PRELIMINARY DRAFT—PLEASE DO NOT REDISTRIBUTE

Differential cohomology via smooth topological field theories.

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1 Notation and terminology.

1.1 The purpose of this section is to fix notation and terminology for some of the notions used in the rest of the paper. It does not attempt to give precise definitions or statements of theorems, for which we refer the reader to the relevant sources.

1.2 The global projective model structure on simplicial (pre)sheaves was introduced by Bousfield and Kan [3, Proposition XI.8.1] and the corresponding local projective model structure on simplicial (pre)sheaves was explored by Beke [6, Example 2.17], Dugger [8], Blander [9]. Beke [7, Remark 3.21] defines model structures on (pre)sheaves of simplicial abelian groups and connective chain complexes.

1.3 In what follows we often have to deal with (presentable) ∞-categories presented by (combinatorial) simplicial model categories. We adopt the following conventions. If X and Y are objects of some category C (possibly with additional structures, such as enrichment over simplicial sets or a model structure), then $\mathrm{Mor}_C(X,Y)$ denotes the set of morphisms (or the corresponding enriched object) from X to Y. If X and Y are objects of some combinatorial simplicial model category C, which presents the corresponding ∞-category D, then $\mathrm{Mor}^\infty_C(X,Y)$ denotes the simplicial set of morphisms from the object presented by X to the object presented by Y in D, which can be computed by cofibrantly replacing X by X', fibrantly replacing Y by Y', and taking the simplicial set $\mathrm{Mor}_C(X',Y')$.

1.4 If X and Y are objects of some closed symmetric monoidal category C (again possibly with additional structures), then $\mathrm{Hom}_C(X,Y)$ denotes the internal hom from X to Y, which itself is an object of C. If X and Y are objects of some closed symmetric monoidal combinatorial simplicial model category C, which presents the corresponding closed symmetric monoidal ∞-category D, then $\mathrm{Hom}^\infty_C(X,Y)$ denotes the internal hom from the object presented by X to the object presented by Y in D, whose representative in C can be computed by cofibrantly replacing X, fibrantly replacing Y, and taking the 1-categorical internal hom in C.

1.5 Denote by sSet the closed symmetric monoidal combinatorial simplicial model category of simplicial sets, where the monoidal structure is given by the categorical product and the simplicial model structure is the standard Quillen structure. Of course, sSet presents the closed symmetric monoidal ∞-category of ∞-groupoids (e.g., Kan complexes) with the ∞-categorical product. We also consider the closed symmetric monoidal combinatorial simplicial model category sSet_* of pointed simplicial sets, where the monoidal structure is given by the smash product (i.e., the left adjoint to the (pointed) internal hom), the simplicial model structure is lifted from sSet (and restricting to pointed maps).

1.6 Denote by sAb the closed symmetric monoidal combinatorial simplicial model category of simplicial abelian groups (we can also think of them as abelian group objects in simplicial sets), with the monoidal structure given by the degreewise tensor product and the simplicial model structure obtained by lifting fibrations and weak equivalences from simplicial sets. Denote by Ch the closed symmetric monoidal combinatorial simplicial model category of nonnegatively graded chain complexes with the monoidal structure given by the tensor product of chain complexes, quasiisomorphisms as weak equivalences, and surjections in nonzero degrees as fibrations. There is an adjoint equivalence between sAb and Ch given by the pair of functors $\mathrm{N}\colon \mathrm{sAb} \to \mathrm{Ch}$ and $\Gamma\colon \mathrm{Ch} \to \mathrm{sAb}$. The functor N sends a simplicial abelian group A to the chain complex $\mathrm{N}A$ whose nth term is the abelian group A_n/DA_n, where DA_n is the abelian subgroup of all degenerate simplices in A_n. The differentials are given by alternating sums of face maps. The functor Γ sends a chain complex A to the simplicial abelian group ΓA such that $\Gamma A_m = \bigoplus_{m\twoheadrightarrow n} A_n$, with the simplicial map corresponding to $l \to m$ and the component A_n indexed by $m \twoheadrightarrow n$ given by taking the composition $l \to m \to n$, factoring it as $l \twoheadrightarrow m' \hookrightarrow n$, and mapping A_n first to $A_{m'}$ and then embedding $A_{m'}$ into ΓA_l as the component indexed by $l \twoheadrightarrow m'$. The map $A_n \to A_{m'}$ is the identity map if $m' \hookrightarrow n$ is an isomorphism. Otherwise it is zero unless the inclusion $m' \hookrightarrow n$ misses only the last element of n, in which case we take the corresponding differential from the chain complex.

1.7 An important property of the adjoint equivalence (N, Γ) is that both functors preserve the entire model structure on the nose (as opposed to Quillen adjoint pairs, where left adjoints preserve cofibrations and acyclic cofibrations and right adjoints preseve fibrations and acyclic fibrations).

1.8 Consider now the category $\mathrm{C}^\infty_{\mathrm{pre}}\mathrm{sSet}$ of presheaves of simplicial sets, which can also be thought of as simplicial objects in the category of presheaves of sets. This category can be equipped with different simplicial model structures that present the ∞-category of ∞-presheaves (valued in ∞-groupoids). We will always use the *global* *projective* simplicial model structure, whose fibrations and weak equivalences are defined componentwise. In particular, fibrant objects are precisely presheaves that are componentwise Kan.

1.9 To obtain the simplicial model category that presents the ∞-category of ∞-sheaves (of ∞-groupoids), we have to localize the above model structure with respect to Čech covers. Here the *Čech cover* associated to a sieve $S\colon U \to X$ (e.g., a sieve that comes from some covering family) is the morphism $F \to X$ of simplicial presheaves, where F is the *Čech nerve* of S, which is defined as the simplicial presheaf whose presheaf of k-simplices is the $(k+1)$-fold product $U \times_X \cdots \times_X U$ and simplicial maps are induced by projections and diagonals. The map $F \to X$ is then induced by the map S. We note in passing that the site Man is hypercomplete, thus using hypercovers does not give us anything new and there is no difference between the Čech projective model structure and the local projective model structure.

1.10 The left Bousfield localization (cofibrations are preserved and we keep those acyclic cofibrations that are local) of the global projective simplicial model structure with respect to Čech covers gives us the *Čech projective* simplicial model structure on simplicial presheaves, which for the case of the site Man coincides with the *local projective* simplicial model structure. The resulting model category is denoted by $\mathrm{C}^\infty\mathrm{sSet}$. Similarly to the 1-categorical case, the localized simplicial model category can be constructed as the full subcategory of the category of simplicial presheaves consisting of objects that satisfy the descent condition, i.e., are local with respect to Čech covers. Such objects can also be described as simplicial objects in the category of sheaves of sets. Fibrant objects in the resulting model category are precisely presheaves valued in Kan complexes that satisfy the descent condition, or alternatively, simplicial objects in the category of sheaves of set that satisfy the Kan condition.

1.11 The Dold—Kan correspondence is applicable to any abelian category, in particular to the abelian categories of presheaves and sheaves of abelian groups. We extend the adjoint equivalence (N, Γ) to these situations. The relevant model structures are still preserved on the nose.

