L^p Fourier transformation on non-unimodular locally compact groups

by

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Abstract. Let G be a locally compact group with modular function Δ and left regular representation λ. We define the L^p Fourier transform of a function $f \in L^p(G)$, $1 \leq p \leq 2$, to be essentially the operator $\lambda(f)\Delta^{1/q}$ on $L^2(G)$ (where $\frac{1}{p} + \frac{1}{q} = 1$) and show that a generalized Hausdorff-Young theorem holds. To do this, we first treat in detail the spatial L^p spaces $L^p(\psi_0)$, $1 \leq p \leq \infty$, associated with the von Neumann algebra $M = \lambda(G)''$ on $L^2(G)$ and the canonical weight ψ_0 on its commutant. In particular, we discuss isometric isomorphisms of $L^2(\psi_0)$ onto $L^2(G)$ and of $L^1(\psi_0)$ onto the Fourier algebra $A(G)$. Also, we give a characterization of positive definite functions belonging to $A(G)$ among all continuous positive definite functions.

Introduction.

Suppose that G is an abelian locally compact group with dual group $\hat{G}$. Then the Hausdorff-Young theorem states that if $f \in L^p(G)$, where $1 \leq p \leq 2$, then its Fourier transform $\mathcal{F}(f)$ belongs to $L^q(\hat{G})$, where $\frac{1}{p} + \frac{1}{q} = 1$ (cf. [23, p. 117]). In the case of Fourier series, i.e. when G is the circle group and $\hat{G}$ the integers, this is a classical result due to F. Hausdorff and W. H. Young [24, p. 101]. An extension of this theorem to all unimodular locally compact groups was given by R. A. Kunze [14]. In this paper we shall treat the case of a general, i.e. not necessarily unimodular, locally compact group.

In order to describe our results, we first briefly recall those of [14]. Suppose that f is an integrable function on a unimodular group G. Then we consider the Fourier transform $\mathcal{F}(f)$ to be the operator $\lambda(f)$ of left convolution by f on $L^2(G)$. (As pointed out by Kunze [14], this point of view is justified by the fact that in the abelian case $\lambda(f)$ is unitarily equivalent to the operator on $L^2(\hat{G})$ of multiplication by the (ordinary) Fourier transform $\hat{f}$.) The Fourier transformation maps $L^1(G)$ into the space $L^\infty(G')$, defined as the von Neumann algebra M generated by $\lambda(L^1(G))$. More generally, one can define $\lambda(f)$ as an (unbounded) operator on $L^2(G)$ even for functions f not in $L^1(G)$. It then turns out that λ maps each $L^p(G)$, $1 \leq p \leq 2$, norm-decreasingly into a certain space $L^q(G')$ of closed densely defined operators on $L^2(G)$ (where $\frac{1}{p} + \frac{1}{q} = 1$). This is the Hausdorff-Young theorem. Kunze introduced the spaces $L^q(G')$ as spaces of measurable operators (in the sense of [21]) with respect to the canonical gage on M [14, p. 533]. An equivalent but simpler way of introducing the $L^q(G')$ is to consider the trace φ_0 on M characterized by $\varphi_0(\lambda(h)^*\lambda(h)) = \|h\|_2^2$ for certain functions h, and then take $L^q(G')$ to be $L^q(M,\varphi_0)$ as defined by E. Nelson [15], viewing it as a space of "φ_0-measurable" operators [15, Theorem 5]. (In either case, the L^q spaces obtained are isomorphic to the abstract L^q spaces of J. Dixmier [5] associated with a trace on a von Neumann algebra.)

In the general (non-unimodular) case, φ_0 is no longer a trace, and the lack of adequate spaces L^q into which the $L^p(G)$ were to be mapped for a long time prevented the formulation of a Hausdorff-Young theorem, except for some special cases ([7, §8], [20, Proposition 15]). In [10], however, U. Haagerup constructed abstract L^p spaces corresponding to an arbitrary von Neumann algebra, and combining methods from [10] with the recent theory of spatial derivatives by A. Connes [2], M. Hilsum has developed a spatial theory of L^p spaces [12]. If M is a von Neumann algebra acting on a Hilbert space H and ψ is a weight on its commutant M', then the elements of $L^p(M,H,\psi)$ are (in general unbounded) operators on H satisfying a certain homogeneity property with respect to ψ. We shall see that when using these spaces (in the particular case of $M = \lambda(G)''$, $H = L^2(G)$, and ψ = the canonical weight on M') and when defining the L^p Fourier transform of an L^p function f to be the operator $\xi \mapsto f*\Delta^{1/q}\xi$ on $L^2(G)$ (where Δ is the modular function of the group), one gets a nice L^p Fourier transformation theory and in particular a Hausdorff-Young theorem.

