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A SPECTRAL SEQUENCE ASSOCIATED WITH A NONLINEAR DIFFERENTIAL EQUATION, AND ALGEBRO-GEOMETRIC FOUNDATIONS OF LAGRANGIAN FIELD THEORY WITH CONSTRAINTS

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This note is a direct continuation of [1]. Here we construct a spectral sequence $\{E_r^{p,q}\}$, one of whose differentials is the operator Er, which associates the corresponding Euler-Lagrange equations to a Lagrangian. As a result, in field theory with constraints it is possible to avoid Lagrange multipliers and to obtain conceptual clarity. The terms $E_1^{p,q}$ are described independently as the Spencer-type homology of the differential operator that is used to find the symmetries (in the sense of [6]) of the constraint equations. When there are no constraints this homology is trivial, which allows us to compute completely the spectral sequence and to compute the cohomology of the complex introduced in [1], prolonging the operator Er (see also [2]).

Below we shall use the definitions and notation of [1], [3]–[6].

1. Some facts from the projective theory of nonlinear differential equations. Let N be a smooth manifold, $\dim N = n + m$, $m > 1$. The term k-jet will designate the class of n-dimensional submanifolds $L \subset N$ that are mutually tangent to order $k \geqslant 0$ at a point $x \in N$. $(L)_x^k$ will denote the k-jet of the submanifold $L \subset N$ at the point x, $N_m^k = \bigcup_{x\in N} N_m^k(x)$, where $N_m^k(x)$ is the set of all k-jets at the point x, and $\pi_{k,l}\colon\ N_m^k \longrightarrow N_m^l$, $k \geqslant l$, is such that $\pi_{k,l}((L)_x^k) = (L)_x^l$; $N_m^\infty = \lim \operatorname{inv}_{k\to\infty} N_m^k$.

We note that $N = N_m^0$. In case $m = 1$ we take as N_1^1 some contact manifold of dimension $2n + 1$ and we set $N_1^k = (N_1^1)_{n+1}^{k-1} \cap \{\bigcup (L)_x^{k-1} \mid x \in N_1^1,\ L \text{ is an integral manifold}\}$. Let $L^n \subset N$ ($L \subset N_1^1$ for $m = 1$) be some submanifold (an integral submanifold for $m = 1$). We set $j_k(L)\colon\ L \longrightarrow N_m^k$, $j_k(L)(x) = (L)_x^k$ ($= (L)_x^{k-1}$ for $m = 1$). We denote by $j(L)$ the inverse limit of the mappings $j_k(L)$ relative to the chain $\pi_{k,k-1}$, $k \longrightarrow \infty$.

Let $m > 1$, let $\alpha\colon\ E^{n+m} \longrightarrow M^n$ be a submersion and let $f\colon\ E \longrightarrow f(E) \subset N$ be a diffeomorphism. Then, setting $f_{(k)}(\sigma_e^k) = (\sigma(M))_{f(e)}^k$, where $\sigma \in \Gamma_{\mathrm{loc}}(\alpha)$, we obtain an injective mapping $f_{(k)}\colon\ J^k(\alpha) \longrightarrow N_m^k$. The pair $(f_{(k)}, J^k(\alpha))$ is called an affine chart on N_m^k, and we introduce on N_m^k a smooth structure that is compatible with all the possible affine charts. We denote by $F_m^k(N)$ the corresponding ring of smooth functions on N_m^k, and we denote by $F_m(N)$ the direct limit of the rings $F_m^k(N)$ relative to the chain of homomorphisms $\pi_{k,k-1}^*$, observing that the subrings $F_m^k(N)$ form a filtration of $F_m(N)$. The inverse limit $f_{(\infty)}\colon\ J^\infty(\alpha) \longrightarrow N_m^\infty$ of the system $f_{(k)}$, $k \longrightarrow \infty$, will be called an affine chart in N_m^∞. If $m = 1$, for the definition of the affine charts $(f_{(k)}, J^k(\alpha))$ we need to start with some contact diffeomorphism $f_{(1)}\colon\ J^1(\alpha) \longrightarrow f_{(1)}(J^1(\alpha)) \subset N_1^1$.

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**Editor's note.* The present translation incorporates corrections made by the author.