2 Categorical preliminaries.

2.1 Consider the site of smooth manifolds Man, whose covering sieves are generated by jointly surjective families of submersions. Denote by $\mathrm{C}^\infty\mathrm{Set}$ the category of sheaves of sets on Man. Objects of $\mathrm{C}^\infty\mathrm{Set}$ can also be called *smooth sets*. Of course, $\mathrm{C}^\infty\mathrm{Set}$ can also be described in many other equivalent ways. For example, we can replace Man with the site Cart of cartesian spaces, whose objects are dualizable real affine spaces, morphisms are smooth maps, and covering families are defined as above.

2.2 The category $\mathrm{C}^\infty\mathrm{Set}$ is a Grothendieck topos and hence has a canonical topology, whose sieves are jointly epimorphic families of morphisms. The canonical topology restricts to the corresponding topologies on Man and Cart and both of these categories are dense in $\mathrm{C}^\infty\mathrm{Set}$. In particular, every sheaf on $\mathrm{C}^\infty\mathrm{Set}$ is representable.

2.3 Denote by $\mathrm{C}^\infty\mathrm{Ab}$ the category of abelian group objects in $\mathrm{C}^\infty\mathrm{Set}$. Alternatively, we can think of $\mathrm{C}^\infty\mathrm{Ab}$ as the category of sheaves with values in the category of abelian groups. Objects of $\mathrm{C}^\infty\mathrm{Ab}$ are also called *smooth abelian groups*. An important example of a smooth abelian group is given by applying the Yoneda embedding to an ordinary abelian Lie group.

2.4 Of particular importance for us is the exact sequence of smooth abelian groups $0 \to \mathbf{P} \to \mathbf{I} \to \mathbf{U} \to 0$, where $\mathbf{P}$ is the (discrete) group of integer multiples of $2\pi i$, $\mathbf{I}$ is the (additive) group of imaginary numbers (real multiples of i), and $\mathbf{U}$ is the (multiplicative) group of complex numbers with absolute value 1. The map $\mathbf{P} \to \mathbf{I}$ is the inclusion and the map $\mathbf{I} \to \mathbf{U}$ is the exponent.

3 The source and its properties.

3.1 **Definition.** Denote by $\mathbf{\Delta}\colon \Delta \to \mathrm{Man}$ the cosimplicial object in the category of manifolds given by composing the free real affine space functor $\mathbf{A}$ with the forgetful functor from finite-dimensional real affine spaces to manifolds.

3.2 **Remark.** The free affine space on a simplex m can be described as the affine subset of the free vector space on m consisting of points whose coordinates sum to 1.

3.3 **Remark.** The above cosimplicial object is sometimes referred to as "extended (smooth) simplices". In fact, for our purposes any reasonable notion of smooth simplices will do (e.g., the usual smooth simplices with corners), but we stick to extended simplices simply for the sake of convenience and because extended smooth simplices do not have boundary, i.e., are manifolds in our sense.

3.4 **Definition.** Given a smooth set X and a natural number n, the smooth set $X^{\mathrm{rank}\le n}$ is defined by setting $X^{\mathrm{rank}\le n}(S)$ to the subset of $X(S)$ (i.e., morphisms from S to X) consisting of maps of rank at most n. A map $S \to X$ from a smooth manifold S to a smooth set X has rank at most n if locally (i.e., after passing to some open cover of S) it factors through a smooth manifold of dimension at most n. If X is a smooth manifold, then this is equivalent to the tangent map having pointwise rank at most n.

3.5 **Definition.** Given a smooth set X, the simplicial smooth set $\mathrm{Sing}\, X$ sends a simplex $m \in \Delta$ to the internal hom $\mathrm{Hom}_{\mathrm{C^\infty Set}}(\mathbf{\Delta}^m, X)$ and a morphism of simplices $m \to n$ to the morphism of internal homs induced by the map $\mathbf{\Delta}^m \to \mathbf{\Delta}^n$.

3.6 **Remark.** The simplicial smooth set $\mathrm{Sing}\, X$ should be contrasted with the smooth singular simplicial set of X, which can be obtained by evaluating $\mathrm{Sing}\, X$ on the smooth manifold pt.

4 The target and its properties.

4.1 **Definition.** Given an smooth abelian group A and a natural number n, define $\mathrm{B}^n A$ to be $\mathrm{U}\Gamma(\Sigma^n A)$. Recall that U forgets the abelian group structure on a simplicial smooth abelian group and Γ turns a chain complex of smooth abelian groups into a simplicial smooth abelian group.

4.2 Of particular interest to us are smooth abelian groups whose $\mathbf{Z}$-module structure comes from a (necessarily unique) $\mathbf{R}$-module structure. It is natural to refer to such groups as *smooth real vector spaces.*

4.3 **Proposition.** If A is a smooth real vector space, then $\mathrm{B}^n A$ is a stack for any $n \ge 0$.

4.4 *Proof.* The descent morphism associated to a cover $c\colon U \to X$ can be written as

$$\mathrm{U}\Gamma\Sigma^n \mathrm{C}^\infty(X, A) \to \operatorname*{holim}_{m\in\Delta} \mathrm{U}\Gamma\Sigma^n \mathrm{C}^\infty(U^m, A),$$

where U^m denotes the m-fold (recall that $m \in \Delta$ is a finite inhabited set) fiber product of U over X. The functor U is a right Quillen adjoint and the cosimplicial diagram $m \in \Delta \mapsto \Gamma\Sigma^n \mathrm{C}^\infty(U^m, A)$ is fibrant in the projective structure, therefore we can move U to the left of the homotopy limit. The functor Γ is an equivalence of categories that preserves the entire model structure, therefore it can also be moved to the left of the homotopy limit. Thus the homotopy mapping space under consideration is $\mathrm{U}\Gamma \operatorname{holim}_{m\in\Delta} \Sigma^n \mathrm{C}^\infty(U^m, A)$. We compute the homotopy limit by taking the connective cover of the totalization of the double complex obtained by applying the normalized chain complex functor to the given cosimplicial diagram. In our case the double complex is concentrated in vertical degree n, where the corresponding horizontal chain complex is precisely $\mathrm{rev}\, \check{\mathrm{C}}(c)$ and the totalization of the double complex is $\Sigma^n \,\mathrm{rev}\, \check{\mathrm{C}}(c)$. The descent morphism can be obtained by applying the functor $\mathrm{U}\Gamma\, \mathrm{Conn}\, \Sigma^n\, \mathrm{rev}$ to the morphism $\mathrm{C}^\infty(X, A) \to \check{\mathrm{C}}(c)$, where $\check{\mathrm{C}}(c)$ is the (normalized) *Čech cochain complex* of the cover $c\colon U \to X$, whose component in degree $m \ge 0$ is $\mathrm{C}^\infty_{\mathrm{norm}}(U^{[m]}, A)$ and the differentials are given by the alternating sums of the face (i.e., projection) maps of $U^{[\bullet]}$, where $[m]$ denotes the linearly ordered set of integers $[0, m]$. In the above formula $\mathrm{C}^\infty_{\mathrm{norm}}$ denotes the abelian group of A-valued smooth functions on $U^{[m]}$ that vanish on the degenerate part of $U^{[k]}$, i.e., the images of the corresponding diagonal maps.

