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FORMAL GEOMETRY OF SYSTEMS OF DIFFERENTIAL EQUATIONS

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INTRODUCTION

0.1. In 1970, I. M. Gelfand introduced the idea of *formal geometry*: By applying the differential-geometric and homological concepts to infinite jet spaces, various objects such as the characteristic classes of differential-geometric structures can be studied systematically. This idea was carried out with success in the study of the characteristic classes of foliations [14] and the combinatorial formulas of the Pontrjagin classes [7]. Furthermore, it seems that this idea was in the background of the *formal calculus of variations* introduced in the study of the conserved densities of the Korteweg-de Vries equation [29].

In this article, by applying this idea to systems of differential equations, we obtain a geometric [2] framework for the study of general systems of differential equations. In spite of its simplicity, this framework turns out to be quite useful.

Now we list two salient aspects of this framework:

▷ We can speak about the "de Rham complex and vector fields" on the solution spaces, which clarify interrelations among various geometric concepts so far introduced in the study of systems of differential equations. For example, such concepts as first integrals, the integral invariants of Poincaré-Cartan, conservation laws, the characteristic classes of foliations, Bott's vanishing theorem and variational operators can be captured uniformly by "the de Rham complex and its cohomology" of the solution spaces of various systems of differential equations. Such concepts as contact transformations and the so-called Lie-Bäcklund transformations can be understood as "vector fields" on the solution spaces. Furthermore, the formal calculus of variations mentioned above, which is a formalization of the functional calculus, is obtained as "the differential calculus" on the solution spaces.

▷ This framework makes the totality of all the systems of differential equations into a category, which seems to give us an advantageous viewpoint in the

[1] This was added in translation and includes some developments which came to light after the publication of the original paper.

[2] In this article, we use the word "geometric" as a synonym of the adjective "manifold theoretic." An approach is called *geometric* when it is based on such concepts as tangent vectors, differential forms, the de Rham complex, vector fields, Lie derivation, etc.

study of various differential correspondences, such as the classical Bäcklund transformations and the Penrose transformation.

0.2. **Basic constituents.** The technical construction of the framework can be summarized as follows.

Let $\mathscr{D}$ be a system of differential equations. Let $R_\infty(\mathscr{D})$ be the space of its formal solutions, which can be introduced either as the infinite jet space of its solutions or by the process of its *infinite prolongation.* This space $R_\infty(\mathscr{D})$ is a fiber bundle over the space B of independent variables.

This fiber bundle has a natural connection H, which is a subbundle of the tangent bundle[3] $TR_\infty(\mathscr{D})$, complementary to the subbundle V formed by the vectors tangent to the fibers. This connection is integrable, that is, $[\Gamma(H),\Gamma(H)] \subset \Gamma(H)$ holds, where $\Gamma(H)$ denotes the space of all C^∞ sections of the vector bundle $H \to R_\infty(\mathscr{D})$ and $[\ ,\ ]$ denotes the bracket product of vector fields. The solutions of $\mathscr{D}$ can be identified with the H-flat[4] sections of the fiber bundle $R_\infty(\mathscr{D}) \to B$. This pair $(R_\infty(\mathscr{D}), H)$ is the basic element of our framework, which plays the role of the solution space of $\mathscr{D}$.

The decomposition of the tangent bundle $TR_\infty = V \oplus H$ produces *the variation bicomplex* $\{\Omega^{*,*}(\mathscr{D}), \delta, \partial\}$ whose total complex is the de Rham complex $\{\Omega^*(R_\infty(\mathscr{D})), d\}$, where $\Omega^{r,s} = \Gamma(\wedge^r V^* \otimes \wedge^s H^*)$ and δ and ∂ denote respectively the $(1,0)$ and $(0,1)$ components of exterior differentiation d.

The spectral sequence $E(\mathscr{D}) = \{E_r^{p,q}, d_r^{p,q}\}$ induced from the filtration by the first index plays the role of the de Rham complex and the de Rham cohomology of the solution space. This spectral sequence was first introduced by A. M. Vinogradov [16].

The Lie algebra $\mathscr{L}(\mathscr{D})$ of all symmetries of $\mathscr{D}$ is defined by

$$\mathscr{L}(\mathscr{D}) = \{X \in \Gamma(V) \mid [X, \Gamma(H)] \subset \Gamma(H)\},$$

whose elements are sometimes called Lie-Bäcklund transformations of $\mathscr{D}$ and are in a sense the vector fields on the solution space.

The following remarks deserve to be kept in mind:

- ▷ It is much easier to go from $\mathscr{D}$ to $R_\infty(\mathscr{D})$ than from $\mathscr{D}$ to its solution space. There is no general method to determine the solution space of a general $\mathscr{D}$, but there does exist an algorithm to determine $R_\infty(\mathscr{D})$. (See the remark in §3.3.)

[3] Throughout this article, we use the following notations. For a C^∞-manifold M we denote its smooth function algebra by $C^\infty(M)$, its tangent bundle by TM, the Lie algebra of its vector fields by $\mathfrak{X}(M)$, the de Rham complex by $\Omega^*(M) := \{\bigoplus \Omega^i(M), d\}$, and the de Rham cohomology by $H^*(M, \mathbf{R})$. The notation "$M = \mathbf{R}_x^n$" is an abbreviation for the sentence that $M = \mathbf{R}^n$ and $x = (x^1, \dots, x^n)$ is the standard system of linear coordinates on M. We write $C^\infty(\mathbf{R}_x^n)$ also as $C^\infty(x^1, \dots, x^n)$. When E is a vector bundle on M, we denote by E^* and $\wedge^p E$, respectively, its dual and its pth exterior product bundle. The space of its C^∞-sections is denoted by $\Gamma(E)$. A vector subbundle of TM is called *a plane field* on M. A plane field H is called *integrable* if $[\Gamma(H), \Gamma(H)] \subset \Gamma(H)$ holds.

[4] A section $s \in \Gamma(\pi)$ is called *H-flat* if $T(s(B)) = H|_{s(B)}$ holds.

▷ If $(R_\infty(\mathscr{D}_1), H) \cong (R_\infty(\mathscr{D}_2), H)$, then the two equations $\mathscr{D}_1$ and $\mathscr{D}_2$ are essentially the same, although they can have quite different appearances.

▷ Each of the following three objects contains "complete information" about a system of differential equations $\mathscr{D}$:

- the pair $(R_\infty(\mathscr{D}), H)$,
- the algebra $C^\infty(R_\infty(\mathscr{D}))$ endowed with the Lie algebra $\Gamma(H)$ of derivations,[5]
- the variation bicomplex $\{\Omega^{*,*}(\mathscr{D}), \delta, \partial\}$.

▷ We can use various concepts and methods of differential geometry in the study of $R_\infty(\mathscr{D})$, since it is the projective limit of finite-dimensional manifolds.

0.3. **Historical background.** Historically, the framework of formal geometry of differential equations is rooted both in the geometric theory and in the formal theory[6] of differential equations. These two are closely interrelated and cannot be separated.

The origin of the geometric approach to differential equations is the idea of regarding the derivatives of unknown functions as independent variables. Although this is an obvious idea, we note that even the concept of differential equation cannot be formulated without it.

It was S. Lie who recognized this idea as an effective method. He made explicit the scheme of geometric approach by introducing the concept of jets, which he called "Flächenelement n^{ter} Ordnung," regarding the solutions as the integral manifolds of the natural system of Pfaff equations[7] on the jet space of unknown functions.

This approach turned out to be quite prolific in the study of the systems of partial differential equations of the first order of one unknown function. Moreover, as was shown by Goursat in [42], it is also effective in the study of systems of partial differential equations of the second order of one unknown

[5] By starting from this, one may elaborate on the framework of "differential algebraic geometry", which is more precise than that of formal geometry. Differential algebraic geometry in Weil's style of algebraic geometry is more or less established in [54]. A. M. Vinogradov develops this framework in a ring-theoretical fashion. The big advantages of this framework seem to lie in that it can relax the regularity condition on systems of differential equations (see the assumption (3.5)), which was absolutely necessary in our framework, and also in that it can thereby treat a wider range of problems of systems of differential equations. For this purpose, however, we must develop differential algebraic geometry systematically in the modern style of algebraic geometry in the Grothendieck fashion. The relation between formal geometry and differential algebraic geometry might be compared to that between manifold theory and algebraic geometry.

[6] We call a theory of systems of differential equations *formal* when it aims at describing the structure of formal power series solutions, which is equivalent to studying the infinite prolongations of given systems of differential equations. Whereas in the real analytic category the formal theory gives automatically the substantial theory by virtue of the Cauchy-Kowalevskaya Theorem, in C^∞ category the formal theory may be said to be rather weak in many aspects, for example, in such questions as the existence of solutions.

[7] Namely, the differential systems generated by differential forms of degree one.

function of two variables. However it seems that the method then available was too primitive to be used in the study of general systems of differential equations.

It was E. Cartan who invented a useful method of studying general systems of differential equations by analyzing deeply systems of Pfaff equations.

First he obtained an existence theorem of local integral manifolds of general analytic systems of Pfaff equations, which is now called the Cartan-Kähler Theorem. This result gives an algorithm to obtain general integral manifolds by applying succesively the Cauchy-Kovalevskaja Theorem, and the space of general integral manifolds is described by the integers which can be easily calculated by linear algebra theory. Furthermore, he gave a precise general definition of characteristic systems in order to describe singular solutions, that is, those which cannot be described by this theorem.

When a system of Pfaff equations $\mathscr{P}$ comes from a system $\mathscr{D}$ of differential equations, only the maximal integral submanifolds on which the independent variables remain functionally independent correspond to the solutions of $\mathscr{D}$. E. Cartan recognized the *involutiveness* as the condition which guarantees that all the maximal integral submanifolds of $\mathscr{P}$ given by the Cartan-Kähler Theorem are in fact the solutions of $\mathscr{D}$ and gave a practical condition for the involutiveness. Moreover, he discovered the *prolongation* procedure, which is an algorithm to reduce a given noninvolutive system of Pfaff equations to an involutive one. The rationalization of this algorithm was rigorously carried out by M. Kuranishi [55].

Using this machinery E. Cartan obtained a geometric theory of involutive systems of partial differential equations of one unknown function of two independent variables [45]: He constructed a *Cartan connection* as a complete system of "invariants" of these systems. [8]

E. Cartan thus obtained practical methods and useful results on the formal sides of systems of differential equations in the differential-geometric framework. His method reduces various general problems about systems of differential equations to rather simple questions in linear algebra and brought about remarkable success in theoretical questions (e.g., the classification of infinite Lie group) and differential-geometric problems (e.g., local equivalence problems of differential-geometric structures).

It should, however, be stressed here that in many kinds of problems the Cartan method turns out rather indirect. In fact, in the analysis of concrete equations (e.g., Einstein's equation), the transcription into systems of Pfaff equations is not always illuminating and gains little in the study of their involutiveness and construction of their prolongations. (Cf. [12] for a translation of the Cartan method to the jet bundle formalism.)

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[8] Differential-geometric studies of differential equations have been aiming at such complete geometrization of differential equations, which is rarely feasible for general systems of differential equations. It thus seems not very productive to pursue geometrization itself in the differential-geometric study of differential equations.

Along with these remarkable contributions of E. Cartan, the formal study of systems of differential equations has been done more directly chiefly by Tresse, Riquier, and Janet. They focussed their study on the direct description of the coefficients of the formal power series of the solutions of systems of differential equations and succeeded in establishing a concept and a result which are essentially the same as the involutiveness and the Cartan-Kähler Theorem, respectively (cf. [13]).

Their method depends heavily on coordinate systems, which fact, it should be remarked, is not a drawback as is usually conceived. In fact, their machineries are powerful and direct in the analysis of concrete systems of differential equations and seem to help us see essential points of various general concepts.

In the 1960s, the formal theory of systems of differential equations based on the concept of jets evolved and became mature through researches such as the generalization of the deformation theory of complex structures and the justification of the Cartan classification of simple infinite Lie groups. Although it was the linear systems of differential equations that were studied intensively, most of the important results such as the Cartan-Kähler Theorem, criteria of involutiveness, and prolongation theorems were generalized to nonlinear systems of differential equations (cf. [10, 53]).

Incidentally, linear systems of differential equations can be studied in the analytic category by a powerful method called *algebraic analysis* initiated and developed by Sato, Kawai, and Kashiwara based on algebraic geometry [58]. This method is applied with remarkable success to various kind of problems.

Before ending this historical survey of the formal geometric framework of the systems of differential equations, we make the following points:

- ▹ What this framework aims at and can do is not to geometrize systems of differential equations but to give a geometric viewpoint to study them.
- ▹ This framework differs from the existent geometric theory of systems of differential equations principally in the acceptance of the infinite jet space as its fundamental ingredient. The merit of the introduction of infinite jets may at first appear only superficial. It, however, simplifies drastically the complications in the treatment of systems of Pfaff equations which are not completely integrable, and thereby makes it possible to introduce concisely various geometric concepts of systems of differential equations.
- ▹ The formal theory of systems of differential equations is not necessary for the development and comprehension of this framework. However, the formal theory is useful in studying concrete systems of differential equations in this framework: For example, it enables us to describe the space $R_\infty(\mathscr{D})$ of formal solutions of a given system $\mathscr{D}$.

0.4. **Main results.** The following are the main results in the formal geometric theory of systems of differential equations:

- ▹ a method of obtaining the conservation laws and Lie-Bäcklund transformations (Theorems 3.5 and 3.6),

- ▷ a generalization of the Noether Theorem,
- ▷ the formal calculus of variations (§4.3),
- ▷ a necessary and sufficient condition for the existence of a global variational problem for a system of differential equations which is locally the Euler-Lagrange equation of a variation problem.

These are corollaries of partial computation of the Vinogradov spectral sequence $E(\mathscr{D})$ (Theorems 3.1 and 3.6). We remark that the formal calculus of variations plays an important role in the study of completely integrable Hamiltonian systems of infinite degree of freedom (cf. §4.4).

We emphasize here again that the point of the formal geometric framework lies not in the machineries it offers but in the clear picture it draws of the world of systems of differential equations. For example, as stated before, this picture enables us to regard the totality of systems of differential equations as a category.

Moreover, this framework suggests various approaches to problems on systems of differential equations: If an argument effective in one concrete system of differential equations is of the formal geometric nature, then it can be potentially applied to every system of differential equations. Such an example is the Bott theorem on the vanishing of the Pontrjagin classes in foliation theory, which gives topological obstructions to the deformability of a tangent plane field (i.e., a homotopic solution) to a foliation (i.e., a real solution). In the formal geometric framework, this can be rephrased as the existence of nonzero elements in $E_\infty^{+,*}(\mathscr{D}) = \bigoplus_{i>0} E_\infty^{i,*}(\mathscr{D})$, where $\mathscr{D}$ denotes the system of differential equations which express the integrability condition of tangent plane fields. Once thus stated, the Bott theorem suggests a general method of obtaining topological obstructions to the deformability of homotopy solutions to real solutions for general systems of differential equations. This method, however, is not yet carried out in other systems of differential equations because of the difficulty of calculating $E_\infty^{+,*}$ (cf. §3.5 and 4.6.f).

Needless to say, there are fundamental and important questions on systems of differential equations which do not fall into the formal geometric framework, among which are, for example, the existence and regularity of solutions, boundary value problems, analysis of singularities of solutions, and the "Galois theory" of systems of differential equations.

It is not yet certain that the formal geometric framework will produce many substantial results, although we can fully expect that it does suggest a good direction in various kinds of studies of concrete systems of differential equations.

0.5. **Outline.** In this article, we try chiefly to clarify the basic construction of the formal geometric framework.

In §1, we apply this to systems of ordinary differential equations and explain in detail how the usual geometric concepts can be understood in our formal geometric concepts.

In §2, we reformulate the result of §1 using the concept of infinite prolongation of systems of differential equations, which we hope provides a psychological introduction to §3. Furthermore, §2 extends the usual differential-geometric terminology to such infinite-dimensional manifolds as jet spaces.

In §3, we explain the main concepts and results of the formal geometric theory of systems of differential equations.

In §4, we comment on various themes which are important but which are not taken up in this article.

0.6. **References.** Finally, we give general remarks on references.

This article is based on the paper [1]. The pair $(R_\infty(\mathscr{D}), H)$ and the spectral sequence $E(\mathscr{D})$ are treated in [2][9] from the viewpoints of algebraic geometry and category theory. The results on the "soliton" equations which fall into the formal geometric framework is expounded in detail in [3] from the standpoint of differential algebra that emphasizes the pair $C^\infty(R_\infty(\mathscr{D}))$ and $\Gamma(H)$.

Fundamentals of formal geometry are given in [4,7,8]. In [8], generalizations of usual differential-geometric concepts to infinite-dimensional manifolds such as $R_\infty(\mathscr{D})$ are explained in detail.