Every system of nonlinear differential equations of order $\leqslant k$ on an n-dimensional submanifold of N^{n+m} can be treated as a submanifold $\mathfrak{E} \subset N^k_m$. We denote by $\mathfrak{E}_{k+l}$ the lth prolongation of the equation $\mathfrak{E}$ and we set $\mathfrak{E}_l = \pi_{k-l}(\mathfrak{E})$ for $l \leqslant k$. It is obvious that $\pi_{s,l}(\mathfrak{E}_s) \subset \mathfrak{E}_l$, so that the inverse limit $\mathfrak{E}_\infty$ of the chain $\cdots \longrightarrow \mathfrak{E}_{s+1} \xrightarrow{\pi_{s+1,s}} \mathfrak{E}_s \longrightarrow \cdots$ is defined as is the direct limit $F(\mathfrak{E})$ of the chain $\cdots \longrightarrow F_s(\mathfrak{E}) \xrightarrow{\pi^*_{s+1,s}} F_{s+1}(\mathfrak{E}) \longrightarrow \cdots$, $F_s(\mathfrak{E}) = C^\infty(\mathfrak{E}_s)$, so that $F(\mathfrak{E})$ is filtered by the images of the subrings $F_s(\mathfrak{E})$. By an affine chart on $\mathfrak{E}_\infty$ we will mean the intersection of an affine chart on N^∞_m with $\mathfrak{E}_\infty$.

In what follows our considerations will originate exclusively in the category of filtered $\mathbf{R}$-differential operators (d.o.) over the filtered ring $F(\mathfrak{E})$. If $\mathfrak{F}$ is the functor of differential calculus in the sense of [3], and Φ is the object representing it in the category that we are considering, then we set

$$\mathscr{C}\Phi=\{\varphi\in\Phi \mid [j(L)(\varphi)](x)=0,\ \forall x\in L^n\subset N,\ (L)_x^\infty\in\mathscr{E}_\infty\},$$
$$\mathscr{C}\mathfrak{F}=\operatorname{Ann}\mathscr{C}\Phi, \quad \bar{\Phi}=\Phi/\mathscr{C}\Phi.$$

The localization of the module $\bar{\Phi}$ to an affine chart is canonically identified with $F(\mathfrak{E}) \otimes_{C^\infty(M)} \Phi(M)$, where $\Phi(M)$ is the representing object for the functor $\mathfrak{F}$ over $C^\infty(M)$ in a geometric subcategory of [5]. In particular, $\bar{\Lambda}^k = 0$ if $k > n$. Further the cohomology of the complex $(\bar{\Lambda}^*, \bar{d})$ is denoted by $\bar{H}^q(\mathfrak{E})$, and the de Rham cohomology of the manifold $\mathfrak{E}$ is denoted by $H^q(\mathfrak{E})$.

2. **Symmetries of nonlinear equations.** An automorphism A of the filtered ring $K = F(\mathfrak{E})$ is called an (intrinsic) symmetry of the equation $\mathfrak{E}$ if $A^* \circ \mathcal{C} = \mathcal{C} \circ A^*$. If $X \in D(K)$ and $X(\mathcal{C}\Phi) \subset \mathcal{C}\Phi$ for all Φ, then X is called an infinitesimal symmetry of the equation $\mathfrak{E}$ and we denote by $\operatorname{Sym}\mathfrak{E}$ the Lie algebra of all such symmetries. Then $\mathcal{C}D(K) \subset \operatorname{Sym}\mathfrak{E}$ and we can consider the quotient $\kappa(\mathfrak{E}) = \operatorname{Sym}\mathfrak{E}/\mathcal{C}D(K)$. We also set $\kappa = \kappa(N^k_m)$, noting that κ does not depend on k.

THEOREM 1. *The following assertions hold*:

1) *Let* $\operatorname{Norm}\mathfrak{E}$ *be the normalizer of the subring* $\mathcal{C}\operatorname{Diff}_* K \subset \operatorname{Diff}_* K$. *Then the natural imbedding* $\kappa(\mathfrak{E}) \longrightarrow \operatorname{Norm}\mathfrak{E}/\mathcal{C}\operatorname{Diff}_* K$ *is an isomorphism.*

2) *Every intrinsic infinitesimal symmetry is the restriction to* $\mathfrak{E}$ *of some external infinitesimal symmetry.*

3) *If* $P = \{P_k\}$ *is a filtered projective module over* K *and the equation* $\varphi = 0$, $\varphi \in P_k$, *gives* $\mathfrak{E} \subset N^k_m$, *then* $\kappa(\mathfrak{E}) = \ker l_\varphi|_{\mathfrak{E}}$, *where* l_φ *is the operator of universal linearization* (*see* [5], [6]), *and* $l_\varphi|_{\mathfrak{E}}$ *is its restriction to* $\mathfrak{E}$.