4.5 It remains to prove that the cochain descent morphism $\mathrm{C}^\infty(X, A) \to \check{\mathrm{C}}(c)$ is a chain homotopy equivalence. We are free to restrict the class of covers $c\colon U \to X$ to any class that generates the Grothendieck topology on the relative site of X. One such class is given by finite etale maps $c\colon U \to X$, which is convenient because it is closed with respect to fiber products over X and for any finite etale map $U \to V$ we can define a pushforward map $\mathrm{C}^\infty(U, A) \to \mathrm{C}^\infty(V, A)$ by multiplying by some fixed partition of unity, i.e., an element in $\mathrm{C}^\infty(U, \mathbf{R})$ whose sum over the fibers is always 1, and summing over the fibers. It is precisely at this moment that we need the $\mathbf{R}$-module structure on A.

4.6 Instead of constructing the inverse cochain map and the corresponding cochain homotopy we can apply the standard augmentation procedure, which inserts $\mathrm{C}^\infty(X = U^\emptyset, A)$ to the left of $\check{\mathrm{C}}(U \to X)$, with the corresponding differential given by the original cochain map in degree 0. The problem now reduces to constructing a cochain homotopy h from the identity map on the augmented Čech cochain complex to the zero map. The degree 0 part $\mathrm{C}^\infty(U, A) \to \mathrm{C}^\infty(X, A)$ of h will then give the inverse cochain map, whereas the other components of h will yield the corresponding cochain homotopy. The degree m component of h is the map $\mathrm{C}^\infty(U^{m+1}, A) \to \mathrm{C}^\infty(U^m, A)$ induced by the last face map, i.e., the projection $U^{m+1} \to U^m$ that discards the component associated to "$+1$" in $m+1$. Observe that this pushforward respects the normalization condition. We now verify the cochain homotopy identity $\mathrm{id}_{\mathrm{C}^\infty(U^m,A)} = h_m d_m + d_{m-1} h_{m-1}$. ▮

4.7 **Proposition.** If F is a smooth set and A is a smooth real vector space, then $\mathrm{Mor}^\infty(F, \mathrm{B}^n A)$ can be computed as $\mathrm{B}^n \mathrm{C}^\infty(F, A)$ for any $n \geq 0$. Likewise, if $h\colon F \to G$ is a morphism of smooth sets, then the morphism $\mathrm{Mor}^\infty(h, \mathrm{B}^n A)\colon \mathrm{Mor}^\infty(G, \mathrm{B}^n A) \to \mathrm{Mor}^\infty(F, \mathrm{B}^n A)$ can be computed as $\mathrm{B}^n \mathrm{C}^\infty(h, A)\colon \mathrm{B}^n \mathrm{C}^\infty(G, A) \to \mathrm{B}^n \mathrm{C}^\infty(F, A)$ for any $n \geq 0$.

4.8 *Proof.* Suppose $W \to F$ is a cofibrant replacement of F in the local projective model structure on simplicial presheaves. The above map induces an equivalence $\mathrm{Mor}^\infty(F, \mathrm{B}^n A) \to \mathrm{Mor}^\infty(W, \mathrm{B}^n A)$, and $\mathrm{Mor}^\infty(W, \mathrm{B}^n A)$ can be computed as $\mathrm{Mor}(W, \mathrm{B}^n A)$, because $\mathrm{B}^n A$ is fibrant by the last proposition. We have $\mathrm{Mor}(W, \mathrm{B}^n A) = \mathrm{Mor}(W, \mathrm{U}\Gamma\Sigma^n A) = \mathrm{Mor}(\mathbf{Z}W, \Gamma\Sigma^n A) = \mathrm{Mor}(\mathrm{N}\mathbf{Z}W, \Sigma^n A)$. Simplicial mapping spaces in connective chain complexes can be computed by applying $\mathrm{U}\Gamma$ to the corresponding internal hom. Thus $\mathrm{Mor}(\mathrm{N}\mathbf{Z}W, \Sigma^n A) = \mathrm{U}\Gamma\,\mathrm{Hom}_{\mathrm{Ch}}(\mathrm{N}\mathbf{Z}W, \Sigma^n A) = \mathrm{U}\Gamma\,\mathrm{Conn}\,\Sigma^n\,\mathrm{Hom}_{\mathrm{Ch}_\mathbf{Z}}(\mathrm{N}\mathbf{Z}W, A) = \mathrm{U}\Gamma\,\mathrm{Conn}\,\Sigma^n\,\mathrm{rev}\,\mathrm{C}^\infty_{\mathrm{norm}}(W, A)$, where $\mathrm{C}^\infty_{\mathrm{norm}}(W, A)$ is the normalized cochain complex of A-valued functions on W. The Dold-Kan correspondence yields a canonical chain homotopy equivalence from $\mathrm{C}^\infty_{\mathrm{norm}}(W, A)$ to $\mathrm{C}^\infty(W, A)$, its nonnormalized cousin. In the next paragraph we prove that $\mathrm{C}^\infty(W, A)$ is quasiisomorphic to $\mathrm{C}^\infty(F, A)$, which is a cochain complex concentrated in degree 0. Assuming this is true, the space $\mathrm{U}\Gamma\,\mathrm{Conn}\,\Sigma^n\,\mathrm{rev}\,\mathrm{C}^\infty(W, A)$ is equivalent to the space $\mathrm{U}\Gamma\,\mathrm{Conn}\,\Sigma^n\,\mathrm{rev}\,\mathrm{C}^\infty(F, A)$, because the functors rev, Σ^n, and Conn preserve quasiisomorphisms of (co)chain complexes, the functor Γ sends quasiisomorphisms of connective chain complexes to weak equivalences of simplicial abelian groups, and the functor U sends weak equivalences of simplicial abelian groups to weak equivalences of simplicial sets. Finally, we have $\mathrm{U}\Gamma\,\mathrm{Conn}\,\Sigma^n\,\mathrm{rev}\,\mathrm{C}^\infty(F, A) = \mathrm{U}\Gamma\,\mathrm{Conn}\,\Sigma^n \mathrm{C}^\infty(F, A) = \mathrm{U}\Gamma\Sigma^n \mathrm{C}^\infty(F, A) = \mathrm{B}^n \mathrm{C}^\infty(F, A)$, as desired. The same computation proves the claim about morphisms, provided that the corresponding morphisms of cochain complexes are quasiisomorphic.

4.9 We now prove that the map $\mathrm{C}^\infty(F, A) \to \mathrm{C}^\infty(W, A)$ given by the pullback along the degree 0 part of $G \to F$ is a quasiisomorphism. So far the choice of cofibrant replacement was irrelevant for the proof, but in order to go further we have to fix a concrete model for W. We choose to resolve F by representables as explained by Lemma 2.7 in Dugger [8]. Concretely, the resulting simplicial presheaf W has as its nth simplicial component the coproduct $\coprod_{X_n \to \cdots \to X_0 \to F} X_n$ of representable presheaves, where the coproduct is taken over all sequences X of n morphisms in Man with an additional morphism $X_0 \to F$ of smooth sets, i.e., all elements in the nth simplicial component of the nerve of the comma category Man/F. The simplicial maps of W are induced by the simplicial maps of the nerve. The corresponding cochain complex $\mathrm{C}^\infty(W, A)$ has $\prod_{X_n \to \cdots \to X_0 \to F} \mathrm{C}^\infty(X_n, A)$ in degree n with differentials being alternating sums of components obtaining by throwing away the corresponding element of X. If we throw away the initial element of X, the resulting element of $\mathrm{C}^\infty(X_{n-1}, A)$ has to be pulled back to X_n via the map $X_n \to X_{n-1}$. In particular, the first differential sends a collection of functions $u_{f: X_0 \to F} \in \mathrm{C}^\infty(X_0, A)$ to the collection of functions $v_{g: X_1 \to X_0, h: X_0 \to F} := u_{gh} - g^* u_h \in \mathrm{C}^\infty(X_1, A)$.