Cartan's book [9] is a standard classical text for his geometric approach to systems of differential equations. Janet's book [13] takes a direct approach to general systems of differential equations covering not only the basic points of formal aspects of systems of differential equations but also useful methods to treat concrete systems of differential equations. The modern formal theory of systems of differential equations is summarized in [10]. [11], written in Japanese, treats both the modern formal theory of systems of differential equations employing the formalism of differential forms but also contains an exposition of many important classical methods of quadrature.

Finally, the articles related to ours are reviewed in *Mathematical Reviews* mostly under the classifications 35A30, 58F05, and 58F07.

0.7. **Summary of Introduction.**

- ▷ This article explains a geometric framework for the study of general systems of differential equations by applying the idea of "formal geometry" introduced by I. M. Gelfand.
- ▷ The theoretical construction of this framework is quite simple.
- ▷ This framework gives a wide viewpoint in the study of systems of differential equations. For example,
  - Various geometric concepts hitherto known about systems of differential equations can be arranged in this framework in such a way that various interrelationships among them can be readily recognized.
  - The totality of systems of differential equations can be considered naturally as a category.

[9] See also [68].

- ▷ One of the points of this framework is the adoption of the infinite jet space as the basic element.
- ▷ Together with the formal theory of systems of differential equations, this framework provides a useful method for studying problems about concrete systems of differential equations.
- ▷ Every concrete study of systems of differential equations necessitates considerations belonging to this framework, which precede all other considerations.
- ▷ There exist important problems about systems of differential equations which cannot be handled within this framework.
- ▷ This framework will play an important rôle in the study of differential correspondence.

1. Formal geometry of the solution space of systems of ordinary differential equations

In this section, we review various well-known geometric concepts in the study of the systems of ordinary differential equations of the first order:

$$\mathscr{D}_1: \qquad \frac{du^i}{dx} = f_i(x, u^1, \dots, u^m), \qquad \text{for} \quad 1 \le i \le m,$$

where the f_i's are C^∞ functions.

In so doing we want to clarify the geometric background of several concepts, later introduced in §3, for general systems of differential equations and to show that formal geometry is very effective in placing various concepts in a relatively simple perspective and making their mutual relationships clearly visible.

1.1. **The phase space** $R_\infty(\mathscr{D}_1)$. Intuitively, it is natural to regard the equation $\mathscr{D}_1$ as a vector field:

$$d_x^0 := f_1(x, u)\frac{\partial}{\partial u^1} + \cdots + f_m(x, u)\frac{\partial}{\partial u^m}$$

on the phase space $P = P(\mathscr{D}_1) := \mathbf{R}_u^m$ which depends generally on the time parameter x. However, when the f_i's actually depend on x, it is more convenient from the differential geometric point of view to regard the system $\mathscr{D}_1$ as the differential system (i.e., the system on Pfaff equations):

$$du^i - f_i(x, u)dx = 0 \qquad (1 \le i \le m) \tag{1.1}$$

on the extended phase space:[10]

$$R_\infty = R_\infty(\mathscr{D}_1) := P(\mathscr{D}_1) \times B \qquad (B := \mathbf{R}_x).$$

The geometric description of $\mathscr{D}_1$ will be even clearer if we consider the connection on the bundle

$$\pi : R_\infty = P \times B \to B$$

[10] The subscript ∞ indicates that it is more natural to regard this space as the infinite prolongation of $\mathscr{D}_1$ from the general point of view comprising systems of partial differential equations (cf. §2.4).

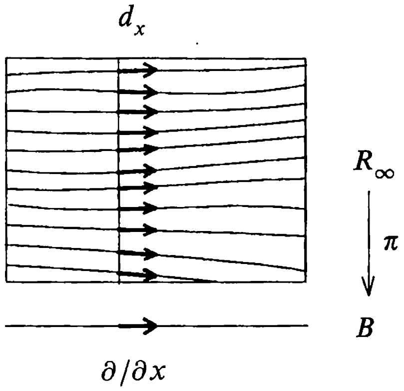

FIGURE 1

induced by the differential system (1.1): Let V be the subbundle of the tangent bundle TR_∞ of R_∞ formed by the vectors tangent to the fibers of π and let H be the subbundle of TR_∞ defined by (1.1). Then

$$TR_\infty \simeq V \oplus H, \tag{1.2}$$

i.e., the subbundle H defines a connection on the bundle π in the most primitive sense. The vector field

$$d_x := \frac{\partial}{\partial x} + d_x^0$$

is a frame of the subbundle H (cf. Figure 1).

A solution of $\mathscr{D}_1$ can be identified with a H-flat section of the bundle π. In fact, the condition that

$$s : x \mapsto (x, s_1(x), \dots, s_m(x))$$

is H-flat is equivalent to

$$s_* \frac{\partial}{\partial x} \in H_{s(x)},$$

i.e.,

$$\begin{aligned} &\frac{\partial}{\partial x} + \sum \frac{\partial s_i}{\partial x}\frac{\partial}{\partial u_i} \in \mathbf{R}.d_x \\ \Longleftrightarrow\ &\frac{ds_i}{dx} = f_i(x, s_1(x), \dots, s_m(x)), \qquad \text{for} \quad 1 \le i \le m, \end{aligned}$$

which means $u^i = s_i(x)$ $(1 \le i \le m)$ is a solution of $\mathscr{D}_1$.

The space of H-flat sections will be denoted by $\mathscr{S}ol(\mathscr{D}_1)$. We may consider the situation as

$$\text{a fiber of } \pi \doteqdot \text{the solution space of } \mathscr{D}_1$$

since a solution of $\mathscr{D}_1$ is uniquely determined by its value at a point b of B. Since we cannot, however, choose b canonically, we adopt the pair (R_∞, H) as a replacement of the solution space of $\mathscr{D}_1$. In the case of ordinary differential equations, this pair (R_∞, H) is simply the original equation $\mathscr{D}_1$ itself, but for a general differential equation $\mathscr{D}$, the pair (R_∞, H) sits in an intermediate place between the original differential equation $\mathscr{D}$ and the solution space since the fiber of R_∞ is roughly the space of all formal solutions of $\mathscr{D}$.

1.2. **Formal geometry of the phase space.** First we explain how the above connection H refines usual differential geometric concepts, where the basic features emerge from the framework of the formal geometry.

1.2.a. **First integrals.** By endowing the algebra $C^\infty(R_\infty)$ of C^∞-functions on the phase space of $\mathscr{D}_1$ with the derivation induced by the vector field $d_x \in \Gamma(H)$, we obtain a differential algebra $A = A(\mathscr{D}_1)$, which contains complete information on $\mathscr{D}_1$.

An element I of A satisfying $d_x I = 0$ is called a *first integral of* $\mathscr{D}_1$. The set of first integrals of $\mathscr{D}_1$ forms an algebra. When $I \in A$ is a first integral, the map

$$\mathscr{S}ol(\mathscr{D}_1) \ni s \mapsto I[s] := s^* I \in C^\infty(B)$$

is constant. In particular, the first integral I induces a real-valued function $s \mapsto I[s]$ on the solution space of $\mathscr{D}_1$. Solving $\mathscr{D}_1$ locally is the same as finding functionally independent m first integrals. By the local existence theorem for systems of ordinary differential equations, we have locally

$$\{\text{ first integrals }\} \fallingdotseq C^\infty(\text{the fiber of } \pi). \tag{1.3}$$

Note that the equation $d_x I = 0$ is a linear first-order partial differential equation for I.

1.2.b. **Differential forms.** Consider now the de Rham complex $\{\Omega^*(R_\infty), d\}$ on R_∞. The decomposition (1.2) of the tangent bundle splits $\Omega^p(R_\infty)$ as follows:

$$\Omega^p(R_\infty) = \bigoplus_{i+j=p} \Omega^{i,j},$$

where $\Omega^{i,j} = \Omega^{i,j}(\mathscr{D}_1) := \Gamma(\wedge^i V^* \otimes \wedge^j H^*)$. Since the plane fields defined by the subbundles V and H are integrable, the exterior differentiation on $\Omega^{i,j}$ decomposes into the (1,0)- and (0,1)-components:

$$d = \delta + (-1)^i \partial.$$

Since $\delta^2 = \partial^2 = 0$ and $\delta\partial = \partial\delta$, we obtain a double complex

$$\{\Omega^{*,*}(\mathscr{D}_1), \delta, \partial\},$$

which will be called the *variation bicomplex of* $\mathscr{D}_1$. The word "variation" indicates that the operator δ describes how the differential forms on the solution space vary when the solutions are deformed (cf. §3.5).

Using the coordinate system $\{x, u^1, \dots, u^m\}$ of R_∞, we can describe this double complex explicitly as follows: As a frame of V^* we take $\delta u^i := du^i - f_i(x,u)dx$ $(1 \le i \le m)$ and dx as a frame of H^*. Then

$$\Omega^{i,j} = A \otimes_{\mathbf{R}} \wedge^i[\delta u^1, \dots, \delta u^m] \otimes_{\mathbf{R}} \wedge^j[dx],$$

where

$$\wedge^i[\theta^1, \dots, \theta^\ell] := \wedge^i(\mathbf{R}.\theta^1 \oplus \dots \oplus \mathbf{R}.\theta^\ell).$$

The differentials δ and ∂ are characterized by the following formulas:

$$\begin{aligned} &\delta(\delta u^i) = \delta(dx) = \partial(dx) = 0, \\ &\partial(\delta u^i) = \sum \frac{\partial f_i}{\partial u^j}\delta u^j \wedge dx, \\ &\partial f = d_x f\, dx, \\ &\delta f = \sum \frac{\partial f}{\partial u^i}\delta u^i \end{aligned} \tag{1.4}$$

for $f \in A$.

1.2.c. **Spectral sequence** $E(\mathscr{D}_1)$. The subbundle H induces the following filtration on the de Rham complex:

$$F^p = F^p\Omega^* := \bigoplus_{p' \ge p} \Omega^{p',*}.$$

Since $dF^p \subset F^p$, we obtain a spectral sequence [11] $E(\mathscr{D}_1) = \{E_r^{p,q}(\mathscr{D}_1), d_r\}$ which converges to the de Rham cohomology $H^*(R_\infty, \mathbf{R})$.

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[11] We recall here briefly the definition and elementary properties of spectral sequences: Let

$$\{\Omega^* = F^0\Omega^* \supset F^1\Omega^* \supset \dots\}$$

be a filtered complex. For $0 \le r \le \infty$, put

$$\begin{aligned} Z_r^{p,q} &:= \{\omega \in F^p\Omega^{p+q} \mid d\omega \in F^{p+q}\Omega^{p+q+1}\}, \\ B_r^{p,q} &:= \{d\omega \in F^p\Omega^{p+q} \mid \omega \in F^{p-q}\Omega^{p+q-1}\}, \end{aligned}$$

where we put

$$F^{-i}\Omega^* := \Omega^* \quad (i \ge 0), \qquad F^\infty := 0,$$

and define

$$E_r^{p,q} := Z_r^{p,q}/F^{p+1} \cap Z_r^{p,q} + B_{r-1}^{p,q}.$$

The original differential d then induces

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

and we can show easily

$$\begin{aligned} E_{r+1}^{p,q} &\cong \mathrm{Ker}(d_r|E_r^{p,q}) / \mathrm{Im}(d_r|E_r^{p-r,q+r-1}), \\ E_\infty^{p,q} &\cong F^pH^{p+q}(\Omega^*) / F^{p+1}H^{p+q}(\Omega^*), \end{aligned}$$

where

$$F^pH^m(\Omega^*) := \mathrm{Im}(H^m(F^p(\Omega^*)) \to H^m(\Omega^*)).$$

We can thereby approximate the total cohomology $H^*(\Omega^*)$ successively. It should be noted that the raison d'être of the spectral sequences lies not only in its usefulness in computation of the total cohomology but also in producing series of invariants of the filtered complex (Ω^*, F).

Remark. The spectral sequence induced by the other filtration $F^p\Omega^* = \bigoplus_{p' \geq p} \Omega^{*,p'}$ is the usual one of the fibering π and does not depend on $\mathscr{D}_1$. □[12]

Consider now the significance of elements of this spectral sequence with regard to $\mathscr{D}_1$.

First of all, by (1.4), we have

$$E_1^{0,0}(\mathscr{D}_1) = \{\text{ first integrals of } \mathscr{D}_1 \} .$$

Furthermore, by definition,

$$E_1^{p,0}(\mathscr{D}_1) = \{\omega \in \Omega^{p,0} \mid \partial\omega = 0\},$$

each element of which is called an *absolute integral invariant of* $\mathscr{D}_1$: Let $x_0 \in B$ and D_0 be a p-dimensional submanifold of the fiber $\pi^{-1}(x_0)$ with a smooth boundary. Move D_0 along the solution curves to a submanifold D_1 in the fiber over $x_1 \in B$. Let N and E be the submanifolds of R_∞ swept by D_0 and δD_0, respectively, when they move from $\pi^{-1}(x_0)$ to $\pi^{-1}(x_1)$ (cf. Figure 2). By the Stokes formula, we have

$$\int_{D_0} \omega - \int_{D_1} \omega = \int_N d\omega - \int_E \omega .$$

But $d\omega = \delta\omega \in \Omega^{p+1,0}$ and $TN \supset H|_N$ imply $i_N^* d\omega = 0 \quad (i_N : N \hookrightarrow R_\infty)$. Similarly, we have $i_E^* d\omega = 0$. Hence

$$\int_{D_0} \omega = \int_{D_1} \omega,$$

i.e., ω is an absolute integral invariant. Hence we obtain the following interpretation:

$$E_1^{p,0}(\mathscr{D}_1) = \{\text{ absolute integral invariants of degree } p \text{ of } \mathscr{D}_1 \} .$$

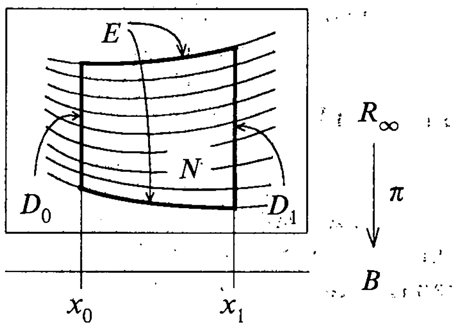

FIGURE 2

[12] This mark indicates the end of the remarks and theorems.

Similarly,

$$E_2^{p-1,1}(\mathscr{D}_1) \cong \{\text{relative integral invariants of degree } p \text{ of } \mathscr{D}_1\}.$$

In fact, let $\omega \in E_2^{p-1,1}$ be represented by $\omega_1 + \omega_2$ $(\omega_1 \in \Omega^{p-1,1}, \omega_2 \in \Omega^{p,0})$ such that $\delta\omega_1 = \partial\omega_2$. Then under the additional condition $\partial D_0 = \varnothing$, we have

$$\int_{D_0} \omega_2 = \int_{D_1} \omega_2 ,$$

which depends only on the class ω. Such differential forms as ω_2 are called *relative integral invariants of* $\mathscr{D}_1$.

What is the significance of the elements of $E_\infty^{p,q}(\mathscr{D}_1)$? An element of $E_\infty^{1,0}$ gives a necessary condition for a section s of π to be deformed to a flat section, i.e., a solution. In fact, suppose $\omega \in E_\infty^{1,0}(\mathscr{D}_1)$ is represented by $\hat{\omega} \in F^1H^1(R_\infty, \mathbf{R})$. Suppose further that $B = S^1$ and $s^*\hat{\omega} \neq 0$. Then there exists no solution in the homotopy class of s. In fact, if s is homotopic to an $s_1 \in \mathscr{S}ol(\mathscr{D}_1)$, then $s_1^*\hat{\omega} = s^*\hat{\omega} \neq 0$; but on the other hand, since $\hat{\omega}$ is represented by an element of $\Omega^{1,0} = \Gamma(H^\perp)$, we have $s_1^*\hat{\omega} = 0$, a contradiction.

The space $E_1^{1,0}(\mathscr{D}_1) \subset \Omega^{1,0}$ can be described more concretely as follows: Let

$$\omega = \omega_1\delta u^1 + \ldots + \omega_m\delta u^m \in \Omega^{1,0}.$$

By definition

$$\partial\omega = \sum_{i=1}^m \left(d_x\omega_i + \sum_{j=1}^m \omega_j \frac{\partial f_j}{\partial u^i} \right) \delta u^i \wedge dx .$$

Hence ω belongs to $E_1^{1,0}$ if and only if

$$d_x\omega_i + \sum_{j=1}^m \omega_j \frac{\partial f_j}{\partial u^i} = 0 \qquad (1 \le i \le m). \tag{1.5}$$

1.2.d. **Vector fields.** Finally, we consider the vector fields on R_∞.