The paper is organized as follows. In Section 1 we fix the notations and describe our set-up. In Section 2, we study the L^p spaces of [12] in our particular case; we give a reformulation of the α-homogeneity property appearing in [2] that does not involve modular automorphism groups and we characterize $L^p(\psi_0)$-operators among all $(-\frac{1}{p})$-homogeneous operators. In Section 3, we treat the case $p = 2$ and obtain explicit expressions for the L^2 Fourier transformation $\mathcal{F}_2 = \mathcal{P}$, called the Plancherel transformation, as well as for its inverse.

Next, in Section 4, we deal with the case of a general $p \in [1,2]$; we define the L^p Fourier transformation $\mathcal{F}_p$, and using interpolation (specifically, the three lines theorem) we prove our version of the Hausdorff-Young theorem.

Finally, in Section 5, we define an L^p Fourier cotransformation $\overline{\mathcal{F}}_p$ taking $L^p(\psi_0)$, $1 \leq p \leq 2$, into $L^q(G)$ and we investigate the relations between cotransformation and Fourier inversion. A detailed study of the $p = 1$ case gives a new characterization of $A(G)_+$ functions among all continuous positive definite functions on G.

1. Preliminaries and notation.

Let G be a locally compact group with left Haar measure dx. We denote by $\mathcal{K}(G)$ the set of continuous functions on G with compact support and by $L^p(G)$, $1 \leq p \leq \infty$, the ordinary Lebesgue spaces with respect to dx. The modular function Δ on G is given by

$$\int f(xa^{-1})\,dx = \Delta(a)\int f(x)\,dx$$

for all $f \in \mathcal{K}(G)$ and $a \in G$. For functions f on G we put

$$\check{f}(x) = f(x^{-1}) \quad , \quad \tilde{f}(x) = \overline{f(x^{-1})} \ ,$$

$$f^*(x) = \Delta^{-1}(x)\overline{f(x^{-1})} \quad , \quad (Jf)(x) = \Delta^{-\frac{1}{2}}(x)\overline{f(x^{-1})} \ ,$$

for all $x \in G$. More generally, for each $p \in [1,\infty]$, we define

$$(J_p f)(x) = \Delta^{-1/p}(x)\overline{f(x^{-1})} \ , \quad x \in G \ .$$

Then in particular $J_1 f = f^*$, $J_2 f = Jf$, $J_\infty f = \tilde{f}$. Note that for each $p \in [1,\infty]$, the operation J_p is a conjugate linear isometric involution of $L^p(G)$.

We shall often make use of the following non-unimodular version of Young's inequalities for convolution:

<u>Lemma 1.1</u>. (Young's convolution inequalities.) Let $p_1, p_2, p \in [1,\infty]$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Assume that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} = 1$. Then for all $f_1 \in L^{p_1}(G)$ and $f_2 \in L^{p_2}(G)$ the convolution product $f_1 * \Delta^{1/q_1} f_2$ exists and belongs to $L^p(G)$, and

$$\| f_1 * \Delta^{1/q_1} f_2 \|_p \le \| f_1 \|_{p_1} \| f_2 \|_{p_2} \ .$$

This theorem is well-known in the unimodular case as well as in the special cases $(p_1, p_2, p) = (p_1, q_1, \infty)$ (where it follows from Hölder's inequality), $(p_1, p_2, p) = (1, p, p)$ or $(p_1, p_2, p) = (p, 1, p)$ [11, (20.14)]. The general case has also been noted [13, Remark 2.2]. It can be proved by modifying the proof of [11, (20.18)] or by interpolation from the special cases mentioned above.

For operators T on the Hilbert space $L^2(G)$ we use the notation $D(T)$ (domain of T), $R(T)$ (range of T), $N(T)$ (kernel of T). If T is preclosed, we denote by $[T]$ the closure of T. If T is a positive self-adjoint operator and P the projection onto $N(T)^\perp$, then by definition T^{it}, $t \in \mathbb{R}$, is the partial isometry coinciding with the unitary $(TP)^{it}$ on $N(T)^\perp$ and 0 on $N(T)$. By convention, when speaking of operators, "bounded" always means "bounded and everywhere defined".

