Jones index as a bicategorical trace.

Dmitri Pavlov.
Mathematical Institute, University of Münster.
dmitri.pavlov@uni-muenster.de.

Abstract.

We construct a shadow (in the sense of Ponto and Shulman) on the bicategory of von Neumann algebras, dualizable bimodules, and their morphisms, which allows us to compute traces of endomorphisms of dualizable bimodules. The trace of the identity endomorphism gives a new invariant of dualizable bimodules, which coincides with the Jones index in the case of type II_1 factors.

The category of von Neumann algebras.

Denote by Ban the category of complex Banach spaces and contractive linear maps (i.e., linear maps that do not increase the norm).

Definition. A *von Neumann algebra* A is a C*-algebra that admits a predual, i.e., a Banach space Z such that Z^* is isomorphic to the underlying Banach space of A in the category Ban.

Definition. A *morphism* $f\colon A\to B$ of von Neumann algebras is a morphism of the underlying C*-algebras that admits a predual, i.e., a morphism of Banach spaces $p\colon Z\to Y$ such that $p^*\colon Y^*\to Z^*$ is isomorphic to the underlying morphism of Banach spaces of f in the category of morphisms of Ban.

Notation. Denote by W^* the category of von Neumann algebras. Denote by $\mathrm{W^*C^*}\colon \mathrm{W}^*\to\mathrm{C}^*$ the faithful forgetful functor from the category of von Neumann algebras to the category of C*-algebras.

Theorem. For any von Neumann algebra A all preduals of A induce the same weak topology on A, which we call the *ultraweak topology.* In particular, the predual is unique up to unique isomorphism and is canonically isomorphic to the dual of A in the ultraweak topology.

Proof. See Corollary 1.13.3 in Sakai [3].

Corollary. The predual of a morphism $f\colon A\to B$ of von Neumann algebras is also unique up to unique isomorphism and is canonically isomorphic to the dual of f in the ultraweak topology.

Corollary. The functor $\mathrm{W^*C^*}$ reflects isomorphisms.

Remark. As explained in the first part of this thesis, the predual of A is canonically isomorphic to $\mathrm{L}_1(A)$.

Notation. Denote by $\mathrm{L}_1\colon \mathrm{W}^{*\mathrm{op}}\to\mathrm{Ban}$ the functor that sends a von Neumann algebra A to its predual, i.e., the dual of A in the ultraweak topology, and likewise for morphisms.

Remark. The predual possesses additional algebraic structures that come from the respective algebraic structures on the original von Neumann algebra A. More precisely, the predual of the unit $\mathbf{k}\to A$ is the Haagerup trace $\mathrm{tr}\colon A_*\to\mathbf{k}$, the predual of the involution $*\colon A\to A$ is the modular conjugation $*\colon A_*\to A_*$, and the predual of the multiplication $A\otimes_{\sigma\mathrm{h}}A\to A$ is the comultiplication $A_*\to A_*\otimes_{\mathrm{eh}}A_*$, where $\otimes_{\sigma\mathrm{h}}$ is the normal Haagerup tensor product and $\otimes_{\mathrm{eh}}$ is the extended Haagerup tensor product. See Section 2 in Effros and Ruan [4] for a discussion of these tensor products. If we refine the codomain of L_1 to the category of involutive comonoids in the category of operator spaces with the involutive monoidal structure coming from the extended Haagerup tensor product, then the resulting functor is fully faithful, in particular it becomes an equivalence of categories if we restrict its codomain to its essential image.

Theorem. The category W^* is complete and the forgetful functor $\mathrm{W^*C^*}\colon\mathrm{W}^*\to\mathrm{C}^*$ preserves and reflects limits.