What is the condition under which the graphs of the solutions are preserved by the local one-parameter transformation group generated by an element $X \in \mathfrak{X}(R_\infty)$? Since the graph of a solution of $\mathscr{D}_1$ is an integral curve of the vector field d_x, that condition can be written as

$$[d_x, X] \in A.d_x = \Gamma(H).$$

Moreover, the graph of each solution is invariant with respect to the one-parameter transformation group generated by each element of $\Gamma(H)$, since it is tangent to the graphs of the solutions. Hence, it is appropriate to consider

$$\mathscr{L}(\mathscr{D}_1) := \frac{\{X \in \mathfrak{X}(R_\infty) \mid [X, \Gamma(H)] \subset \Gamma(H)\}}{\Gamma(H)}$$

as the space of the infinitesimal transformations of the solution space. Since $\Gamma(H)$ is an ideal of its normalizer in $\mathfrak{X}(R_\infty) = \Gamma(T)$, $\mathscr{L}(\mathscr{D}_1)$ inherits a Lie algebra structure from $\mathfrak{X}(R_\infty)$. An element of $\mathscr{S}(\mathscr{D}_1)$ is called a *symmetry*[13] *of* $\mathscr{D}_1$ and the Lie algebra $\mathscr{S}(\mathscr{D}_1)$ is called the *Lie algebra of the symmetries of* $\mathscr{D}_1$. Note that this algebra is determined solely by H independently of the choice of its complement V in T.

By using the coordinates, the elements of $\mathscr{S}(\mathscr{D}_1)$ are described as follows: By (1.2), an element of $\mathscr{S}(\mathscr{D}_1)$ is represented by an element $X = \xi_1 \partial/\partial u^1 + \cdots + \xi_m \partial/\partial u^m$ of $\Gamma(V)$. Since

$$[X, d_x] = \sum_{i=1}^m \left(\sum_{j=1}^m \xi_j \frac{\partial f_i}{\partial u^j} - d_x \xi_i \right) \frac{\partial}{\partial u^i},$$

the condition $[X, d_x] \in \Gamma(H)$ is equivalent to $[X, d_x] = 0$, i.e.,

$$d_x \xi_i - \sum_{j=1}^m \xi_j \frac{\partial f_i}{\partial u^j} = 0 \qquad (1 \le i \le m). \tag{1.6}$$

This equation is called the *characteristic equation of the symmetries of* $\mathscr{D}_1$.

Note the similarity of (1.6) with the variational equation of $\mathscr{D}_1$ at $s \in \mathscr{S}ol(\mathscr{D}_1)$ (i.e., the linearization of $\mathscr{D}_1$ at s):

$$\frac{\partial v_i}{\partial x} = \sum_{j=1}^m v_j \left. \frac{\partial f_i}{\partial u^j} \right|_{u^k = s^k(x)}. \tag{1.7}$$

If (ξ_i) is a solution of (1.6), then, for all $s \in \mathscr{S}ol(\mathscr{D}_1)$,

$$v = (v_i), \qquad (v_i := \xi_i|_{u^k = s^k})$$

is a solution of (1.7). Thus, it is reasonable to call (1.6) the *universal linearization equation of* $\mathscr{D}_1$ (cf. [2]). Note also that (1.6) is the adjoint equation of (1.5).

The Lie algebra $\mathscr{S}(\mathscr{D}_1)$ acts naturally on the spectral sequence $E(\mathscr{D}_1)$: Let $X \in \Gamma(V)$ represent an element of $\mathscr{S}(\mathscr{D}_1)$. Since $[X, d_x] = 0$, the action of the Lie derivation L_X on $\Omega^{*,*}$ preserves the bidegree, commutes with δ and ∂, and hence induces an action on $E(\mathscr{D}_1)$. In particular, the symmetries of $\mathscr{D}_1$ act on the spaces of the first integrals and the invariant integrals of $\mathscr{D}_1$.

1.3. Examples.

1.3.a. Trivial equations. Consider the case where $f_1 = \cdots = f_m = 0$ in $\mathscr{D}_1$:

$$\mathscr{D}_2: \qquad \frac{du^i}{dx} = 0 \qquad (1 \le i \le m).$$

Since $H(\mathscr{D}_2)$ is a connection of the product bundle

$$R_\infty = P \times B \to B,$$

[13] In this article, we call infinitesimal transformations, that is, vector fields, simply *transformations* or *symmetries*.

we have $d_x = \partial/\partial x$. Hence, we have

$$(1.8)\qquad \begin{aligned} E_1^{p,q}(\mathscr{D}_2) &\cong \Omega^p(P)\otimes H^q(B,\mathbf{R}),\\ E_2^{p,q}(\mathscr{D}_2) &\cong H^p(P,\mathbf{R})\otimes H^q(B,\mathbf{R}),\\ \mathscr{L}(\mathscr{D}_2) &\cong \mathfrak{X}(P), \end{aligned}$$

etc. As $B = \mathbf{R}$ and $P = \mathbf{R}^m$, we obtain

$$E_1^{p,1} = E_2^{p,1} \cong (0) \quad (\forall p),$$

$$E_2^{p,0} \cong \begin{cases} \mathbf{R}, & (p=0),\\ (0), & (p>0). \end{cases}$$

Remark. Locally, $\mathscr{D}_1$ can be transformed to $\mathscr{D}_2$ by a change of local coordinates, although the construction of such a new system of local coordinates amounts to the same thing as solving $\mathscr{D}_1$ locally. Hence, the "local parts"[14] of $E(\mathscr{D}_1)$ and $\mathscr{L}(\mathscr{D}_1)$ have simple structures as described by (1.8). However, it is generally rather difficult to use this local information to obtain global results concerning $E(\mathscr{D}_1)$ and $\mathscr{L}(\mathscr{D}_1)$; this is one of the hard points related to these invariants of $\mathscr{D}_1$. The situation is similar to that of the de Rham cohomology, where what is locally trivial gives rise to nontrivial cohomology due to a global topological property of the space. □

1.3.b. **Linear equations.** Consider now

$$\mathscr{D}_3:\qquad \frac{du^i}{dx} = \sum_{j=1}^m a^i_j(x)u^j \qquad (1\le i\le m).$$

First we note that there is a one-to-one correspondence between a solution $s = (s^i(x))$ of $\mathscr{D}_3$ and an element of $\mathscr{L}(\mathscr{D}_3)$ of the form

$$\xi_1(x)\frac{\partial}{\partial u^1} + \cdots + \xi_m(x)\frac{\partial}{\partial u^m}$$

defined by $\xi_i(x) = s^i(x)$ $(1\le i\le m)$. This allows us to regard $\mathscr{S}ol(\mathscr{D}_3)$ as a subset of $\mathscr{L}(\mathscr{D}_3)$. Denote by $\mathscr{L}_{\mathrm{lin}}(\mathscr{D}_3)$ the subset of $\mathscr{L}(\mathscr{D}_3)$ formed by the elements of the form

$$X_b = \sum b^i_j(x)u^j\frac{\partial}{\partial u^i}.$$

Then $\mathscr{L}_{\mathrm{lin}}(\mathscr{D}_3)\oplus\mathscr{S}ol(\mathscr{D}_3)$ is a subalgebra of $\mathscr{L}(\mathscr{D}_3)$ and $\mathscr{S}ol(\mathscr{D}_3)$ is its ideal. Moreover, we obviously have

$$[X_b, s] = -b(s),$$

where $b(s)^i := \sum_j b^i_j s^j$. It is easy to see that the condition $X_b \in \mathscr{L}(\mathscr{D}_3)$ is given by

$$\frac{db}{dx} = [a,b],$$

where $a = (a^i_j)$.

[14] This means the sheafification of $E(\mathscr{D}_1)$ and $\mathscr{L}(\mathscr{D}_1)$.

1.3.c. **Autonomous systems.** Now suppose that f_i's are independent of x, i.e.,

$$\mathscr{D}_4: \qquad \frac{du^i}{dx} = f_i(u^1, \dots, u^m) \qquad (1 \le i \le m)$$

and reconsider the notions introduced in §1.2. For such a system, it is more convenient to regard the vector field

$$X = d_x^0 := f_1 \frac{\partial}{\partial u^1} + \cdots + f_m \frac{\partial}{\partial u^m}$$

on the phase space $p = \mathbf{R}_u^m$ as the basic object. The trivial lifting of X to $R_\infty = P \times B$ commutes with $\partial/\partial x$, whence we can extract, from those spaces introduced in §1.2, the elements which are $\partial/\partial x$-invariant, i.e., do not depend on x. We shall denote those subspaces by putting bars over the symbols of the total spaces. For example,

$$\overline{A} := \left\{ I \in A \mid \frac{\partial I}{\partial x} = 0 \right\} \cong C^\infty(P),$$
$$\overline{\Omega}^{p,q} = \{\omega \in \Omega^{p,q} \mid L_{\partial/\partial x}\omega = 0\} \cong \Omega^p(P) \otimes \wedge^q[dx].$$

The differentials δ and ∂ on $\overline{\Omega}^{*,*}$ are given by

$$\delta\omega = d_P\omega, \quad \partial\omega = L_X\omega \wedge dx \quad (\omega \in \Omega^p(P)),$$

where d_P denotes the exterior differentiation on P. Hence, the spectral sequence $\overline{E}$ generated from $\{\overline{\Omega}^{*,*}, \delta, \partial\}$ endowed with the induced filtration from $\Omega^{*,*}$ is given by

$$\overline{E}_1^{0,0} \cong \{f \in C^\infty(P) \mid Xf = 0\},$$
$$\overline{E}_1^{p,0} \cong \{\omega \in \Omega^p(P) \mid L_X\omega = 0\},$$
$$\overline{E}_2^{p,1} \cong \frac{\{\omega + \eta \wedge dx \mid \omega \in \Omega^{p+1}(P), \eta \in \Omega^p(P), L_X\omega = d\eta\}}{\{d\zeta + L_X\zeta \wedge dx \mid \zeta \in \Omega^p(P)\}},$$

etc. On the other hand, the symmetries are described as

$$\overline{\mathscr{S}}(\mathscr{D}_4) \cong \{\xi \in \mathfrak{X}(P) \mid [X, \xi] = 0\}.$$

Obviously, $X \in \overline{\mathscr{S}}(\mathscr{D}_4)$. Note that, whereas $\mathscr{S}(\mathscr{D}_4) \doteq \mathfrak{X}(P)$, it can happen that $\overline{\mathscr{S}}(\mathscr{D}_4) = \mathbf{R}.X$. Thus, we might say that $\overline{\mathscr{S}}(\mathscr{D}_4)$ reflects more accurately the specific features of the equation $\mathscr{D}_4$.

1.3.d. **Ordinary differential equations of higher order.** Consider

$$\mathscr{D}_5: \qquad u^{(m)} = f(x, u, u', \dots, u^{(m-1)}),$$

where $u^{(i)} := d^iu/dx^i$ and f is a smooth function of $x, u, \dots, u^{(m-1)}$. When $m > 1$, this equation has new aspects not present in the case $m = 1$.

First we rewrite $\mathscr{D}_5$ in the form of $\mathscr{D}_1$:

$$\mathscr{D}_5' : \quad \begin{cases} \dfrac{du^i}{dx} = u^{i+1} & (0 \le i \le m-2), \\ \dfrac{du^{m-1}}{dx} = f(x, u^0, u^1, \dots, u^{m-1}). \end{cases}$$

Then according to §§1.1 and 1.2, we can construct $R_\infty(\mathscr{D}_5) = \{\mathscr{R}_\infty(\mathscr{D}_5), H(\mathscr{D}_5)\}$, $A(\mathscr{D}_5)$, $\{\Omega^{*,*}(\mathscr{D}_5), \delta, \partial\}$, $E(\mathscr{D}_5)$, and $\mathscr{L}(\mathscr{D}_5)$.

The system $\mathscr{D}_5'$ and hence $\mathscr{D}_5$ may be considered as a $\mathscr{D}_1$ endowed with a new structure. By exploiting the special form of d_x:

$$d_x = \frac{\partial}{\partial x} + u^1 \frac{\partial}{\partial u^0} + \cdots + u^{m-1} \frac{\partial}{\partial u^{m-2}} + f \frac{\partial}{\partial u^{m-1}},$$

we can express $E_1^{1,0}$ and $\mathscr{L}$ concisely as follows. The characteristic equations (1.5) and (1.6) are, respectively,

$$d_x\omega_0 + \omega_{m-1}\frac{\partial f}{\partial u^0} = 0,$$

$$d_x\omega_i + \omega_{i-1} + \omega_{m-1}\frac{\partial f}{\partial u^i} = 0 \qquad (1 \le i \le m-1)$$

and

$$d_x\xi_i - \xi_{i+1} = 0 \qquad (0 \le i \le m-2),$$

$$d_x\xi_{m-1} - \sum_{i=0}^{m-1} \xi_i \frac{\partial f}{\partial u^i} = 0.$$

These are equivalent respectively to

$$\omega_i = \ell_i\, \omega_{m-1} \qquad (0 \le i \le m-2),$$

$$\ell^+ \omega_{m-1} = 0,$$

and

$$\xi_i + d_x^2\xi_0 = 0 \qquad (0 \le i \le m-2),$$

$$\ell\, \xi_0 = 0,$$

where

$$\ell_i := (-d_x)^{m-i-1} - \sum_{j=1}^{m-i-1} (-d_x)^{m-i-j-1} \circ \frac{\partial f}{\partial u^{m-j}} \qquad (0 \le i \le m-2),$$

$$\ell = \ell(\mathscr{D}_5) := d_x^m - \sum_{i=0}^{m-1} \frac{\partial f}{\partial u^i} d_x^i,$$

$$\ell^+ = \ell^+(\mathscr{D}_5) := (-d_x)^m - \sum_{i=0}^{m-1} (-d_x)^i \circ \frac{\partial f}{\partial u^i}.$$

Hence, if we put, for $g \in A$,

$$\omega_g := \sum_{i=0}^{m-2} \ell_i g\, \delta u^i + g\, \delta u^{m-1} \in \Omega^{1,0},$$

$$X_g := \sum_{i=0}^{m-1} d_x^i g \frac{\partial}{\partial u^i} \in \Gamma(V),$$

then we have the following theorem.

Theorem 1.1.

$$E_1^{1,0}(\mathscr{D}_5) \cong \{\omega_g \mid g \in A(\mathscr{D}_5),\ \ell^+(\mathscr{D}_5)\, g = 0\}, \tag{1.9}$$

$$\mathscr{L}(\mathscr{D}_5) \cong \{X_g \mid g \in A(\mathscr{D}_5),\ \ell(\mathscr{D}_5)\, g = 0\}. \tag{1.10}$$

We call g the *generating function of* X_g and of ω_g.

Among the elements of $\mathscr{L}(\mathscr{D}_5)$ are those classically called *point transformations* and *contact transformations,* which will be taken up again in §2.3. We note that equations (1.9) and (1.10) imply the Noether Theorem for the Euler-Lagrange equation of a variational problem (cf. §2.5).

1.4. **Remarks on methods of quadrature.** The notions introduced in §1.2 are deeply related to the classical theory of quadrature which tries to find the algorithms to solve explicitly differential equations. Here we explain, using $\mathscr{L}(\mathscr{D}_1)$ and $E(\mathscr{D}_1)$, some of the geometric methods of quadrature found by S. Lie, E. Cartan, etc.

Finding the elements of $\mathscr{L}(\mathscr{D}_1)$ amounts to the same thing as solving the linear system (1.6) of partial differential equations of the first order, whose difficulty is almost the same as that in finding the first integrals of $\mathscr{D}_1$. However, for concrete $\mathscr{D}_1$, some elements of $\mathscr{L}(\mathscr{D}_1)$ can often be found easily, for example, from obvious symmetries of $\mathscr{D}_1$. These symmetries enable us to transform $\mathscr{D}_1$ to a system of ordinary differential equations with fewer unknown functions. This is the gist of Lie's method (cf. [18, 21]).

For example, suppose an $X_1 \in \mathscr{L}(\mathscr{D}_1)$ is given. Furthermore, suppose that by solving the system of ordinary differential equations corresponding to X_1 we have obtained a mapping $\lambda : R_\infty \to \overline{R}_\infty$ whose fibers are exactly the orbits of the vector field X_1. Then there exists a vector field $\overline{d} \in \mathfrak{X}(\overline{R}_\infty)$ satisfying

$$\lambda_* d_x = \overline{d}_{\overline{x}} \qquad (\overline{x} = \lambda x),$$

and hence $\mathscr{D}_1$ is reduced to the system of ordinary differential equations on $\overline{R}_\infty$ given by $\overline{d}$. Lie tried to enumerate systematically those systems of ordinary differential equations to which the above method of quadrature is applicable. He started from X_1 and then tried to find $\mathscr{D}_1$ having X_1 as its symmetry, i.e., $X_1 \in \mathscr{L}(\mathscr{D}_1)$.

E. Cartan attacked the problem of solving $\mathscr{D}_1$, i.e., of determining $E_1^{0,0}(\mathscr{D}_1)$ through the study of $E_1^{1,0}(\mathscr{D}_1)$, the space of integral invariants. He found a remarkable method: When the differential system (1.1) has a class of privileged frames (of $\Gamma(V^*)$) with an asymmetry (or, in modern terminology, defines a G-structure), then one can exploit this asymmetry to produce elements of $E_1^{1,0}(\mathscr{D}_1)$ (as the components of the curvature of the G-structure), which in turn makes the asymmetry greater (or reduce the group G), and then one can repeat the process. This method explained systematically many of the geometric methods of quadrature known at that time. This method of quadrature is basically the same as the method of solution of the local equivalence problem of G-structures invented by E. Cartan.