We denote by λ and ρ the left and right regular representations of G on $L^2(G)$, i.e. the unitary representations given by

$$(\lambda(x)f)(y) = f(x^{-1}y) ,$$

$$(\rho(x)f)(y) = \Delta^{\frac{1}{2}}(x)f(yx) ,$$

for all $x,y \in G$ and $f \in L^2(G)$. The corresponding representations of the algebra $L^1(G)$ (as in [4, 13.3]) are given by

$$\lambda(h)f = h*f ,$$
$$\rho(h)f = f*\Delta^{-\frac{1}{2}}\check{h} ,$$

for all $h \in L^1(G)$ and $f \in L^2(G)$.

We denote by M the von Neumann algebra of operators on $L^2(G)$ generated by $\lambda(G)$ (or $\lambda(\mathcal{K}(G))$, or $\lambda(L^1(G))$). In other words, M is the left von Neumann algebra of $\mathcal{K}(G)$, where $\mathcal{K}(G)$ is considered as a left Hilbert algebra [3, Definition 2.1] with convolution, involution $*$, and the ordinary inner product in $L^2(G)$. The commutant M' of M is the von Neumann algebra generated by $\rho(G)$, and $M' = JMJ$.

A function $\xi \in L^2(G)$ is called left (resp. right) bounded if left (resp. right) convolution with ξ on $\mathcal{K}(G)$ extends to a bounded operator on $L^2(G)$, i.e. if there exists a bounded operator $\lambda(\xi)$ (resp. $\lambda'(\xi)$) such that $\forall k \in \mathcal{K}(G): \lambda(\xi)k = \xi * k$ (resp. $\lambda'(\xi)k = k*\xi$). The set of left (resp. right) bounded $L^2(G)$-functions is denoted $\mathfrak{A}_\ell$ (resp. $\mathfrak{A}_r$). Obviously, $\mathcal{K}(G) \subseteq \mathfrak{A}_\ell$, $\mathcal{K}(G) \subseteq \mathfrak{A}_r$, and for $\xi \in \mathcal{K}(G)$ we have $\lambda'(\xi) = \rho(\Delta^{-\frac{1}{2}} \check{\xi})$. Note that $\xi \in L^2(G)$ is left bounded if and only if the operator $\eta \mapsto \lambda'(\eta)\xi: \mathfrak{A}_r \to L^2(G)$ extends to a bounded operator on $L^2(G)$; if this is the case, we have $\lambda(\xi)\eta = \lambda'(\eta)\xi$ for all $\eta \in \mathfrak{A}_r$. (Our definition of left-boundedness therefore agrees with [1, Définition 2.1]). If $\xi \in \mathfrak{A}_\ell$ and $T \in M$, then $T\xi \in \mathfrak{A}_\ell$ and $\lambda(T\xi) = T\lambda(\xi)$.

We denote by φ_0 the canonical weight on M [1, Définition 2.12]. Then the weight ψ_0 on M' given by $\psi_0(y) = \varphi_0(JyJ)$ for all $y \in (M')_+$ is called the canonical weight on M'. The corresponding modular automorphism groups are given by

$$\sigma_t^{\varphi_0}(x) = \Delta^{it} x \Delta^{-it}, \quad x \in M,$$

$$\sigma_t^{\psi_0}(y) = \Delta^{-it} y \Delta^{it}, \quad y \in M',$$

for all $t \in \mathbb{R}$. Here, Δ denotes the multiplication operator on $L^2(G)$ by the function Δ (note that we shall not distinguish in our notation between the function Δ and the corresponding multiplication operator). With this definition, Δ is in fact the modular operator of $\mathcal{K}(G)$ (as defined in [3, Lemma 2.2]).

It follows from the defining property of φ_0 [1, Théorème 2.11] that for all $y \in M'$ we have

$$\psi_0(y^*y) = \begin{cases} \|\eta\|_2^2 & \text{if } y = \lambda'(\eta) \text{ for some } \eta \in \mathfrak{A}_r , \\ \infty & \text{otherwise} \end{cases} .$$

We identify the Hilbert space completion H_{ψ_0} of $n_{\psi_0} = \{y \in M' \mid \psi_0(y^*y) < \infty\}$ with $L^2(G)$ via $\eta \mapsto \lambda'(\eta)$.