4.10 On the other hand, an element of $\mathrm{C}^\infty(F, A)$ is by definition a morphism of presheaves $F \to A$, i.e., a choice of an element $u_f \in A(X_0)$ for each map $f\colon X_0 \to F$ such that for each map $g\colon X_1 \to X_0$ we have $u_{gf} = g^* u_f$. We observe that this is precisely the definition of 0-cocycle in the above chain complex, and the map $\mathrm{C}^\infty(F, A) \to \mathrm{C}^\infty(W, A)$ sends an element of $\mathrm{C}^\infty(F, A)$ to the corresponding family u in $\mathrm{C}^\infty(W, A)_0$. Thus the map $\mathrm{C}^\infty(F, A) \to \mathrm{C}^\infty(W, A)$ is a cochain map and it induces an isomoprhism in homology in degree 0.

4.11 It remains to prove that the cochain complex $\mathrm{C}^\infty(W, A)$ is exact in nonzero degrees. ▮

5 Summary of the main results.

5.1 The main result of the next section is as follows:

5.2 **Theorem.** For any smooth set X and smooth real vector space A the space $\mathrm{Mor}^\infty_{\mathrm{C^\infty sSet}}(\mathrm{Sing}\, X, \mathrm{B}^n A)$ is equivalent to $\mathrm{U}\Gamma\,\mathrm{Conn}\,\Sigma^n\,\mathrm{rev}\,\mathrm{S}(X,A)$ for all $n \geq 0$. Here U and Γ are as above, Conn takes the connective cover of an unbounded chain complex, Σ^n shifts an unbounded chain complex in degree by n, rev turns a cochain complex into a chain complex by negating all degrees, and $\mathrm{S}(X,A)$ denotes the smooth A-valued (normalized) singular cochain complex of X. The latter by definition has $\mathrm{C}^\infty_{\mathrm{norm}}(\mathrm{Sing}_m X, A)$ in degree m, where $\mathrm{C}^\infty_{\mathrm{norm}}$ denotes smooth maps that vanish on simplicially degenerate smooth singular simplices, whereas the differentials are given by alternating sums of simplicial face maps.

5.3 **Remark.** The space $\mathrm{Mor}^\infty_{\mathrm{C^\infty sSet}}(\mathrm{Sing}\, X, \mathrm{B}^n A)$ is precisely the degree n abelian sheaf cohomology of $\mathrm{Sing}\, X$ with coefficients in the sheaf A.

5.4 **Remark.** The definition of $\mathrm{S}(X,A)$ should be contrasted with the usual definition of a singular A-valued cochain (on smooth simplices), which is not required to map a smooth family of simplices to a smooth A-valued function.

5.5 **Remark.** If we worked with sheaves of spectra instead of sheaves of spaces, then in the above theorem $\mathrm{U}\Gamma\,\mathrm{Conn}$ would be replaced by the Eilenberg—MacLane functor (sometimes denoted by H) from unbounded chain complexes to spectra. In our case, however, $\mathrm{B}^n A$ is a smooth homotopy n-type, and mapping into a homotopy n-type necessarily discards information about higher homotopy groups of the source (more precisely, the source can be replaced by its homotopy n-truncation by applying the functor $\pi_{\leq n}$), which explains the necessity of taking the connective cover in the above theorem.

5.6 The subsequent section will then prove that the smooth singular cochain complex is related to the familiar de Rham complex.

5.7 **Theorem.** For any smooth set X and smooth real vector space A the smooth (normalized) singular cochain complex $\mathrm{S}(X,A)$ is chain homotopy equivalent to the A-valued de Rham complex $\Omega(X,A)$.

5.8 **Remark.** By the Dold—Kan correspondence, the smooth normalized singular cochain complex is chain homotopy equivalent to the smooth unnormalized singular cochain complex, i.e., the one that uses $\mathrm{C}^\infty(\mathrm{Sing}_m X, A)$ instead of $\mathrm{C}^\infty_{\mathrm{norm}}(\mathrm{Sing}_m X, A)$.

5.9 Finally, the following minor result treats the rank-restricted case:

5.10 **Proposition.** For any smooth set X and any natural number n the de Rham complex of $X^{\mathrm{rank}\leq n}$ is precisely the de Rham complex of X truncated above degree n.

5.11 **Remark.** When we use this result in the above theorem, we have two integer parameters in the corresponding homotopy mapping space: $\mathrm{Mor}^\infty_{\mathrm{C^\infty Set}}(\mathrm{Sing}(X^{\mathrm{rank}\leq m}), \mathrm{B}^n A)$. For applications to differential cohomology the case $m = n$ is the most relevant one.

6 Computation of the homotopy space of maps.

6.1 **Theorem.** For any smooth set X and $n \geq 0$ the simplicial set $\mathrm{Mor}^\infty(\mathrm{Sing}\, X, \mathrm{B}^n A)$ is equivalent to the simplicial set $\mathrm{U}\Gamma\,\mathrm{Conn}\,\Sigma^n\,\mathrm{rev}\,\mathrm{S}(X,A)$. Here $S(X,A)$ is the smooth singular cochain complex of X, consisting of the abelian group $\mathrm{C}^\infty_{\mathrm{norm}}(\mathrm{Sing}_m X, A)$ (i.e., smooth A-valued functions on $\mathrm{Sing}_m X$ that vanish on simplicially degenerate simplices) in degree m and differentials given by alternating sums of maps induced by face maps of $\mathrm{Sing}_m X$.

6.2 *Proof.* In the model category of simplicial presheaves with the local projective structure homotopy colimits can be computed componentwise. Recall that any simplicial set T can be identified with the homotopy colimit of the diagram of simplicial sets given by the composition $\Delta^{\mathrm{op}} \to \mathrm{Set} \to \mathrm{sSet}$, where the first functor is X itself and the second functor sends a set to the corresponding constant simplicial set. (This can be deduced from the fact that the homotopy colimit of a simplicial diagram of simplicial sets can be computed by taking the diagonal of the corresponding bisimplicial set, because every simplicial diagram of simplicial sets is Reedy cofibrant.) Combining these two facts we obtain that $\mathrm{Sing}\, X$ is equivalent to $\mathrm{hocolim}_{m\in\Delta^{\mathrm{op}}} \mathrm{dis}(\mathrm{Sing}_m X)$, where dis sends a presheaf of sets to the corresponding presheaf of constant simplicial sets. The universal property of homotopy colimits and homotopy mapping spaces tells us that $\mathrm{Mor}^\infty(\mathrm{hocolim}_{m\in\Delta^{\mathrm{op}}} \mathrm{dis}(\mathrm{Sing}_m X), \mathrm{B}^n A) = \mathrm{holim}_{m\in\Delta} \mathrm{Mor}^\infty(\mathrm{dis}(\mathrm{Sing}_m X), \mathrm{B}^n A)$. Furthermore, $\mathrm{Sing}_m X$ is a smooth set and A is a smooth real vector space, hence we have $\mathrm{Mor}^\infty(\mathrm{dis}(\mathrm{Sing}_m X), \mathrm{B}^n A) = \mathrm{B}^n\mathrm{C}^\infty(\mathrm{Sing}_m X, A)$. The original mapping space is therefore equivalent to $\mathrm{holim}_{m\in\Delta} \mathrm{B}^n\mathrm{C}^\infty(\mathrm{Sing}_m X, A)$. Thus we passed from the world of simplicial presheaves to the world of simplicial sets.