1.5. **Remarks.** Note that it is only the flat bundle H that is necessary for the construction of the spectral sequence $E(R_\infty)$ and the Lie algebra of symmetries $\mathscr{L}(R_\infty)$ as well as for the definition of the action of $\mathscr{L}(R_\infty)$ on $E(R_\infty)$. These invariants can thus be defined generally for a pair (M, H) of a smooth manifold M and an integral plane field, i.e., a foliation H.

Let $\mathscr{F}ol$ denote the category whose objects are foliated manifolds $\mathscr{R} = (M, H)$ and whose morphisms $\phi:(M_1, H_1) \to (M_2, H_2)$ are smooth mappings $\phi: M_1 \to M_2$ satisfying $\phi_* H_1 \subset H_2$. Note that we do not fix the rank of the vector bundles H. Then it is easy to see that the correspondence $\mathscr{R} \mapsto E(\mathscr{R})$ is a contravariant functor from $\mathscr{F}ol$ to the category of spectral sequences.

The main theme in the remainder of this paper is to generalize the theory in this section to the category $\mathscr{F}ol^\infty$ which is an extension of $\mathscr{F}ol$ by admitting certain kinds of infinite-dimensional manifolds as the underlying foliated manifolds.

2. Formal geometry of function spaces

In this section, we take up the space of all infinite jets of functions of one variable. The main themes are twofold: We first clarify the *differential analysis* on such infinite-dimensional spaces and then explain how the phase space $R_\infty(\mathscr{D}_5)$ of the system of ordinary differential equations $\mathscr{D}_5$ of higher rank can be identified with an infinite prolongation of $\mathscr{D}_5$. We hope this section will enable the readers to envisage how the differential geometric concepts for the ordinary differential equations developed in §1 will be generalized for the systems of partial differential equations in §3.

2.1. **The infinite jet space of functions of one variable.** A function $\varphi(x)$ of one variable x is in one-to-one correspondence with a solution of the system of an infinite number of ordinary differential equations:

$$\mathscr{D}_6: \qquad \frac{du^i}{dx} = u^{i+1} \qquad (i = 0, 1, \ldots)$$

by $u^i = d^i\varphi/dx^i$ $(i = 0, 1, 2, \ldots)$. When the construction of §1.1 is applied formally to $\mathscr{D}_6$, we first obtain a fibering

$$J_\infty N := P_\infty \times B \xrightarrow{\pi} B \qquad (B := \mathbf{R}_x,\ P_\infty := \mathbf{R}_u^\infty)$$

and the system $\mathscr{D}_6$ is regarded as the flat connection $TJ_\infty N = V \oplus H$, where H is the subbundle of $TJ_\infty N$ spanned by

$$d_x := \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u^{i+1} \frac{\partial}{\partial u^i} .$$

2.2. **Differential calculus on $J_\infty N$.** Let us explain that most of the fundamental terminology in the finite-dimensional manifold theory makes sense even when it is applied to infinite-dimensional spaces like $J_\infty N$ and $\mathbf{R}_u^\infty$. (For more details see [8]).

For the sake of brevity, we consider

$$M_\infty := \mathbf{R}_z^\infty = \{(z^1, z^2, \dots)\} .$$

If we put $M_k := \mathbf{R}_z^k = \{(z^1, \dots, z^k)\}$, we obtain a projective system of usual C^∞ manifolds:

$$\{\cdots \to M_{k+1} \to M_k \to M_{k-1} \to \cdots\}$$

by the natural projections. As a topological space, M_∞ is the limit of this projective system. Thus, we are forced to define

$$C^\infty(M_\infty) := \operatorname{ind} \lim C^\infty(M_k) = \bigcup_{k=1}^{\infty} C^\infty(z^1, \dots, z^k) . \tag{2.1}$$

Similarly, we put

$$\Omega^p(M_\infty) := \operatorname{ind} \lim \Omega^p(M_k).$$

In other words, a form on M_∞ is by definition a finite sum of the terms with the form

$$f\, dz^{i_1} \wedge \cdots \wedge dz^{i_p} \qquad (f \in C^\infty(M_\infty)) .$$

As for vector fields, we need some care. Define a vector field as a derivation of the algebra $C^\infty(M_\infty)$. It is easy to see then that this can be rewritten as an infinite sum

$$X = \sum_{i=1}^{\infty} \xi_i \frac{\partial}{\partial z^i} \qquad (\xi_i \in C^\infty(M_\infty)),$$

where X acts on $C^\infty(M_\infty)$ by

$$X f = \sum_{i=1}^{\infty} \xi_i \frac{\partial f}{\partial z^i},$$

which is actually a finite sum because of $f \in C^\infty(M_k)$ $(\exists k < \infty)$. Thus, we define

$$\mathfrak{X}(M_\infty) := \left\{ \sum_{i=1}^{\infty} \xi_i \frac{\partial}{\partial z^i} \;\middle|\; \xi_i \in C^\infty(M_\infty) \right\}.$$

This can be expressed similarly to (2.1) by

$$\mathfrak{X}(M_\infty) = \operatorname{proj}\lim_k \left(\operatorname{ind}\lim_{l \geq k} \operatorname{Der}(C^\infty(M_k), C^\infty(M_\ell)) \right) ,$$

where, for algebras A and B, we denote by $\mathrm{Der}(A, B)$ the space of all the derivations from A to B.

An element of $\mathfrak{X}(M_\infty)$ is said to be *integrable* if an integral curve passes through every point of M_∞. Every vector field on a finite-dimensional manifold is integrable in this sense, but on M_∞ nonintegrable vector fields exist. In fact, $\sum z^i \partial / \partial z_{i+1}$ is not integrable. It is easy to see that X is integrable if and only if

$$X.C^\infty(M_k) \subset C^\infty(M_k) \qquad \text{for all } k \geq 1 .$$

Now we consider vector bundles on M_∞. For a vector bundle of finite rank, we define its smoothness and smooth cross sections in the obvious way. There are two types of vector bundles of infinite rank: the projective limit type and the inductive limit type. Referring to [8] for general definitions, we explain here typical bundles: the tangent bundle TM_∞ and the cotangent bundle T^*M_∞.

Let $\pi_k : M_\infty \to M_k$ denote the natural projection and put $E_k := \pi_k^* T_k M$. Then we have the projective system

$$\mathscr{E} = \{\cdots \to E_{k+1} \to E_k \to \cdots\}$$

and the inductive system

$$\mathscr{E}^* = \{\cdots \leftarrow E_{k+1}^* \leftarrow E_k^* \leftarrow \cdots\}$$

of vector bundles of finite rank on M_∞. TM_∞ and T^*M_∞ are defined as the limits of $\mathscr{E}$ and $\mathscr{E}^*$, respectively. The fibers of TM_∞ and T^*M_∞ are isomorphic, respectively, to the direct product and the direct sum of the countable number of $\mathbf{R}$'s. The spaces of C^∞ sections $\Gamma(TM_\infty)$ and $\Gamma(T^*M_\infty)$ are defined, respectively, as the limit of the projective system

$$\{\cdots \to \Gamma(E_{k+1}) \to \Gamma(E_k) \to \cdots\}$$

and the inductive system

$$\{\cdots \leftarrow \Gamma(E_{k+1}^*) \leftarrow \Gamma(E_k^*) \leftarrow \cdots\} .$$

It turns out that $\mathfrak{X}(M_\infty) = \Gamma(TM_\infty)$ and $\Omega^1(M_\infty) = \Gamma(T^*M_\infty)$.

As for the spaces of C^∞ maps, we define for example

$$C^\infty(\mathbf{R}_z^\infty, \mathbf{R}_w^\infty) = \operatorname{ind}\lim_k \left(\operatorname{proj}\lim_{\ell \geq k} C^\infty(\mathbf{R}_z^k, \mathbf{R}_w^\ell) \right) .$$

In other words, a C^∞ map $\varphi : \mathbf{R}_z^\infty \to \mathbf{R}_w^\infty$ is given by

$$w^i = \varphi_i(z), \qquad i = 1, 2, \dots ,$$

where $\varphi_i(z) \in C^\infty(\mathbf{R}_z^\infty)$.

We note that, although we used the well ordering of $\mathbf{N} = \{1, 2, \dots\}$ in the definition of vector bundles and their cross sections, only the Fréchet filter $\{X \subset \mathbf{N} \mid \mathbf{N} \backslash X \text{is finite}\}$ of $\mathbf{N}$ suffices for that purpose: for example,

$$C^\infty(\mathbf{R}_z^{\mathbf{N}}) = \lim_{X \subset \mathbf{N}} C^\infty(\mathbf{R}_z^X) .$$

2.3. **Formal geometry on** $(J_\infty N, H)$. Now we can generalize §1.2 to $(J_\infty N, H)$ word for word.

2.3.a. First of all, the differential algebra

$$\left\langle A := C^\infty(x, u^0, u^1, u^2, \dots), d_x \right\rangle$$

contains the differential algebra of differential polynomials

$$\left\langle \mathbf{R}[u^0, u^1, u^2, \dots], d_x \right\rangle,$$

which plays a fundamental role in the theory of differential algebras (cf. [56]).

2.3.b. Put

$$\delta u^i := du^i - u^{i+1} dx, \quad \text{for } i = 0, 1, 2, \dots .$$

Then

$$\Omega^{p,q}(\mathscr{D}_6) \cong A \otimes \wedge^p[\delta u^0, \delta u^1, \dots] \otimes \wedge^q[dx]$$

and ∂ and δ are characterized by

$$\begin{aligned} \delta(\delta u^i) &= \delta(dx) = \partial(dx) = 0, \\ \partial(\delta u^i) &= \delta u^{i+1} \wedge dx, \\ \partial f &= d_x f\, dx, \\ \delta f &= \sum \frac{\partial f}{\partial u^i} \delta u^i \end{aligned}$$

for $f \in A$.

2.3.c. It is easy to calculate the spectral sequence $E(\mathscr{D}_6)$ (cf. §3.6). The E_1-terms are given by

$$\begin{aligned} E_1^{p,0} &\cong \begin{cases} (0), & \text{for } p > 0, \\ \mathbf{R}, & \text{for } p = 0, \end{cases} \\ E_1^{0,1} &\cong \frac{A}{d_x A} \qquad (\text{by } [f\,dx] \leftrightarrow [f]), \\ E_1^{1,1} &\cong A \qquad (\text{by } [f\,\delta u \wedge dx] \leftrightarrow f]) . \end{aligned}$$

Let us calculate the differential $d_1 : E_1^{0,1} \to E_1^{1,1}$:

$$\begin{aligned} d(f\,dx) &= \delta f \wedge dx \\ &= \sum_{j=0}^{\infty} \frac{\partial f}{\partial u^j} \delta u^j \wedge dx \\ &= \frac{\delta f}{\delta u} \delta u \wedge dx + \sum_{j \geq 1} \partial \left\{ -\sum_{k=0}^{j-1} (-d_x)^k \frac{\partial f}{\partial u^j} \delta u^{j-k-1} \right\}, \end{aligned}$$

where

$$\frac{\delta f}{\delta u} := \sum_{k \geq 0} (-d_x)^k \frac{\partial f}{\partial u^k} .$$

Hence,

$$d_1 [f\,dx] = \left[\frac{\delta f}{\delta u} \delta u \wedge dx \right].$$

By the Poincaré Lemma, we know

$$E_\infty^{p,q} \cong \begin{cases} (0), & \text{for } (p,q) \neq (0,0), \\ \mathbf{R}, & \text{for } (p,q) = (0,0). \end{cases}$$

Hence, we obtain

$$E_2^{p,q} \cong \begin{cases} (0), & \text{for } (p,q) \neq (0,0), \\ \mathbf{R}, & \text{for } (p,q) = (0,0). \end{cases}$$

This implies the exactness of the following sequence:

$$0 \to \mathbf{R} \to A \xrightarrow{d_x} A \xrightarrow{\delta/\delta u} A \xrightarrow{d_1} E_1^{1,2} \to \cdots .$$

In particular, we obtain

Theorem 2.1. *For* $f \in A$,

(2.2) $$\frac{\delta f}{\delta u} = 0 \Longleftrightarrow f \in d_x A,$$

(2.3) $$f \in \operatorname{Im}\frac{\delta}{\delta u} \Longleftrightarrow \ell_f = \ell_f^+,$$

where

$$\ell_f := \sum \frac{\partial f}{\partial u^i} d_x^i, \qquad \ell_f^+ := \sum (-d_x)^i \circ \frac{\partial f}{\partial u^i}$$

are endomorphisms of A.

2.3.d. How is $\mathscr{L}(\mathscr{D}_6)$ described? If $X = \sum \xi_i \partial/\partial u^i \in \Gamma(V)$ satisfies $[X, d_x] \in \Gamma(H)$, then $[X, d_x] = 0$. Hence,

$$[X, d_x] = \sum_{i=0}^{\infty} (\xi_{i+1} - d_x \xi_i) \frac{\partial}{\partial u^i} = 0,$$

i.e.,

$$\xi_i = d_x^i \xi_0 \qquad (\forall i \geq 0).$$

Therefore, we obtain a natural bijection

$$X_\xi \longleftrightarrow \xi \in C^\infty(J_\infty N),$$

where

$$X_\xi := \sum d_x^i \xi \frac{\partial}{\partial u^i}.$$

We call ξ the *generating function of* X_ξ. The Lie algebra structure of $\mathscr{L}(\mathscr{D}_6)$ is given by

$$[X_\xi, X_\eta] = X_{\{\xi,\eta\}},$$

where $\{\xi, \eta\} := X_\xi \eta - X_\eta \xi$. In particular, $A = C^\infty(J_\infty N)$ turns out to be a Lie algebra by the operation $(\xi, \eta) \mapsto \{\xi, \eta\}$.

An element X of $\mathscr{L}(\mathscr{D}_6)$ is said to be *integrable* if it is represented by an integrable $X_0 \in \mathfrak{X}(J_\infty N)$. For example, $[X_{u_1}]$ is integrable since

$$[X_{u_1}] = \left[\sum_{i=0}^{\infty} u^{i+1}\frac{\partial}{\partial u^i}\right] = \left[-\frac{\partial}{\partial x}\right].$$

It is known classically that we can identify integrable elements of $\mathscr{L}(\mathscr{D}_6)$ with the infinitesimal contact transformations. In fact, let

$$X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial u^0} + h\frac{\partial}{\partial u^1}$$

be a contact vector field on the contact manifold

$$\langle J_1 N = \mathbf{R}^3_{x,u^0,u^1},\ \omega = du^0 - u^1\,dx\rangle .$$

By definition, X preserves ω, i.e., $L_X\omega \equiv 0 \pmod{\omega}$, which is equivalent to

$$X = X_W^{\mathrm{ct}} := -\frac{\partial W}{\partial u^1}\frac{\partial}{\partial x} + \left(W - u^1\frac{\partial W}{\partial u^1}\right)\frac{\partial}{\partial u^0} + \overline{d}_x W\frac{\partial}{\partial u^1},$$

for some $W \in C^\infty(x, u^0, u^1)$, where $\overline{d}_x := \partial/\partial u + u^1\partial/\partial u^0$.

Lemma. *A unique* $\widetilde{X} \in \mathfrak{X}(J_\infty N)$ *exists which satisfies*

$$(\pi_1)_*\widetilde{X} = X, \qquad [\widetilde{X}, d_x] \equiv 0 \pmod{d_x},$$

where $\pi_1 : J_\infty N \to J_1 N$ *is the natural projection.*

Proof. Put $\widetilde{X} = X_\eta + \zeta d_x \quad (\eta, \zeta \in A)$. Then

$$\widetilde{X} = \zeta\frac{\partial}{\partial x} + (u^1\zeta + \eta)\frac{\partial}{\partial u^0} + (u^2\zeta + d_x\eta)\frac{\partial}{\partial u^1} + \cdots .$$

Hence, we must solve

$$\zeta = f = -\frac{\partial W}{\partial u^1},$$
$$u^1\zeta + \eta = g = W - u^1\frac{\partial W}{\partial u^1},$$
$$u^2\zeta + d_x\eta = h = \overline{d}_x W .$$

From this we obtain $\eta = W$ and $\zeta = -\partial W/\partial u^1$, and hence $\widetilde{X} = X_W - \partial W/\partial u^1\, d_x$ is the unique element satisfying the condition. □

The above $\widetilde{X}$ is called the *extension of* X. We call X_W^{ct}, X_W, and $[X_W]$ the *contact transformations generated by* W, and W is called their *generating function.* Note that $[\widetilde{X}] = [X_W]$ is obviously integrable. Conversely, all the integral elements of $\mathscr{L}(\mathscr{D}_6)$ are obtained in this way.

We summarize the above arguments in the following theorem.