Proof. Suppose $D\colon I\to\mathrm{W}^*$ is a small diagram. Denote by A the limit of $\mathrm{W^*C^*}D$. Denote by Z the colimit of L_1D. From the construction of limits in C^* it follows that A is a von Neumann algebra and Z is its predual. Now for any cone from B to D we have a canonical morphism $A_*\to B_*$. Dualizing this morphism we obtain a morphism of von Neumann algebras and their corresponding cones from B to A. This morphism is unique because A is the limit of $\mathrm{W^*C^*}D$. Since W^* is complete and $\mathrm{W^*C^*}$ preserves limits and reflects isomorphisms, $\mathrm{W^*C^*}$ also reflects limits.

We now summarize the categorical properties of bimodules over von Neumann algebras.

Theorem. The category of von Neumann algebras and their isomorphisms together with the category of right L_0-bimodules and their morphisms forms a framed double category in the sense of Shulman [5].

Proof. Brouwer wrote up a full account of the bicategory of von Neumann algebras in [6]. By Theorem 4.1 in Shulman [5] the additional structure of a framed double category is given by assigning to every isomorphism of von Neumann algebras $A \to B$ the A-B-bimodule A and the B-A-bimodule B together with the corresponding morphisms of bimodules. These are given by the identities and the isomorphism $A \to B$, which satisfy the relevant identities for trivial reasons.

The categorical tensor product of von Neumann algebras.

The categorical tensor product was constructed in 1966 by Guichardet [2]. Unlike the spatial tensor product it has a nice universal property, which allows us to construct various kinds of bimodules below.

Definition. Suppose that $f: A \to C$ and $g: B \to C$ are morphisms of von Neumann algebras. We say that f and g *commute* if for all $a \in A$ and $b \in B$ we have $f(a)g(b) = g(b)f(a)$.

Theorem. For any von Neumann algebras A and B the following functor $T: \mathrm{W}^* \to \mathrm{Set}$ is representable: T sends a von Neumann algebra E to the set of all pairs of commuting morphisms $f: A \to E$ and $g: B \to E$. It sends a morphism $h: E \to F$ of von Neumann algebras to the morphism of sets that sends a pair (f, g) to (hg, hg). We denote the representing object by $A \otimes B$ and call it the *categorical tensor product* of A and B.

Proof. First we prove that the functor T preserves limits. Consider a small diagram $D: I \to \mathrm{W}^*$ together with its limit E and the corresponding cone. To prove the universal property of limit for the image of this cone under T it is enough to consider one-element sets. A cone from a one-element set to TD is a compatible system of pairs of commuting morphisms, which can be interpreted as a pair of cones from A respectively B to D. By the universal property of limit we obtain a pair of morphisms from A respectively B to E. This pair commutes because composing this pair with all possible projections from the limit to individual objects of the diagram D gives a commuting pair of morphisms.

Remark. By the universal property of the categorical tensor product we have a canonical epimorphism $A \otimes B \to A \bar{\otimes} B$ for any von Neumann algebras A and B, where $\bar{\otimes}$ is the spatial tensor product of von Neumann algebras. (The universal map is an epimorphism because the spatial tensor product is generated by the images of A and B.) This epimorphism is an isomorphism if and only if A or B is type I with atomic center. See Lemme 8.2, Proposition 8.5, and Proposition 8.6 in Guichardet [2].

Remark. In the notation of the above theorem the element of $T(A \otimes B)$ corresponding to the identity morphism of $A \otimes B$ gives us a canonical pair of commuting morphisms $i_{A,B}: A \to A \otimes B$ and $j_{A,B}: B \to A \otimes B$. Precomposing a morphism $A \otimes B \to C$ with $i_{A,B}$ or $j_{A,B}$ allows one to extract the underlying pair of commuting morphisms $A \to C$ and $B \to C$.

Remark. The universal property of the categorical tensor product implies that the categorical tensor product is generated by the images of $i_{A,B}$ and $j_{A,B}$.

