Shadows and traces in the bicategory of von Neumann algebras

Dmitri Pavlov

Department of Mathematics, University of California, Berkeley

https://dmitripavlov.org/

Abstract. We construct a shadow (in the sense of Ponto and Shulman) on the bicategory of von Neumann algebras, dualizable bimodules, and their morphisms, which allows us to compute traces of endomorphisms of dualizable bimodules. The trace of the identity endomorphism gives a new invariant of dualizable bimodules, which coincides with the Jones index in the case of type II_1 factors.

1 Shadows and refinement of the Jones index

In this section it is more convenient to use density $1/2$ bimodules instead of W*-bimodules (density 0 bimodules). Equivalently, we can talk about correspondences of von Neumann algebras, i.e., pairs of commuting representations on a Hilbert space. Recall that these two categories are equivalent via the algebraic tensor product and the algebraic inner hom with $L_{1/2}$ as explained in the first part of this thesis.

Recall that W^* denotes the category of von Neumann algebras and their morphisms, whereas $\hat{W}^*$ denotes the bicategory of von Neumann algebras, bimodules, and intertwiners. Dualizable bimodules (in the categorical sense of dualizable 1-morphisms) in $\hat{W}^*$ are precisely finite index bimodules, see for example the paper by Bartels, Douglas, and Henriques [BDH] for the case of von Neumann algebras with finite-dimensional centers. We refer the reader to the paper by Ponto and Shulman [PoSh] for the general theory of shadows and traces in bicategories, which we take here for granted.

Theorem 1.1. Suppose M is a von Neumann algebra and X is a dualizable M-M-bimodule. Then the map $X \mapsto L_{1/2}(_M \operatorname{Hom}_M(\operatorname{id}_M, X))$ defines a shadow on the bicategory of von Neumann algebras, dualizable bimodules, and their morphisms with values in the category of Hilbert spaces, with the cyclic isomorphism supplied by Frobenius reciprocity. Recall that $_M \operatorname{Hom}_M(\operatorname{id}_M, X)$ is a corner of the von Neumann algebra $_M \operatorname{End}_M(\operatorname{id}_M \oplus X)$, hence we can talk about its $L_{1/2}$-space.

Proof. Suppose M and N are von Neumann algebras, X is a dualizable M-N-bimodule, Y is a dualizable N-M-bimodule. The Frobenius reciprocity (see Theorem 23 in Yamagami [YaB]) immediately yields a canonical cyclic isomorphism: $_M \operatorname{Hom}_M(\operatorname{id}_M, X_N \otimes {_N}Y) = {_N}\operatorname{Hom}_M(X^*, Y) = {_N}\operatorname{Hom}_N(\operatorname{id}_N, Y_M \otimes {_M}X)$. Associativity and unitality follow from the proof of the theorem cited above. ∎

Theorem 1.2. The shadow of the identity bimodule over a von Neumann algebra M is canonically isomorphic to $L_{1/2}(Z(M))$ as a **C**-**C**-bimodule.

Proof. We have $L_{1/2}{_M}\operatorname{Hom}_M(\operatorname{id}_M, \operatorname{id}_M) = L_{1/2}{_M}\operatorname{End}_M(\operatorname{id}_M) = L_{1/2}(Z(M))$, because the endomorphism algebra of id_M is $Z(M)$. ∎

Remark 1.3. All definitions depend only on the underlying bicategory of von Neumann algebras, hence the end result is independent of the choice of a particular model of bimodules. In particular, the space $L_{1/2}(Z(M))$ pops up even in the case of density 0 bimodules, because the result has to be a **C**-**C**-bimodule, i.e., a Hilbert space.

Remark 1.4. If M is a type II_1 factor, then the shadow of a dualizable M-M-bimodule X is canonically isomorphic to the **C**-**C**-bimodule $X/[X, M]$ (here X must have density $1/2$ and $[X, M]$ denotes the closure of the linear span of all commutators of the form $xm - mx$). Connes in [CoC] showed that every dualizable M-M-bimodule X (of density $1/2$) canonically decomposes in a direct sum $Z(X) \oplus [X, M]$, where $Z(X)$ is the set of all elements in X such that for all $m \in M$ we have $xm = mx$. Thus the shadow of X is canonically isomorphic to $Z(X)$ as a **C**-**C**-bimodule, i.e., a Hilbert space.

Theorem 1.5. Suppose M is a type II_1 factor and N is a finite index subfactor of M. Denote by X the associated M-N-bimodule, which is $L_{1/2}(M)$ equipped with the standard left action of M and the right action of N coming from the inclusion of N into M. The the trace of the identity endomorphism of X is equal to the Jones index of X.

Proof. We use the language of density 0 bimodules. Identify (the density 0 counterpart of) X with M as an M-N-bimodule, where the right inner product is given by the canonical conditional expectation associated

to the morphism $N \to M$. Choose a Pimsner-Popa basis R for X. The trace of id_X is the composition of the shadow of the coevaluation map of X, the cyclic morphism, and the shadow of the evaluation map. We identify shadows of bimodules with their central elements. The coevaluation map sends $1 \in \mathrm{id}_N$ to $\sum_{r \in R} r^* \otimes r$. The cyclic morphism sends this element to $\sum_{r \in R} r \otimes r^*$ and the evaluation maps sends it to $\sum_{r \in R} rr^*$, which is the Jones index of the inclusion $N \to M$. See Théorème 3.5 in Baillet, Denizeau, and Havet [BaDeHa] for the relevant facts about index. ▌

2 Acknowledgments

3 References

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