6.3 Recall that B^n was defined as $\mathrm{U}\Gamma\Sigma^n$, where Σ^n sends an abelian group to the corresponding chain complex concentrated in degree n. Thus the above expression can be rewritten as $\mathrm{holim}_{m\in\Delta} \mathrm{U}\Gamma\Sigma^n\mathrm{C}^\infty(\mathrm{Sing}_m X, A)$. The functor U is a right Quillen adjoint and the cosimplicial diagram $m \in \Delta \mapsto \Gamma\Sigma^n\mathrm{C}^\infty(\mathrm{Sing}_m X, A)$ is fibrant in the projective structure, therefore we can move U to the left of the homotopy limit. The functor Γ is an equivalence of categories that preserves the entire model structure, therefore it can also be moved to the left of the homotopy limit. Thus the homotopy mapping space under consideration is $\mathrm{U}\Gamma\,\mathrm{holim}_{m\in\Delta} \Sigma^n\mathrm{C}^\infty(\mathrm{Sing}_m X, A)$. It remains to prove that $\mathrm{holim}_{m\in\Delta} \Sigma^n\mathrm{C}^\infty(\mathrm{Sing}_m X, A)$ can be computed as $\mathrm{Conn}\,\Sigma^n\,\mathrm{rev}\,\mathrm{S}(X,A)$. We compute the homotopy limit by taking the connective cover of the totalization of the double complex obtained by applying the normalized chain complex functor to the given cosimplicial diagram. In our case the double complex is concentrated in vertical degree n, where the corresponding horizontal chain complex is precisely $\mathrm{rev}\,\mathrm{S}(X,A)$. The totalization of this double complex is $\Sigma^n\,\mathrm{rev}\,\mathrm{S}(X,A)$ and its connective cover gives the desired answer. ∎

7 Deformation retraction of the smooth singular cochain complex to a smaller subcomplex.

7.1 In this section we construct a cochain endomorphism A of the smooth singular cochain complex $\mathrm{S}(X,A)$ such that $\mathrm{A}^2 = \mathrm{A}$. We also construct a cochain homotopy Aha between A and $\mathrm{id}_{\mathrm{S}(X,A)}$. Thus $\mathrm{S}(X,A)$ is chain homotopy equivalent to the image of A. The latter can be characterized as the kernel of $\mathrm{id}_{\mathrm{S}(X,A)} - \mathrm{sd}(X,A)$, where $\mathrm{sd}(X,A)$ is the barycentric subdivision map on A-valued smooth singular cochains in X. In other words, the image of A consists of cochains that are additive under barycentric subdivision. In the next section we identify such cochains with the de Rham complex.

7.2 Observe that $\mathrm{S}(X,A) = \mathrm{Hom}(\mathcal{S}(X), A)$, where $\mathcal{S}(X)$ is the (integral) smooth singular chain complex of X, i.e., the chain complex of smooth abelian groups that has $\mathbf{Z}[\mathrm{Sing}_m X]$ in degree m and differentials given by the alternating sums of face maps. We construct the map $\mathrm{A}\colon \mathrm{S}(X,A) \to \mathrm{S}(X,A)$ by taking the limit of the maps $\mathrm{sd}^n(X,A) = \mathrm{Hom}(\mathrm{Sd}(X)^n, A)$ as n goes to infinity, where $\mathrm{Sd}(X)\colon \mathcal{S}(X) \to \mathcal{S}(X)$ is the barycentric subdivision map on singular chains in X. As we explain below, the limit exists because our singular cochains are smooth (as opposed to continuous or arbitrary, in which case the argument fails).

7.3 The chain homotopy $\mathrm{Aha}\colon \mathrm{S}(X,A) \to \Sigma\mathrm{S}(X,A)$ can be constructed as the infinite sum $\sum_{k\geq 0} \mathrm{Hom}(\mathrm{Sh}\,\mathrm{Sd}^k, A)$, where $\mathrm{Sh}\colon \mathcal{S}(X) \to \Omega\mathcal{S}(X)$ is the canonical chain homotopy from $\mathrm{id}_{\mathcal{S}(X)}$ to $\mathrm{Sd}(X)$. Again, the sum exists because cochains are smooth. Σ and Ω denote the suspension and loop functors on chain complexes.

7.4 The original reference for the following definitions is Eilenberg and Steenrod [2, Theorem VII.8.2] or the original paper [1, Section 22] by Eilenberg on singular homology.

7.5 **Definition.** Given an arbitrary affine space A (in fact, a convex structure is sufficient because we only use positive combinations) we define the following three linear maps on the smooth singular chain complex of A on individual (families of) smooth n-simplices $\sigma\colon \mathbf{\Delta}^n \to A$ ($n \geq 0$ is arbitrary) of A and then extend them by linearity. For a fixed element $a \in A$ the *join map* $\mathrm{J}_a\colon \mathcal{S}(A) \to \Omega\mathcal{S}(A)$ sends σ to the $(n+1)$-simplex $\mathrm{J}_a\sigma$ defined as the unique simplex that sends the 0th vertex to a and whose 0th face is σ. The *subdivision map* $\mathrm{Sd}\colon \mathcal{S}(X) \to$

$\mathcal{S}(X)$ sends σ to itself if $n = 0$ and to $\mathrm{J}_b \operatorname{Sd}(\partial\sigma)$ otherwise. The *subdivision homotopy* $\operatorname{Sh}: \mathcal{S}(X) \to \Omega\mathcal{S}(X)$ sends σ to 0 if $n = 0$ and to $\mathrm{J}_b(\sigma - \operatorname{Sd}(\sigma) - \operatorname{Sh}(\partial\sigma))$ otherwise. Here b is the *barycenter* of σ, i.e., the affine combination with equal coefficients of individual vertices of σ.

7.6 **Proposition.** We have the following identities: $\partial \mathrm{J}_b + \mathrm{J}_b \partial = \mathrm{id}$ (in this identity for $n = 0$ we use the augmented singular chain complex coming from the augmented cosimplicial object $\mathbf{\Delta}$ given by $\mathbf{\Delta}^{-1} = \emptyset$), $\partial \operatorname{Sd} = \operatorname{Sd} \partial$, $\partial \operatorname{Sh} + \operatorname{Sh} \partial = \mathrm{id} - \operatorname{Sd}$.

7.7 *Proof.* The proof is given by Eilenberg and Steenrod in Lemma VII.5.3 and Theorem VII.8.2 of [2]. ∎

7.8 We extend the maps Sd and Sh defined above to smooth singular chains on an arbitrary smooth set X. By linearity it is sufficient to deal with the case of a smooth S-indexed family $\sigma: S \times \mathbf{\Delta}^n \to X$ of n-simplices over X. Applying the corresponding map to the singular n-simplex in $\mathbf{\Delta}^n$ given by the identity map, we obtain a singular m-chain in $\mathbf{\Delta}^n$ (here $m = n$ for Sd and $m = n + 1$ for Sh). For each m-simplex in this chain take the product with the identity map of S, obtaining a smooth map $\tau: S \times \mathbf{\Delta}^m \to S \times \mathbf{\Delta}^n$ and postcompose it with the map σ, obtaining a map $S \times \mathbf{\Delta}^m \to X$. The sum of the resulting m-simplices gives us the answer. Observe that for $A = X$ the new maps coincide with the old ones, hence we keep the same notation for the new maps. The identities for Sd and Sh proved above also remain in force. We observe that all simplices produced by these constructions are obtained by precomposing with some affine map $\mathbf{\Delta}^m \to \mathbf{\Delta}^n$.

