Étale Localic Groupoids are Identical to Complete Distributive Inverse Semigroups

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Abstract

1 Introduction

Inverse semigroups are a special class of semigroups where unique inverses make a sense. Also, they can be thought of as a generalization of pseudogroups who capture local symmetries of topological spaces and smooth manifolds more sensitively. In this paper we will discuss how a nice class of inverse semigroups is in a correspondence with localic étale groupoids.

An inverse semigroup is a set S with an associative binary operation $S \times S \xrightarrow{\cdot} S$ and an involutive unary operation $S \xrightarrow{\star} S$ such that every $a \in S$ satisfies

$$a = aa^{\star}a.$$

An inverse monoid is an inverse semigroup that has a multiplicative identity, which is usually denoted by 1. Here are some prototypical examples for inverse semigroups.

1. Any group is an inverse semigroup under the obvious operations.
2. Any meet-semilattice is a commutative idempotent inverse semigroup under meet and identity operations. Furthermore, if the lattice is bounded then its greatest element makes it an inverse monoid.
3. For any set X, the set $\mathbf{SymInv}(X)$ of partial bijections on X form an inverse semigroup. The composition of partial bijections is their composite as relations (or partial functions). This structure is called the *symmetric inverse semigroup* on X. In particular, this structure has both zero and a unit element.
4. If X is a topological space, the groupoid consisting of homeomorphisms between open subsets of X is a sub-inverse-semigroup of $\mathbf{SymInv}(X)$.

5. When H is a Hilbert space, the set of partial isometric operators and their adjoints forms an inverse semigroup. Also, the self-adjoint elements inside it form a sub-inverse-semigroup. For defining identities, recall that An operator T on H is said to be a partial isometry if it is an isometry on the orthogonal complement of its kernel. Then for any $y \in \ker(T)^{\perp} = \overline{\operatorname{im}(T^*)}$ we have
$$\langle T^*TT^*x, y\rangle = \langle TT^*x, Ty\rangle = \langle T^*x, y\rangle$$
and therefore $T^*TT^* = T^*$, and similarly the adjoin identity. In fact T being a partial isometry is equivalent to satisfying one of theses identities.

6. In a similar vein to the last two examples, we can consider partial diffeomorphisms on a smooth manifold. There is a well known equivalence between the category of pseudogroups (of transformations) and the category of effective étale Lie groupoids. The purpose of this essay is a generalization of this equivalence to facilitate arbitrary étale localic groupoids.

7. Let $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ be a (discrete) groupoid. For any two subsets $U, V \subseteq \mathcal{G}_1$ define pointwise operations
$$UV = \{\ xy \mid x \in U, y \in V, s(x) = t(y)\}, \qquad U^\star = \{x^{-1} \mid x \in U\}.$$
Subsets $U \subseteq \mathcal{G}_1$ for which the restrictions $s\big|_U, t\big|_U$ both injective (equivalently, $UU^\star, U^\star U \subseteq \mathcal{G}_0$) are called $\mathcal{G}$-*slices*, and they form an inverse monoid with a unit $\mathcal{G}_0$. In addition, a topological groupoid $\mathcal{G}$ is an étale groupoid if and only if open $\mathcal{G}$-slices form a sub-inverse-monoid of the inverse monoid of all $\mathcal{G}$-slices. Moreover, one can identify the latter inverse monoid with the inverse monoid of local bisections of $\mathcal{G}$.

8. Given an inverse semigroup S, one can construct a canonical groupoid $\mathbf{Ind}(S)$ analogous to the delooping construction: The set of objects of $\mathbf{Ind}(S)$ is $\{ss^\star \mid s \in S\}$, the set of morphisms in S, and $s \in S$ is a morphism $s^\star s \xrightarrow{s} ss^\star$. Composition and inversion of morphisms in $\mathbf{Ind}(S)$ are given by the product operation and star operation in the inverse semigroup.[1] However, this construction lost some information of S and in order to keep track of these lost data, we need to consider the so-called inductive groupoids, which we will discuss later in details.

There is an analogue of Cayley's theorem for inverse semigroups known as the Wagner–Preston representation theorem. More precisely, it says that every inverse semigroup S can be realized as a semigroup of partial bijections on the underlying set of S.

Wagner–Preston Representation Theorem: Given an inverse semigroup S, the function $\Phi : S \to \mathbf{SymInv}(S)$ given by $a \mapsto \Phi(a)$, where $\Phi(a)$ is is the partial map (between left cosets):
$$a^\star S \xrightarrow{\Phi(a)} aS$$
$$b \longmapsto ab$$

[1]The topology given by downward closed subsets in the natural ordering of S make this groupoid into an étale groupoid over the space of idempotents.

is a faithful (injective) representation of S. Also we can strengthen this theorem to identify S with a sub inverse semigroups of partial isometries on a Hilbert space. First we can embeds S into the inverse semigroup convolution $*$-algebra

$$\mathbb{C}[S] = \left\{ S \xrightarrow{f} \mathbb{C} : \text{supp}(f) \text{ is finite } \right\}$$

by identifying $a \in S$ with the indicator function $\chi_a \in \mathbb{C}[S]$, which is $\chi_a(a) = 1$ and 0 elsewhere. Here convolution product and involution are given by

$$(f.g)(a) = \sum_{a=bc} f(b)g(c), \qquad f^*(a) = \overline{f(a^\star)}$$

extends inverse semigroup operations as $\chi_a.\chi_b = \chi_{ab}$ and $\chi_a^* = \chi_{a^\star}$. The completion of $\mathbb{C}[S]$ under ℓ^2-norm is the Hilbert space

$$\ell^2(S) = \left\{ S \xrightarrow{f} \mathbb{C} : \sum_{a\in S} |f(a)|^2 < \infty \right\}, \qquad \langle f, g\rangle = \sum_{a\in S} f(a)\overline{g(a)}.$$

Said differently, we can construct $\ell^2(S)$ by taking S as an orthonormal basis over $\mathbb{C}$ and replacing finitely supported functions with finite formal linear combinations of the form $\sum_{a\in S} af(a)$ etc. Now we can lift the partial bijection constructed in Wagner–Preston representation to a partial isometry of $\ell^2(S)$ such that

$$\overline{\text{span}(a^\star S)} \xrightarrow{\Phi(a)} \overline{\text{span}(aS)}$$

extend linearly, continuously there and by zero on the closed complement of span$(a^\star S)$. That is, any element $a \in S$ can be represented as a partial isometry on $\ell^2(S)$ via the $*$-representation $\Phi : S \to B(\ell^2(S))$ given by

$$\Phi(a) \left(\sum_{b\in S} bf(b) \right) = \sum_{bb^\star \leq a^\star a} abf(b)$$

targeted in closed linear span of $aS = \{b \in S \mid bb^\star \leq aa^\star\}$. In literature, this is known as the *left regular representation* of S on $\ell^2(S)$. [2]

1.1 Idempotents, Partial Ordering and Morphisms

Recall that an element $e \in S$ of a semi-group is called an *idempotent* if it equals to its own square: $e^2 = e$. Let's observe the following simple but non-trivial properties of idempotents of an inverse semigroup, which induce a nice structure on them.

- If $x \in S$ is an idempotent, then $x^\star = x$.
- For any $x \in S$, both $x^\star x$ and $xx^\star$ are idempotents. In fact, all idempotents are of this form.

[2]It should not be hard to develop this in to an equivalence between the category of inverse semigroups and the category of partial isometries of Hilbert spaces. Objects of latter are pairs $(\mathcal{H}, U)$, where $\mathcal{H}$ is a Hilbert space and U is a sub inverse semigroup of partial isometries of $\mathcal{H}$. Also a morphism $(T, \varphi) : (\mathcal{H}_1, U_1) \to (\mathcal{H}_2, U_2)$ is a continuous linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ and a semigroup homomorphism $\varphi : U_1 \to U_2$ such that $Ta = \varphi(a)T$ for all $a \in U_1$.

- $x^\star y$ is an idempotent if and only if $xz = yz$ for $z = y^\star y x^\star x$.
- Idempotents are closed under multiplication, and they commute with each other.

With these properties, we can conclude that idempotent elements of an inverse semigroup form a commutative sub inverse semigroup, which is typically denoted by $E(S)$. In addition, we can deduce that the operation $\star$ is an anti-involution, i.e., $(xy)^\star = y^\star x^\star$ for all $x, y \in S$.

For any two x, y in the inverse semigroup S, the following four conditions are equivalent; in which case we write $x \leq y$.

1. There is an idempotent e such that $x = ey$.
2. $x = xx^\star y$.
3. There is an idempotent f such that $x = yf$.
4. $x = yx^\star x$.

The relation $\leq$ is a partial ordering on S, called the *natural partial ordering* induced by idempotents. Contrast to the other classes of semigroups, such as regular semigroups, the natural partial ordering interacts nicely with the inverse semigroup structure and sub-semigroup of idempotents. In particular, it is compatible with both multiplication and inversion. On $E(S)$, the partial order $e \leq f$ is equivalent to $e = ef$, and hence the product of idempotents becomes their binary meet, i.e., for any $e_1, e_2 \in E(S)$ we have $e_1 \wedge e_2 = e_1 e_2$. Therefore, $E(S)$ is a meet semilattice as well as a downward closed subset of S.

A homomorphism $\varphi : S \to H$ of inverse semigroups that is commute with multiplication, i.e., a map that makes the following diagram commute:

$$\begin{array}{ccc} S \times S & \longrightarrow & S \\ {\scriptstyle \varphi\times\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ H \times H & \longrightarrow & H. \end{array}$$

So, $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in S$. It is clear that homomorphisms of inverse semigroups preserve inverses, idempotents, and respect the partial ordering. That is,

$$\varphi(x^\star) = (\varphi(x))^\star \quad \forall x \in S, \qquad \varphi(E(S)) \subseteq E(H), \qquad x \leq y \implies \varphi(x) \leq \varphi(y).$$

A representation of S on a set (or a topological space) X is a homomorphism $\rho : S \to \mathbf{SymInv}(X)$ of inverse semigroups. This can be thought of as a partial action of S on the space X, and to be a total action ρ should preserve unit and zero whenever they exist.

1.2 Complete and Distributive Inverse Semigroups

It is straightforward to prove when the join $\bigvee A = \bigvee_{a\in A} a$ exists for some $A \subseteq S$ under the natural partial ordering we have $\bigvee A^\star = (\bigvee A)^\star$ and as a consequence idempotents are closed under existing joins, But in general it is not guaranteed to have all possible joins. Here let's investigate inverse semigroups those with all possible joins. Suppose $x, y \leq z$ for any three elements of S, then $xy^\star \leq zz^\star$ and $x^\star y \leq z^\star z$, and therefore a necessary condition for x and y to have a join (or any upper bound) is that both $xy^\star$ and $x^\star y$ are idempotents. The relation $x \sim y$ if $xy^\star$ and $x^\star y$ are idempotents is a compatibility (reflexive and symmetric, but not necessarily transitive) relation on S. On the other hand, when x and y are compatible they necessarily (and sufficient) have the binary meet $x \wedge y = xy^\star y = yx^\star x$. Any subset of S is called *compatible* if each pair of elements in that set is compatible, and S is called *complete* if every compatible subset has a join in S. Complete inverse semigroups experience nice algebraic proprieties such as:

- The join of the empty set $\bigvee \emptyset = 0$ is the least (and absorbing) element satisfying $0s = s0 = 0$ for all $s \in S$.
- The compatible set $E(S)$ has a join if and only if S is an inverse monoid, in that case $\bigvee E(S) = 1$ is the identity of S and it makes $E(S)$ an order ideal.

In a (symmetric) inverse semigroup $\mathbf{SymInv}(X)$, an idempotent is the identity map on some subset of X and the natural partial ordering becomes $x \leq y$ if and only if $\mathrm{dom}(x) \subseteq \mathrm{dom}(y)$ and $x = y\big|_{\mathrm{dom}(x)}$. Hence the natural partial ordering on symmetric inverse semigroups is the restriction order on partial bijections/homeomorphisms. Also in this setting, two partial maps are compatible if they coincide on the intersection of their domains.

Next, another important property of any symmetric (topological) inverse semigroup $\mathbf{SymInv}(X)$ is that there multiplication distributes over all joins that exist. Accordingly, we adopt the following definition: An inverse semigroup S is (infinitely) *distributive*, if for all $s \in S$ and for all subsets $A \subseteq S$ for which $\bigvee A$ exists, the join $\bigvee(sA)$ exists and $s(\bigvee A) = \bigvee(sA)$. In other words,

$$s\left(\bigvee_{a\in A} a\right) = \bigvee_{a\in A} (sa).$$

It is worth to note that we need only left (or right) distributivity, because the other can derive using the distributivity of inverses over joins. We can relate the distributivity of sub inverse semigroup $E(S)$ to that of S via the following important results:

1. S is distributive if and only if $E(S)$ is distributive.
2. A homomorphism $\varphi : S \to H$ of inverse semigroups preserves joins if and only if the restriction $\varphi\big|_{E(S)} : E(S) \to E(H)$ preserves joins.