Theorem 2.2. *An element* $[X_\xi]$ *of* $\mathscr{L}(\mathscr{D}_6)$ *is integrable if and only if* $\xi \in C^\infty(x, u^0, u^1)$. *In this case,*

$$X_\xi - \frac{\partial \xi}{\partial u^1} d_x \in \mathfrak{X}(J_\infty N)$$

is the unique integrable element belonging to the class represented by X_ξ *and is the extension of the contact transformation* X_ξ^{ct}.

We remark that when $\xi = g - u^1 f$ $(f, g \in C^\infty(x, u^0))$, X_ξ^{ct} is an extension of

$$X^0 = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial u^0} \in \mathfrak{X}(\mathbf{R}^2_{x, u^0}) .$$

In such cases, $X^0, X_\xi^{\mathrm{ct}}, X_\xi$, and $[X_\xi]$ are called *point transformations.*

2.4. **The infinite prolongation of the ordinary differential equation** $\mathscr{D}_5$. The phase space (R_∞, H) of the ordinary differential equation $\mathscr{D}_5$ of higher ranks can also be described as its infinite prolongation. This description is more suitable for the systems of partial differential equations.

For $\mathscr{D}_5$, define

$$F := u^m - f(x, u^0, u^1, \dots, u^{m-1}) \in C^\infty(J_\infty N) .$$

The ordinary differential equations

$$\mathscr{D}_{5,i} \qquad d_x^i F|_{u^n = u^{(n)}} = 0$$

($i = 1, 2, \dots$) are called the *prolongations of* $\mathscr{D}_5$. The solutions of $\mathscr{D}_5$ also satisfy $\mathscr{D}_{5,i}$ $(\forall i)$. For example, $\mathscr{D}_{5,1}$ is the ordinary differential equation given by

$$u^{(m+1)} - \frac{\partial f}{\partial x}(x, u, \dots, u^{(m-1)}) - \sum_{i=0}^{m-1} u^{(i+1)} \frac{\partial f}{\partial u^i}(x, u, \dots, u^{(m-1)}) = 0 .$$

Now let I_∞ be the ideal of $C^\infty(J_\infty N)$ generated by

$$\{d_x^i F \mid i = 0, 1, \dots\}$$

and denote its zero set by R'_∞. Since every u^i $(i \geq m)$ can be expressed on R'_∞ by $x, u^0, u^1, \dots, u^{m-1}$, the set R'_∞ is a finite-dimensional manifold. Moreover, the natural projection

$$J_\infty N = \mathbf{R}_u^\infty \times \mathbf{R}_x \to R_\infty(\mathscr{D}_5) = P \times \mathbf{R}_x \qquad (P = \mathbf{R}^m_{(u^0, u^1, \dots, u^{m-1})})$$

is a diffeomorphism on R'_∞. Since $d_x I_\infty \subset I_\infty$, the vector field d_x is tangent to R'_∞, whence

$$H' = H|_{R'_\infty} \subset TR'_\infty .$$

Furthermore, the above diffeomorphism maps H' to $H(\mathscr{D}_5)$. We thereby obtain an isomorphism

$$(R'_\infty, H') \cong (R_\infty(\mathscr{D}_5), H(\mathscr{D}_5)) .$$

Thus, the phase space $R_\infty(\mathscr{D}_5)$ turns out to be the infinite prolongation of $\mathscr{D}_5$.

2.5. **Noether Theorem.** By (1.9), (1.10) and (2.3), we have

Theorem 2.3. *Suppose $\mathscr{D}_5$ is the Euler-Lagrange equation of a functional with local density, i.e., it is written as*

$$\frac{\delta L}{\delta u} = 0$$

with $L \in C^\infty(J_\infty N)$. Then the correspondence $X_\xi \leftrightarrow \omega_\xi$ gives an isomorphism:

$$\mathscr{L}(\mathscr{D}_5) \cong E_1^{1,0}(\mathscr{D}_5)\,.$$

This was proved by Gelfand and Dikii by using the Hamiltonian formalism.

3. Formal geometry of the solution space of systems of partial differential equations

3.1. **Infinite jet bundles.** Let $\pi_k : J_k N \to B$ be the kth jet bundle of π (cf. [51]) and $\pi_\infty : J_\infty N := \operatorname{proj\,lim} J_k N \to B$ the infinite jet bundle of π. The jet extension map $\Gamma(N) \to \Gamma(J_k N)\,(1 \le k \le \infty)$ will be denoted by j_k. As in §2.1, the projection π_∞ has a flat connection $H \subset TJ_\infty N$ with

$$TJ_\infty N = V \oplus H\,.$$

Here V is the subbundle of $TJ_\infty N$ consisting of all the vectors tangent to the fibers of π_∞, and the fiber of H at $\xi = j_\infty s(x)\,(s \in \Gamma(N), x \in B)$ is defined to be the image of $d_x(j_\infty s) : T_x B \to T_\xi J_\infty N$, which does not depend on the choice of s. The connection H defines a lifting map

$$\mathfrak{X}(B) \ni X \mapsto \widetilde{X} \in \Gamma(H) \subset \mathfrak{X}(J_\infty N)\,,$$

which is characterized by

$$\widetilde{X}\varphi \circ j_\infty s = X(\varphi \circ j_\infty s), \quad \text{for } \varphi \in C^\infty(J_\infty N),\ s \in \Gamma(N)\,. \tag{3.1}$$

This implies

$$[\Gamma(H)\,, \Gamma(H)] \subset \Gamma(H),$$

which means H is flat.

The following can be shown easily:

$$\widetilde{X}.C^\infty(J_m N) \subset C^\infty(J_{m+1} N) \quad \text{for } m \ge 0,\ X \in \mathfrak{X}(B), \tag{3.2}$$

$$\Gamma(N) \cong \{s \in \Gamma(J_\infty N) \mid s \text{ is flat with respect to } H\}, \tag{3.3}$$

where we consider $C^\infty(J_m N)$ as included in $C^\infty(J_\infty N)$.

3.2. **Definition of systems of differential equations.** Let m be a natural number. A subset $\mathscr{D}$ of $C^\infty(J_m N)$ is called a *system of differential equations of rank* $\le m$ *on* $\Gamma(N)$.[15] An $s \in \Gamma(N)$ satisfying $\varphi \circ j_m s = 0\,(\forall \varphi \in \mathscr{D})$ is called a *solution of* $\mathscr{D}$. The set of all the solutions of $\mathscr{D}$ will be denoted by $\mathscr{S}ol(\mathscr{D})$.

[15] From the practical point of view, it is better to define a system of differential equations as a pair of an open subset U of $J_m N$ and a subset $\mathscr{D}$ of $C^\infty(U)$. The solution then is defined to be a section $s \in \Gamma(N)$ which satisfies $j_m s(B) \subset U$ and $\varphi \circ j_m s = 0\,(\forall \varphi \in \mathscr{D})$. It is straightforward to extend the results of this section to systems of differential equations in this broader sense.

For example, if we put

$$\mathscr{D}_5' = \{u^m - f(x, u^0, u^1, \ldots, u^{m-1})\},$$

then $u = \varphi(x)$ is a solution of $\mathscr{D}_5$ if and only if φ is a solution of $\mathscr{D}_5'$ when considered as a section of $\mathbf{R}^2_{x,u_0} \to \mathbf{R}^1_x$.

3.3. **Prolongations of systems of differential equations.** For $k \geq 1$, the system of differential equations of rank $\leq m+p$,

$$\mathscr{D}_p := \mathscr{D} \cup \{\tilde{X}_1 \ldots \tilde{X}_\ell \varphi \mid 1 \leq \ell \leq p, X_i \in \mathfrak{X}(B), \varphi \in \mathscr{D}\},$$

a subset of $C^\infty(J_{m+p}N)$, is called the *pth prolongation of* $\mathscr{D}$. The union $\mathscr{D}_\infty := \bigcup_{p=0}^\infty \mathscr{D}_p$ is called the *infinite prolongation of* $\mathscr{D}$. The zero set of $\mathscr{D}_\infty$ is denoted by $R_\infty(\mathscr{D})$ $(\subset J_\infty N)$ and is called the *space of formal solutions of* $\mathscr{D}$. By (3.1), we have, for $s \in \Gamma(N)$,

$$s \in \mathscr{S}ol(\mathscr{D}) \iff j_\infty s(B) \subset R_\infty(\mathscr{D}). \tag{3.4}$$

Hence, if $\pi_\infty|_{R_\infty}$ is not surjective, then $\mathscr{D}$ has no solution. In this case, we call $\mathscr{D}$ *incompatible.*

From now on, we assume that $\mathscr{D}$ is *regular* [16], i.e., the following condition is satisfied:

(3.5) *$R_\infty(\mathscr{D})$ is a manifold and the mapping $\pi_\infty|_{R_\infty}$ has maximal rank.*

Remark. You may ask whether such objects as $R_\infty(\mathscr{D})$ are effective from the practical point of view since they require in their definition an infinite number of operations. We can then answer at three different levels:

(i) $R_\infty(\mathscr{D})$ is useful in giving a geometric insight into various concepts on systems of differential equations even if it cannot be computed.

(ii) For concrete systems such as evolutionary systems, the space $R_\infty(\mathscr{D})$ can be easily described explicitly.

(iii) Actually, the Cartan-Kuranishi Prolongation Theorem gives an algorithm for determining $R_\infty(\mathscr{D})$ for any system of differential equations $\mathscr{D}$. More precisely, for every $\mathscr{D}$, by carrying out a mechanical procedure for a determined number of times, we can construct another system of differential equations $\mathscr{D}'$ with $R_\infty(\mathscr{D}') = R_\infty(\mathscr{D})$ which is either incompatible or involutive (cf. [10] for example). We thereby obtain a finite description of $R_\infty(\mathscr{D})$, since there are no hidden compatibility conditions for involutive systems of differential equations. □

By definition, the defining ideal of $R_\infty(\mathscr{D})$, i.e., the ideal I_∞ generated by $\mathscr{D}_\infty$ satisfies

$$X.I_\infty \subset I_\infty, \quad \text{for } X \in \Gamma(H),$$

[16] This assumption is not essential for the definition of the Vinogradov spectral sequence and the Lie algebra of symmetries. These can be defined by adopting an algebraic formalism such as that developed in [16]. Such formalism, however, seems to obscure the point of our framework, which is very simple in essence.

whence H is tangent to $R_\infty(\mathscr{D})$, i.e.,

$$H(\mathscr{D}) := H|_{R_\infty(\mathscr{D})} \subset TR_\infty(\mathscr{D}) .$$

This subbundle defines a flat connection of $R_\infty(\mathscr{D}) \to B$. By (3.3) and (3.4), we have

$$\mathscr{S}ol(\mathscr{D}) = \{s \in \Gamma(R_\infty(\mathscr{D})) \mid s \text{ is flat with respect to } H(\mathscr{D})\} .$$

Thus, the relation between $\mathscr{D}$ and the pair

$$\mathscr{R}_\infty(\mathscr{D}) := \{R_\infty(\mathscr{D}) \to B \,,\, H(\mathscr{D})\}$$

is the same as that in the case of systems of ordinary differential equations.

3.4. **Formal geometry of $\mathscr{R}_\infty(\mathscr{D})$.** From now on, we proceed in the same way as in §1.2.

3.4.a. **Filtered differential algebras.** The algebra

$$A(\mathscr{D}) := C^\infty(R_\infty(\mathscr{D}))$$

has a natural structure of $\mathfrak{X}(B)$-algebra. In fact, there exists a Lie algebra homomorphism from $\mathfrak{X}(B)$ to the Lie algebra of derivations of $A(\mathscr{D})$ defined by

$$X.f = \tilde{X}f \qquad (X \in \mathfrak{X}(B)\,, f \in A(\mathscr{D})) .$$

In contrast to the case of the systems of ordinary differential equations, the algebra $A(\mathscr{D})$ is multiplicatively infinitely generated in general. However, when the action of $\mathfrak{X}(B)$ is taken into consideration, it is usually finitely generated.

In addition, the algebra $A(\mathscr{D})$ has another structure. If we put

$$A_k(\mathscr{D}) := (C^\infty(J_kN) + I_\infty)/I_\infty \subset A(\mathscr{D}) ,$$

$A_k(\mathscr{D})$ is a smooth function algebra of a finite-dimensional manifold and satisfies the condition:

$$\mathfrak{X}(B).A_k(\mathscr{D}) \subset A_{k+1}(\mathscr{D}) .$$

In the case of the systems of ordinary differential equations, this filtration $\{A_k(\mathscr{D})\}$ turns out to be "trivial."

The pair of $\mathfrak{X}(B)$-algebra structure of $A(\mathscr{D})$ and this filtration contains all the information about the system $\mathscr{D}$. For example, various kinds of characteristics can be defined algebraically by using this filtration (cf. §4.6.c).

3.4.b. **Variation bicomplex and the spectral sequence.** The variation bicomplex $\{\Omega^{*,*}(\mathscr{D})\,, \delta\,, \partial\}$ and the spectral sequence $E(\mathscr{D}) = \{E_r^{p,q}(\mathscr{D})\,, d_r\}$ are defined in the same way as in §1.2.b. This spectral sequence was introduced by Vinogradov [16], where it is called the *$\mathscr{C}$-spectral sequence*. Hereafter, we call $E(\mathscr{D})$ the *Vinogradov spectral sequence* of the equation $\mathscr{D}$.

3.4.c. **Symmetries.** We put

$$\mathscr{L}(\mathscr{D}) := \frac{\{X \in \mathfrak{X}(R_\infty(\mathscr{D})) \mid [X, \Gamma(H)] \subset \Gamma(H)\}}{\Gamma(H)}$$

and call its elements *generalized symmetries* or *Lie-Bäcklund transformation* of the equation $\mathscr{D}$. A generalized symmetry is called a *Lie transformation* if it is represented by an integrable vector field, and we denote by $\mathscr{L}_0(\mathscr{D})$ the Lie subalgebra of $\mathscr{L}(\mathscr{D})$ of all the Lie transformations of $\mathscr{D}$. We remark that a nonintegrable element of $\mathscr{L}(\mathscr{D})$ can often be integrable when restricted to a large subset of $\mathscr{S}ol(\mathscr{D})$.

The Lie algebra $\mathscr{L}(\mathscr{D})$ acts naturally on the spectral sequence $E(\mathscr{D})$.

3.4.d. **Remark.** When the equation $\mathscr{D}$ is a system of algebraic or analytic equations, then the above construction can be carried out in the algebraic or analytic category, respectively. If D is invariant under an action of a group G, then we can consider only the G-invariant elements of $E(\mathscr{D})$ and $\mathscr{L}(\mathscr{D})$. Through these amplifications, the framework of the formal geometry encompasses the work of Gelfand and Fuks [5] on the cohomology of the Lie algebra of formal vector fields and Gilkey's result [6] on the combinatorial characterization of characteristic classes. It seems that originally the idea of the formal geometry was introduced by taking these two results into consideration (cf. [4]).

3.5. **Interpretations of the Vinogradov spectral sequence..** Let $\mathscr{D}$ be a system of differential equations on the sections of a bundle $\pi: N \to B$ and consider a smooth family $S: X \to \mathscr{S}ol(\mathscr{D})$ of its solutions. Define a map $\widetilde{S}: X \times B \to R_\infty(\mathscr{D})$ by

$$\widetilde{S}(x, b) = (j_\infty S(x))(b) \qquad (x \in X, b \in B),$$

which induces a double complex homomorphism

$$\{\Omega^{*,*}(\mathscr{D}), \delta, \partial\} \to \{\Omega^{*,*}(X \times B), d_X, d_B\},$$

where $\Omega^{i,j}(X \times B) := \Gamma(\Lambda^i T^* X \times \Lambda^j T^* B)$. Thus, we obtain a spectral sequence homomorphism:

$$S_r^*: E_r^{p,q}(\mathscr{D}) \to E_r^{p,q}(X \times B) \qquad (r \geq 0).$$

Here $\{E_r^{p,q}(X \times B), d_r\}$ denotes the spectral sequence obtained from the complex $\Omega^*(X \times B)$ with the filtration $F^p\Omega^*(X \times B) := \sum_{p' \geq p} \Omega^{p',*}$.

We obtain interpretations of various elements of $E(\mathscr{D})$ from the following well-known isomorphisms:

$$\{E_1^{*,q}(X \times B), d_1\} \cong \{\Omega^*(X) \otimes H^q(B, \mathbf{R}), d_X \otimes 1\},$$
$$E_2^{p,q}(X \times B) \cong H^p(X, \mathbf{R}) \otimes H^q(B, \mathbf{R}).$$

For example, an element ω of $E_1^{0,q}$ induces a mapping

$$\overline{\omega}: \mathscr{S}ol(\mathscr{D}) \to H^q(B, \mathbf{R})$$

by $\overline{\omega}(s) := s_1^*(\omega) \in E_1^{0,q}(\{s\} \times B) = H^q(B, \mathbf{R})$. We call $\overline{\omega}(s)$ the *ω-characteristic classes* of the solution s. Furthermore, if $\omega \in E_2^{0,q}(\mathscr{D})$, then the mapping $\overline{\omega}$ is constant on the "arcwise connected component" of $\mathscr{S}ol(\mathscr{D})$. When the equation $\mathscr{D}$ is an evolutionary equation, the elements of $E_1^{0,n-1}$ ($n = \dim B$) correspond to the equivalence classes of the *conservation laws* of $\mathscr{D}$.