7.9 **Definition.** Define $\operatorname{Sd}^0 = \mathrm{id}$ and $\operatorname{Sh}^0 = 0$. For $n > 0$ define $\operatorname{Sd}^n = \operatorname{Sd}^{n-1} \operatorname{Sd}$ and $\operatorname{Sh}^n = \operatorname{Sh}^{n-1} \operatorname{Sd} + \operatorname{Sh}$.

7.10 **Proposition.** For any $n \geq 0$ the map Sd^n is a chain map, i.e., $\partial \operatorname{Sd}^n = \operatorname{Sd}^n \partial$, and the map Sh^n is a chain homotopy from id to Sd^n, i.e., $\partial \operatorname{Sh}^n + \operatorname{Sh}^n \partial = \mathrm{id} - \operatorname{Sd}^n$.

7.11 **Definition.** Denote $\operatorname{sd}^n(X, A) = \operatorname{Hom}(\operatorname{Sd}^n(X), A)$ and $\operatorname{sh}^n(X, A) = \operatorname{Hom}(\operatorname{Sh}^n(X), A)$. If the superscript is suppressed, then we assume $n = 1$. Of course, $\operatorname{sd}^n(X, A)$ is a cochain map and $\operatorname{sh}^n(X, A)$ is a cochain homotopy from the identity map to $\operatorname{sd}^n(X, A)$.

7.12 The following observation is due to Barr [5], who constructed in his paper a cochain homotopy equivalence from singular cochains to *antisymmetric* singular cochains (see below) and observed that his map in low degrees looks exactly like the barycentric subdivision map. We give a proof of this statement here.

7.13 **Proposition.** Any cochain in the image of sd (hence sd^n for $n \geq 1$) is antisymmetric, i.e., permuting the vertices of a singular simplex σ multiplies the value of the cochain on σ by the sign of the permutation.

7.14 *Proof.* It is sufficient to prove that after transposing any two consecutive vertices of a singular k-simplex σ the value of a given smooth singular k-cochain $\operatorname{sd}(c)$ on σ changes sign. By definition, $\operatorname{sd}(c) = c(\mathrm{J}_b(\operatorname{Sd}(\partial\sigma)))$, where b is the barycenter of the domain of σ. The term $\partial\sigma$ consists of $k+1$ terms, which are singular $(k-1)$-simplices. We extract from $\partial\sigma$ the two consecutive terms corresponding to the chosen vertices. By definition of ∂ these simplices have different signs. When we apply the permutation, these two terms are exchanged, hence their sum is multiplied by -1. The other $k - 1$ terms of $\partial\sigma$ contain both vertices and $c\mathrm{J}_b \operatorname{Sd}$ changes sign on them by induction. ∎

7.15 An important observation is that all singular k-simplices occurring in Sd^n and Sh^n applied to some fixed singular k-simplex $\sigma: \mathbf{\Delta}^k \to X$ are obtained by precomposition with some affine map $\mathbf{\Delta}^k \to \mathbf{\Delta}^k$. Thus they are uniquely determined by $k + 1$ vertices in $\mathbf{\Delta}^k$, i.e., they can be parametrized by the finite-dimensional smooth manifold $(\mathbf{\Delta}^k)^{k+1}$. In particular, the tangent space $\mathrm{T}((\mathbf{\Delta}^k)^{k+1}) = (\mathrm{T}(\mathbf{\Delta}^k))^{\oplus(k+1)}$ is finite-dimensional. The $k + 1$ components of this tangent space can be seen as specifying direction in which the particular vertex is supposed to go.

7.16 A smooth A-valued k-cochain restricted to the simplices described above can then be seen as a smooth map $(\mathbf{\Delta}^k)^{k+1} \to A$. This map is antisymmetric if the original cochain is antisymmetric. The jets of such maps at constant simplices satisfy an important property, described in the next proposition, which plays the key role in establishing convergence of the corresponding limits and defining the differential form corresponding to a given smooth cochain.

7.17 **Proposition.** Given a smooth antisymmetric map $f\colon (\mathbf{\Delta}^k)^{k+1} \to A$, consider the diagonal map $C\colon \mathbf{\Delta}^k \to (\mathbf{\Delta}^k)^{k+1}$, whose image consists of constant simplices. The jets of f of order less than k restricted to C vanish and the kth jet is determined by a function $a\colon \mathbf{\Delta}^k \to A$. The value of the kth jet on a $(k+1)$-tuple v of tangent vector fields at C (here we interpret the kth jet as a polynomial function $\mathrm{T}((\mathbf{\Delta}^k)^{k+1}) = (\mathrm{T}(\mathbf{\Delta}^k))^{\oplus(k+1)} \to A$ of degree k) can be computed by multiplying $k!a$ by the (fiberwise) determinant $\det((1, v_i))$ of $k+1$ vectors in $\mathbf{R} \oplus \mathrm{T}(\mathbf{\Delta}^k)$ of the form $(1, v_i)$. This determinant can also be computed as the sum $\sum_{0\le i\le k}(-1)^i \det(\hat{v}_i)$, where $\hat{v}_i$ denotes all vertices in v except for the ith vertex. Finally, for any given point $p \in \mathbf{\Delta}^k$ identified via C with its image in $(\mathbf{\Delta}^k)^{k+1}$ the function f can be represented in the form $f(\sigma) = (\mathrm{J}^k f)(p)(\sigma - p) + h(\sigma)(\sigma - p)$, where h is some smooth *symmetric* $(k+1)$-form on $(\mathbf{\Delta}^k)^{k+1}$. In other words, the expansion of any such a map f around the diagonal has a term of degree k given by the (appropriately interpreted) volume multiplied by a together with some higher order term.

7.18 *Proof.* By induction, if we already proved that jets of order less than j vanish, then the jet of order j is a symmetric linear map $(\mathrm{T}((\mathbf{\Delta}^k)^{k+1}))^{\otimes j} = ((\mathrm{T}(\mathbf{\Delta}^k))^{\oplus(k+1)})^{\otimes j} \to A$, which is determined by its individual components of the form $(\mathrm{T}(\mathbf{\Delta}^k))^{\otimes j}$ indexed by j vertices. The value of the jet on a decomposable element of $(\mathrm{T}(\mathbf{\Delta}^k))^{\otimes j}$ depends only on the value of f on affine simplices that can be obtained from a constant simplex by moving the vertices that appear in the corresponding index in the direction of the corresponding vector in $\mathrm{T}(\mathbf{\Delta}^k)$. Since $(k+1) - j \ge 2$, at least two vertices stay in place, thus the resulting simplices are linearly degenerate, hence the function f must vanish on them because of the antisymmetry.