The distributivity of products over joins has much stronger consequences, as it implies the distributivity of binary meets over joins. Here is the exact statement: Let S be a distributive inverse semigroup. Let $s \in S$ and $\{a_i\}_{i\in I} \subseteq S$ be such that the join $\bigvee_{i\in I} a_i$ and the meet $s \wedge (\bigvee_{i\in I} a_i)$ exists. Then for any $i \in I$ the meet $s \wedge a_i$, and the join $\bigvee_{i\in I}(s \wedge a_i)$ exists. Furthermore, we have

$$s \wedge \left(\bigvee_{i\in I} a_i\right) = \bigvee_{i\in I} (s \wedge a_i)\,.$$

Hence, the distributivity of $E(S)$ implies that of S both with respect to multiplication and binary meets, which in $E(S)$ are the same. Moreover, when S is both complete and distributive we have the following nice result:

Proposition: Given a complete distributive inverse semigroup, idempotents form a locale.

It should be noted that every homomorphisms of inverse semigroups preserve all finite meets, but in general, they does not preserve (existing) joins. Therefore, in order to produce a nice category, we require homomorphisms between complete and distributive inverse semigroups to preserve arbitrary joins.

A particular nice subclass of homomorphisms of complete distributive inverse semigroups is locally isomorphic homomorphisms which we refer as total étale homomorphisms, and we will use a modified version of them in the construction of étale localic groupoid. In addition to preserve joins we require total étale homomorphisms to induce isomorphisms on locales of principal order ideals, i.e., given an total étale homomorphism $\varphi : S \to H$ of complete distributive inverse semigroups

1. $\varphi\left(\bigvee_{i\in I} a_i\right) = \bigvee_{i\in I} \varphi(a_i)$ for any compatible family $\{a_i\}_{i\in I} \subseteq S$
2. $\downarrow a$ and $\downarrow \varphi(a)$ are isomorphic as posets for any $a \in S$

In particular, the second condition is equivalent to an isomorphism of locales

$$E(S) \xrightarrow{\varphi|_{E(S)}} \downarrow \varphi(1_S) \subseteq E(H).$$

1.3 Completion and Maximum Group Image of Inverse Semigroups

The forgetful functor from the category of complete distributive inverse semigroups (together with join-preserving homomorphisms) to the category of inverse semigroups has a free left adjoint, called the completion functor. Here we will briefly sketch the construction leaving some details incomplete.

A subset A of an inverse semigroup S is *permissible* if it is compatible and downward closed. The set of all permissible subsets of S is denoted $C(S)$. It is not difficult to prove that $C(S)$ is a complete distributive inverse semigroup under the pointwise multiplication and pointwise inversion of subsets. The idempotents of $C(S)$ are exactly permissible subsets of $E(S)$, and $C(S)$ is ordered by inclusion order. The function $\iota : S \hookrightarrow C(S)$ given by $s \mapsto \{t : t \leq s\} = \downarrow \{s\}$ is an injective homomorphism of semigroups, and every element of $C(S)$ is a join of a subset of $\iota(S)$. It should be emphasized that, in this construction, ι preserve all existing meets, but almost no joins. For example, suppose $A \in C(S)$ has no greatest element but $a = \bigvee A \in S$. Then $\iota(\bigvee A) = \iota(a)$ while $\bigvee \iota(A) = \bigvee_{a_i \in A} \iota(a_i) = \bigcup_{a_i \in A} \downarrow \{a_i\} = A$.

However, this isomorphic embedding ι has the following universal property: If $\sigma : S \to T$ is some homomorphism to a complete inverse semigroup, then there is a unique join-preserving lifting homomorphism $\Sigma : C(S) \to T$ such that $\Sigma\iota = \sigma$. Diagrammatically

$$\begin{array}{ccc} S & \xrightarrow{\quad\sigma\quad} & T. \\ & {\scriptstyle\iota}\searrow \quad \nearrow{\scriptstyle\Sigma} & \\ & C(S) & \end{array}$$

Therefore, the category of complete and distributive inverse semigroups together with join-preserving homomorphisms forms a reflective (but not full) subcategory of the category of inverse semigroups and ordinary homomorphisms.

1.4 Clifford Inverse Semigroups

A regular semigroup in which idempotents commute with idempotents become an inverse semigroup. If we further enrich this inverse semigroup allowing idempotents to commute with any semigroup element the resulting structure is called a Clifford inverse semigroup. These can be identify as persheaves of groups over meet-semilattices (of idempotents). This subsection is about the representation of Clifford inverse semigroup as semilattice of groups.

Let $\mathcal{L}$ be a meet-semilattice and let $\mathcal{F} : \mathcal{L}^{\mathbf{op}} \to \mathbf{Group}$ be a presheaf. Then we can equip the disjoint union $\bigsqcup_{L \in \mathcal{L}} \mathcal{F}(L)$ with the following operations to construct a Clifford inverse semigroup:

- $(a, L_1)(b, L_2) = (\mathcal{F}(\iota_1)(a)\mathcal{F}(\iota_2)(b), L_1 \wedge L_2)$, where $L_1 \xleftarrow{\iota_1} L_1 \wedge L_2 \xrightarrow{\iota_2} L_2$ are morphisms in the lattice, and $a \in \mathcal{F}(L_1), b \in \mathcal{F}(L_2)$ are group elements.
- $(a, L)^\star = (a^{-1}, L)$, since $\mathcal{F}(\mathrm{id}_L) = \mathrm{id}_{\mathcal{F}(L)}$ this is compatible with the above composition law.
- Idempotents are of the form $(e_{\mathcal{F}(L)}, L)$, where $e_{\mathcal{F}(L)}$ is the identity element of the group $\mathcal{F}(L)$ and therefore they commute with any other semigroup element.

Conversely, set of idempotents $E(S)$ of any inverse semigroup S has the structure of a meet-semilattice and the assignment $\mathcal{F} : E(S)^{\mathbf{op}} \to \mathbf{Set}$ given by $e \mapsto \{a \in S : a^\star a = e\}$ is a persheaf. In particular, when $E(S)$ is central, the presheaf is valued in groups. Furthermore, S is abelian if and only if each of these component group is abelian. Based on this classification we can give the following simple characterization for Clifford inverse semigroups.

Proposition: $E(S)$ is central if and only if $a^\star a = aa^\star$ for all $a \in S$.

Any inverse semigroup S admits a free Cliffordification S_C that has the universal property induced by the left adjoint to the forgetful functor **CliffInvSemiGrp** $\to$ **InvSemiGrp**, in other words **CliffInvSemiGrp** is a reflective subcategory of **InvSemiGrp**. Therefore given any Clifford inverse semigroup T with a semigroup homomorphism $\varphi : S \to T$ there exists a unique semigroup homomorphism $\varphi_C : S_C \to T$ which makes the following diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{\quad\varphi\quad} & T. \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\varphi_C} & \\ & S_C & \end{array}$$

One way to make this construction is form the minimal Clifford congruence,[3] as we did in the maximal group image construction. Let ρ be the intersection of all congruences containing the set $\{(a^\star a, aa^\star) \mid a \in S\}$. Then S/ρ is a Clifford inverse semigroup and, more importantly it has this universal factorization property. Also, we can use a similar congruence to find the universal abelianization of an inverse semigroup, which we will not discuss here.

1.5 Inductive Groupoids

We know that "delooping" an inverse semigroup provides a groupoid. However this construction completely ignores the natural partial ordering, and therefore in order to recover an inverse semigroup from a groupoid we must at least star with an internal groupoid in the category of posets. The Ehresmann–Schein–Nambooripad theorem asserts an equivalence between the category of inverse semigroups and a subcategory of the category of ordered groupoids (with a restriction structure) known as inductive groupoids. In this section, we define inductive groupoids and describe the equivalence between said categories.

An ordered groupoid is an internal groupoid in the finitely complete category of posets, i.e., a groupoid whose set of morphisms is equipped with a partial ordering $\leq$, and all structure maps respect this ordering. An inductive groupoid is an ordered groupoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ with the following additional properties:

- The set of object $\mathcal{G}_0$ admits binary meets, i.e., the partially ordered set of identities forms a meet-semilattice.
- Given $x \in \mathcal{G}_1$ and $e \in \mathcal{G}_0$ such that $e \leq s(x)$ there exists a unique $(x|_\star e) \in \mathcal{G}_1$ such that $(x|_\star e) \leq x$ and $s((x|_\star e)) = e$, which is called the *restriction of x to e.*
- Given $x \in \mathcal{G}_1$ and $f \in \mathcal{G}_0$ such that $f \leq t(x)$ there exists a unique $(f_\star|x) \in \mathcal{G}_1$ such that $(f_\star|x) \leq x$ and $t((f_\star|x)) = f$, which is called the *corestriction of x to f.*

In fact the last two conditions in this definition are equivalent. A morphism of inductive groupoids $\mathcal{G} \to \mathcal{G}'$ is an internal functor from $\mathcal{G}$ to $\mathcal{G}'$. Next we shall describe the component functors between the categories of inverse semigroups and inductive groupoids.

Recall that the groupoid $\mathbf{Ind}(S)$ attached to an inverse semigroup S is the core of the category of idempotents of S. Objects in the category of idempotents are idempotents of S, and an arrow $e \xrightarrow{x} f$ is a triple (e, x, f) of elements in S, where e, f are idempotents and x is any element of S such that $xe = x = fx$. Such an arrow is invertible precisely when $e = x^\star x$ and $f = xx^\star$. Thus the core consists of arrows of the form $x^\star x \xrightarrow{x} xx^\star$. Diagrammatically, we can write identities and inverse of this arrow as

$$x^\star x \circlearrowright x^\star x \underset{x^\star}{\overset{x}{\rightleftarrows}} xx^\star \circlearrowleft xx^\star$$

[3] In general, a congruence on a semigroup S is an equivalence relation ρ on S that respect to its multiplication such that $(a, b), (c, d) \in \rho$ implies $(ac, bd) \in \rho$. An equivalence relation is a congruence if and only if it is both left and right multiplicative, that is $(a, b) \in \rho, c \in S$ implies $(ca, cb) \in \rho$ and $(ac, bc) \in \rho$.

Clearly, **Ind**(S) is an inductive groupoid. Conversely, for an inductive groupoid $\mathcal{G}$, a tensor product $\otimes : \mathcal{G}_1 \times \mathcal{G}_1 \to \mathcal{G}_1$ may be defined by the rule

$$x \otimes y = (x|_\star e_{xy}) \cdot (e_{xy\star}|y),$$

where $e_{xy} = \mathrm{dom}(x) \wedge \mathrm{codom}(y)$ and $\cdot$ indicates composition in $\mathcal{G}$. It may be shown that $(\mathcal{G}_1, \otimes)$ is an inverse semigroup **Inv**$(\mathcal{G})$ and the two notions are equivalent.

Ehresmann-Schein-Nambooripad Theorem: There are canonical natural isomorphisms $S \to \mathbf{Inv}(\mathbf{Ind}(S))$ and $\mathcal{G} \to \mathbf{Ind}(\mathbf{Inv}(\mathcal{G}))$, providing an adjoint equivalence of categories

$$\mathbf{InvSemiGrp} \simeq \mathbf{IndGrpd}.$$

It is easy to see that, under this description complete inverse semigroups correspond to a special class of inductive groupoids whose space of morphisms have all compatible joins and distributivity proprty. In particular, its space of objects (identity morphisms) form a locale. Among many other things, this theorem can also be use to formulate the Lie theory of inverse semigroups.

1.6 C^*-Algebra, Universal Groupoid and Hopf Algebroid

Similar to case of groups and groupoids, there are multiple C^*-algebras that we can assign to an inverse semigroup. Recall the construction of $\ell^2(S)$ that we used to discribe inverse semigroups as partial isometries. Completion of inverse semigroup (convolution) algebra $\mathbb{C}[S]$ under ℓ^1-norm is a Banach $*$-algebra, denoted by $\ell^1(S)$, accompanied with same convolution product and $*$-involution. At first these constructions seems bit ad hoc, but recall the free-forgetful adjunction

$$\mathbf{Group} \underset{(-)^\times}{\overset{\mathbb{C}[-]}{\rightleftarrows}} \mathbf{Alg}_{\mathbb{C}}.$$

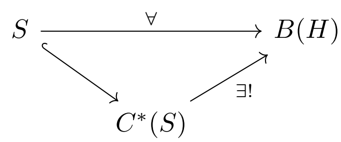
The group ring construction is interesting because every representation of a group G over a $\mathbb{C}$-vector space can be realize as a $\mathbb{C}[G]$-module and vice-versa. Also, this adjunction has few variations,

1. for locally compact, Hausdorff groups via Haar measure that defines a convolution product on the space $C_c(G)$ of continuous complex valued functions on G with compactly support.
2. convolution (discrete) groupoid algebra $\mathbb{C}[\mathcal{G}]$ in which

   $$(f.g)(a) = \sum_{b \in \mathcal{G}} f(ab^{-1})g(b)$$

   as usually, with the caveat that $(a, b) \mapsto ab^{-1}$ is a partial mapping.
3. the adjunction between groups/groupoids and C^*-algebras essentially works in the same way. Moreover, this construction reveals non-commutative phenomenons which are not visible in the groupoid-algebra level.