Now we embed the categorical tensor product into a symmetric monoidal structure on the category of von Neumann algebras. We identify the relevant data and properties in the following list:

- If $f: A \to E$ and $g: B \to F$ are morphisms of von Neumann algebras, then the commuting pair of morphisms $(i_{E,F} f, j_{E,F} g)$ defines a morphism $f \otimes g: A \otimes B \to E \otimes F$.
- By the universal property of the categorical tensor product for any von Neumann algebras A and B we have $\mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A \otimes B}$, because both morphisms come from the commuting pair of morphisms $(i_{A,B}, j_{A,B})$. Likewise, if $e: A \to B$, $f: B \to C$, $g: E \to F$, and $h: F \to G$ are morphisms of von Neumann algebras, then $fe \otimes hg = (f \otimes h)(e \otimes g)$, because both morphisms come from the commuting pair of morphisms $(i_{C,G} fe, j_{C,G} hg)$.
- The field of scalars $\mathbf{k}$ is the monoidal unit: For any von Neumann algebra A we have canonical isomorphisms $\lambda_A: \mathbf{k} \otimes A \to A$ and $\rho_A: A \otimes \mathbf{k} \to A$ (left and right unitor respectively). These isomorphisms are functorial: $\lambda_B(\mathrm{id}_{\mathbf{k}} \otimes f) = f \lambda_A$ and $\rho_B(f \otimes \mathrm{id}_{\mathbf{k}}) = f \rho_A$ for any morphism of von Neumann algebras $f: A \to B$.

- If A, B, and C are von Neumann algebras, then the commuting pair of morphisms $(k, j_{A,B\otimes C}j_{B,C})$ defines a canonical isomorphism (the associator) $\alpha_{A,B,C}\colon (A\otimes B)\otimes C\to A\otimes(B\otimes C)$, whose inverse is constructed in a similar way. Here $k\colon A\otimes B\to A\otimes(B\otimes C)$ is the morphism defined by the commuting pair $(i_{A,B\otimes C}, j_{A,B\otimes C}i_{B,C})$.
- The associator is functorial: For any morphisms of von Neumann algebras $f\colon A\to E$, $g\colon B\to F$, and $h\colon C\to G$ we have $\alpha_{E,F,G}((f\otimes g)\otimes h) = (f\otimes(g\otimes h))\alpha_{A,B,C}$, because the underlying triple of commuting morphisms is in both cases $(i_{E,F\otimes G}f, j_{E,F\otimes G}i_{F,G}g, j_{E,F\otimes G}j_{F,G}h)$.
- Unitors are compatible with the associator: For any von Neumann algebras A and B we have $(\mathrm{id}_A\otimes\lambda_B)\alpha_{A,\mathbf{k},B} = \rho_A\otimes\mathrm{id}_B$, because the underlying triple of commuting morphisms is in both cases $(i_{A,B}, \mathbf{k}\to A\otimes B, j_{A,B})$.
- The associator satisfies the pentagon identity: For any von Neumann algebras A, B, E, and F we have $\alpha_{A,B,E\otimes F}\alpha_{A\otimes B,E,F} = (\mathrm{id}_A\otimes\alpha_{B,E,F})\alpha_{A,B\otimes E,F}\alpha_{A,B,E\otimes F}$, because the underlying 4-tuple of commuting morphisms is in both cases $(i_{A,B\otimes(E\otimes F)}, j_{A,B\otimes(E\otimes F)}i_{B,E\otimes F}, j_{A,B\otimes(E\otimes F)}j_{B,E\otimes F}i_{E,F}, j_{A,B\otimes(E\otimes F)}j_{B,E\otimes F}j_{E,F})$.
- If A and B are two von Neumann algebras, then the commuting pair of morphisms $(j_{B,A}, i_{B,A})$ defines a canonical morphism (the braiding) $\gamma_{A,B}\colon A\otimes B\to B\otimes A$.
- The braiding is functorial: For any morphisms of von Neumann algebras $f\colon A\to E$ and $g\colon B\to F$ we have $(g\otimes f)\gamma_{A,B} = \gamma_{E,F}(f\otimes g)$, because the underlying pair of morphisms is in both cases $(j_{F,E}f, i_{F,E}g)$.
- The braiding is symmetric: $\gamma_{B,A}\gamma_{A,B} = \mathrm{id}_{A\otimes B}$, because both morphisms are represented by the commuting pair $(i_{A,B}, j_{A,B})$.
- The braiding satisfies the hexagon identities: $(\mathrm{id}_B\otimes\gamma_{A,C})\alpha_{B,A,C}(\gamma_{A,B}\otimes\mathrm{id}_C) = \alpha_{B,C,A}\gamma_{A,B\otimes C}\alpha_{A,B,C}$, because both morphisms are represented by the commuting triple $(j_{B,C\otimes A}j_{C,A}, i_{B,C\otimes A}, j_{B,C\otimes A}i_{C,A})$. The other hexagon identity follows automatically because the braiding is symmetric.