Just as in §1.2.c, the elements of general $E_r^{p,q}$ can be considered as integral invariants. For example, let $\omega \in E_r^{p,q}$ and suppose $\overline{\omega} \in F^p\Omega^{p+q}$ represents ω. If $S : X \to \mathscr{S}ol(\mathscr{D})$ is a smooth family of solutions with $\dim X = p+r-1$, then $\omega_S := \widetilde{S}^*\overline{\omega}$ belongs to $F^p\Omega^{p+q}(X \times B)$ and satisfies $d\omega_S \in F^{p+r}$. Hence, if $Y \subset X \times B$ is a compact submanifold of dimension $p+q$, then $\int_Y \omega_S$ depends only on the classes

$$[Y] \in H_*(X, \mathbf{R}) \otimes H_*(B, \mathbf{R})$$

and $\omega = [\overline{\omega}]$.

Furthermore, similarly to the discussion at the end of §1.2.c, the elements of

$$\bigoplus_{p+q\leq n,\, p\geq 1} E_\infty^{p,q}(\mathscr{D})$$

give necessary conditions for a homotopy solution (that is, an element of $\Gamma(R_\infty(\mathscr{D}))$) to be deformed to a genuine solution. This may be considered as a *formal* generalization of the Bott Vanishing Theorem in the foliation theory.

Remark. When a system of differential equations $\mathscr{D}$ expresses the integrability condition of the plane fields of codimension q on a manifold, we can construct a linear map

$$H^i(W_q, o_q; S^j W_q^*) \to E_1^{j,i}(\mathscr{D})$$

(cf. [61]), where W_q is the topological Lie algebra of all the formal vector fields of q variables, o_q the subalgebra of linear vector fields which generate orthogonal transformations, $S^j W_q^*$ the jth symmetric product of the dual space W_q^*, considered as a W_q-module. From this mapping, we obtain various notions and results in the foliation theory such as the characteristic classes of foliations and their deformations and the Bott Vanishing Theorem. □

3.6. **Trivial equations.** When $\mathscr{D} = \{0\}$, then $R_\infty(\mathscr{D}) = J_\infty N$, and it is not difficult to calculate $E(\mathscr{D})$ and $\mathscr{L}(\mathscr{D})$. We state here only the results and refer to [1] for the proof.

3.6.a. **Computation of the spectral sequence** $E(\mathscr{D})$.

Theorem 3.1. *Let* $n := \dim B$. *Then*

(i) *When* $r \geq 1$,

$$E_r^{p,q} \cong (0) \quad \text{if } p > 0 \text{ and } q \neq n,$$
$$E_r^{0,q} \cong H^q(N, \mathbf{R}) \quad \text{if } q < n.$$

(ii) *When* $r \geq 2$,

$$E_r^{p,n} \cong H^{p+n}(N, \mathbf{R})$$

for all nonnegative p.

Suppose in addition $N = \mathbf{R}_u^m \times \mathbf{R}_x^n \to B = \mathbf{R}_x^n$ and put $A := C^\infty(J_\infty N)$. We choose as a system of coordinates on $J_\infty N$

$$\{x^i, u_\alpha^j \mid 1 \leq i \leq n, 1 \leq j \leq m, \alpha \in \mathbf{N}^n\},$$

where $u = (u^1, \dots, u^m)$ and, for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$, the symbol u_α^i denotes the function on $J_\infty N$ corresponding to the partial derivative $\partial^{|\alpha|} u^i / \partial x^\alpha$.

Theorem 3.2. (i) *When* $r \geq 1$,

$$E_r^{0,0} \cong \mathbf{R},$$
$$E_r^{p,q} \cong (0) \quad \text{if } q < n \text{ and } (p,q) \neq (0,0).$$

(ii) *When* $r \geq 2$, $E_r^{p,q}$ *vanishes whenever* $(p,q) \neq (0,0)$.

(iii) $E_1^{0,n} \cong A/d_1 A + \dots + d_n A$.

(iv) *The correspondence*

$$\alpha : (f_1, \dots, f_m) \mapsto \left[\sum f_i \delta u^i \wedge dx^1 \wedge \cdots \wedge dx^n\right]$$

gives an isomorphism $A^m \xrightarrow{\cong} E_1^{1,n}$.

(v) $E_1^{p,n} \cong A \otimes B_p$ *for* $p \geq 2$.

Here $d_i := (\partial/\partial x^i)^\sim \in \Gamma(H)$, $\delta u^i := du^i - \sum u_{(j)}^i dx^j$, *the symbol* (j) *denoting the element of* $\mathbf{N}^n$ *with* 1 *on the* j*th component and* 0 *on the other ones. Furthermore, the module* B_p *is defined as follows. Let* V *denote the* $\mathbf{R}$*-vector space* $\mathbf{R}^n$ *and* $S_* V$ *the symmetric algebra on* V. *Considered as a commutative Lie algebra,* V *acts on* $W_p := \wedge^p(\mathbf{R}^m \otimes S^* V)$ *naturally and* B_p *denotes the quotient module* $W_p / V.W_p$.

By these results, we can easily generalize Theorem 2.1 as follows. For $F = (F_1, \dots, F_m) \in A^m$, define linear maps ℓ_F and $\ell_F^+ : A^m \to A^m$ as $\ell_F(g_i) = (h_i)$, $\ell_F^+(g_i) = (h_i')$, where

$$h_i = \sum_j \frac{\partial F_i}{\partial u_\alpha^j} d_x^\alpha g_j, \qquad h_i' = \sum_j (-d_x)^\alpha \left(\frac{\partial F_j}{\partial u_\alpha^i} g_j \right).$$

Corollary 3.3. (i) *For* $L \in A$,

$$\frac{\delta L}{\delta u} = 0 \iff L \in d_1 A + \cdots + d_n A.$$

(ii) *For* $F \in A^m$,

$$F \in Im \frac{\delta}{\delta u} \iff \ell_F = \ell_F^+.$$

3.6.b. **Description of the symmetry algebra.** For $\xi = (\xi_1, \dots, \xi_m) \in A^m$ put

$$X_\xi := \sum_{1\le i\le m} \sum_{\alpha\in\mathbf{N}^n} d_x^\alpha \xi_i \frac{\partial}{\partial u_\alpha^i} \in \mathfrak{X}(J_\infty N).$$

Then just as in §2, we can prove the following theorem easily.

Theorem 3.4. (i) *The correspondence $\xi \mapsto X_\xi$ gives an isomorphism: $A^m \cong \mathscr{L}(\mathscr{D})$.*

(ii) *When $m \ge 2$, a necessary and sufficient condition for X_ξ to be integrable is that there exist $g_i, f_r \in C^\infty(N)$ $(1 \le i \le m, 1 \le r \le n)$ such that*

$$\xi_i = g_i - \sum_r f_r u_{(r)}^i .$$

Furthermore, if X_ξ is actually integrable, then it is the extension of the vector field $\sum_r f_r \partial/\partial x_r + \sum_i g_i \partial/\partial u^i$ on N.

(iii) *When $m = 1$, a necessary and sufficient condition for X_ξ to be integrable is that ξ belongs to $C^\infty(J_1 N)$. Moreover, X_ξ is then the extension of the contact vector field*

$$X_\xi^{ct} := -\sum \frac{\partial \xi}{\partial u_i}\frac{\partial}{\partial x_i} + \left(\xi - \sum u_i \frac{\partial \xi}{\partial u_i}\right)\frac{\partial}{\partial u} + \sum \overline{d}_i \xi \frac{\partial}{\partial u_i}$$

$(u_i = u_{(i)}, \overline{d}_i := \partial/\partial x_i + u_i \partial/\partial u)$ on $J_1 N$.

3.7. **A method of computing $\mathscr{L}(\mathscr{D})$.** Let $N = \mathbf{R}_u^m \times \mathbf{R}_x^n \to \mathbf{R}_x^n$ be the product bundle and $\mathscr{D}$ a system of differential equations of rank $\le k$ on $\Gamma(N)$, i.e., $\mathscr{D} \subset C^\infty(J_k N)$. For $F \in \mathscr{D}$, define a linear mapping $\ell_F : A^m \to A$ by

$$\ell_F(g_i) := \sum \frac{\partial F}{\partial u_\alpha^i} d_x^\alpha g_i .$$

It can easily be verifed that, for $\xi \in A^m$, the element $X_\xi \in \mathfrak{X}(J_\infty N)$ is tangent to R_∞ if and only if $\ell_F \xi = 0$ $(\forall F \in \mathscr{D})$ and the following holds.

Theorem 3.5.

$$\begin{aligned} \mathscr{L}(\mathscr{D}) &\cong \{X_\xi|_{R_\infty(D)} \mid \xi \in A^m, \ell_F\xi = 0 \quad (\forall F \in \mathscr{D})\} \\ &\cong \{\xi \in A(\mathscr{D})^m \mid \ell_F^{\mathscr{D}} \xi = 0 \quad (\forall F \in \mathscr{D})\}, \end{aligned}$$

where $\ell_F^{\mathscr{D}} : A(\mathscr{D})^m \to A(\mathscr{D})$ is induced from $\ell_F : A^m \to A$.

Remark. Besides solving the equation $\ell_F^{\mathscr{D}} \xi = 0$ $(\forall F \in \mathscr{D})$, one can determine the space $\mathscr{L}_0(\mathscr{D})$ of all the Lie transformations by Cartan's method for the equivalence problem (cf. [44]), which also enables us to find the structure of the Lie algebra. When $\mathscr{L}_0(\mathscr{D})$ is infinite-dimensional, only the latter method is possible. It is conceivable that a similar method exists for the problem of determining $\mathscr{L}(\mathscr{D})$. □

3.8. **A method of computing** $E_1^{1,n-1}(\mathscr{D})$**.** Suppose further that $\mathscr{D}$ is *determined.* This means that after an appropriate change of the coordinate of $B = \mathbf{R}_x^n$, it is given as $\mathscr{D} = \{F_1, \dots, F_m\}$ with $F_i = u^i_{k_i(1)} - K_i$ and that

$$\frac{\partial K_i}{\partial u^j_\alpha} = 0 \qquad (1 \le j \le m, \alpha_1 \ge k_i)$$

holds. Define $\ell^+_{\mathscr{D}} \xi := (g_i) \in A(\mathscr{D})^m$ for $\xi = (\xi_1, \dots, \xi_m) \in A(\mathscr{D})^m$ where

$$g_i := \sum_{\alpha, j} (-d_x)^\alpha \left(\frac{\partial F_j}{\partial u^i_\alpha} \xi_j \right) .$$

We then have the following theorem.

Theorem 3.6.
(i)

$$E_1^{1,n-1}(\mathscr{D}) \cong \operatorname{Ker} \ell^+_{\mathscr{D}} .$$

(ii) *If* $n \ge 2$, *then*

$$E_1^{0,0}(\mathscr{D}) \cong \mathbf{R},$$
$$E_1^{p,q}(\mathscr{D}) = (0) \qquad ((p,q) \ne (0,0), q \le n-2) .$$

(iii) *If* $n \ge 2$, *then*

$$E_2^{0,n-1}(\mathscr{D}) = E_2^{1,n-1}(\mathscr{D}) = (0).$$

Remark. If $\mathscr{D}$ is not only a dėtermined system but also the Euler-Lagrange equation of a variational problem, then (ii) of Corollary 3.3 and Theorems 3.5 and 3.6 imply

$$E_1^{1,n-1}(\mathscr{D}) \cong \mathscr{L}(\mathscr{D}) . \tag{3.6}$$

Since $d_1 : E_1^{0,n-1}(\mathscr{D}) \to E_1^{1,n-1}(\mathscr{D})$ is injective when $n \ge 2$ and has the kernel $\mathbf{R}$ when $n = 1$, the above isomorphism (3.6) gives the extension of the Noether Theorem to the Lie-Bäklund transformations (cf. [1] for more details). □

3.9. **The Yang-Mills equation.** As a concrete example of the system of differential equations , we take up the Yang-Mills equation $\mathscr{D}_{YM}$ on the Minkowski space

$$M = (\mathbf{R}^4, -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2) .$$

Let $\mathfrak{g}$ be a Lie algebra and $\omega = \sum B_\nu dx^\nu \in \Omega^1(M) \otimes \mathfrak{g}$ a gauge field on M, that is, a $\mathfrak{g}$-valued differential 1-form on M. The Yang-Mills equation $\mathscr{D}_{YM}$ is the following system of differential equations of rank 2 for ω:

$$\mathscr{D}_{YM}: \qquad \sum_{\nu=0}^{3} (\partial_\nu F^{\nu\mu} + [B_\nu, F^{\nu\mu}]) = 0 \qquad (0 \le \mu \le 3),$$

where for $0 \le \mu, \nu \le 3$,

$$F_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] ,$$
$$F^{\mu\nu} := \sum_{\mu', \nu'} g^{\mu\mu'} g^{\nu\nu'} F_{\mu'\nu'} ,$$

and $(g^{\mu\nu})$ is the diagonal matrix with diagonal components $-1, 1, 1, 1$.

Put for $1 \le k \le 3$

$$E_k := F^{k,0} ,$$
$$H_k := -\frac{1}{2} \sum_{1 \le i, j \le 3} \varepsilon_{kij} F^{ij} ,$$

where, for a map $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$, we denote

$$\varepsilon_\sigma := \varepsilon_{\sigma(1)\sigma(2)\sigma(3)} := \begin{cases} \operatorname{sgn} \sigma & \text{if } \sigma \text{ is bijective,} \\ 0, & \text{otherwise.} \end{cases}$$

We can then take as a coordinate system on $R_\infty(\mathscr{D}_{YM})$ the following variables:

$$\begin{array}{ll} x^\lambda & (0 \le \lambda \le 3), \\ B^\sigma_{0,I} & (I \in \mathbf{N}^3[0, 3]), \\ B^\sigma_{3,J}, E^\sigma_{1,J}, E^\sigma_{2,J}, H^\sigma_{1,J}, H^\sigma_{2,J} & (J \in \mathbf{N}^3[1, 3]), \\ B^\sigma_{2,K}, E^\sigma_{3,K}, H^\sigma_{3,K} & (K \in \mathbf{N}^2[1, 2]), \\ B^\sigma_{1,n} & (n \in \mathbf{N}^1[1, 1]) \end{array}$$

$(1 \le \sigma \le \dim \mathfrak{g})$. Here B^σ denotes the σth component of B and $B^\sigma_{2,K}$ $(K = (p, q) \in \mathbf{N}^2[1, 2])$ denotes the function on $R_\infty(\mathscr{D}_{YM})$ corresponding to the partial derivative $\partial^{p+q} B^\sigma_2 / \partial x_1^p \partial x_2^q$. As an algebra of coefficients $A \subset C^\infty(R_\infty(\mathscr{D}_{YM}))$, take the algebra of polynomials of $B^\cdot_{\cdot,\cdot}$, $E^\cdot_{\cdot,\cdot}$, and $H^\cdot_{\cdot,\cdot}$ with coefficients in $C^\infty(M)$. We have then the following theorem.

Theorem 3.7. (i) *For* $r \ge 1$,

$$E_r^{0,0} \cong \mathbf{R},$$
$$E_r^{p,q} = (0) \qquad (0 \le q \le 2, (p, q) \ne (0, 0)),$$

(ii) $$E_1^{1,3} \cong \frac{\{Y = (Y_0, Y_1, Y_2, Y_3) \in A^4 \otimes \mathfrak{g} \mid Y \text{ satisfies (3.7) and (3.8)}\}}{\{(\nabla_0 Z, \nabla_1 Z, \nabla_2 Z, \nabla_3 Z) \mid Z \in A \otimes \mathfrak{g}\}},$$

where $\nabla_\lambda := d_\lambda + \operatorname{ad}(B_\lambda)$ $(0 \le \lambda \le 3)$, *and*

(3.7) $$\sum_{i=1}^{3} (\nabla_i^2 Y_0 - \nabla_0 \nabla_i Y_i) = 2 \sum_{i=1}^{3} [Y_i, E_i],$$

$$(3.8)\quad \sum_{0\le\lambda,\mu\le 3} g^{\lambda\mu}(\nabla_\lambda\nabla_\mu Y_i - \nabla_i\nabla_\lambda Y_\mu)$$
$$= 2\left([Y_0, E_i] - \sum_{1\le j,k\le 3}\varepsilon_{ijk}[Y_j, H_k]\right) \qquad (1\le i\le 3).$$

(iii) $\mathscr{L}(\mathscr{D}_{YM}) \cong \{Y \in A^4 \otimes \mathfrak{g} \mid Y \ \textit{satisfies}\ (3.7) \text{ and } (3.8)\}$.

