}}(-, \mathrm{U}(1))). \end{aligned}$$

(Brylinski–McLaughlin 1996, Fiorenza–Sati–Schreiber 2013)

Step 2′ Even better: can compute the whole space $\mathbf{R}\,\mathrm{Map}(\mathrm{B}_\nabla G, W)$.

Example: the prequantum Chern–Simons theory (2)

Step 1 Result: $\mathcal{V}_3^\times = (\mathrm{B}^3\mathrm{U}(1))_3^\times = \mathrm{B}^3\mathrm{C}^\infty_{\mathsf{fconst}}(-, \mathrm{U}(1))$.

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Quantization of functorial field theories

X: the prequantum geometric structure
Y: the quantum geometric structure (e.g., a point)

$$\begin{array}{ccc} \mathsf{FFT}_{d,\mathcal{V}}(X) & \xrightarrow[\simeq]{\mathsf{GCH}} & \mathbf{R}\,\mathrm{Map}(X, \mathcal{V}_d^\times) \\ \Big\downarrow \int & & \Big\downarrow Q \\ \mathsf{FFT}_{d,\mathcal{V}}(Y) & \xrightarrow[\mathsf{GCH}]{\simeq} & \mathbf{R}\,\mathrm{Map}(Y, \mathcal{V}_d^\times) \end{array}$$

$d = 1$: recover the Spin^c geometric quantization when X is a smooth manifold, $Y = \mathrm{Riem}_{1|1}$, $\mathcal{V} =$ Fredholm complexes.