The above constructions and proofs can be summarized as follows:

Theorem. The category of von Neumann algebras equipped with the additional structures described above is a symmetric monoidal category.

The categorical external tensor product of W*-modules and W*-bimodules.

Recall the following universal property of the W*-category of W*-modules over a von Neumann algebra (see Theorem 7.13 in Ghez, Lima, and Roberts [1]):

Theorem. The W*-category of W*-modules over a von Neumann algebra M is a free W*-category with direct sums and sufficient subobjects on one object with endomorphism algebra M. More precisely, consider the W*-category $*_M$ with one object whose endomorphism algebra is M. Embed this category into the W*-category Mod_M of W*-modules over M by sending the only object to M_M and the endomorphisms of this object to the left action of M on M_M. Suppose C is a W*-category with direct sums and sufficient subobjects. The restriction W*-functor from the W*-category of W*-functors from Mod_M to C to the W*-category of W*-functors from $*_M$ to C given by the embedding $*_M\to\mathrm{Mod}_M$ is a W*-equivalence of W*-categories.

The categorical external tensor product of W*-modules will be a W*-functor $\mathrm{Mod}_M\otimes\mathrm{Mod}_N\to\mathrm{Mod}_{M\otimes N}$. First we extend the categorical tensor product of von Neumann algebras (i.e., W*-categories with one object) to arbitrary W*-categories.

Definition. Suppose C and D are W*-categories. The categorical tensor product $C\otimes D$ is defined as follows: Objects of $C\otimes D$ are pairs of objects of C and D. We denote the object corresponding to the pair (c,d) as $c\otimes d$. If $a\otimes b$ and $c\otimes d$ are two objects in $C\otimes D$, then $\mathrm{Hom}(a\otimes b, c\otimes d) = \mathrm{Hom}(a,c)\otimes\mathrm{Hom}(b,d)$. Here we use the categorical tensor product of corners of von Neumann algebras, which is defined as follows. Consider the von Neumann algebra M of 2×2-matrixes with entries in $\mathrm{End}(a)$, $\mathrm{Hom}(a,c)$, $\mathrm{Hom}(c,a)$, and $\mathrm{End}(c)$. We have $\mathrm{Hom}(a,c) = pMq$ for some projections p and q in M such that $p+q=1$. Do the same trick with $\mathrm{Hom}(b,d) = rNs$. Now $\mathrm{Hom}(a,c)\otimes\mathrm{Hom}(b,d) := (p\otimes r)(M\otimes N)(q\otimes s)$. The resulting category is a W*-category because the algebra of 2×2-matrixes corresponding to every pair of objects is a W*-algebra. More precisely, in the above notation it is the algebra $(p\otimes r+q\otimes s)(M\otimes N)(p\otimes r+q\otimes s)$.

Proposition. If C is a W*-category with a generator A and D is a W*-category with a generator B, then $A \otimes B$ is a generator of $C \otimes D$.