THEOREM 2 (B. A. KUPERŠMIDT). *If* $\mathfrak{E}_\infty = J^\infty(\alpha)$, *then* $\pi(E)$ *is equal to the module of all evolution differentiations of the ring* $\operatorname{dif}_*(\alpha, 1_M)$.

For the proof see [5].

3. **A $\mathcal{C}$-spectral sequence.** In what follows Λ^i denotes the object representing the functors D_i in our category of differential operators over $F(\mathfrak{E}) = K$. It is not difficult to show that $\mathcal{C}\Lambda^i = \mathcal{C}\Lambda^1 \wedge \Lambda^{i-1}$ and $\mathcal{C}\Lambda^*$ is an ideal in Λ^*. We denote by $\mathcal{C}^k\Lambda^*$ the kth power of the ideal $\mathcal{C}\Lambda^*$ and we consider the filtration of the K-module Λ^* by these ideals: $\Lambda^* = \mathcal{C}^0\Lambda^* \supset \mathcal{C}^1\Lambda^* \supset \cdots \mathcal{C}^k\Lambda^* \supset \cdots$. The ideal $\mathcal{C}\Lambda^*$ is stable relative to the exterior differentiation d, since any natural operator obviously commutes with the operation $\mathcal{C}$. Therefore the $\mathcal{C}^k\Lambda^*$ are subcomplexes of Λ^* and the filtration indicated above

leads to a spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ which converges to the de Rham cohomology algebra of the "manifold" $\mathcal{E}_\infty$. Here p is the filtration index and $p+q$ is the degree. For "good" equations $\mathcal{E}$, for example for formally integrable ones, this cohomology is identical with $H^*(\mathcal{E})$.

Let $\mathcal{C}_{(k)}\mathrm{Diff}^{(+)}\hat{P}$ denote the kth exterior power of the K-module $\mathcal{C}\,\mathrm{Diff}_*^{(+)}\hat{P}$, $\hat{P} = \mathrm{Hom}_K(P, \overline{\Lambda}^n)$ and the equation $\mathcal{E}$ has the form $\varphi = 0$, $\varphi \in P$, where the module $P = \{P_k\}$ is projective, and φ is such that functions of the form $\gamma(\varphi)$, $\gamma \in P_k^*$, generate the ideal of the submanifold $\mathcal{E}$. We recall also that every operator $\Delta \in \mathrm{Diff}_*(P, Q)$ generates a mapping on the chain level $\overline{S}_*(\Delta)$: $\overline{S}_*(P) \longrightarrow \overline{S}_*(Q)$, where $S_*(W)$ is the Spencer complex

$$\cdots \to \dot{D}_i(\mathrm{Diff}_*{}^+W) \to \cdots \to \dot{D}(\mathrm{Diff}_*{}^+W) \to \mathrm{Diff}_*W.$$

In Theorem 3 we shall use Spencer complexes in the $\mathcal{C}$-category of filtered differential operators over K, in which we consider only operators of the form $\mathcal{C}\,\mathrm{Diff}_*(P, Q)$. In this category all the general constructions of differential calculus in the sense of [3] hold in a natural way, so we need only make the substitutions $\mathfrak{F} \longrightarrow \mathcal{C}\,\mathfrak{F}$, $\Phi \longrightarrow \overline{\Phi}$.

THEOREM 3. 1) $E_0^{0,q} = \overline{\Lambda}^q$, $d_0 = \overline{d}$.

2) $E_0^{p,q}$, $p \geqslant 1$, *is canonically isomorphic to the cokernel of the homomorphism*

$$\mathscr{C}_{(p-1)}\,\mathrm{Diff}_*^+\hat{\varkappa}_{\mathscr{E}} \otimes \mathscr{C}\dot{D}_{n-q}(\mathscr{C}\,\mathrm{Diff}_*^+\hat{P}) \xrightarrow{1\otimes\overline{S}_{n-q}(l_\varphi^*)|_{\mathscr{E}}} \mathscr{C}_{(p-1)}\,\mathrm{Diff}_*^+\hat{\varkappa}_{\mathscr{E}} \otimes \mathscr{C}\,\dot{D}_{n-q}(\mathscr{C}\,\mathrm{Diff}_*^+\hat{\varkappa}_{\mathscr{E}}),$$

where $\overline{S}_r(\Delta)$ is the restriction of $\overline{S}_*(\Delta)$ to $\dot{D}_r(\mathrm{Diff}_*^+ P)$. Here the differential $d_0^{p,q}$ is identified with the operator $1 \otimes S_*^{n-q}$, where S_*^i: $\dot{D}_i(\mathrm{Diff}_*^+) \longrightarrow \dot{D}_{i-1}(\mathrm{Diff}_*^+)$ is the Spencer operator.