Naturally, one can expect a similar extended adjunction between inverse semigroups and C^*-algebras and it does exist. Though, the Banach $*$-algebra $\ell^1(S)$ is the free $\mathbb{C}$-algebra associated to S, is not a C^*-algebra, but its C^*-enveloping algebra is precisely the construction that give rise to this adjunction. By the universal property, any representation of S on a complex Hilbert space H uniquely factor along the inclusion $S \hookrightarrow C^*(S)$ as



where $B(H)$ is the C^*-algebra of bounded linear operators on H. Taking the left regular representation of S on $\ell^2(S)$ we can construct a map $C^*(S) \to B(\ell^2(S))$ and image of this map is another C^*-algebra reduced inverse semigroup C^*-algebra, denoted by $C^*_r(S)$. Its closure in the (ultra) weak* topology $VN(S) = \overline{C^*_r(S)}$ is the (left regular representation) von Neumann algebra of S. Clearly the reduced C^*-algebra is a quotient of $C^*(S)$ and they are far from being isomorphic in general. In particular, when G is a (locally compact) group $C^\star_r(G) \cong C^\star(G)$ if and only if G is amenable.

In [1], Paterson assigned to any inverse semigroup S an étale groupoid $G(S)$, which he called its universal groupoid, and showed that the both full and reduced C^*-algebras of S and $G(S)$ coincide. Briefly, the construction consists of two steps:

- Recall that a partial action of S on a topological space X is a semigroup homomorphism $\rho : S \to \mathbf{SymInv}(X)$, with a given such action one can define the “germ” equivalence relation on
$$S \rtimes_\rho X = \{(a, x) \mid a \in S, x \in \mathrm{dom}(\rho(a))\}$$
by $(a, x) \sim (b, y)$ if $x = y$ and there is $e \in E(S)$ such that $ae = be$. The quotient $S \rtimes_\rho X/\sim$ is naturally an étale groupoid under the germ topology, with the unit space X, where X being identified with a subset of $S \rtimes_\rho X/\sim$ via the injection $x \mapsto \{[e, x] \mid e \in E(S), x \in \mathrm{dom}(\rho(e))\}$, and structure maps define in the most obvious way.

  An interesting example of this construction is when $X = \{\star\}$ is a singleton set on which S acts trivially. It is then straightforward to show that $\mathcal{M}(S) = S \rtimes_\rho \{\star\}/\sim$ is really the maximal group image of S.

- Next, construct a topological space $X = \widehat{E}(S)$ intrinsic to S following the philosophy of Stone duality. Consider the spectrum (or the space of nonzero semi-characters)
$$\widehat{E}(S) = \{E(S) \xrightarrow{\chi} \{0, 1\} \mid \chi \text{ is a nonzero semigroup homomorphism}\}$$
of the meet semi-lattice $E(S)$ of idempotents. This space has the canonical subspace topology induced by the product topology of the Cantor cube $\{0, 1\}^E$. Also, under that $\widehat{E}(S)$ is totally disconnected, locally compact and Hausdorff. Any semi-character χ uniquely determine an $E(S)$ valued filter by $\chi^{-1}(1) = \{e \in E(S) \mid \chi(e) = 1\}$ and vice versa. Moreover, we have a canonical partial action $\rho : S \to \mathbf{SymInv}(\widehat{E}(S))$ with $\mathrm{dom}(\rho(a)) = \{\chi \in \widehat{E}(S) \mid \chi(a^\star a) = 1\}$ and $\mathrm{codom}(\rho(a)) = \{\chi \in \widehat{E}(S) \mid \chi(aa^\star) = 1\}$ such that $\rho(a)(\chi)(e) = \chi((ea)^\star(ea)) = \chi(a^\star ea)$ for any $a \in S, e \in E(S)$.

With these two ingredients we have the universal groupoid $G(S) = S \rtimes_\rho \widehat{E}(S)/\sim$.

2 Display Space of a Sheaf over a Locale

2.1 The Construction of Display Locale

The definitions of presheaves (and sheaves) on a topological space X do not involve the underlying set of points of X, but only the lattice structure of its open subsets. It is therefore almost superfluous to mention that one can define presheaves and sheaves on an arbitrary locale. The display locale $\Psi(\mathcal{F})$ is essentially the étale space of an ordinary presheaf $\mathcal{F}$ of sets over a locale. However, the well known construction of étale spaces is not directly transportable to this setting as it uses the stalk and germs at a point of the underlying topological space, and therefore we need an equivalent point-free construction.

First, fix a locale X and often we shall write $\mathbf{Open}(X)$ for the locale itself seen as a frame of open subsets of downward-closed subsets of its elements. Consider the functor $\Gamma : \mathbf{Locale}/X \to \mathbf{Presh}(\mathbf{Open}(X), \mathbf{Set})$ that sends a locale over X to the presheaf of sections, i.e.,

$$(T \xrightarrow{p} X) \longmapsto (U \mapsto \{U \xrightarrow{q} T \mid pq = \iota\}),$$

where $U \overset{\iota}{\hookrightarrow} X$ is the open inclusion. Since (co)limits of presheaves compute objectwise, Γ preserves limits and it also satisfies the solution set condition. So, by the general adjoint functor theorem it has a left adjoint, which we denote by Ψ and refer to as the *display locale functor*. Another way to describe Ψ is, since $\mathbf{Locale}/X$ admits all colimits (and all limits), by the universal property of presheaves, the inclusion functor $\iota : \mathbf{Open}(X) \to \mathbf{Locale}/X$ given by $U \mapsto (U \hookrightarrow X)$ factors uniquely (up to a unique isomorphism) through the Yoneda embedding of $\mathbf{Open}(X)$ into $\mathbf{Presh}(\mathbf{Open}(X), \mathbf{Set})$. This Yoneda extension to ι is precisely the display locale functor:

$$\begin{array}{ccc} \mathbf{Open}(X) & \xrightarrow{\quad \iota \quad} & \mathbf{Locale}/X. \\ {\scriptstyle y}\downarrow & \nearrow_{\Psi = \mathrm{Lan}_y \, \iota} & \\ \mathbf{Presh}(\mathbf{Open}(X), \mathbf{Set}) & & \end{array}$$

To see the equivalence between these two constructions, observe that for any $U \in \mathbf{Open}(X)$,

$$\begin{aligned} \hom(\Psi(y(U)), (T \to X)) &\cong \hom(y(U), \Gamma(T \to X)), && \text{by the adjunction} \\ &\cong \Gamma(T \to X)(U), && \text{by the Yoneda lemma.} \end{aligned}$$

The essential image of Ψ in $\mathbf{Locale}/X$ is, $\mathbf{EtLoc}/X$, the complete and cocomplete subcategory of $\mathbf{Locale}/X$ spanned by local homeomorphisms over X. The adjoint pair $\Psi \dashv \Gamma$ restricts to an adjoint equivalence

$$\mathbf{Sh}(\mathbf{Open}(X), \mathbf{Set}) \underset{\Gamma}{\overset{\Psi}{\rightleftarrows}} \mathbf{EtLoc}/X.$$

Given a presheaf $\mathcal{F} : \mathbf{Open}(X)^{\mathbf{op}} \to \mathbf{Set}$, by co-Yoneda lemma, we can write it as a colimit of representable presheaves over the comma category, and for representable presheaves the display locale is trivial. In general, we can write the pointwise Kan extension formula for an explicit construction of the display locale as

$$\Psi(\mathcal{F}) = (\mathrm{Lan}_y \, \iota)(\mathcal{F}) = \operatorname*{colim}_{\mathcal{D}} \iota(U),$$

where $\mathcal{D}$ is the comma category whose objects are $U \in \mathbf{Open}(X)$ with a natural transformation $y(U) \to \mathcal{F}$ (or equivalently an element of $\mathcal{F}(U)$), and morphisms are commutative triangles of the form

$$\begin{array}{ccc} y(V) & & \\ \downarrow & \searrow & \\ & & \mathcal{F} \\ & \nearrow & \\ y(W). & & \end{array}$$

It is well known that any colimit is a coequalizer of coproducts. In other words, we can obtain the display locale as the coequalizer of the cofork

$$\bigsqcup\nolimits_{V \leq W} V \times \mathcal{F}(W) \underset{W \times \mathcal{F}(W)}{\overset{V \times \mathcal{F}(V)}{\rightrightarrows}} \bigsqcup\nolimits_{U \in \mathbf{Open}(X)} U \times \mathcal{F}(U)$$

is the display locale of the presheaf $\mathcal{F}$, which will be denoted by $\Psi(\mathcal{F})$ from here on. The category of elements $\bigsqcup_{U \in \mathbf{Open}(X)} U \times \mathcal{F}(U)$ of $\mathcal{F}$ over $\mathbf{Open}(X)$ is a poset category with the induced partial ordering given by $(U', a') \leq (U, a)$ whenever $U' \leq U$ and $a' = a\big|_{U'}$, but it is not a locale in general. The free locale generated by it is the poset of downsets of ordered by inclusion, and since in the display locale two parallel maps identify downward closed families

$$\bigsqcup\nolimits_{W} \{V : V \leq W\} \times \mathcal{F}(W),$$

open subsets of the display locale $\Psi(\mathcal{F})$ are unions of sets of the above form under the pasting compatible sections. Moreover, open subsets are generated (under joins) by the principal downward closed subsets of the form $\downarrow (U, a) = \{(U', a') \mid (U', a') \leq (U, a)\}$, and therefore form a basis, i.e., each element in display locale is a join of elements of this sublattice. The lattice structure of open subsets is given by the inclusion order with joins being unions and meets being intersections.

On the other hand, since locales and frames are opposite categories of each other, the above coequalizer is equivalently an equalizer of frames:

$$\prod\nolimits_{V \leq W} \prod\nolimits_{y \in \mathcal{F}(W)} \downarrow V \underset{V \wedge \varphi_{W,y}}{\overset{\varphi_{V,y}\big|_V}{\leftleftarrows}} \prod\nolimits_{U} \prod\nolimits_{x \in \mathcal{F}(U)} \downarrow U,$$

and this leads us to an alternative description of open subsets of $\Psi(\mathcal{F})$. First note that an element of the right hand side product is a family of the form $(\varphi_{U,x})_{U \in \mathbf{Open}(X),\, x \in \mathcal{F}(U)}$, where $\varphi_{U,x} \leq U$ represents some open element in U and the restriction of the section $x \in \mathcal{F}(U)$ to that subset. Now, in the equalizer we must have

$$\varphi_{V,y\big|_V} = V \wedge \varphi_{W,y} \qquad \text{for all } V \leq W \text{ and } y \in \mathcal{F}(W),$$

and this property characterizes factors of open subsets of the locale. Also, with this formulation we can compute finite meets and joins of open subsets as

$$\bigvee_{i\in I}(\varphi^i_{U,x})_{U,\,x} = \left(\bigvee_{i\in I}\varphi^i_{U,x}\right)_{U,\,x} \qquad \bigwedge_{\substack{i\in I\\ \text{finite } I}}(\varphi^i_{U,x})_{U,\,x} = \left(\bigwedge_{\substack{i\in I\\ \text{finite } I}}\varphi^i_{U,x}\right)_{U,\,x}$$

and it is not hard to show that these sets are indeed open.