Proof. By Proposition 7.3 in Ghez, Lima, and Roberts [1] an object E of a W*-category is a generator if and only for any object F there is a family of partial isometries $r\colon K \to \mathrm{Hom}(F,E)$ such that $\mathrm{id}_F = \sum(k \in K \mapsto r_k^* r_k \in \mathrm{End}(F))$. Consider an arbitrary object $X \otimes Y$ of $C \otimes D$. Choose families of partial isometries $p\colon I \to \mathrm{Hom}(X,A)$ and $q\colon J \to \mathrm{Hom}(Y,B)$ that satisfy the above property. Then the family $p \otimes q\colon (i,j) \in I \times J \mapsto p_i \otimes q_j \in \mathrm{Hom}(X \otimes Y, A \otimes B)$ satisfies $\sum(i,j) \in I \times J \mapsto (p_i \otimes q_j)^*(p_i \otimes q_j) \in \mathrm{End}(X \otimes Y) = \sum(i,j) \in I \times J \mapsto (p_i^* p_i \otimes q_j^* q_j) \in \mathrm{End}(X \otimes Y) = (\sum i \in I \mapsto p_i^* p_i \in \mathrm{End}(X)) \otimes \left(\sum j \in J \mapsto q_j^* q_j \in \mathrm{End}(Y)\right) = \mathrm{id}_X \otimes \mathrm{id}_Y = \mathrm{id}_{X \otimes Y}$.

Theorem. For any two von Neumann algebras M and N there is a unique W*-functor $\odot\colon \mathrm{Mod}_M \otimes \mathrm{Mod}_N \to \mathrm{Mod}_{M \otimes N}$ that sends $M_M \odot N_N$ to $(M \otimes N)_{M \otimes N}$ and maps the endomorphism algebra of $M_M \odot N_N$ isomorphically to $M \otimes N$. We call it the *categorical external tensor product functor*.

Proof. The W*-category $\mathrm{Mod}_M \otimes \mathrm{Mod}_N$ has a generator $M_M \odot N_N$ and the W*-category $\mathrm{Mod}_{M \otimes N}$ has direct sums and sufficient subobjects, hence the conditions of the above theorem are satisfied and there is an equivalence of W*-categories $\mathrm{Hom}(\mathrm{Mod}_M \otimes \mathrm{Mod}_N, \mathrm{Mod}_{M \otimes N}) \to \mathrm{Hom}(M_M \odot N_N, \mathrm{Mod}_{M \otimes N})$ given by the restriction to the full subcategory of $\mathrm{Mod}_M \otimes \mathrm{Mod}_N$ consisting of $M_M \odot N_N$.

Theorem. The above functor extends to bimodules as follows:

$$\odot\colon {}_K\mathrm{Bimod}_M \otimes {}_L\mathrm{Bimod}_N \to {}_{K \otimes L}\mathrm{Bimod}_{M \otimes N}.$$

Proof. An object of ${}_K\mathrm{Bimod}_M$ is an object X of Mod_M equipped with a morphism $K \to \mathrm{End}(X)$. For ${}_L\mathrm{Bimod}_N$ we have a similar pair $(Y, L \to \mathrm{End}(Y))$. We define $(X, K \to \mathrm{End}(X)) \odot (Y, L \to \mathrm{End}(Y)) = (X \odot Y, K \otimes L \to \mathrm{End}(X) \otimes \mathrm{End}(Y) = \mathrm{End}(X \odot Y))$. A morphism of bimodules is a morphism of the underlying right modules that commutes with the left action. The external tensor product of morphisms of the underlying right M-modules and N-modules commutes with the left action of $K \otimes L$, hence we get a functor $\odot\colon {}_K\mathrm{Bimod}_M \otimes {}_L\mathrm{Bimod}_N \to {}_{K \otimes L}\mathrm{Bimod}_{M \otimes N}$.

Remark. The external tensor product of two invertible bimodules is again an invertible bimodule.

Definition. Denote by Bimod the category whose objects are triples (M, N, X), where M and N are von Neumann algebras and X is an M-N-bimodule. Consider two arbitrary objects ${}_K X_M$ and ${}_L Y_N$. A morphism from X to Y is a triple (f, g, h), where $f\colon K \to L$ and $g\colon M \to N$ are isomorphisms of von Neumann algebras, and h is a morphism of K-M-bimodules from X to ${}_f Y_g$, where ${}_f Y_g$ denotes Y with the left action composed with f, the right action composed with g, and the inner product composed with g^{-1}. We extend the categorical external tensor product to Bimod.