COROLLARY 1. $E_1^{0,q} = \overline{H}^q(\mathcal{E})$, *and the term $E_1^{p,q}$, $p > 0$, is equal to the $(n-q)$th Spencer homology group* (*see* [3]) *of the operator $l_\varphi^*|_{\mathcal{E}}$ with coefficients in $\mathcal{C}_{(p-1)}\mathrm{Diff}_*^+\hat{\kappa}_{\mathcal{E}}$.*

4. **The absolute case:** $\mathcal{E}_\infty = N_m^\infty$ **or** $J^\infty(\alpha)$. In this situation we can set $P = 0$ in the notation of Theorem 3, which allows us to use the following result, where $\mathcal{H}_s(P)$ denotes the s-dimensional homology of the Spencer complex $\overline{S}_*(P)$ (see [3]) in the $\mathcal{C}$-category.

THEOREM 4. $\mathcal{H}_s(P) = 0$, $s > 0$, *and* $\mathcal{H}_0(P) = P$ *for an arbitrary equation $\mathcal{E}$ and an arbitrary filtered K-module P.*

COROLLARY 2. *In the absolute case $E_1^{0,q} = \overline{H}^q(N_m^\infty)$, $E_1^{p,q} = 0$, $p > 0$, $q \neq n$, and $E_1^{p,n}$ is identified with the set of all skew $\hat{\kappa}$-valued $(p-1)$-forms w on the K-module κ such that $Y_1 \lrcorner (\cdot\cdot\cdot(Y_{p-2} \lrcorner w)\cdots)$ as a 1-form of $Y \in \kappa$ is an antisymmetric (relative to $*$) differential operator in the $\mathcal{C}$-category for any $Y_i \in \kappa$. Here the operator $d_1 = d_1^{p,n}$: $E_1^{p,n} \longrightarrow E_1^{p+1,n}$, restricted to an affine chart, acts according to the formula*

$$(d_1w)(f_1,\ldots,f_k) = \sum_s (-1)^{s+1}\partial_{f_s}(w(f_1,\ldots,\hat{f}_s,\ldots,f_k))$$

$$+\sum_{s<t}(-1)^{s+t}w(\{f_s,f_t\},\ldots,\hat{f}_s,\ldots,\hat{f}_t,f\ldots,f_k)$$

$$+\frac{1}{k}\sum_s\{f_s, w(f_1,\ldots,\hat{f}_s,\ldots,f_k)\}^*,$$

where we write $w(f, \ldots)$ *in place of* $w(\partial_f, \ldots)$ *and* $\{f, g\}^* = l_g^*(f) - l_f^*(g)$.

COROLLARY 3. *If* $i < n$, *then* $\overline{H}^i(N_m^\infty) = H^i(N_m^\epsilon)$, *where* $\epsilon = 1$ *for* $m > 1$, *and* $\epsilon = 2$ *for* $m = 1$; $E_2^{j,n} = H^{j+n}(N_m^\epsilon)$ *if* $j > 0$. *Furthermore, for* $i < n$, *we have* $\overline{H}^i(J^\infty(\alpha)) = H^i(E)$ (*where* $E = J^0(\alpha)$) *and* $E_2^{j,n} = H^{j+n}(E)$ *if* $j > 0$.

5. **Absolute Lagrangian formalism.** A form $\omega \in \overline{\Lambda}^n$ is called a Lagrangian density, and its $\overline{d}$-cohomology class $\mathfrak{L} = [\omega] \in \overline{H}^n(N_m^\infty)$ is called a Lagrangian. Thus, the set of all Lagrangians on N is $E_1^{0,n}$. Further, the Euler equation corresponding to $\mathfrak{L}$ will be understood to be the element $d_1(\mathfrak{L}) \in E_1^{1,n} = \hat{\kappa}$. More precisely, let $d_1(\mathfrak{L}) \in \hat{\kappa}_k$ ($\{\hat{\kappa}_k\}$ is the filtration of $\hat{\kappa}$). Then $L \subset N$ is an extremal for $\mathfrak{L}$ if the restriction of $d_1(\mathfrak{L})$ to the submanifold im $j_k(L)$ is equal to zero.