Before we move into the next section, it would be beneficial to see that above two descriptions of open subsets of $\Psi(\mathcal{F})$ are the same, via a monotonic bijection between them. First, given a principal downward closed set $\downarrow(W,y) \in \bigsqcup_{U\in\mathbf{Open}(X)} U\times\mathcal{F}(U)$, where the section $y\in\mathcal{F}(W)$, we can take $\varphi_{W,y} = \bigvee_{W'\le W} W$ to produce an element of $\prod_U\prod_{x\in\mathcal{F}(U)}\downarrow U$. Then, for any $V\le W$

$$\begin{aligned} V\wedge\varphi_{W,y} &= V\wedge\left(\bigvee_{W'\le W} W'\right)\\ &= \bigvee_{W'\le W}(V\wedge W')\\ &= \bigvee_{V'\le V} V'\\ &= \varphi_{V,y|_V}. \end{aligned}$$

In addition it is easy to see that $\varphi_{V,y|_V} \le \varphi_{W,y}$. Conversely, given a family $(\varphi_{U,x})_{U\in\mathbf{Open}(X),\,x\in\mathcal{F}(U)}$ where $\varphi_{U,x}\le U$ and $V\wedge\varphi_{W,y} = \varphi_{V,y|_V}$ the family

$$\{(U,x)\mid \varphi_{U,x} = U\} \in \bigsqcup_{U\in\mathbf{Open}(X)} U\times\mathcal{F}(U)$$

is downward closed, because $T = T\wedge U = T\wedge\varphi_{U,x} = \varphi_{T,x|_T}$ for all $T\le U$ and $x\in\mathcal{F}(U)$. Also, any compatible subfamily $(U_i,x_i)_{i\in I}$ satisfying $x_i|_{U_i\wedge U_j} = x_j|_{U_i\wedge U_j}$ has a unique $x\in\mathcal{F}\left(\bigvee_{i\in I}U_i\right)$ such that $x_i = x|_{U_i}$, due to the sheaf property of $\mathcal{F}$. Therefore the subfamily has a unique gluing

$$\bigvee_{i\in I}(U_i,x_i) = \left(\bigvee_{i\in I}U_i, x\right)$$

with $U_j\wedge\varphi_{\bigvee_{i\in I}U_i,x} = \varphi_{U_j,x_j} = U_j$ for all $j\in I$ and $\varphi_{\bigvee_{i\in I}U_i,x}\le\bigvee_{i\in I}U_i$, hence

$$\varphi_{\bigvee_{i\in I}U_i,x} = \bigvee_{i\in I}U_i.$$

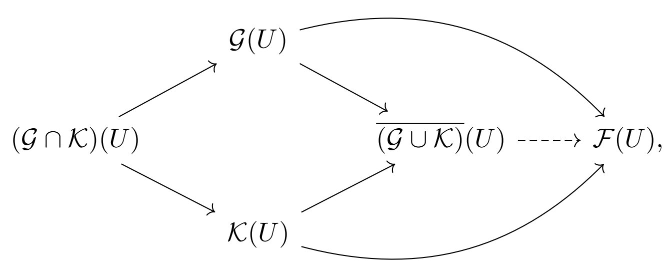
In fact, any open subset in this locale defines a subsheaf of $\mathcal{F}$ adding a new interpretation to display locale construction. Including it there are at least two other descriptions of display locale in literature.

2.2 Localic Groupoid of Subsheaves

An alternative way to describe the localic groupoid constructed in the last theorem is to look at the lattice of subsheaves of $\mathcal{F}$. In general, this lattice is isomorphic to the display locale and when $\mathcal{F}$ is given as in the theorem we can describe a localic groupoid structure on it.

Theorem 1. *The lattice of open subsets of the display locale $\Psi(\mathcal{F})$ is isomorphic to the locale of subsheaves of $\mathcal{F}$. Moreover, the locale of subsheaves of the sheaf obtained by a complete distributive inverse semigroup has a canonical groupoid structure.*

Proof. First lets show that subsheaves an arbitrary sheaf $\mathcal{F}$ of sets over a locale X is again a locale and it is isomorphis to the display locale. Recall that a subpresheaf of $\mathcal{F}$ is a presheaf $\mathcal{G} : \mathbf{Open}(X)^{\mathbf{op}} \to \mathbf{Set}$ together with a natural transformation $\mathcal{G} \Rightarrow \mathcal{F}$ such that the components $\mathcal{G}(U) \to \mathcal{F}(U)$ are monic. Therefore, a subpresheaf $\mathcal{G}$ of $\mathcal{F}$ can be given by $\mathcal{G}(U) \subseteq \mathcal{F}(U)$ for all U, with the induced restriction maps. Being a subsheaf is more subtle, because for every compatible family, a subsheaf should contain the unique gluing of this family, which is also the gluing of the family in $\mathcal{F}$. Therefore $\mathcal{G}$ is a subsheaf of $\mathcal{F}$ if and only if for every covering $U = \bigvee_{i \in I} U_i$ and every $a \in \mathcal{F}(U)$ such that $aU_i \in \mathcal{G}(U_i)$ we have $a \in \mathcal{G}(U)$. Recall the partial order relation on subpresheaves, $\mathcal{G} \leq \mathcal{K}$ if and only if $\mathcal{G}(U) \subseteq \mathcal{K}(U)$ for all $U \in \mathbf{Open}(X)$. The category of presheaves is complete and cocomplete, with both limits and colimits computed pointwise. According to above local characterization, a pointwise intersection of subsheaves is again a subsheaf. However, even though a pointwise union of subsheaves is a presheaf, it is not necessarily a subsheaf. So, subsheaves of $\mathcal{F}$ form an inf lattice, hence also a complete lattice under the inclusion ordering.



where $\overline{(\mathcal{G} \cup \mathcal{K})}$ is the sheafification of the pointwise union of $\mathcal{G}$ and $\mathcal{K}$, given by

$$\overline{(\mathcal{G} \cup \mathcal{K})}(U) = \left\{ a \in \mathcal{F}(U) \mid \text{ there exists } I \text{ such that } U = \bigvee_{i \in I} U_i \text{ and } aU_i \in (\mathcal{G} \cup \mathcal{K})(U_i) \right\}.$$

Here we are using the multiplicative notation instead of restriction for notational simplicity, which should not confuse with multiplication in inverse semigroups. Let's prove that the set of subsheaves $\text{sub}(\mathcal{F})$ is a complete Heyting algebra, which implies that it is also a locale, without going through sheafification. Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be three subsheaves of $\mathcal{F}$. Define

$$(\mathcal{B} \Rightarrow \mathcal{C})(U) = \{a \in \mathcal{F}(U) \mid \forall V \leq U \quad aV \in \mathcal{B}(V) \implies aV \in \mathcal{C}(V)\}.$$

Clearly, for all $a \in (\mathcal{B} \Rightarrow \mathcal{C})(U)$ and for all $V \leq U$, we have $aV \in (\mathcal{B} \Rightarrow \mathcal{C})(V)$. Therefore $\mathcal{B} \Rightarrow \mathcal{C}$ is a subpresheaf of $\mathcal{F}$. To prove this is a subsheaf, take $U = \bigvee_{i \in I} U_i$ in $\mathbf{Open}(X)$ and $a \in \mathcal{F}(U)$ such that $aU_i \in (\mathcal{B} \Rightarrow \mathcal{C})(U_i)$ for all $i \in I$. Given any $V \leq U$,

$$\begin{aligned} aV \in \mathcal{B}(V) &\Rightarrow a(V \wedge U_i) \in \mathcal{B}(V \wedge U_i) \quad \forall i \in I \\ &\Rightarrow a(V \wedge U_i) \in \mathcal{C}(V \wedge U_i) \quad \forall i \in I \\ &\Rightarrow aV \in \mathcal{C}(V) \qquad\qquad\qquad \text{(because } \mathcal{C} \text{ is a subsheaf of } \mathcal{F}\text{)}, \end{aligned}$$

and hence $a \in (\mathcal{B} \Rightarrow \mathcal{C})(U)$. Next, we need to show the duality of meets and implication. If $\mathcal{A} \leq \mathcal{B} \Rightarrow \mathcal{C}$, then for any $a \in (\mathcal{A} \cap \mathcal{B})(U)$ we have $a \in (\mathcal{B} \Rightarrow \mathcal{C})(U) \cap \mathcal{B}(U)$ and hence $a \in \mathcal{C}(U)$. Conversely, suppose $\mathcal{A} \cap \mathcal{B} \leq \mathcal{C}$ then for every $V \leq U$ and $a \in \mathcal{A}(U)$ we have

$$aV \in \mathcal{B}(V) \implies aV \in (\mathcal{A} \cap \mathcal{B})(V) \implies aV \in \mathcal{C}(V).$$

Next, we shall show that $\mathrm{sub}(\mathcal{F})$ is isomorphic to $\Psi(\mathcal{F})$ as posets. Recall that an open subset of $\Psi(\mathcal{F})$ is a downward closed subset of $\bigsqcup_{U \in \mathbf{Open}(X)} \mathcal{F}(U)$ that satisfies the gluing property. This disjoint union with its poset structure is the category of elements of the (pre)sheaf $\mathcal{F}$ and therefore, by the equivalence between presheaves and discrete fibrations[4], we should be able to relate open subsets to subsheaves of $\mathcal{F}$. Given any subsheaf $\mathcal{G} \leq \mathcal{F}$, the set

$$P_{\mathcal{G}} = \{(U, a) : a \in \mathcal{G}(U), U \in \mathbf{Open}(X)\}$$

is clearly a downward closed subset with the gluing property. Conversely, given an open element P of the display locale, there is a restricted projection $\pi : P \to \mathbf{Open}(X)$ induced by the canonical discrete fibration $\bigsqcup_{U \in \mathbf{Open}(X)} U \times \mathcal{F}(U) \to \mathbf{Open}(X)$. The assignment

$$\mathcal{G}_P(U) = \{a : (U, a) \in P\} \hookrightarrow \mathcal{F}(U)$$

defines a subsheaf of $\mathcal{F}$. For any $(V, b) \leq (U, a)$ we have $(V, b) \in P$ and $b = aV \in \mathcal{G}_P(V)$ gives the restriction map induced by $\mathcal{F}$. Also for any compatible family in P, i.e., $\{(U_i, a_i)\}_{i \in I}$ such that $a_i U_j = a_j U_i$ for all i, j, there is a unique $a \in \mathcal{F}\left(\bigvee_{i \in I} U_i\right)$ such that $a_i = aU_i$ and $\left(\bigvee_{i \in I} U_i, a\right) \in P$. This proves that $\mathcal{G}_P$ is indeed a subsheaf. These constructions induce an order-preserving bijection between posets $\mathrm{sub}(\mathcal{F})$ and $\Psi(\mathcal{F})$.

Alternatively, we could look at families $(\varphi_{U,a})_{U \in \mathbf{Open}(X),\, a \in \mathcal{F}(U)}$ satisfying $\varphi_{T,aT} = T \wedge \varphi_{U,a}$ for all $T \leq U$ and $a \in \mathcal{F}(U)$. Given a subpresheaf $\mathcal{G} \leq \mathcal{F}$, we construct an open element of $\Psi(\mathcal{F})$ by

$$\varphi_{U,a} = \bigvee_{\substack{V \leq U \\ aV \in \mathcal{G}(V)}} V.$$

[4]For any category $\mathcal{C}$, the category of presheaves of sets over $\mathcal{C}$ and the category of discrete fibrations over $\mathcal{C}$ are equivalent. This is the 1-categorical version of classical Grothendieck construction.

To see this is in fact an open element in display locale, observe that

$$\begin{aligned}
T \wedge \varphi_{U,a} &= T \wedge \left(\bigvee_{\substack{V \le U \\ aV \in \mathcal{G}(V)}} V \right) \\
&= \bigvee_{\substack{V \le U \\ aV \in \mathcal{G}(V)}} (T \wedge V) \\
&= \bigvee_{aV \in \mathcal{G}(V)} (T \wedge V) \\
&= \bigvee_{\substack{S \le T \\ aTS \in \mathcal{G}(S)}} S \\
&= \varphi_{T,aT} \qquad\qquad \text{for all } T \le U,\ a \in \mathcal{F}(U).
\end{aligned}$$

Since $\{aV \in \mathcal{G}(V) : a \in \mathcal{F}(U)\}_{V \le U}$ is a compatible family, when $\mathcal{G}$ happens to be a subsheaf, we get $\varphi_{U,a} = U$. On the other hand, given an open element $(\varphi_{U,a})_{U \in E(S),\, a \in \mathcal{F}(U)}$ of the display locale, we can construct a unique subsheaf $\mathcal{G}$ by declaring, $a \in \mathcal{G}(U)$ if $\varphi_{U,a} = U$. With this condition, for any $T \le U$ we have

$$\varphi_{T,aT} = T \wedge \varphi_{U,a} = T \wedge U = T$$

and this implies $aT \in \mathcal{G}(T)$, and hence $\mathcal{G}$ is a presheaf. Now take any $U = \bigvee_{i \in I} U_i$ and $a \in \mathcal{F}(U)$ such that $aU_i \in \mathcal{G}(U_i)$, then

$$\begin{aligned}
\varphi_{U,a} &= U \wedge \varphi_{U,a} \\
&= \left(\bigvee_{i \in I} U_i \right) \wedge \varphi_{U,a} \\
&= \bigvee_{i \in I} (U_i \wedge \varphi_{U,a}) \\
&= \bigvee_{i \in I} \varphi_{U_i, aU_i} \\
&= \bigvee_{i \in I} U_i \\
&= U,
\end{aligned}$$

proves the sheaf property of $\mathcal{G}$. Hence, again, the map $\mathrm{sub}(\mathcal{F}) \to \Psi(\mathcal{F})$ given by

$$\mathcal{G} \mapsto (\varphi_{U,a})_{U \in \mathbf{Open}(X),\, a \in \mathcal{G}(U)}, \qquad \varphi_{U,a} = U$$

is an order-preserving bijection.