Now we embed the categorical external tensor product into a symmetric monoidal structure on the category Bimod of bimodules over von Neumann algebras. For the sake of brevity we often establish necessary properties for the case of modules, automatically extending them to bimodules. We also work with morphisms of bimodules over the same pair of algebras, automatically extending them to other morphisms. We identify the relevant data and properties in the following list:

- The identity bimodule $\mathrm{id}_{\mathbf{k}}$ over the field of scalars $\mathbf{k}$ is the monoidal unit: For any bimodule X we have canonical isomorphisms $\lambda_X\colon \mathrm{id}_{\mathbf{k}} \odot X \to X$ and $\rho_X\colon X \odot \mathbf{k} \to X$ (left and right unitor respectively). These isomorphisms are functorial: $\lambda_Y(\mathrm{id}_{\mathbf{k}} \odot f) = f\lambda_X$ and $\rho_Y(f \odot \mathrm{id}_{\mathbf{k}}) = f\rho_X$ for any morphism of bimodules $f\colon X \to Y$.
- The associator is the unique natural isomorphism that satisfies the identity $(L_L \odot M_M) \odot N_N = L_L \odot (M_M \odot N_N)$. The inverse associator is constructed in a similar way.
- The associator is functorial: For any morphisms of bimodules $f\colon U \to X$, $g\colon V \to Y$, and $h\colon W \to Z$ we have $\alpha_{X,Y,Z}((f \odot g) \odot h) = (f \odot (g \odot h))\alpha_{U,V,W}$, because $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ as an element of $\mathrm{Hom}(U,X) \otimes \mathrm{Hom}(V,Y) \otimes \mathrm{Hom}(W,Z) = \mathrm{Hom}(U \odot V \odot W, X \odot Y \odot Z)$.
- Unitors are compatible with the associator: For any bimodules X and Y we have $(\mathrm{id}_X \odot \lambda_Y)\alpha_{X,\mathbf{k},Y} = \rho_X \odot \mathrm{id}_Y$.
- The associator satisfies the pentagon identity: For any bimodules W, X, Y, and Z we have $\alpha_{W,X,Y \odot Z}\,\alpha_{W \odot X,Y,Z} = (\mathrm{id}_W \odot \alpha_{X,Y,Z})\alpha_{W,X \odot Y,Z}\alpha_{W,X,Y \odot Z}$.

- The braiding γ is defined as the unique morphism of the corresponding functors that sends $M_M \odot N_N$ to $N_N \odot M_M$.
- The braiding is functorial: For any morphisms of bimodules $f: U \to X$ and $g: V \to Y$ we have $(g \odot f)\gamma_{U,V} = \gamma_{X,Y}(f \odot g)$.
- The braiding is symmetric: $\gamma_{Y,X}\gamma_{X,Y} = \mathrm{id}_{X\odot Y}$.
- The braiding satisfies the hexagon identities: $(\mathrm{id}_Y \odot \gamma_{X,Z})\alpha_{Y,X,Z}(\gamma_{X,Y} \odot \mathrm{id}_Z) = \alpha_{Y,Z,X}\gamma_{X,Y\odot Z}\alpha_{X,Y,Z}$. The other hexagon identity follows automatically because the braiding is symmetric.

The above constructions and proofs can be summarized as follows:

Theorem. The category Bimod has a symmetric monoidal structure given by the structures defined above.

Symmetric monoidal framed double category of von Neumann algebras.

Theorem. The symmetric monoidal category $\hat{\mathrm{W}}^*$ of von Neumann algebras and their isomorphisms together with the symmetric monoidal category of W*-bimodules and their morphisms forms a (strong) symmetric monoidal framed double category.