We shall say that the Lagrangian L is trivial if $d_1(\mathfrak{L}) = 0$.

COROLLARY 4. *The vector space of trivial Lagrangians in the projective (respectively affine) case is* $H^n(N_m^\epsilon)$ (*respectively* $H^n(J^0(\alpha))$).

REMARK. The consideration of variational problems with boundary conditions is carried out according to the scheme described above. In this case it is necessary to introduce, in a suitable manner, a submanifold $B_\infty \subset N_m^\infty$, which will realize the boundary conditions, and then to consider the $\mathcal{C}$-spectral sequence generated by the filtration of the relative complex $\Lambda^*(N_m^\infty, B_\infty)$ by powers of the ideal $\mathcal{C}\Lambda^*(N_m^\infty, B_\infty)$.

6. **The $\mathcal{C}$-spectral sequence of the equation $\mathcal{E}$.** In this case, in the notation of Theorem 3 we have

THEOREM 5. 1) $E_1^{1,q} = \overline{H}^q(\mathcal{E})$;
2) $E_1^{p,q} = 0$ *if* $p > 0$, $q \neq n-1, n$;
3) $E_1^{1,n-1} = \ker l_\varphi^*|_\mathcal{E}$, $E_1^{1,n} = \operatorname{coker} l_\varphi^*|_\mathcal{E}$.

COROLLARY 5. $E_3^{p,q} = E_\infty^{p,q}$.

We note that the group $E_1^{0,n-1} = \overline{H}^{n-1}(\mathcal{E})$ is the group of conservation laws of the equation $\mathcal{E}$, and $\ker l_\varphi|_\mathcal{E} = \operatorname{Sym} \mathcal{E}$ [5], [6].

COROLLARY 6. *If* $l_\varphi|_\mathcal{E} = l_\varphi^*|_\mathcal{E}$ (*i.e.* $\mathcal{E}$ *is self- or anti-conjugate*), *then* $d_1^{0,n-1}\rho$ *is an infinitesimal symmetry of* $\mathcal{E}$ *for every conservation law* ρ *of* $\mathcal{E}$.

COROLLARY 7. *If* $\mathcal{E}$ *is formally integrable, then* $\overline{H}^q(\mathcal{E}) = H^q(\mathcal{E})$, $q \leqslant n-2$, *and* $\ker d_1^{0,n-1} = H^{n-1}(\mathcal{E})$.

7. **Lagrangian theory with constraints.** Such a theory in the language used by us is formulated exactly as in the case in which there are no constraints. More precisely, suppose that we have a variational problem with constraints, i.e. the varying quantities satisfy some equation $\mathcal{E}$ (the equation of the constraints) and we are required to find the extremals of the Lagrangian $\mathfrak{L}$ under such variations. Then it is natural to understand the Lagrangian $\mathfrak{L}$ as an element of the group $\overline{H}^n(\mathcal{E}) = E_1^{0,n}$, and the "Euler-Lagrange equation" corresponding to it as an element $d_1^{0,n}(\mathfrak{L}) \in E_1^{1,n}$. If $d_1(\mathfrak{L})$ is an element of the k-th filtration, then a solution $L \subset N$ to the equation $\mathcal{E}$ is an extremal of the variational problem under consideration if and only if the value of $d_1(\mathfrak{L})$ on the submanifold im $j_k(L)$ is equal to zero. It is natural to call the condition on $L \subset N$ arising in this way an Euler-Lagrange equation.

The details of the theory described here and its variants will be given elsewhere.

REMARK.* The proposed conception of the Euler-Lagrange equation is natural in the category of nonlinear partial differential equations (see [6]). For example, Euler-Lagrange equations are invariance under morphisms $\mathcal{E}'_\infty \longrightarrow \mathcal{E}_\infty$ which corresponds to nonlinear differential operators defined on solutions of $\mathcal{E}'$ with values in solutions of $\mathcal{E}$. This variance property is well known for transformations of the set of dependent and independent variables.

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* *Added in translation.*