Next, we shall describe the localic groupoid structure on $\mathrm{sub}(\mathcal{F})$ when $\mathcal{F}$ is the sheaf described in theorem 1. □

2.3 Points of the Display Locale

As we already discussed, a point of the display locale $\Psi(\mathcal{F})$ is a localic continuous map from the abstract point (terminal object in the category of locales) $\mathbf{2} \to \Psi(\mathcal{F})$ (or dual of a frame homomorphism $\mathbf{Open}(\Psi(\mathcal{F})) \to \mathbf{Open}(\mathbf{2}) = \{0 < 1\}$), and they are on a bijective correspondence with completely prime filters in the frame of open elements. Let F be a completely prime filter in $\mathbf{Open}(X)$ (correspond to some point x of the base locale X). We define the stalk of $\mathcal{F}$ at F by

$$\mathcal{F}_F = \operatorname*{colim}_{U \in F} \mathcal{F}(U) = \bigsqcup\nolimits_{U \in F} \mathcal{F}(U) / \sim,$$

where $\sim$ is the equivalence relation $(a \in \mathcal{F}(V)) \sim (b \in \mathcal{F}(W))$ if there is $U \in F$ with $U \leq V \wedge W$ and $a\big|_U = b\big|_U$. Elements (equivalence classes) of the stalk are the germs of sections at the filter F, and explicitly the germ of $a \in \mathcal{F}(V)$ at F is given by $\mathrm{germ}_{V \in F}(a) = \{b \in \mathcal{F}(W) : W \in F \text{ and } a \sim b\}$. This is the point-free reformulation of classical germ of a section (at a point). This germ corresponds to a completely prime filter of $\mathbf{Open}(\Psi(\mathcal{F}))$ given by

$$\uparrow \bigsqcup \{W \leq V \mid W \in F\} \times \{b \in \mathcal{F}(W) \mid b \sim a\},$$

hence a point in $\Psi(\mathcal{F})$. Conversely, let $\hat{F}$ be any completely prime filter in $\mathbf{Open}(\Psi(\mathcal{F}))$. Then the collection $F = \uparrow \{V \in \mathbf{Open}(X) \mid \downarrow (V, a) \in \hat{F} \text{ for some } a \in \mathcal{F}(V)\}$ is a completely prime filter in $\mathbf{Open}(X)$. Moreover if $\downarrow (V, a), \downarrow (W, b) \in \hat{F}$, using its downward closeness we can show that, the sections $a \in \mathcal{F}(V)$ and $b \in \mathcal{F}(W)$ have the same germ at F. In other wards, the point $\hat{F}$ defines a unique germ in the display locale. Therefore points of display locale (completely prime filters) are in a bijective correspondence with germs.

3 Étale Groupoid - Inverse Semigroup Correspondence

3.1 Localic Groupoid of a Complete Distributive Inverse Semigroup

From the section on "Complete and Distributive Inverse Semigroups", it is clear that when the inverse semigroup S is distributive and complete, its set of idempotents $E(S)$ forms a suplattice satisfying the distributivity law

$$x \wedge \left(\bigvee_{y \in Y} y \right) = \bigvee_{y \in Y} (x \wedge y)$$

for any $x \in E(S)$ and $Y \subseteq E(S)$. Hence the set of idempotents canonically inherits the structure of a frame/locale, and therefore ought to be thought as the lattice of open subsets of some hypothetical topological space. Making use of this locale, one can construct an equivalence between the category of complete distributive inverse semigroups and the category of localic étale groupoids via the correspondence between localic sheaf of sets - étale display locale construction. We will investigate their relationship in the following series of propositions.

Theorem 2. *Given a complete distributive inverse semigroup S, the persheaf of sets on the locale of idmpotents $\mathcal{F} : \mathbf{Open}(E(S))^{\mathbf{op}} \to \mathbf{Set}$ given by*

$$U \mapsto \{a \in S : a^\star a = U\}$$

with the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ for $V \leq U$ defined by $a \mapsto aV$ is a sheaf on the locale $E(S)$. The corresponding display locale $\Psi(\mathcal{F})$ has a canonical structure of localic étale groupoid.

Proof. The restrictions maps are clearly functorial and thus $\mathcal{F}$ is a presheaf. The fact that this is a sheaf rather than just a presheaf is precisely equivalent to distributivity and completeness of S. Let $U \in E(S)$, and let $\{U_i\}_{i \in I}$ be idempotents such that $U = \bigvee_{i \in I} U_i$, i.e., $\{U_i\}_{i \in I}$ is an open cover of U. If $a, b \in \mathcal{F}(U)$ are such that $aU_i = bU_i$ for all $i \in I$, then by distributivity

$$\begin{aligned} a &= aU \\ &= a\left(\bigvee_{i \in I} U_i\right) \\ &= \bigvee_{i \in I} (aU_i) \\ &= \bigvee_{i \in I} (bU_i) \\ &= b\left(\bigvee_{i \in I} U_i\right) \\ &= bU \\ &= b. \end{aligned}$$

On the other hand, if there are $a_i \in \mathcal{F}(U_i)$ such that $a_i U_j = a_j U_i$ for all $i, j \in I$, then the family $\{a_i\}_{i \in I}$ is compatible. In other words,

$$a_i(a_j^\star a_j a_i^\star a_i) = a_j(a_j^\star a_j a_i^\star a_i), \qquad a_i^\star(a_j a_j^\star a_i a_i^\star) = a_j^\star(a_j a_j^\star a_i a_i^\star).$$

Hence, by completeness, it has a join $a = \bigvee_{i\in I} a_i$ in S such that

$$\begin{aligned} a^\star a &= \left(\bigvee_{i\in I} a_i\right)^\star \left(\bigvee_{i\in I} a_i\right) \\ &= \bigvee_{i\in I} (a_i^\star a_i) \\ &= \bigvee_{i\in I} U_i \\ &= U, \end{aligned}$$

which implies $a \in \mathcal{F}(U)$. Also we have

$$\begin{aligned} aU_j &= \left(\bigvee_{i\in I} a_i\right) U_j \\ &= \bigvee_{i\in I} (a_i U_j) \\ &= \bigvee_{i\in I} (a_j U_i) \\ &= a_j \left(\bigvee_{i\in I} U_i\right) \\ &= a_j U \\ &= a_j \qquad \text{for any } j \in I. \end{aligned}$$

Observe that, distributivity implies being a separated presheaf and completeness implies that $\mathcal{F}$ has the gulability property. Together they proves the sheaf property of $\mathcal{F}$.

Next, let's show that the display locale $\Psi(\mathcal{F})$ of the sheaf $\mathcal{F}$ associated to the complete distributive inverse semigroup S is a localic étale groupoid over the space of units $E(S)$. Recall that,

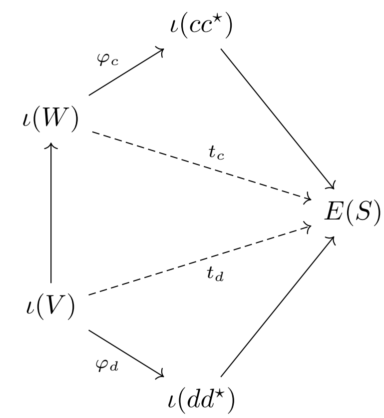
$$\Psi(\mathcal{F}) = \operatorname*{colim}_{\mathcal{D}} \iota(U),$$

where $\mathcal{D}$ is the indexing diagram with objects being $U \in E(S)$ with a natural transformation $y(U) \to \mathcal{F}$ and morphisms being commutative triangles

$$\begin{array}{ccc} y(V) & \longrightarrow & y(W) \\ & \searrow \quad \swarrow & \\ & \mathcal{F} & \end{array}$$

when $V \leq W$. Now in order to specify source and target maps $\Psi(\mathcal{F}) \rightrightarrows E(S)$ it is enough to define both maps separately from each component $\iota(U) \to E(S)$ when there is a natural transformation $y(U) \to \mathcal{F}$ (or equivalently $\mathcal{F}(U) \neq \emptyset$, by Yoneda lemma). First let's define the target map: given $a \in \mathcal{F}(U)$ the target map $\iota(U) \xrightarrow{t_a} E(S)$ is given by the composition $\iota(U) \xrightarrow{\varphi_a} \iota(aa^\star) \xrightarrow{\text{inclusion}} E(S)$.

As maps between frames $\iota(aa^\star) \xrightarrow{\varphi_a} \iota(U)$ is $\varphi_a(b) = (ba)^\star(ba) = a^\star b^\star ba$ for any $b \leq aa^\star$, and intuitively this is the map that restrict the domain of the partial bijection a. The inclusion map on idempotents $E(S) \to \iota(aa^\star)$ maps $e \mapsto eaa^\star$. Clearly both maps (and hence the composition) are frame homomorphisms. Suppose we have an open inclusion $\iota(V) \hookrightarrow \iota(W)$ in **Locale**$/E(S)$ with $c \in \mathcal{F}(W)$ and $d \in \mathcal{F}(V)$ with $d = cV$, then the diagram of locales

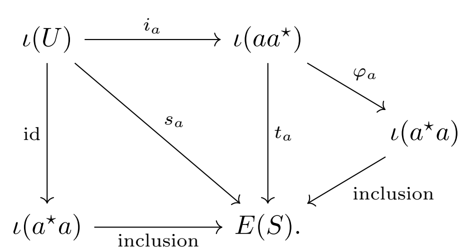


commutes, because

$$(ed)^\star(ed) = (ecV)^\star(ecV) = (ec)^\star(ec)V$$

for any $e \in E(S)$. Hence the target map is compatible with open inclusions. The source map is defined in a similar way but, since $a^\star a = U$ for all $a \in \mathcal{F}(U)$, it is more trivial. So, given $a \in \mathcal{F}(U)$ has component of the source map $\iota(U) \xrightarrow{s_a} E(S)$ given by the composition of localic maps $\iota(U) \xrightarrow{\text{id}} \iota(aa^\star) \xrightarrow{\text{inclusion}} E(S)$ and it is readily varify that this map is compatible with open inclusions.

The identity assigning map $E(S) \to \Psi(\mathcal{F})$ is inherently different from source and target maps as it is a map into the display locale (which is a colimit in the category of locales over $E(S)$). There is a canonical map from $E(S)$ to $\mathcal{D}$ that sends $U \mapsto (\iota(U), U \in \mathcal{F}(U))$, and from each of these components we have a universal localic map into $\Psi(\mathcal{F})$. This composition assigns an identity to each idempotent in a way that is consistent with open inclusions. Also, both source and targets of $U \in E(S)$ is $\iota(U) \xrightarrow{\text{id}} \iota(U) \xrightarrow{\text{inclusion}} E(S)$ as we expected. Next, we will take the inversion map $\Psi(\mathcal{F}) \to \Psi(\mathcal{F})$ as a map from each component $(\iota(U), a \in \mathcal{F}(U))$ to underlying diagram $\mathcal{D}$ to be $\iota(U) \xrightarrow{i_a = \varphi_a} \iota(aa^\star)$ which we already defined as a part of the target map. In fact, as a consequence we have $s_a(i_a) = t_a$ and in addition, commutativity of the diagram



shows that $s_a = t_a(i_a)$. After formally reversing arrows to obtain frame homomorphisms, commutativity of this diagram is equivalent to the straightforward identity

$$((ea^\star)^\star(ea^\star)a)^\star((ea^\star)^\star(ea^\star)a) = ea^\star a$$

for any $e \in E(S)$ and by letting $e \in \iota(U)$ we can deduce that $i_a^2 = \mathrm{id}_{\iota(U)}$. Similarly we can derive that the inverse of the identity of an idempotent is identity itself. Finally, for any open inclusion $V \leq W$ together with $c \in \mathcal{F}(W), d \in \mathcal{F}(V)$ such that $d = cV$ we have the commutative square of locales

$$\begin{array}{ccc} \iota(V) & \xrightarrow{i_d} & \iota(dd^\star) \\ \downarrow & & \downarrow \\ \iota(W) & \xrightarrow{i_c} & \iota(cc^\star). \end{array}$$

For the multiplication map, first note that the indexing comma category $\mathcal{D}$ is filtered, and therefore distributivity of colimits over pullbacks

$$\Psi(\mathcal{F}) \underset{E(S)}{\times} \Psi(\mathcal{F}) = \operatorname*{colim}_{\mathcal{D}} \iota(U) \underset{E(S)}{\times} \operatorname*{colim}_{\mathcal{D}} \iota(V) = \operatorname*{colim}_{U\in\mathcal{D}} \operatorname*{colim}_{V\in\mathcal{D}} \left(\iota(U) \underset{E(S)}{\times} \iota(V) \right)$$

holds. The pullback $\iota(U) \times_{E(S)} \iota(V)$ consists of pairs of the form $((ea)^\star(ea), eb^\star b)$, where $e \in E(S)$ and $a \in \mathcal{F}(U), b \in \mathcal{F}(V)$. Similar to how we defined the inversion map, it is enough to define the multiplication map from each pair of components $((\iota(U), a \in \mathcal{F}(U)), (\iota(V), b \in \mathcal{F}(V)))$ to some other component of the diagram $\mathcal{D}$. This is simply $\iota(U) \times_{E(S)} \iota(V) \xrightarrow{m_{ab}} \iota((ba)^\star(ba))$ which is the frame homomorphism $c \mapsto (c, aca^\star)$ corresponding to $e = (ac)(ac)^\star = aca^\star$. Now it is not difficult to derive all the identities subject to an internal groupoid.