By (ii) and (iii) of this theorem, we obtain a surjection

$$\mathscr{L}(\mathscr{D}_{YM}) \to E_1^{1,3}(\mathscr{D}_{YM}),$$

which, however, is not injective. This comes from the fact that, although $\mathscr{D}_{YM}$ is the Euler-Lagrange equation of a variational problem, it is not a determined system. The kernel of this surjection is

$$\left\{\sum \nabla_\lambda Z^\sigma \frac{\delta}{\delta B_\lambda^\sigma}\ \middle|\ Z \in A \otimes \mathfrak{g}\right\},$$

an element of which may be called a *generalized infinitesimal gauge transformation*, because it is an ordinary infinitesimal gauge transformation when $Z \in C^\infty(M) \otimes \mathfrak{g}$. In particular, the conservation laws corresponding to generalized infinitesimal gauge transformations by the Noether Theorem are trivial.

We remark that if Y satisfies (3.7) and (3.8) and its components belong to $C^\infty(x^\lambda, B_\lambda^\sigma, E_i^\sigma, H_i^\sigma)$, i.e., it is a Lie symmetry, then it is nothing but an infinitesimal conformal transformation (cf. [52]) of M. Whereas this is proved in [55] by using a computer, it is rather easily verified by solving equations (3.7) and (3.8).

We remark, finally, that in the action of $\mathscr{L}(\mathscr{D}_{YM})$ on $E_1^{1,3}(\mathscr{D}_{YM})$ the generalized infinitesimal gauge transformations operate trivially.

4. Remarks

We comment briefly on some of the important topics that are not treated in this paper.

4.1. **Structure of $R_\infty(\mathscr{D})$.** For an arbitrary system of partial differential equations $\mathscr{D}$, we first defined $\mathscr{R}_\infty(\mathscr{D}) = \{R_\infty(\mathscr{D}), H(\mathscr{D})\}$ by the infinite prolongation procedure. From this pair, we introduced the Vinogradov spectral sequence $E(\mathscr{D})$ and the Lie algebra of symmetries $\mathscr{L}(\mathscr{D})$, as the "invariants" of $\mathscr{D}$, which correspond to various geometric concepts about $\mathscr{D}$ including those classically known. In spite of the simplicity of their definition, the analysis of these invariants is rather difficult. The first problem we encounter is describing $\mathscr{R}_\infty(\mathscr{D})$ when a concrete $\mathscr{D}$ is given. We consider $\mathscr{R}_\infty(\mathscr{D})$ to be adequately described when a system of coordinates is constructed by which we can describe the action of $\mathfrak{X}(B)$ concisely.

When $\mathscr{D}$ is a linear system, the study of the structure of $\mathscr{R}_\infty(\mathscr{D})$ amounts to the same thing as analyzing the representation of the Lie algebra $\mathscr{L}(\mathscr{D})$ on the linear space which is the fiber of the vector bunde $R_\infty(\mathscr{D}) \to B$. The latter is equivalent to investigating the structure of finitely generated modules over the algebra of differential operators on B, which is by now very deeply studied by the so-called algebraic analysis initiated by M. Sato.

Even when $\mathscr{D}$ is nonlinear, we may say that the structure of $\mathscr{R}_\infty(\mathscr{D})$ is known as far as a formal geometric study is concerned. We should, however, underline that even if we have complete knowledge on the structure of $\mathscr{R}_\infty(\mathscr{D})$, there exists a different kind of difficulty in studying the invariants $E(\mathscr{D})$ and $\mathscr{L}(\mathscr{D})$.

In some rare cases, the structure of $\mathscr{R}_\infty(\mathscr{D})$ may be well understood by a completely different method, a typical example of which is given by the result of M. Sato and his collaborators on the series of the Kadomtsev-Petviashvili equations (cf. [57]). They discovered an extension $\mathscr{D}$ (cf. §4.2) of it such that the fiber of $\mathscr{R}_\infty(\mathscr{D}) \to B$ is an infinite-dimensional "Grassmann manifold" and that the flat connection H corresponds to a concrete "linear" flow on it.

4.2. **Bäcklund transformation.** There exist many transformations on the solution space of a system differential equations which cannot be understood by the concept of the Lie-Bäcklund transformation: the classical Bäcklund transformation cannot be deformed to the identity transformation and infinitesimal transformations are known which are not Lie-Bäcklund transformations. How can we handle these in the framework of formal geometry?

Consider the category $\mathscr{F}ol^\infty$ whose objects are such pairs as $\mathscr{R}_\infty(\mathscr{D}) = (R_\infty, H)$ (cf. §1.5) [17]. A Bäcklund transformation between two systems of differential equations $\mathscr{D}_1$ and $\mathscr{D}_2$ can be defined as a correspondence in this category, that is, a diagram

$$\mathscr{R}_\infty(\mathscr{D}_1) \xleftarrow{\pi_1} \mathscr{R}_\infty \xrightarrow{\pi_2} \mathscr{R}_\infty(\mathscr{D}_2)$$

with π_1 and π_2 surjective.

When $\mathscr{R}_\infty \to \mathscr{R}_\infty(\mathscr{D})$ is an *extension* of $\mathscr{D}$, that is, a surjective morphism in the category $\mathscr{F}ol^\infty$, then its fiber corresponds to the *pseudopotential* in the sense of Whalquist-Estabrook [41] and an element of "$\mathscr{L}(\mathscr{R}_\infty)$" may be considered as an infinitesimal transformation of $\mathscr{D}$ whose "generating function" is a nonlocal functional of the solutions.

It is a significant but laborious problem to obtain extensions of a given $\mathscr{D}$ and to construct Bäcklund transformations from it to other systems of differential equations. A method of constructing the "universal" extension was given by Whalquist and Estabrook in [41]. Essentially, it consists in solving the equation

$$\partial\omega - \tfrac{1}{2}[\omega, \omega] = 0$$

whose unknown objects are the Lie algebra $\mathfrak{g}$ and the $\mathfrak{g}$-valued (0,1)-form $\omega \in \Omega^{1,0}(\mathscr{D}) \otimes \mathfrak{g}$. Unfortunately, it seems not easy to solve this equation except for special kinds of equations.

[17] See §5.2.

4.3. **Formal calculus of variation.** The results of §3.6 enable us to formalize the "differential analysis" on the infinite-dimensional space $\Gamma(N)$, a concept first introduced by Gelfand and Dikii in [29]. The formalization is roughly described by the following translation:

$$\begin{array}{ccc} `C^\infty(\Gamma(N))' & \cdots\to & E_1^{0,n} \cong A/d_1A + \cdots + d_nA, \\ `\Omega^1(\Gamma(N))' & \cdots\to & E_1^{1,n} \cong A^m, \\ \text{`exterior differentiation'} & \cdots\to & d_1 \cong \frac{\delta}{\delta u}, \\ `\mathfrak{X}(\Gamma(N))' & \cdots\to & \mathscr{L} \cong A^m. \end{array}$$

We can also define naturally the pairing between vector fields and 1-forms and also Lie differentiation of 1-forms by vector fields. We can thereby give clear meaning to many concepts about manifolds of finite dimension when applied to section spaces which are of infinite dimension.

4.4. **Hamiltonian formalism.** The Hamiltonian formalism is indispensable for understanding the relationship between symmetries and conserved quantities. Although it has been so far based on nondegenerate closed 2-forms, that is, symplectic forms, it has been recently recognized that it is more natural and more general to take contravariant 2-forms as the starting point of the formalism as follows. Let M be a C^∞-manifold and $H \in \Gamma(\wedge^2 T)$, which defines a map

$$H \circ d : C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{H} \mathfrak{X}(M) .$$

For $f, g \in C^\infty(M)$, put

$$\{f, g\}_H := (H\,df).g .$$

When the Scouten bracket $[H, H] \in \Gamma(\wedge^3 T)$ vanishes, the bracket $\{\ ,\ \}_H$ defines a Lie algebra structrue on $C^\infty(M)$ and Hd is a Lie algebra homomorphism. The vector field $Hdf \in \mathfrak{X}(M)$ is called the *evolution equation defined by the Hamiltonian* f. We can thereby simplify and generalize the Hamiltonian formalism.

When we use the formal calculus of variation (cf. §4.3), the above formalism can be applied to the space of infinite dimensions. In this situation the most important problem is to construct an operator $H \in A \otimes_{\mathbf{R}} \mathrm{Hom}_{\mathbf{R}}(A, A)$ satisfying $[H, H] = 0$, which is called the *Hamiltonian operator.* Gelfand and Dorfman construct many examples of Hamiltonian operators in [33]. Interestingly, it turned out that Hamiltonian operators of a special class are simply the solutions of the classical Yang-Baxter equation (cf. [34]).

4.5. **Completely integrable systems.** The recent explosive progress in the research of evolutionary nonlinear systems of differential equations was triggered by the discovery of "complete integrability" of various concrete equations such as the Korteweg-de Vries equation, the Toda lattice, the Sine Gordon equation, etc. In the context of formal geometry, complete integrability presents itself as the existence of an infinite number of independent conserved quantities (i.e., $\dim E_1^{0,n-1} = \infty$), and as the existence of an infinite number of mutually commutative Lie-Bäcklund transformations (in particular, $\dim \mathscr{L}(\mathscr{D}) = \infty$) or of a one-parameter family of commutative Bäcklund transformations. Thus, the formal geometric invariants $E_1^{0,n-1}$, $\mathscr{L}(\mathscr{D})$, etc. offer tests by which we can check whether a given system $\mathscr{D}$ is completely integrable or not.

In the context of Hamiltonian formalism, complete integrability appears in another geometric setting, which was discovered independently by many researchers and the basic elements of which are called *hereditary symmetry, bihamilton structure, Nijenhuis operator*, etc. The main feature can be summarized in the case of finite degrees of freedom as follows.

Let N be a C^∞-manifold and suppose the tensors $H \in \Gamma(\wedge^2 T)$ and $N \in \Gamma(T \otimes T^*)$ satisfy the following condition. When $N : \mathfrak{X}(M) \to \mathfrak{X}(M)$ and $\tilde{N} : \Omega^1(M) \to \Omega^1(M)$ denote the $C^\infty(M)$-linear maps induced by N, then

$$NH = H\tilde{N}, \qquad [H, H] = [N, H] = [N, N] = 0 . \tag{4.1}$$

Here the (3,0) tensor $[N, N] \in \Gamma(\wedge^3 T^* \otimes T)$ is the Nijenhuis product and the (2,1) tensor $[N, H] \in \Gamma(T^* \otimes \wedge^2 T)$ is a differential concommitant of H and N which is defined under condition (4.1). We have then $K := NH \in \Gamma(\wedge^2 T)$ and (K, H) gives a *Hamiltonian pair*, that is, every linear combination $tK + sH$ $(t, s \in \mathbf{R})$ is a Hamiltonian operator.

Suppose further a function $f \in C^\infty(M)$ satisfies $d\tilde{N}df = 0$. Then under the condition $H^1(M, \mathbf{R}) = 0$, we have a sequence of functions $f_i \in C^\infty(M)$ $(i = 1, 2, \dots)$ and the following commutative diagram:

$$\begin{tikzcd}
f_0 = f \arrow[d, "d"] & f_1 \arrow[d, "d"] & f_2 \arrow[d, "d"] & \\
df_0 \arrow[r, "\tilde{N}"] \arrow[d, "H"'] \arrow[dr, "K"'] & df_1 \arrow[r, "\tilde{N}"] \arrow[d, "H"'] \arrow[dr, "K"'] & df_2 \arrow[r, "\tilde{N}"] \arrow[d, "H"'] \arrow[dr, "K"'] & \cdots \\
X_{f_0} \arrow[r, "N"'] & X_{f_1} \arrow[r, "N"'] & X_{f_2} \arrow[r, "N"'] & \cdots
\end{tikzcd}$$

Furthermore, we have the commutativity,

$$\{f_i, f_j\}_H = 0, \quad [X_{f_i}, X_{f_j}] = 0 \qquad (i, j \geq 0),$$

where $X_f := H\,df$.

This mechanism gives, in the context of the formal calculus of variations, the diagram found, for example, by Lax, Magri, Gelfand and Dorfman, etc. However, it is very difficult to find a (1,1) tensor $N \in \mathrm{Hom}_{\mathbf{R}}(A, A)$ which satisfies $[N, N] = 0$. The mere check of the vanishing of the Schouten bracket of a concrete N requires enormous amounts of computation.

4.6. **Problems.** Although the formal geometric studies of systems of differential equations have a long history, substantial progress has started only recently. We thus have many important problems ranging from concrete questions to theoretically foundational investigations, among which the following should be mentioned.

4.6.a. First we consider the theory of the category $\mathscr{F}ol^{\infty}$. Establish a theory of correspondences and extensions (cf. §4.2) in the category of the pairs (R_∞, H). For this purpose, we must first analyze the formal geometric aspects of the well-known interrelationship between concrete systems of differential equations such as Bäcklund transformations, "Ansatz," Twistor, and various methods of quadrature.

4.6.b. Find an effective method to the formal geometric study of the pair (R_∞, H). For example, is it possible to extend to (R_∞, H) the Cartan method of moving frames?

4.6.c. Analyze the theory of characteristic systems and methods of quadrature in the formal geometric framework. Study various kinds of characteristics starting from the filtered differential algebra (A, H, F) (F being the filtration of the algebra A introduced in §3.4.a).

4.6.d. Find a method of computing $E(\mathscr{D})$ and $\mathscr{L}(\mathscr{D})$.[18] For the determined system, Theorems 3.5 and 3.6 reduced the computation to the solution of a linear differential equations. For the over-determined system there exists no systematic method of their computation.

4.6.e. Compute $E(\mathscr{D})$ and $\mathscr{L}(\mathscr{D})$ of concrete systems of differential equations, such as the Yang-Mills equation and the system $[N, N] = 0$ for $(1,1)$ tensor fields N.

4.6.f. Compute the cohomology spaces

$$H^*(W_{2n}, L_n^{\mathbf{C}}; R) \quad \text{and} \quad H^*(W_q, o_q; S^* W_q^*)$$

(cf. §3.5). Here $L_n^{\mathbf{C}}$ is the Lie algebra of all the formal holomorphic vector fields on $\mathbf{C}^n = \mathbf{R}^{2n}$ considered as a Lie subalgebra of W_{2n}.

5 Addendum[19]

5.1. **A method of calculating E_1-terms and $\mathscr{L}(\mathscr{D})$.** We explain here briefly a method of computing E_1-terms and $\mathscr{L}(\mathscr{D})$ when $\mathscr{D}$ is an involutive system of differential equations. Details will be published elsewhere.

First we extend to $\mathscr{R}_\infty = (R_\infty, H)$ *along* H the usual differential-geometric constructions.

[18] Added in translation. For involutive systems of differential equations, there is a general homological procedure to compute E_1-terms and $\mathscr{L}(\mathscr{D})$, which includes as a special case Theorems 3.4 and 3.5. See §5.1.

[19] Added in translation.

Let $\pi : E \to R_\infty$ be a vector bundle on R_∞. An H-connection on E is a subbundle K of TE such that the differential $d\pi$ satisfies

$$d\pi|_K : T_e E \xrightarrow{\simeq} H_{\pi(e)} \qquad \text{for } e \in E .$$

An H-connection is called *flat* if it satisfies $[\Gamma(K), \Gamma(K)] \subset \Gamma(K)$.

If $\mathscr{E}$ is a vector bundle with a flat connection K, we can construct a semi-exact sequence of $C^\infty(R_\infty)$-modules just as in the finite-dimensional situation:

$$\Gamma(\mathscr{E}) \overset{\partial^H}{\to} \Omega^{0,1}(\mathscr{E}) \overset{\partial^H}{\to} \Omega^{0,2}(\mathscr{E}) \to \cdots , \tag{5.1}$$

where we put $\mathscr{E} := J_\infty^H E$ and $\Omega^{0,j}(\mathscr{E}) := \Gamma(\mathscr{E} \otimes \wedge^j H^*)$. We denote the ith cohomology of this complex by $H^i(R_\infty, \mathscr{E})$. Note that

$$\mathscr{L}(\mathscr{D}) \cong H^0(R_\infty, V), \qquad E_1^{1,i} \cong H^i(R_\infty, V^*) .$$

On the other hand, for a vector bundle E of finite rank over R_∞, we can construct H-jet bundles $J_k^H E$ for $k = 1, 2, \dots, \infty$, which are characterized by the following properties. When H is a flat connection on a fibering $R_\infty \to M$ and $\widetilde{E}$ is a lifting of a vector bundle E on M, then for every flat section s of $R_\infty \to M$, the induced bundle $s^* J_k^H \widetilde{E}$ is isomorphic to the usual jet bundle $J_k E$.