7.19 We now analyze the jet of order k. We decompose it as above into individual components $(\mathrm{T}(\mathbf{\Delta}^k))^{\otimes k} \to A$ indexed by k vertices. If at least two vertices in an index coincide, then the corresponding component vanishes for reasons explained in the previous paragraph. Furthermore, due to the symmetry of the jet we can restrict our attention to components whose vertices are ordered, which explains the factor of $k!$. There is exactly one such component for every vertex and the corresponding index consists of all vertices except for the given one. We now observe that due to the antisymmetry of the function f all $(k+1)$ components identified above coincide up to the sign given by the parity of the excluded vertex. Furthermore, the antisymmetry tells us that the map $(\mathrm{T}(\mathbf{\Delta}^k))^{\otimes k} \to A$ is itself antisymmetric. Hence all information about the kth jet is contained in the map $\Lambda^k(\mathrm{T}(\mathbf{\Delta}^k)) \to A$. The source of this map can be canonically identified with the trivial line bundle $\mathbf{R}$, thus the entire information about the kth jet is captured by a smooth map $\mathbf{\Delta}^k \to A$. The formula for the jet then follows from multilinearity and antisymmetry.

7.20 Recall that the ordinary Hadamard lemma says that for a smooth function f on a finite-dimensional real vector space V that vanishes at 0 one can find a smooth 1-form g on V such that $f(x) = (g(x))(x)$. The lemma is proved by integrating the derivative of the function $t \in [0,1] \mapsto f(tx)$. The derivative can be computed as $(df)(tx)(x)$, which allows us to set $g(x)$ to the integral of $(df)(tx)$ over $[0,1]$, proving the lemma. By induction, we obtain a higher version of this theorem, which requires all jets of f at 0 of order less than m to vanish and produces a smooth *symmetric* m-form g on V such that $f(x) = (g(x))(x^{\otimes m})$. The form $g(x)$ can be obtained by integrating the mth jet (which is well-defined because V is a vector space, even though lower-order jets might be nonvanishing away from 0) of the function $t \in [0,1]^m \mapsto (f(x \cdot \prod_i t_i))$ with respect to the canonical measure on $[0,1]^m$. Applying the higher Hadamard lemma to the function $x \mapsto f(x) - (g(0))(x)$, whose jets of order at most m vanish, we represent f in the form $f(x) = (g(0))(x) + (h(x))(x)$, where g is as above and h is a smooth symmetric $(m+1)$-form. Observe that $g(0)$ is simply the mth jet of f at 0.

7.21 We apply the refined higher Hadamard lemma to the function $f\colon (\mathbf{\Delta}^k)^{k+1} \to A$, taking the constant simplex at some fixed point $p \in \mathbf{\Delta}^k$ as the origin, thus endowing the affine space $(\mathbf{\Delta}^k)^{k+1}$ with the structure of a vector space. We obtain that $f(x) = (\mathrm{J}^k f)(p)(x) + (h(x))(x)$, where h is some smooth symmetric $(k+1)$-form. ▮

7.22 Although smooth singular chains by definition can have only a finite number of simplices, once we pass to cochains we can take the limit of the above maps.

7.23 **Proposition.** The limits $\lim_n \mathrm{sd}^n(X, A)$ and $\lim \mathrm{sh}^n(X, A)$ exist and define a cochain map $\mathrm{A}\colon \mathrm{S}(X, A) \to \mathrm{S}(X, A)$ and a cochain homotopy $\mathrm{Aha}\colon \mathrm{S}(X, A) \to \Sigma\mathrm{S}(X, A)$ from id to A, i.e., $\mathrm{d}\,\mathrm{Aha} + \mathrm{Aha}\,\mathrm{d} = \mathrm{id} - \mathrm{A}$.

7.24 *Proof.* We observe first that the identities involving A and Aha ($\mathrm{d}\,\mathrm{A} = \mathrm{A}\,\mathrm{d}$, $\mathrm{d}\,\mathrm{Aha} + \mathrm{Aha}\,\mathrm{d} = \mathrm{id} - \mathrm{A}$) follow automatically once the convergence is proven, because the corresponding finite identities are satisfied for all $n \ge 0$. To prove the convergence, we estimate the differences $\mathrm{sd}^n(X, A) - \mathrm{sd}^{n+1}(X, A) = \mathrm{Hom}((\mathrm{id} - \mathrm{Sd}(X))\,\mathrm{Sd}^n X, A)$ and $\mathrm{sh}^{n+1}(X, A) - \mathrm{sh}^n(X, A) = \mathrm{Hom}(\mathrm{Sh}\,\mathrm{Sd}^n(X), A)$. Observe that for $n = 0$ both terms vanish, hence we can assume that $n > 0$. The crucial point is that the value of any smooth cochain on $(\mathrm{id} - \mathrm{Sd})(s)$ and $\mathrm{Sh}(s)$ for some singular chain s can be bounded linearly with respect to the maximum size of individual simplices in s. In our case s is in the image of Sd^n and the size of simplices in $\mathrm{Sd}^n(t)$ decreases exponentialy with n for a fixed singular chain t, which proves convergence in both cases.

7.25 We now estimate the value of some smooth antisymmetric singular k-cochain c on $(\mathrm{id}-\mathrm{Sd})(\sigma)$ for some smooth singular k-simplex σ. We apply the formula obtained in the previous proposition, taking the barycenter of $\boldsymbol{\Delta}^k$ for p. Observe that the term of order k vanishes. Indeed, the term in A is the same in both cases because the basepoint p is the same. By the formula in the previous proposition the coefficient in $c(\sigma)$ is an alternating sum of volumes of simplices that are joins of the barycenter and one of the faces, i.e., the total volume of σ. For $c(\mathrm{Sd}(\sigma))$ we obtain a signed sum of volumes of all simplices in the barycentric subdivision. By induction, the sign is positive if and only if the corresponding simplex in the barycentric subdivision has positive orientation, thus the sum of all terms is again the total volume of σ. Thus the term of order k vanishes and the remainder is a sum of terms of order greater than k.

7.26 The estimate for $\mathrm{Sh}(\sigma)$ is even simpler. By definition, all simplices in the image of Sh are obtained by precomposing σ with some affine map $\boldsymbol{\Delta}^{k+1} \to \boldsymbol{\Delta}^k$. In particular, all these simplices have volume 0, thus the term of order k vanishes. The remainder is estimated in the same fashion as above. ∎

7.27 **Proposition.** We have $\mathrm{A}\,\mathrm{sd} = \mathrm{sd}\,\mathrm{A} = \mathrm{A}$. Furthermore, $\mathrm{A}\,\mathrm{A} = \mathrm{A}$ and the kernel of $\mathrm{id} - \mathrm{A}$ coincides with the kernel of $\mathrm{id} - \mathrm{sd}$. In particular, the image of A is precisely the equalizer of id and sd.

7.28 *Proof.* The map $\mathrm{A}\,\mathrm{sd}$ is defined by the same limit as A, but shifted by one, hence we have $\mathrm{A}\,\mathrm{sd} = \mathrm{A}$. Likewise, we obtain $\mathrm{sd}\,\mathrm{A} = \mathrm{A}$. This implies that $\mathrm{A}\,\mathrm{A} = \mathrm{A}$. If $c = \mathrm{A}(c)$, then $c = \mathrm{A}(c) = \mathrm{sd}\,\mathrm{A}(c) = \mathrm{sd}(c)$. Conversely, if $c = \mathrm{sd}(c)$, then all terms in the limit for $\mathrm{A}(c)$ are the same, hence $c = \mathrm{A}(c)$. ∎

7.29 **Remark.** Additive cochains are automatically antisymmetric and vanish on all simplicially degenerate simplices, i.e., they are automatically normalized. Indeed, terms in the limit $\lim_n \mathrm{sd}_n(X, A)$ are antisymmetric as long as $n \geq 1$. In fact, additive cochains vanish on all linearly degenerate singular simplices, i.e., simplices obtained by precomposing a singular n-simplex $\sigma\colon \boldsymbol{\Delta}^n \to X$ with an affine map induced by a nonsurjective map $n \to n$ of simplices, because any linearly degenerate simplex can be obtained from a simplicially degenerate simplex via some permutation of vertices.