Since the essential image of the display space functor is precisely the étale locales, the source map of the localic groupoid constructed in this way is indeed étale. Also, one can directly prove that it is étale using the basis of principal downward closed subsets $\{(U,a)\}_{U\in E(S), a\in\mathcal{F}(U)}$ which has the properties:

1. $\bigvee_{U\in E(S), a\in\mathcal{F}(U)}(U,a) = \Psi(\mathcal{F})$
2. source map $\Psi(\mathcal{F}) \xrightarrow{s} E(S)$ restricts to an isomorphism between the open sublocales generated $\downarrow (U,a) \longrightarrow \downarrow U$.

This completes the proof. □

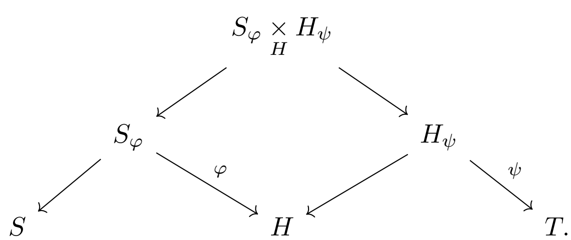
3.2 Category of Étale Complete Distributive Inverse Semigroups

Since étale maps between locales encoded local information, we should take it into account and modify the ordinary notion to get appropriate étale maps between inverse semigroups to reflect this local nature. The rationale behind the following definition of étale homomorphisms is that a map between inverse semigroups $\Gamma(\mathcal{G}) \to \Gamma(\mathcal{G}')$ of open $\mathcal{G}$-slices of étale groupoids $\mathcal{G}, \mathcal{G}'$, that we introduced in first page, make sense only on $\mathcal{G}$-slices that map injectively to $\mathcal{G}'_0$. This forces the étale homomorphism to be a partial map, but still carry enough information to reconstruct a functor between groupoids.

An étale (locally isomorphic partial) homomorphism $\varphi : S \to H$ of complete distributive inverse semigroups is a partial mapping whose domain consists of "sufficiently small elements" $S_\varphi \subseteq S$ which has following properties:

- S_φ is a lower set (downward closed) and closed under inverses.
- S_φ is dense in S, i.e., for each $a \in S$, we have $a = \bigvee_{\substack{s \leq a \\ s \in S_\varphi}} s$.
- φ preserves all inverses and existing products, joins of S_φ.
  In particular, any bounded family $\{U_i\}_{i \in I} \subseteq S$ such that $U_i \leq s$ for all $i \in I$ for some $s \in S_\varphi$ is compatible and therefore the join $\bigvee_{i \in I} U_i \leq s$ exist in S_φ. Then $\varphi\left(\bigvee_{i \in I} U_i\right) = \bigvee_{i \in I} \varphi(U_i)$.
- φ reflects joins of bounded families in S_φ.
  i.e., for any $s \in S_\varphi$, let $\{U_i\}_{i \in I} \subseteq S_\varphi$ be some family whose all elements are bounded by s, ($U_i \leq s$ for all $i \in I$), then $s = \bigvee_{i \in I} U_i$ provided that $\varphi(s) = \bigvee_{i \in I} \varphi(U_i)$ in H.
- For each $s \in S_\varphi$, the restriction $\varphi\big|_{\downarrow s}$ is an isomorphism between posets $\{a \in S \mid a \leq s\}$ and $\{b \in H \mid b \leq \varphi(s)\}$.

Observe that an (ordinary) injective and join preserving homomorphism $\varphi : S \to H$ that induces an isomorphism between idempotents is étale with $S_\varphi = S$. In particular all identity homomorphisms $S \to S$ are étale. To prove that complete inverse semigroups with étale maps form a category we need them to closed under composition. One way to think about the étale homomorphism $\varphi : S \to H$ as a span of total maps $S \leftarrow S_\varphi \xrightarrow{\varphi} H$ with certain properties, in particular the left leg is injective. Then the composition of two such spans is simply the pullback



Note that, since monomorphisms are closed under base changes and compositions, this span makes sense and the following proposition completes the construction.

Proposition The composite of two étale homomorphisms $S \xrightarrow{\varphi} H \xrightarrow{\psi} T$ of complete inverse semigroups (as partial functions) is again étale.

Proof. Let's prove that the $S_{\psi\varphi} = S_\varphi \times_H H_\psi = S_\varphi \cap \varphi^{-1}(H_\psi)$ by verifying that it has all the properties mentioned in the definition.

- Take any $a \in S_\varphi \cap \varphi^{-1}(H_\psi)$ and $b \leq a$. Then since S_φ is downward closed $b \in S_\varphi$, and since $\downarrow a \cong \downarrow \varphi(a)$ we have $\varphi(b) \in H_\psi$. Hence $S_\varphi \cap \varphi^{-1}(H_\psi)$ is downward closed.

- Next, note that $a = \bigvee_{\substack{s \leq a \\ s \in S_\varphi}} s$ for any $a \in S$, and $\varphi(s) = \bigvee_{\substack{h \leq \varphi(s) \\ h \in H_\psi}} h$ for any $s \in S_\varphi$. Again, since $\downarrow s \cong \downarrow \varphi(s)$, for each $h \leq \varphi(s)$ there is a unique $s_h \leq s$ such that $h = \varphi(s_h)$. This implies

$$\begin{aligned} \implies \quad \varphi(s) &= \bigvee_{\substack{s_h \leq s \\ S_\varphi \cap \varphi^{-1}(H_\psi)}} \varphi(s_h) \\ \implies \quad s &= \bigvee_{\substack{s_h \leq s \\ S_\varphi \cap \varphi^{-1}(H_\psi)}} s_h \\ \implies \quad a &= \bigvee_{\substack{s_h \leq a \\ S_\varphi \cap \varphi^{-1}(H_\psi)}} s_h. \end{aligned}$$

- Let $a, b \in S_\varphi$ be such that $\varphi(a), \varphi(b) \in H_\psi$. Then $\psi\varphi(ab) = \psi(\varphi(a)\varphi(b)) = \psi\varphi(a)\psi\varphi(b)$. Similarly, $\psi\varphi$ preserve inverses and all existing joins of $S_\varphi \cap \varphi^{-1}(H_\psi)$.

- Suppose $\{U_i\}_{i \in I} \subseteq S_\varphi \cap \varphi^{-1}(H_\psi)$ is a set of elements bounded by $s \in S_\varphi \cap \varphi^{-1}(H_\psi)$ and $\psi\varphi(s) = \bigvee_{i \in I} \psi\varphi(U_i)$. Note that $\varphi(U_i) \leq \varphi(s)$ for all $i \in I$, since ψ is étale, then we have $\varphi(s) = \bigvee_{i \in I} \varphi(U_i)$. Similarly, since φ is étale, $s = \bigvee_{i \in I} U_i$.

- Finally, for any $a \in S_\varphi \cap \varphi^{-1}(H_\psi)$ it is easy to see that $\downarrow a \cong \downarrow \varphi(a) \cong \downarrow \psi\varphi(a)$ as posets under maps induced by φ and ψ respectively.

□

This proves that complete distributive inverse semigroups together with étale homomorphisms form a category. Moreover, étale homomorphisms are stable under base changes, and have two-out-of-three property. We will not prove these facts here, but let the desired equivalence between étale localic groupoids and complete distributive inverse semigroups to imply them finally. In fact, the étale category of complete distributive inverse semigroups will be complete and cocomplete for the same reason.

Closes thing to étale maps of complete distributive inverse semigroup in the known literature is étale morphisms of pseudogroups. Let X be a topological space and $\mathcal{H}$ be the groupoid whose objects are the open subsets of X and morphisms are the homeomorphisms (or diffeomorphisms, isometries, symplectomorphisms, any local automorphisms in the context of inverse semigroups) between them. A pseudogroup (of local transformations) on X is a wide sub-groupoid $\mathfrak{P}$ of $\mathcal{H}$ satisfying the below sheaf property:

- If $f : U \to V$ is a homeomorphism and $\{U_i\}_{i \in I}$ is a cover of U, then $f \in \mathfrak{P}$ if and only if each restriction $f\big|_{U_i} : U \to f(U_i)$ is in $\mathfrak{P}$.

In his study of foliations, A. Haefliger considered étale morphisms of pseudogroups $\Phi : \mathfrak{P} \to \mathfrak{P}'$ defined as the maximal collection of homeomorphisms satisfying the following conditions:

1. The domains of elements of Φ covers X.
2. If $\varphi \in \Phi$ and $f \in \mathfrak{P}, f' \in \mathfrak{P}'$, then $f'\varphi f \in \Phi$.

3. If $\varphi, \psi \in \Phi$, then $\varphi\psi^{-1} \in \mathfrak{P}'$.

Even though we didn't investigate their relationship in details, it appears that the category of étale complete distributive inverse semigroups contains this category of étale pseudogroups as a full subcategory.

The forgetful functor $\mathcal{U} : \mathbf{CompDisInvSemiGrp} \to \mathbf{Set_{partial}}$ into the category of sets and partial functions between them, which is equivalent to but not isomorphic with the category of pointed sets and point-preserving maps, is faithful functor. Therefore it is concretizable, and as in such categories any morphism is a monomorphism (an epimorphism) when the underlying (partial) function is injective (surjective). For the future reference, we shall completely characterize monomorphisms and epimorphisms of the current category.

Lemma An étale morphism $\varphi : S \to H$ of complete distributive inverse semigroups is

i) a monomorphism if and only if it is injective.

ii) an epimorphism if and only if $h = \bigvee_{\substack{\varphi(s) \leq h \\ s \in S_\varphi}} \varphi(s)$ for all $h \in H$.

Proof.

i) Here the sufficient part is very straightforward. Therefore it is enough to prove only the necessity. Assume φ is a momomorphism and $a, b \in S$ there are such that $\varphi(a) = \varphi(b)$. That induces an isomorphism $\downarrow a \xrightarrow{\varphi\big|_{\downarrow b}^{-1} \varphi\big|_{\downarrow a}} \downarrow b$, and produces a cofork

$$\langle a \rangle \underset{j}{\overset{\iota}{\rightrightarrows}} S \xrightarrow{\varphi} H$$

where $\langle a \rangle \subseteq S$ is the sub-complete inverse semigroup in S generated by $\downarrow a$ and j is the total étale homomorphism that extends $\varphi\big|_{\downarrow b}^{-1} \varphi\big|_{\downarrow a}$. Since $\varphi\iota = \varphi j$, we have $\iota = j$ and hence $a = b$.

ii) Let φ be an epimorphism, then

$$S \xrightarrow{\varphi} H \underset{\mathrm{id}}{\overset{f}{\rightrightarrows}} H$$

□

Next, given an étale homomorphism of complete distributive inverse semigroups $\varphi : S \to H$, we shall expect a functor between corresponding étale localic groupoids $\Psi(\varphi) : \Psi(\mathcal{F}_S) \to \Psi(\mathcal{F}_H)$ internal to the ambient category **Locale** restricted to étale maps. As a first step towards this, we show that étale localic groupoids of S and S_φ are isomorphic.

Lemma

i) Let $\theta : I \to J$ be a functor between two small categories, and let $F : I \to C$ and $G : J \to C$ respectively be two diagrams in a cocomplete category C such that $F = G\theta$. Then there is a canonical morphism $\theta_* : \mathrm{colim}_I(F) \to \mathrm{colim}_J(G)$ between colimits of diagrams. Moreover, if θ is an equivalence, then colimits are isomorphic.

ii) Let I be a small category and $I' \overset{\iota}{\hookrightarrow} I$ be a dense subcategory, i.e., every object in I is canonically a colimit of objects in I'. For any cocontinuous $F : I \to C$ into a cocomplete category, we have $\operatorname{colim}_I F \cong \operatorname{colim}_{I'} F\iota$.

Proof.

i) A cocone (c, t) for the diagram $G : J \to C$ [5] induces a cocone $(c, t\theta)$ for $F : I \to C$ since compositions $t_{\theta(i)} : F(i) = G(\theta(i)) \to C$ respects the structure of I by the functoriality of θ. In particular, a colimit of G induces a cocone of F. Since a colimit $\operatorname{colim}_I(F)$ of F is a universal cocone for F, we must have a unique morphism $\theta_* : \operatorname{colim}_I(F) \to \operatorname{colim}_J(G)$ commuting with the injective morphisms of both colimits. In addition, if θ happens to be an equivalence of categories, θ^{-1} induces a unique morphism $\theta_*^{-1} : \operatorname{colim}_J(G) \to \operatorname{colim}_I(F)$ and therefore colimits are isomorphic.

ii) Part i) provides a unique morphism $\operatorname{colim}_{I'} F\iota \to \operatorname{colim}_I F$ that is compatible with all injective morphisms of both diagrams. On the other hand, Any $i \in I$ is the colimit over the diagram $I'/i \to I$ and therefore any cocone for the diagram $F\iota : I' \to C$ is also a cocone for $F : I \to C$ which continue to compatible with morhisms of I. This provides a unique morphism $\operatorname{colim}_I F \to \operatorname{colim}_{I'} F\iota$ and hence they are isomorphic.