Proof. We only have to show that the source, target, identity, and composition functors are strong symmetric monoidal functors and both unitors and the associator are symmetric monoidal natural isomorphisms. The source and target functors are symmetric monoidal by definition. The identity functor is symmetric monoidal because $\mathrm{id}_M \odot \mathrm{id}_N = \mathrm{id}_{M\otimes N}$. Finally, the composition functor is symmetric monoidal because the categorical tensor product of von Neumann algebras is a functor.

Shadows and refinement of the Jones index.

In this section it is more convenient to use density 1/2 bimodules instead of W*-bimodules (density 0 bimodules). Equivalently, we can talk about correspondences of von Neumann algebras, i.e., pairs of commuting representations on a Hilbert space. Recall that these two categories are equivalent via the algebraic tensor product and the algebraic inner hom with $\mathrm{L}_{1/2}$ as explained in the first part of this thesis.

Recall that W^* denotes the category of von Neumann algebras and their morphisms, whereas $\hat{\mathrm{W}}^*$ denotes the bicategory of von Neumann algebras, bimodules, and intertwiners. Dualizable bimodules (in the categorical sense of dualizable 1-morphisms) in $\hat{\mathrm{W}}^*$ are precisely finite index bimodules, see for example the paper by Bartels, Douglas, and Henriques [11] for the case of von Neumann algebras with finite-dimensional centers. We refer the reader to the paper by Ponto and Shulman [8] for the general theory of shadows and traces in bicategories, which we take here for granted.

Theorem. Suppose M is a von Neumann algebra and X is a dualizable M-M-bimodule. Then the map $X \mapsto \mathrm{L}_{1/2}({}_M\operatorname{Hom}_M(\mathrm{id}_M, X))$ defines a shadow on the bicategory of von Neumann algebras, dualizable bimodules, and their morphisms with values in the category of Hilbert spaces, with the cyclic isomorphism supplied by Frobenius reciprocity. Recall that ${}_M\operatorname{Hom}_M(\mathrm{id}_M, X)$ is a corner of the von Neumann algebra ${}_M\operatorname{End}_M(\mathrm{id}_M \oplus X)$, hence we can talk about its $\mathrm{L}_{1/2}$-space.

Proof. Suppose M and N are von Neumann algebras, X is a dualizable M-N-bimodule, Y is a dualizable N-M-bimodule. The Frobenius reciprocity (see Theorem 23 in Yamagami [10]) immediately yields a canonical cyclic isomorphism: ${}_M\operatorname{Hom}_M(\mathrm{id}_M, X_N \otimes_N Y) = {}_N\operatorname{Hom}_M(X^*, Y) = {}_N\operatorname{Hom}_N(\mathrm{id}_N, Y_M \otimes_M X)$. Associativity and unitality follow from the proof of the theorem cited above.

Theorem. The shadow of the identity bimodule over a von Neumann algebra M is canonically isomorphic to $\mathrm{L}_{1/2}(\mathrm{Z}(M))$ as a $\mathbf{C}$-$\mathbf{C}$-bimodule.

Proof. We have $\mathrm{L}_{1/2\,M}\operatorname{Hom}_M(\mathrm{id}_M, \mathrm{id}_M) = \mathrm{L}_{1/2\,M}\operatorname{End}_M(\mathrm{id}_M) = \mathrm{L}_{1/2}(\mathrm{Z}(M))$, because the endomorphism algebra of id_M is $\mathrm{Z}(M)$.

Remark. All definitions depend only on the underlying bicategory of von Neumann algebras, hence the end result is independent of the choice of a particular model of bimodules. In particular, the space $\mathrm{L}_{1/2}(\mathrm{Z}(M))$

pops up even in the case of density 0 bimodules, because the result has to be a **C**-**C**-bimodule, i.e., a Hilbert space.