The infinite H-jet bundle $J_\infty^H E$ has a natural flat H-connection and it is easily verified that if $\mathscr{E} = J_\infty^H E$, then the following complex is exact:

$$0 \to \Gamma(E) \overset{j_\infty^H}{\to} \Gamma(\mathscr{E}) \overset{\partial^H}{\to} \Omega^{0,1}(\mathscr{E}) \overset{\partial^H}{\to} \Omega^{0,2}(\mathscr{E}) \to \cdots .$$

For the dual bundle $J_\infty^H E^*$, it is the following sequence which is exact:

$$0 \to \Omega^{0,0}(\mathscr{E}^*) \to \cdots \to \Omega^{0,n-1}(\mathscr{E}^*) \to \Omega^{0,n}(\mathscr{E}^*) \overset{j_\infty^*}{\to} \Omega^{0,n}(E^*) \to 0,$$

where n is the rank of H and j_∞^* is the adjoint of j_∞^H characterized by

$$\langle j_\infty^H s, \omega \rangle \equiv \langle s, j_\infty^* \omega \rangle \pmod{\partial \Omega^{0,n-1}},$$

for $s \in \Gamma(E)$ and $\omega \in \Omega^{0,n}(\mathscr{E}^*)$.

Based on the H-jet bundles, we can define the notion of *H-differential operators.* For example, when E and F are vector bundles of finite rank over R_∞, a *linear H-differential operator* from $\Gamma(E)$ to $\Gamma(F)$ is a linear operator given as the composition of the H-jet extension map $\Gamma(E) \to \Gamma(J_k^H E)$ and the linear map $\Gamma(J_k^H E) \to \Gamma(F)$ induced from a vector bundle map $J_k^H E \to F$.

Now suppose $R \subset J_k N$ is an involutive system of differential equations on sections of fibering $N \to M$, which can be nonlinear, and R_∞ is its infinite prolongation. Then the lifting E_0 of TR is an *involutive* system of H-linear differential equations on the sections of the lifting E_1 of TN and we can construct its Janet resolution by linear H-differential operators:

$$0 \to \Gamma(E_0) \longrightarrow \Gamma(E_1) \xrightarrow{D_1} \Gamma(E_2) \xrightarrow{D_2} \cdots ,$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 & \Omega^{1,0} \arrow[l] \arrow[d, "\partial"] & \Omega^{0,0}(\mathscr{E}_1^*) \arrow[l] \arrow[d, "\partial"] & \Omega^{0,0}(\mathscr{E}_2^*) \arrow[l] \arrow[d, "\partial"] & \cdots \arrow[l] \\
0 & \Omega^{1,1} \arrow[l] \arrow[d, "\partial"] & \Omega^{0,1}(\mathscr{E}_1^*) \arrow[l] \arrow[d, "\partial"] & \Omega^{0,1}(\mathscr{E}_2^*) \arrow[l] \arrow[d, "\partial"] & \cdots \arrow[l] \\
 & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \\
0 & \Omega^{1,n-1} \arrow[l] \arrow[d, "\partial"] & \Omega^{0,n-1}(\mathscr{E}_1^*) \arrow[l] \arrow[d, "\partial"] & \Omega^{0,n-1}(\mathscr{E}_2^*) \arrow[l] \arrow[d, "\partial"] & \cdots \arrow[l] \\
0 & \Omega^{1,n} \arrow[l] \arrow[d] & \Omega^{0,n}(\mathscr{E}_1^*) \arrow[l] \arrow[d, "j_H^*"] & \Omega^{0,n}(\mathscr{E}_2^*) \arrow[l] \arrow[d, "j_H^*"] & \cdots \arrow[l] \\
 & 0 & \Omega^{0,n}(E_1^*) \arrow[l, "D_0^+"'] \arrow[d] & \Omega^{0,n}(E_2^*) \arrow[l, "D_1^+"'] \arrow[d] & \cdots \arrow[l, "D_2^+"'] \\
 & & 0 & 0 &
\end{tikzcd}
$$

DIAGRAM 1

where the E_i's are vector bundles on R_∞ of finite rank. This complex induces an exact sequence of H-flat vector bundles:

(5.2) $$0 \to V \to J^H_\infty E_1 \to J^H_\infty E_2 \to J^H_\infty E_3 \to \cdots ,$$

and by dualizing

$$0 \leftarrow V^* \leftarrow J^H_\infty E_1^* \leftarrow J^H_\infty E_2^* \leftarrow J^H_\infty E_3^* \leftarrow \cdots ,$$

where all the vector bundle homomorphisms preserve the H's.

By constructing the resolution (5.1) of each bundle of this sequence, we obtain the double complex of Diagram 1, where $\mathscr{E}_i := J^H E_i$. Since all the vertical complexes except the utmost left one and all the horizontal complexes except the bottom one are exact, the obvious diagram chasing gives the following isomorphism.

Theorem 5.1. *For* $0 \leq j \leq n$,

$$E_1^{1,j} \cong \frac{\operatorname{Ker} D^+_{n-j}}{\operatorname{Im} D^+_{n-j+1}}. \quad \square$$

When the equation is determined and $j = n-1$, we obtain by virture of $E_i^* = (0)$ for $i \geq 2$,

$$E_1^{1,n-1} \cong \operatorname{Ker} D_1^+,$$

which is simply Theorem 3.4. Furthermore, Theorem 3.7 can be proved by this theorem.

A similar double complex constructed from (5.2) proves the following theorem.

Theorem 5.2. *For* $i = 0, 1, 2, \ldots$,

$$H^i(R_\infty, V) \cong \frac{\operatorname{Ker} D_{i+1}}{\operatorname{Im} D_i},$$

where $D_0 = 0$. $\square$

For $i = 0$, this generalizes Theorem 3.5. We note that the space $H^1(R_\infty, V)$ might be considered as the space of infinitesimal deformations of R_∞.

5.2. **Classes of morphisms in** $\mathscr{F}ol^\infty$. Let $\mathscr{R}_{\infty i} = (R_{\infty i}, H_i)$ $(i = 1, 2)$ be two objects in $\mathscr{F}ol^\infty$ (cf. §4.2). A C^∞-map $\varphi : R_{\infty 1} \to R_{\infty 2}$ is called a *morphism* of the category $\mathscr{F}ol^\infty$ if it satisfies $d\varphi(H_1) \subset H_2$.

Morphisms φ for which $d\varphi|_{H_1}$ is of constant rank can be roughly classified by the types of two linear maps: $d^H\varphi :=d\varphi|_{H_1}$ and $d^V\varphi : TR_{\infty 1}/H_1 \to TR_{\infty 2}/H_2$ induced by $d\varphi$, respectively called the *horizontal differential* and the *vertical differential.*

Note that since the complementary direction to H "parametrizes" the space of solutions, that is, maximal leaves of the foliated manifold (R_∞, H), the vertical differential is injective, bijective, and surjective, respectively, when the map between leaf spaces induced by φ is *approximately* injective, bijective, and surjective. This factor is already present in usual differential geometry. What refines the situation in formal geometry is the type of horizontal differential.

Note also that a general morphism can be decomposed into two morphisms; one has the surjective or bijective horizontal and vertical differentials and the other has the injective or bijective horizontal and veritcal differentials. Thus, case (A) in Table 1 can be considered as a composite of the others.

In the most interesting cases, the horizontal differentials are bijective or surjective and are tabulated in Table 1.

TABLE 1

	$d\varphi^H$: bijection	$d\varphi^H$: surjection
$d\varphi^V$: inj	solutions	(A)
$d\varphi^V$: bij	isomorphism	Cauchy characteristic
$d\varphi^V$:surj	extension, reduction, differential elimination	Monge characteristic

Note further that the significance of each case varies according to which $\mathscr{R}_{\infty i}$'s are parametric. Here an object of $\mathscr{F}ol^{\infty}$ is called *parametric* when it is isomorphic to $(J_{\infty}N, H)$ for some fibering $N \to M$.

We explain briefly how to understand Table 1. To make the situation more concrete, we assume that the objects $\mathscr{R}_{\infty i}$ $(i = 1, 2)$ have spaces of independent variables. Namely, they are fiberings over finite-dimensional manifolds,

$$\pi_i : R_{\infty i} \to M_i$$

and the differentials $d\pi_i$ are isomorphic when restricted to the fibers of H_i. We assume further that φ preserves the spaces of the independent variables, i.e., the following diagram commutes:

$$\begin{array}{ccc} R_{\infty 1} & \xrightarrow{\varphi} & R_{\infty 2} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow[f]{} & M_2 \end{array}$$

for some smooth map f.

First suppose $d\varphi^H$ is an isomorphism. If $d\varphi^V$ is injective, the image $\varphi(R_{\infty 1})$ satisfies the following condition. When an integral submanifold of $R_{\infty 2}$ intersects with this image, then it is completely contained in it. Such a subset is usually called an *intermediate integral* and in the extreme case becomes a solution of $R_{\infty 2}$.

If $d\varphi^V$ is surjective, then every solution s of $R_{\infty 2}$ gives an intermediate integral $\varphi^{-1}s$ of $R_{\infty 1}$. One can thus decompose the solution of $R_{\infty 1}$ into *simpler* equations $R_{\infty 2}$ and $\varphi^{-1}s$, whence the "reduction" in the table. The explanation of "extension" is given in §4.1. "Differential elimination" is explained as follows. When $R_{\infty 2}$ is parametric and $R_{\infty 3}$ is an intermediate integral of $R_{\infty 1}$, one is asked to give the defining equation of the image $\varphi(R_{\infty 3})$.

Suppose now $d\varphi^H$ is strictly surjective. If $d\varphi^V$ is an isomorphism, a solution s_1 of $R_{\infty 1}$ and a solution s_2 of $R_{\infty 2}$ are in one-to-one correspondence by $s_2 = \varphi(s_1)$ and that s_1 can be obtained from s_2 simply by solving systems of ordinary differential equations. Thus, the problem of solving a system of partial differential equations is reduced to solving one with fewer independent variables and ordinary differential equations. Such a situation arises exactly when the equation $R_{\infty 1}$ has the nontrivial *Cauchy characteristics.*

If $d\varphi^V$ is strictly surjective, then a solution s_2 of $R_{\infty 2}$ gives an intermediate integral $\varphi^{-1}(s_2)$. It should be noted that when given $R_{\infty 1}$, it is very rare that it admits a morphism of this type to a parametric $R_{\infty 2}$. The Monge characteristics of systems of partial differential equations of second order of one unknown function give rise to such morphisms and the main point of Darboux's method of integrating such systems is to integrate two independent Monge characteristics, which can be rephrased in our terminology generally as follows. Let R_∞ be the object of $\mathscr{F}ol^\infty$ corresponding to the system considered. Then there are two morphisms $\varphi: R_\infty \to R_{\infty i}$ $(i = 1, 2)$ with parametric $R_{\infty i}$'s which have rank one, and the product morphism

$$\varphi_1 \times \varphi_2 : R_\infty \to R_{\infty 1} \times R_{\infty 2}$$

has finite-dimensional fibers. Thus, given two arbitrary functions of one variable, general solutions of R_∞ can be obtained by solving systems of ordinary differential equations.

The case of injective horizontal differentials seems not so interesting but can be dealt with in the Cauchy-Kovalevskaya Theorem. The injectivity of the horizontal differential means roughly that f is an immersion and every solution of $\mathscr{R}_{\infty 1}$, a function on M_1, can be "extended" to a solution of $\mathscr{R}_{\infty 2}$, a function on M_2.

References

There are many books and articles on the geometric theory of systems of differential equations. We list some of them referred to in this paper. See also §0.6 for general remarks.

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Formal geometry

The idea of formal geometry seems to stem from works [5–7].

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Classical geometric approach to systems of differential equations : prolongation, involutiveness, Cartan-Kähler Theorem, prolongation theorem, etc.

[9] is useful in giving an overall picture of the theory of differential systems, whereas [10] gives a systematic exposition of the modern theory of the systems of differential equations based on jet formalism. The relationship between these two approaches is clearly analyzed in [12]. [13] gives a method for choosing good systems of coordinates on the infinite-dimensional spaces R_∞.

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Variation bicomplexes

The terminology "variation bicomplex" was first introduced in [14], where the concept of "difference bicomplex" was also introduced. In [15], the variation bicomplex when $R_\infty = J_\infty N$ is considered.

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Vinogradov spectral sequence $E(\mathscr{D})$, integral invariants

The Vinogradov spectral sequence was first introduced in [16]. The interpretation of the elements of the spectral sequence $E(\mathscr{D})$ as characteristic classes and integral invariants was given in [1]. [17] explains in detail the relationship between the integral invariants and the method of quadrature.

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Symmetries

For point transformations see [18, 21] and for contact transformations [19, 20, 21]. The terminology "Lie-Bäcklund transformation" was introduced in [22].

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Computation of $E(\mathscr{D})$ and $\mathscr{L}(\mathscr{D})$ of concrete systems of differential equations

There are many papers devoted to computing $E_1^{1,n-1}(\mathscr{D})$ and $\mathscr{L}(\mathscr{D})$ by solving the characteristic equations (cf. Theorems 3.5 and 3.6), examples of which are the following. Many others can been found in Math. Reviews (58F07).

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Variational operator

Many people have independently obtained the result (Corollary 3.3) on the kernel and image of the variational operator, e.g., [26, 28]. [27] takes up the question of when a system of differential equations which is locally the Euler-Lagrange equation of a functional with local density is globally so. A solution can, however, be easily given from our result $E_1^{1,n} \cong H^{n+1}(N, \mathbf{R})$ (Theorem 3.1). [28] gives a useful formula of Taylor type for the variational operator.

26. B. Kupershmidt, *Geometry of jet bundles and the structure of Lagrangian and Hamilton formalism*, Lecture Notes in Math., vol. 775, Springer, Berlin, 1980, pp. 162–218.
27. I. M. Anderson and T. Duchamp, *On the existence of global variational principles*, Amer. J. Math. **102** (1980), 781–868.
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Formal calculus of variations

The following develops systematically the formal calculus of variation, where "Gelfand-Dikii" transformation is introduced as a tool to verify various identities on differential polynomials. It deserves, however, to be noted that this tool is not indispensable for the formal calculus of variation.

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Hamiltonian formalism, Hamilton operator, complete integrability

For classical Hamiltonian formalism see [30], for example. The Hamilton operator is treated in [31–33, 36]. [31–35] study the complete integrability.

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36. A. S. Fokas and B. Fuchssteiner, *The structure of symplectic operators and hereditary symmetries*, Lett. Nuovo Cimento (2) **28** (1980), 299–303.

Bäcklund transformation, pseudopotentials, etc.

For examples of Bäcklund transformations see [37]. An interesting observation on the role of Bäcklund transformations is given in [38]. [39, 40] are among the few which argue Bäcklund transformations generally. [41] is an important paper which introduced a general method of constructing Bäcklund transformations, whose potentiality has not yet been fully developed.

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Characteristic systems, method of quadrature, Cartan connections, etc.

See [42, 46] for the Darboux method of quadrature of systems of partial differential equations of one unknown function, which is in a sense the ultimate of the classical methods of quadrature. E. Cartan gave in [43] a method of quadrature for the completely integrable systems of Pfaff equations, which evolved later into his moving frame method of solving the local equivalence problem of geometric structures and of constructing the Cartan connections of various kinds of geometric structures. In [45], he "geometrized" the involutive systems of differential equations of one unknown function and two independent variables by constructing their canonical Cartan connections and thereby gave a method of quarature of such systems. [46] extends it to a class of second order systems of differential equations and constructed their canonical Cartan connections, where the idea of N. Tanaka [47] is used. [49] treats the results of [45] rigorously from the modern point of view. See [11, 42, 50], etc. for the theory of characteristic of systems of differential equations.

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After the publication of this paper, many papers and books were published which are related to our formalism. We list only a portion of those which came to my attention, in which further references can be found. [62] contains a brief introduction to the formal theory of systems of differential equations, which is enough to enable us to calculate the symmetries and conservation laws of evolutionary equations. [63, 68] give a systematic exposition of an algebro-geometric theory of systems of differential equations, introduced briefly in [16]. This theory may be called "differential algebraic geometry" and has conceptually a much wider range of applicability than ours, but the basic construct seems more sophisticated and more difficult to assimilate if we have no experience in algebraic geometry. [64] is a textbook covering a wide range of knowledge concerning symmetries of systems of differential equations. [65] intends to unite the theory of differential algebra and the formal theory of systems of differential equations and to establish thereby the Galois theory of systems of differential equations. It also gives the idea of Janet resolutions of involutive systems of differential equations, which is practically much more useful than the Spencer resolutions and, in fact, when suitably modified, plays a fundamental role in the calculation of the Vinogradov spectral sequences as indicated in the Addendum. [66] gives a method for treating the functionals called *polynomial functionals* on $\mathscr{S}ol(\mathscr{D})$, which are more general than those having local densities. Its idea can be traced back to the Gelfand-Fuks' method of calculation of the cohomology spaces of the Lie algebra of vector fields on manifolds (cf. [4]). [67] extends the usual formal theory of differential equations to systems of super differential equations. It uses the idea of Gröbner basis in the theory of differential algebra and recognizes the involutiveness of a system as the existence of a special Gröbner basis for the differential ideal generated by it. This idea seems new even when applied to the usual systems of differential equations (i.e., in the pure boson case). It also gives a basis for constructing the "Janet resolution" of concrete equations, which will be explained elsewhere.

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