□

Recall that, the associated localic groupoid $\Psi(\mathcal{F}_S)$ of a complete distributive inverse semigroups S is

$$\Psi(\mathcal{F}_S) = \operatorname*{colim}_{\mathcal{D}_S} \iota_S,$$

where $\iota_S : \mathcal{D}_S \to \mathbf{EtLoc}/E(S)$ and indexing diagram $\mathcal{D}_S$ is the poset category whose objects being pairs $(U \in E(S), a \in \mathcal{F}(U))$. Composing ι_S with the forgetful functor $\mathfrak{U}_S : \mathbf{EtLoc}/E(S) \to \mathbf{EtLoc}$ we get a diagram in **EtLoc**, and since colimit in a slice category computed as a colimit in the underlying category, that still has the same colimit $\Psi(\mathcal{F}_S)$. Now, given an étale homomorphism of complete distributive inverse semigroups $\varphi : S \to H$, it is easy to see that $\mathcal{D}_{S_\varphi}$ is dense in $\mathcal{D}_S$ and hence by the second part of lemma $\Psi(\mathcal{F}_S) \cong \Psi(\mathcal{F}_{S_\varphi})$ as étale locales.

Next, We can observe that φ induces a functor, $\theta_\varphi : \mathcal{D}_{S_\varphi} \to \mathcal{D}_H$ by $(U, a) \mapsto (\varphi(U), \varphi(a))$ that satisfies $\mathfrak{U}_S \iota_S = \mathfrak{U}_H \iota_H \theta_\varphi$, between indexing comma categories of corresponding étale localic groupoids. This produces a morphism between étale localic groupoids as explained in first part of the lemma. Note that here commutativity condition is equivalent to saying $\downarrow a \cong \downarrow \varphi(a)$ as posets for any $a \in S_\varphi$ which is exactly the étale condition for homomorphisms of complete distributive inverse semigroups. Now, according to the last lemma we have an induced étale map of display locales $\Psi(\varphi) : \Psi(\mathcal{F}_{S_\varphi}) \to \Psi(\mathcal{F}_H)$ which is compatible with injective maps of diagrams. One should note that, since S_φ may be an inverse semigroup, the presheaf $\mathcal{F} : \mathbf{Open}(E(S_\varphi))^{\mathbf{op}} \to \mathbf{Set}$ does not make sense. But as **EtLoc** is cocomplete still the $\operatorname{colim}_{\mathcal{D}_{S_\varphi}} \iota_{S_\varphi}$ exist and it is isomorphic to the étale localic groupoid $\Psi(\mathcal{F}_S)$. It only remains to show that this induced map is indeed a groupoid homomorphism.

[5] that is, an object $c \in C$ and a collection of morphisms from each vertex $t_j : G(j) \to c$ compatible with all the morphisms in J

3.3 Complete Distributive Inverse Semigroup of a Localic Groupoid

To construct a complete distributive inverse semigroup associated to a given localic étale groupoid $\mathcal{G}$, we can use local bisections of $\mathcal{G}$. It is well known that the global bisections of a Lie groupoid form a infinite dimensional Lie group under pointwise composition, which can be develop to a functor between said categories. A (global) bisection of a Lie groupoid is a smooth section $\sigma : \mathcal{G}_0 \to \mathcal{G}_1$ of the source map such that $t\sigma : \mathcal{G}_0 \to \mathcal{G}_0$ is a diffeomorphism. Set of all bisections form a group, sometimes called the guage group of $\mathcal{G}$, with respect to composition

$$(\rho\sigma)(x) = \rho(t(\sigma(x)))\sigma(x),$$

and inversion

$$\sigma^{-1}(x) = i(\sigma((t\sigma)^{-1}(x)))$$

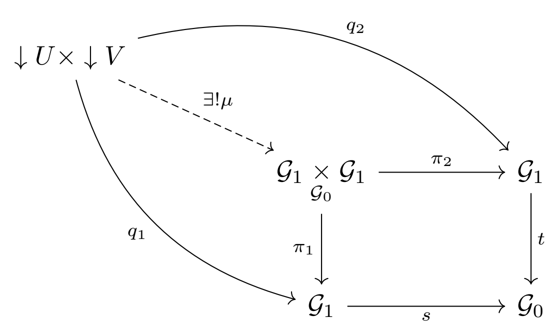
and identity being the unit (identity assigning) map of $\mathcal{G}$. The group of bisections $\mathrm{Bisec}(\mathcal{G})$ form a (infinite dimensional) Lie group and it has a natural action on $\mathcal{G}$ induce by the post composition map $\mathrm{Bisec}(\mathcal{G}) \xrightarrow{t\_} \mathrm{Aut}(\mathcal{G})$. These operations can easily modify to obtain the inverse semigroup of local bisections of a Lie groupoid.

Similarly, for a localic étale groupoid, by taking local bisections, we can functorially associate a complete distributive inverse semigroup as stated in the following theorem.

Theorem 3. *Let $(\mathcal{G}, s, t, u, i, m) = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ be a localic étale groupoid in traditional notation. The set of local bisections of $\mathcal{G}$, denoted by $\Gamma(\mathcal{G})$, form a complete distributive inverse semigroup. Moreover, it defines a functor Γ :* **EtLocGrpd** $\to$ **CompDisInvSemiGrp**.

Proof. A local bisection of $\mathcal{G}$ is an open sublocales $\downarrow U \hookrightarrow \mathcal{G}_1$ such that both $s, t : \downarrow U \to \mathcal{G}_0$ are open monomorphism of locales. Since the inversion i is a global homeomoephism of $\mathcal{G}_1$, it is sufficient to require this only for the source map s. Also, since $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ is étale, there is a basis of $\mathcal{G}_1$ on which both source and target maps are local homeomorphisms, therefore it contained in $\Gamma(\mathcal{G})$. In the case of $\mathcal{G}$ is spatial, one can identify these local bisections with open $\mathcal{G}$-slices that we introduced in the first page. Operations on the guage group of a Lie groupoid extends to a partial composition rule and an inversion on the set of all local bisections $\Gamma(\mathcal{G})$ making it in to an inverse monoid with a zero element. Furthermore, idempotents of this inverse semigroup are precisely restrictions of the unit map to open sublocales of $\mathcal{G}_0$.

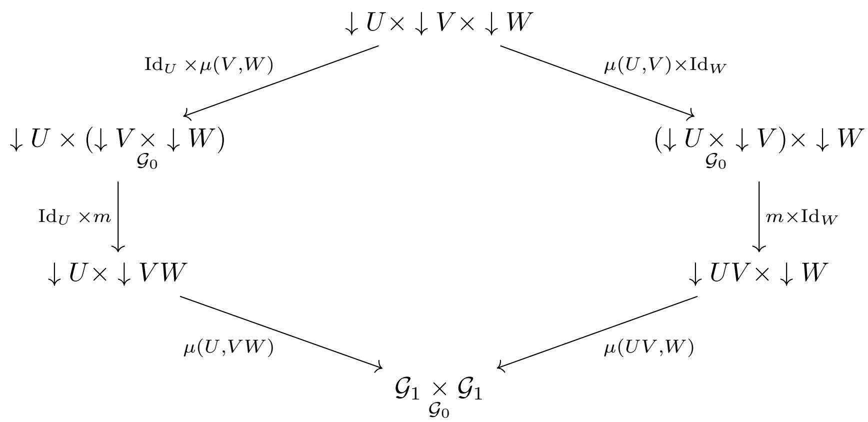
Note that, given any two open sublocales $\downarrow U, \downarrow V \hookrightarrow \mathcal{G}_1$, the inclusion $\downarrow U \times \downarrow V \hookrightarrow \mathcal{G}_1 \times \mathcal{G}_1$ uniquely factor through $\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1$ as

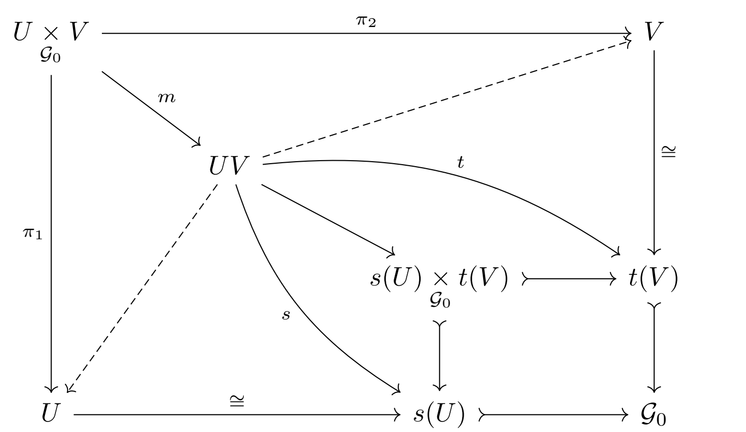


by the universal property of fibered products. In particular, all the maps in this diagram are étale and therefore open. Composing this unique étale map μ with the multiplication map $\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \xrightarrow{m} \mathcal{G}_1$ we get another open sublocale of $\mathcal{G}_1$, which is precisely the product UV, i.e.,

$$UV = m\mu(U, V).$$

Observe that $s(UV) = sm\mu(U, V) = s\pi_2\mu(U, V)$ and similarly $t(UV) = tm\mu(U, V) = t\pi_1\mu(U, V)$. Associativity of multiplication of $\Gamma(\mathcal{G})$ follows for that of $\mathcal{G}$, and it can present as commutativity of the diagram





Next, since inversion i is a homeomorphism on $\mathcal{G}_1$, we define

$$U^\star = i(U).$$

Clearly, $U^\star$ is again a local bisection for any such $U \in \Gamma(\mathcal{G})$ as it is just a different embedding of U into $\mathcal{G}_1$. Furthermore, $\mu(U^\star, U) = (U^\star, U)$ and therefore $s(U^\star U) = s(U) = t(U^\star) = t(U^\star U)$ for all $U \in \Gamma(\mathcal{G})$. The open sublocale $\downarrow U^\star U \to \mathcal{G}_1$ factors through $\mathcal{G}_0 \to \mathcal{G}_1$ given by the unit assigning map if and only if U is a local bisection.

Note that the source map s is open, and therefore $s(\mathcal{G}_1) = \mathcal{G}_0$ is open in $\mathcal{G}$. Hence $\mathcal{G}_0$ is a local bisection with $\mathcal{G}_0^\star = \mathcal{G}_0$. Moreover $\mu(U, \mathcal{G}_0) = (U, s(U)), \mu(\mathcal{G}_0, U) = (t(U), U)$ implies $U\mathcal{G}_0 = U = \mathcal{G}_0 U$ for any $U \in \Gamma(\mathcal{G})$. Similarly, in general

$$U = \begin{cases} UV, & \text{if } s(U) = U^\star U \leq V \\ \\ VU, & \text{if } t(U) = UU^\star \leq V \end{cases}$$

and especially $\emptyset$, the least element of $\mathcal{G}_0$, is the local bisection that act as the zero. Any local bisection inside $\mathcal{G}_0$ is clearly an idempotent under this multiplication. Conversely, if $U = U^2$, then $s(U) = t(U)$ and $U^\star U = UU^\star$, hence $U^\star U = U^\star UU = UU^\star U = U$ shows that $\downarrow U$ lie inside $\mathcal{G}_0$. Consequently, the idempotent meet-semilattice of $\Gamma(\mathcal{G})$ consists precisely of the open sublocales of $\mathcal{G}_0$ with the natural ordering of local bisections being the order induced by $\mathcal{G}_1$.

Since

$$UU^\star U = U, \qquad U^\star UU^\star = U^\star$$

and all idempotents commute with each others, we have an inverse semigroup structure on $\Gamma(\mathcal{G})$. Furthermore, join of a compatible collection of local bisections $\{U_i\}_{i \in I}$ is again a local bisection. To see this, observe that, $\bigvee_{i \in I}(\downarrow U_i) = \downarrow (\bigvee_{i \in I} U_i)$ exist in $\mathcal{G}$ and inclusions $\downarrow U_i^\star U_i \to \mathcal{G}_0 \to \mathcal{G}_1$ induce a unique factorization of the inclusion of their colimit such that

$$\begin{array}{ccc} \bigvee_{i \in I}(\downarrow U_i^\star U_i) & \longrightarrow & \mathcal{G}_1. \\ & \searrow \quad \nearrow_{u} & \\ & \mathcal{G}_0 & \end{array}$$

Then

$$\bigvee_{i \in I} (\downarrow U_i^\star U_i) = \downarrow \left(\bigvee_{i \in I} U_i \right)^\star \left(\bigvee_{i \in I} U_i \right)$$

proves that $\bigvee_{i \in I} U_i$ is again a local bisection of $\mathcal{G}$. Also, $\Gamma(\mathcal{G})$ is distributive as a consequence of distributivity of the locale $\mathcal{G}_1$. Collectively, $\Gamma(\mathcal{G})$ form a complete distributive inverse semigroup.