7.30 **Summary.** For any smooth set X and any smooth real vector space A there is a canonical cochain deformation retraction of the smooth singular A-valued cochain complex $\mathrm{S}(X, A)$ of X onto its subcomplex of *subdivision additive* cochains, i.e., the equalizer of the identity map and the barycentric subdivision map sd on cochains.

7.31 **Remark.** The (subdivision) additivity should be distinguished from *cocycle additivity.* A smooth singular cocycle c is additive in a certain restricted sense, namely for any simplex σ the sum of values of c on the faces of σ vanishes. There are plenty of additive cochains that are not cocycles, e.g., take the image of any cochain that is not a cocycle under the map A. We also have counterexamples for the other inclusion: there are cocycles (in fact, coboundaries) that are not (subdivision) additive, for example, one can take $c = \mathrm{d}\,f$, where f is a 1-cochain on $\mathbf{R}$ that sends a smooth 1-simplex $\sigma\colon [0,1] \to \mathbf{R}$ to the real number $(\sigma(1) - \sigma(0))^2$. By definition, c is a coboundary, hence it is also a cocycle. However, c is far from being (subdivision) additive. Henceforth by additivity we always mean subdivision additivity.

8 Identification of additive smooth singular cochains and differential forms.

8.1 In this section we construct for any smooth set X mutually inverse cochain isomorphisms $\mathrm{D}\colon \mathrm{S}_{\mathrm{add}}(X,A) \to \Omega(X,A)$ and $\textstyle\int\colon \Omega(X,A) \to \mathrm{S}_{\mathrm{add}}(X,A)$ between the complex $\mathrm{S}_{\mathrm{add}}(X,A)$ of subdivision additive smooth A-valued singular cochains on X and the A-valued de Rham complex $\Omega(X,A)$ of X. A similar result for singular cubes can be found in Section 3 of Félix and Lavendhomme [4], see Definition 2 and Theorem 15 there. Félix and Lavendhomme need three properties to establish their theorem and two of them (additivity and vanishing on degenerate cubes) appear in our formulation as subdivision additivity and normality. The third property is antisymmetry, which is satisfied automatically in our case. Furthermore, normality also follows automatically from subdivision additivity, as was demonstrated in the previous section. Thus subdivision additivity is the only property that ensures that a given smooth singular cochain comes from a differential form.

8.2 **Definition.** The morphism of cochain complexes $\textstyle\int\colon \Omega(X,A) \to \mathrm{S}_{\mathrm{add}}(X,A)$ sends a differential form $\omega \in \Omega^k(X,A)$ for $k \le n$ to the cochain $\textstyle\int \omega \in \mathrm{S}^k_{\mathrm{add}}(X,A)$ whose value on a smooth S-family $\sigma\colon S \times \mathbf{\Delta}^k \to X$ of k-simplices is obtained by integrating the k-form $\sigma^*\omega$ with respect to the map $S \times \mathbf{\Delta}^k \to S$ using the standard orientation of $\mathbf{\Delta}^k$, obtaining a smooth function on S: $(\textstyle\int \omega)(\sigma) := \int_{S\times\mathbf{\Delta}^k\to S} \sigma^*\omega$. This map is a cochain map because of Stokes's theorem: the integral of a form along the boundary of a simplex equals the integral of its differential along the simplex itself. The cochain $\textstyle\int \omega$ is subdivision additive because the integration of forms is subdivision additive. In particular, $\textstyle\int \omega$ is antisymmetric, hence also normalized. This can be seen directly by observing that integration is invariant under diffeomorphisms up to a sign given by the orientation.

8.3 **Remark.** If $\textstyle\int \omega$ is a cocycle, then ω itself is a cocycle, i.e., it is closed. Indeed, the cocycle condition for smooth cochains posits that the sum of values of all faces of any simplex in Sing^n_{k+1} is zero, which via Stokes's theorem implies that the form $\mathrm{d}\,\omega$ has to integrate to zero along any such simplex, hence $\mathrm{d}\,\omega = 0$ and ω is closed.

8.4 **Remark.** The map $\textstyle\int$ is injective in all degrees because integration along smooth singular k-simplices distinguishes differential k-forms.

8.5 **Definition.** The *disintegration map* $\mathrm{D}\colon \mathrm{S}_{\mathrm{add}}(X,A) \to \Omega(X,A)$ sends a cochain $c \in \mathrm{S}^k_{\mathrm{add}}(X,A)$ to the differential form $\mathrm{D}c$ whose value

8.6 **Remark.** A priori, the disintegration map D depends on a choice of a Riemannian metric on X. However, we will see that D is the inverse of $\textstyle\int$, which proves that D is independent of the choice of a Riemannian metric.

8.7 **Remark.** We cannot get a k-form from a k-cochain for $k > n$ because the tangent bundle (more accurately, the tangent cone) restricted to constant simplices no longer contains all affine vector fields, but only those with rank at most n, and the corresponding jets vanish by the above lemma.

8.8 **Proposition.** We have $\mathrm{D} \textstyle\int = \mathrm{id}_{\Omega(X,A)}$.

8.9 *Proof.* By definition, $\textstyle\int \omega$ evaluated at some simplex σ integrates the pullback of ω to σ. The latter pullback can be canonically identified with a smooth function on the domain of σ using the canonical volume form. The cochain $\mathrm{D} \textstyle\int \omega$ evaluated at some k-tuple of tangent vectors v takes a simplex σ whose directions pointing from the 0th vertex are precisely v and then takes the corresponding jet, i.e., the kth partial derivative with respect to each of the directions. The partial derivative simply recovers the value of the original integrand at 0. By definition, this is the value of ω at the k-vector v. ∎

8.10 **Proposition.** We have $\textstyle\int \mathrm{D} = \mathrm{id}_{\mathrm{S}_{\mathrm{add}}(X,A)}$.

8.11 *Proof.* For $c \in \mathrm{S}^k_{\mathrm{add}}(X,A)$ we have $\mathrm{sd}^n c = c$. Thus to prove that $c = \textstyle\int \mathrm{D}c$ it is sufficient to establish that $\mathrm{sd}^n c - \textstyle\int \mathrm{D}c$ approaches 0 as n increases. This is done using the technique developed in the previous section. Since c and $\textstyle\int \mathrm{D}c$ are additive, it suffices to see that for a single simplex σ the value of $c(\sigma) - (\textstyle\int \mathrm{D}c)(\sigma)$ approaches zero faster than the volume of σ. Using the machinery developed in the previous section we restrict to the space of affine simplices that map to the domain of σ and estimate the difference between the two resulting functions. Indeed, the terms of order k corresponding to c and $\textstyle\int \mathrm{D}c$ are by definition the same. The remainder has the required growth rate. ∎

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