Remark. If M is a type II_1 factor, then the shadow of a dualizable M-M-bimodule X is canonically isomorphic to the **C**-**C**-bimodule $X/[X, M]$ (here X must have density 1/2 and $[X, M]$ denotes the closure of the linear span of all commutators of the form $xm - mx$). Connes in [9] showed that every dualizable M-M-bimodule X (of density 1/2) canonically decomposes in a direct sum $\mathrm{Z}(X) \oplus [X, M]$, where $\mathrm{Z}(X)$ is the set of all elements in X such that for all $m \in M$ we have $xm = mx$. Thus the shadow of X is canonically isomorphic to $\mathrm{Z}(X)$ as a **C**-**C**-bimodule, i.e., a Hilbert space.

Theorem. Suppose M is a type II_1 factor and N is a finite index subfactor of M. Denote by X the associated M-N-bimodule, which is $\mathrm{L}_{1/2}(M)$ equipped with the standard left action of M and the right action of N coming from the inclusion of N into M. The the trace of the identity endomorphism of X is equal to the Jones index of X.

Proof. We use the language of density 0 bimodules. Identify (the density 0 counterpart of) X with M as an M-N-bimodule, where the right inner product is given by the canonical conditional expectation associated to the morphism $N \to M$. Choose a Pimsner-Popa basis R for X. The trace of id_X is the composition of the shadow of the coevaluation map of X, the cyclic morphism, and the shadow of the evaluation map. We identify shadows of bimodules with their central elements. The coevaluation map sends $1 \in \mathrm{id}_N$ to $\sum_{r \in R} r^* \otimes r$. The cyclic morphism sends this element to $\sum_{r \in R} r \otimes r^*$ and the evaluation maps sends it to $\sum_{r \in R} rr^*$, which is the Jones index of the inclusion $N \to M$. See Théorème 3.5 in Baillet, Denizeau, and Havet [7] for the relevant facts about index.

Acknowledgments.

I thank André Henriques for suggesting that there should be a refinement of the Jones index for von Neumann algebras with non-trivial center, numerous fruitful discussions about von Neumann algebras, and useful feedback on the contents of this paper, Stephan Stolz and Ryan Grady for discussions about traces and shadows, Michael Hartglass and David Penneys for various discussions about subfactors, Vaughan Jones for helpful comments about the contents of this paper, Arthur Bartels for pointing out an incorrect statement in an early version of this paper, and last and most importantly my advisor Peter Teichner for introducing me to this area, conducting numerous invaluable discussions, and supporting me through the course of my research.

References.

[1] P. Ghez, Ricardo Lima, John Roberts. W*-categories. Pacific Journal of Mathematics 120:1 (1985), 79–109.

[2] Alain Guichardet. Sur la catégorie des algèbres de von Neumann. Bulletin des Sciences Mathématiques [2e Série] 90 (1966), 41–64. Available at http://math.berkeley.edu/∼pavlov/scans/guichardet.pdf.

[3] Shôichirô Sakai. C*-algebras and W*-algebras. Springer, 1971.

[4] Edward G. Effros, Zhong-Jin Ruan. Operator space tensor products and Hopf convolution algebras. Journal of Operator Theory 50 (2003), 131–156.

[5] Michael Shulman. Framed bicategories and monoidal fibrations. Theory and Applications of Categories 20:18 (2008), 650–738. arXiv:0706.1286v2 [math.CT].

[6] Rachel Brouwer. A bicategorical approach to Morita equivalence for von Neumann algebras. arXiv: math/0301353v1 [math.OA]. Abridged version appeared in Journal of Mathematical Physics 44:5 (2003), 2206-2214.

[7] Michel Baillet, Yves Denizeau, Jean-François Havet. Indice d'une espérance conditionelle. Compositio Mathematica 66:2 (1988), 199–236.

[8] Kate Ponto, Michael Shulman. Shadows and traces in bicategories. arXiv:0910.1306v2 [math.CT].

[9] Alain Connes. Correspondences. Manuscript, 1980.

[10] Shigeru Yamagami. Modular theory for bimodules. Journal of Functional Analysis 125:2 (1994), 327–357.

[11] Arthur Bartels, Christopher L. Douglas, André Henriques. Dualizability and index of subfactors. arXiv:1110.5671v1 [math.OA].