Let $\mathcal{F} : \mathcal{G} \to \mathcal{G}'$ be an internal functor between étale localic groupoids. Lets show that, its functorialy induces an étale homomorphisms between corresponding complete distributive inverse semigroups of local bisections. Define the mapping $\Gamma(\mathcal{F}) : \Gamma(\mathcal{G}) \to \Gamma(\mathcal{G}')$ by $\Gamma(\mathcal{F})(U) = \mathcal{F}(U)$ for any local bisection $U \in \Gamma(\mathcal{G})$. Since every morphism of locales has a left adjoint, they preserve monomorphisms and hence $\mathcal{F}(U) \xrightarrow{s} \mathcal{F}(\mathcal{G}_0)$ is a monomorphism. But the composition $\mathcal{F}(U) \to \mathcal{F}(\mathcal{G}_0) \to \mathcal{G}'_0$ may not be a monomorphisms in general, and therefore we cannot implement $\Gamma(\mathcal{F})$ to a total homomorphism. Recall that an étale homomorphism $\Gamma(\mathcal{F}) : \Gamma(\mathcal{G}) \to \Gamma(\mathcal{G}')$ is a partial mapping whose domain $\Gamma(\mathcal{G})_{\Gamma(\mathcal{F})}$ has certain nice properties. Also, by the definition of étaleness, there is a maximal family $\mathfrak{F} = \{a_i\}_{i \in I} \subseteq \mathcal{G}_1$ such that

1. $\bigvee_{a_i \leq U} a_i = U$ for any $U \in \mathcal{G}_1$
2. for every $i \in I$, $\mathcal{F}$ restricts to an isomorphism $\downarrow a_i \to \downarrow \mathcal{F}(a_i)$ between open sublocales.

Let $\Gamma(\mathcal{G})_{\Gamma(\mathcal{F})}$ be the set

$$\{U \in \Gamma(\mathcal{G}) \cap \mathfrak{F} \mid \downarrow \mathcal{F}(U) \xrightarrow{s} \mathcal{G}'_0 \text{ is a monomorphism}\}.$$

Then the mapping $\Gamma(\mathcal{F}) : \Gamma(\mathcal{G})_{\Gamma(\mathcal{F})} \to \Gamma(\mathcal{G}')$ given by $\Gamma(\mathcal{F})(U) = \mathcal{F}(U)$ is well defined. Next, we should show that $\Gamma(\mathcal{G})_{\Gamma(\mathcal{F})}$ has all the desired properties.

- All the sets $\Gamma(\mathcal{G}), \mathfrak{F}$ and $\{U \in \mathcal{G}_1 \mid \downarrow \mathcal{F}(U) \xrightarrow{s} \mathcal{G}'_0 \text{ is a monomorphism}\}$ are downward closed. Hence $\Gamma(\mathcal{G})_{\Gamma(\mathcal{F})}$ also has the same property.
- Since $\Gamma(\mathcal{G}')$ form a basis of $\mathcal{G}'_1$, for any $U \in \Gamma(\mathcal{G})$ we have $\mathcal{F}(U) = \bigvee_{\substack{U' \leq \mathcal{F}(U) \\ U' \in \Gamma(\mathcal{G}')}} U'$. By the isomorphism $\downarrow U \cong \downarrow \mathcal{F}(U)$, for each $U' \leq \mathcal{F}(U)$ we can find a unique $V \leq U$ such that $\downarrow \mathcal{F}(V) = \downarrow U' \to \mathcal{G}'_0$ is a monomorphism. Here $U = \bigvee_{\substack{V \leq U \\ \mathcal{F}(V) \in \Gamma(\mathcal{G}')}} V$ and we can cover each V by open subsets of $\Gamma(\mathcal{G}) \cap \mathfrak{F}$, hence

$$U = \bigvee_{\substack{a \leq U \\ a \in \Gamma(\mathcal{G})_{\Gamma(\mathcal{F})}}} a.$$

- Suppose the product $U_1 U_2$ exist in $\Gamma(\mathcal{G})_{\Gamma(\mathcal{F})}$. Then

$$\begin{aligned} \Gamma(\mathcal{F})(U_1U_2) &= \mathcal{F}(U_1U_2) \\ &= \mathcal{F}(m\mu(U_1, U_2)) \\ &= m(\mathcal{F}(\mu(U_1, U_2))) \\ &= m\mu(\mathcal{F}(U_1), \mathcal{F}(U_2)) \\ &= \mathcal{F}(U_1)\mathcal{F}(U_2) \\ &= \Gamma(\mathcal{F})(U_1)\Gamma(\mathcal{F})(U_2) \end{aligned}$$

  shows that $\Gamma(\mathcal{F})$ preserve existing products. Also, by being a morphism of locales $\mathcal{F}$ preserve all joins and inverses.
- Let $\{U_i\}_{i \in I}$ be a family of elements in $\Gamma(\mathcal{G})$ bounded by some $U \in \Gamma(\mathcal{G})_{\Gamma(\mathcal{F})}$.
- Since $\Gamma(\mathcal{G})_{\Gamma(\mathcal{F})} \subseteq \mathfrak{F}$, for any $a \in \Gamma(\mathcal{G})_{\Gamma(\mathcal{F})}$ we have an isomorphism $\downarrow a \to \downarrow \mathcal{F}(a)$ of open sublocales.

Now, it is immediate that Γ is a functor from the category of étale localic groupoids to the étale category of complete distributive inverse semigroups.

□

For the sake of completion of this construction, we should discuss the analogue of natural transformations for étale complete distributive inverse semigroups. Suppose there are two functors $\mathcal{F}, \mathcal{F}' : \mathcal{G} \rightrightarrows \mathcal{G}'$ and a natural transformation $\theta : \mathcal{F} \Rightarrow \mathcal{F}'$. For brevity, lets take $\varphi, \psi : S \rightrightarrows H$ to be the corresponding inverse semigroups and their homomorphisms. It is clear that $S_\varphi \cap S_\psi$ is downward closed and dense in S, while $\downarrow \varphi(a) \cong \downarrow \psi(a)$ for all $a \in S_\varphi \cap S_\psi$. Components of the natural transformation create $\theta_e \in H$ for any idempotent $e \in S_\varphi \cap S_\psi$ such that $\theta_e^\star \theta_e = \varphi(e)$ and $\theta_e \theta_e^\star = \psi(e)$. Moreover, the naturality square

$$\begin{array}{ccc} s\mathcal{F}(U) & \xrightarrow{\mathcal{F}(U)} & t\mathcal{F}(U) \\ {\scriptstyle\theta_{s(U)}}\downarrow & & \downarrow{\scriptstyle\theta_{t(U)}} \\ s\mathcal{F}'(U) & \xrightarrow{\mathcal{F}'(U)} & t\mathcal{F}'(U) \end{array}$$

implies that any $a \in S_\varphi \cap S_\psi$ with $a^\star a, aa^\star \in S_\varphi \cap S_\psi$ satisfies

$$\theta_{aa^\star}\varphi(a) = \psi(a)\theta_{a^\star a}.$$

3.4 Adjoint Equivalence of EtLocGrpd and CompDisInvSemiGrp

Now that we have studied construction of an étale localic groupoid of a complete distributive inverse semigroup and vise versa, we are ready to establishes an adjunction between the category of étale localic groupoids **EtLocGrpd** and the category of complete distributive inverse semigroups **CompDisInvSemiGrp** which will be in fact an equivalence. Consider the pair of functors:

$$\mathbf{CompDisInvSemiGrp} \underset{\Gamma}{\overset{\Psi}{\rightleftarrows}} \mathbf{EtLocGrpd}.$$

Now it is enough to establishes the unit and counit natural transformations, and show that they satisfy triangle identities.

Let $\eta : 1_{\mathbf{CompDisInvSemiGrp}} \Rightarrow \Gamma\Psi$ be the natural transformation define component-wise by $\eta_S : S \to \Gamma\Psi(S)$ the principal ideal mapping $a \mapsto \downarrow a$. This is clearly an étale morphism of complete distributive inverse semigroups, and commutativity of

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & H \\ {\scriptstyle\eta_S}\downarrow & & \downarrow{\scriptstyle\eta_H} \\ \Gamma\Psi(S) & \xrightarrow{\Gamma\Psi(\varphi)} & \Gamma\Psi(H), \end{array}$$

which equivalent to $\Gamma\Psi(\varphi)(a) = \downarrow \varphi(a)$ for all $a \in S$, shows the naturality of η. Moreover, η_S is clearly an isomorphism of inverse semigroups.

Next, by the equivalence of sheaves and étale spaces, any étale groupoid $\mathcal{G}$ is a sheaf over its space of objects $\mathcal{G}_0$ and hence there is a natural bijection

$$\hom_{\mathbf{EtLocGrpd}}(\Psi\Gamma(\mathcal{G}), \mathcal{G}) \cong \hom_{\mathbf{EtLoc}/\mathcal{G}_0}(\Gamma(\mathcal{G}), \Gamma(\mathcal{G})),$$

The map in the right hand side dual to $\mathrm{id}_{\Gamma(\mathcal{G})} : \Gamma(\mathcal{G}) \to \Gamma(\mathcal{G})$ provides a canonical map $\Psi\Gamma(\mathcal{G}) \to \mathcal{G}$. Explicitly, this is the map induced by $\mathcal{D}_{\Gamma(\mathcal{G})} \to \mathcal{G}$ with $(U^\star U, U) \mapsto U$, and it give rise to a natural transformation $\epsilon : \Psi\Gamma \Rightarrow 1_{\mathbf{EtLocGrpd}}$ whose components are exactly the canonical map $\epsilon_{\mathcal{G}} : \Psi\Gamma(\mathcal{G}) \to \mathcal{G}$. Naturality of ϵ

$$
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\mathcal{F}} & \mathcal{G}' \\
\uparrow{\scriptstyle \epsilon_{\mathcal{G}}} & & \uparrow{\scriptstyle \epsilon_{\mathcal{G}'}} \\
\Psi\Gamma(\mathcal{G}) & \xrightarrow{\Psi\Gamma(\mathcal{F})} & \Psi\Gamma(\mathcal{G}')
\end{array}
$$

is implied by $\Psi\Gamma(\mathcal{F})(U) = \mathcal{F}(U)$, and it is easy to see that this is also a natural isomorphism.

Al though the constructions of display locale and the inverse semigroup of a localic groupoid are quit involved, unit and couint maps are seemingly trivial. In addition, they satisfy triangle identities for trivial reasons.

3.5 Étale Bundles of Groups over a Locale

Previously we established an adjoint equivalence between the category of complete distributive inverse semigroups and that of localic étale groupoids via the display locale construction of a particular sheaf valued in sets. In this section we further enrich that construction by replacing sheaf of sets with a sheaf of groups, i.e., a local system valued in groups. The resulting inverse semigroups are the Clifford inverse semigroups that we studied in section 1.4 and they are in a correspondence with étale bundle of groups on locales.

According to our previous discussions, provided the completeness and distributivity the Clifford inverse semigroup become a sheaf of groups over the locale of idempotents, or simply a locale of groups.

References

[1] A. Haefliger, Leaf closures in Riemannian foliations, in: T. Mizutani, Y. Matsumoto, S. Morita (Eds.), A Fête of Topology, Academic Press, Boston, MA, 1988, pp. 3–32.

[2] A.L.T. Paterson, Groupoids, inverse semigroups, and their operator algebras, Birkhauser, 1999.

[3] B. M. Schein, Completions, translational hulls, and ideal extensions of inverse semigroups, Czechoslovak Journal of Mathematics 23 (1973), 575-610.

[4] Francis Borceux, Handbook of Categorical Algebra, Cambridge University Press, 1994.

[5] I. Moerdijk, J. Mr˘cun, Introduction to Foliations and Lie Groupoids, Cambridge University Press, 2003.

[6] Jorge Picado, Ales Pultr, Frames and Locales — Topology without points, Springer, 2012.

[7] Mark V. Lawson, Inverse semigroups — The Theory of Partial Symmetries, World Scientific, 1998.

[8] P.T. Johnstone, Stone Spaces, Cambridge Stud. Adv. Math., vol. 3, Cambridge Univ. Press, 1982.

[9] P.T. Johnstone, Sketches of an Elephant — A Topos Theory Compendium, vol. 2, Oxford Logic Guides 44, Oxford University Press, 2002.

[10] Pedro Resende, Lectures on étale groupoids, inverse semigroups and quantales. https://www.math.tecnico.ulisboa.pt/ pmr

[11] Ryu Exel, Inverse semigroups and combinatorial C*-algebras, https://arxiv.org/pdf/math/0703182.pdf.