Now recall that by definition [2, Definition 1], $D(L^2(G),\psi_0)$ is the set of $\xi \in L^2(G)$ such that $y \mapsto y\xi : n_{\psi_0} \to L^2(G)$ extends to a bounded operator $R^{\psi_0}(\xi): H_{\psi_0} \to L^2(G)$, i.e., in view of the identification of H_{ψ_0} with $L^2(G)$, such that $\eta \mapsto \lambda'(\eta)\xi : \mathfrak{A}_r \to L^2(G)$ extends to a bounded operator on $L^2(G)$. Thus $D(L^2(G),\psi_0) = \mathfrak{A}_\ell$, and for all $\xi \in D(L^2(G),\psi_0)$ we have $R^{\psi_0}(\xi) = \lambda(\xi)$.

If φ is a normal semi-finite weight on M , then by definition [2], $\frac{d\varphi}{d\psi_0}$ is the unique positive self-adjoint operator T satisfying

$$\forall \xi \in \mathfrak{A}_\ell : \varphi(\lambda(\xi)\lambda(\xi)^*) = \begin{cases} \|T^{\frac{1}{2}}\xi\|^2 & \text{if } \xi \in D(T^{\frac{1}{2}}) \\ \infty & \text{otherwise} \end{cases}$$

and

$$T^{\frac{1}{2}} = [T^{\frac{1}{2}}|_{\mathfrak{A}_\ell \cap D(T^{\frac{1}{2}})}] .$$

In particular, we have

$$\frac{d\varphi_0}{d\psi_0} = \Delta$$

(cf. [2, Lemma 10 (b)] together with the proof of [2, Lemma 10 (a)]).

If φ is a functional, then by the definition of $\frac{d\varphi}{d\psi_0}$ we have $\mathfrak{a}_\ell \subseteq D\left(\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}}\right)$ and $\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}} = \left[\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}} \Big| \mathfrak{a}_\ell\right]$.

Finally, we note that the predual space M_* of the von Neumann algebra M may be viewed as a space of functions on the group in the following manner: for each $\varphi \in M_*$, define $u: G \to \mathbb{C}$ by

$$u(x) = \varphi(\lambda(x)) \ , \quad x \in G \ .$$

Then u is a continuous function on the group determining φ completely. The linear space of such functions, normed by $\|u\| = \|\varphi\|$, is exactly the Fourier algebra $A(G)$ of G introduced by P. Eymard [6] (this follows from [6, Théorème (3.10)]). The identification of $A(G)$ with M_* is such that

$$<\varphi, \lambda(f)> = \int \varphi(x) f(x)\, dx$$

for all $\varphi \in M_* \simeq A(G)$ and all $f \in L^1(G)$.

Recall that by [4, 13.4.4] a continuous function φ on G is positive definite if and only if

$$\forall \xi \in \mathcal{K}(G): \int \varphi(x) (\xi * \xi^*)(x)\, dx \geq 0$$

i.e., if and only if

$$\forall \xi \in \mathcal{K}(G): \iint \varphi(yx^{-1}) \xi(y) \overline{\xi(x)}\, dy\, dx \geq 0 \ .$$

If $\varphi \in A(G)$, then φ is positive definite if and only if the corresponding functional $\varphi \in M_*$ is positive. We denote by $A(G)_+$ the set of positive definite $\varphi \in A(G)$.

2. Homogeneous operators on $L^2(G)$ and the spaces $L^p(\varphi_0)$.

Definition. Let $\alpha \in \mathbb{R}$. An operator T on $L^2(G)$ is called α-homogeneous if

$$\forall x \in G: \rho(x)T \subseteq \Delta^{-\alpha}(x)T\rho(x) .$$

Remarks. (1) The 0-homogeneous operators are precisely the operators affiliated with M .

(2) If T is α-homogeneous, then actually $\rho(x)T = \Delta^{-\alpha}(x)T\rho(x)$ for all $x \in G$ (to see this, replace x by x^{-1} in the definition).

(3) If T and S are both α-homogeneous, then $T+S$ is α-homogeneous. If T is α-homogeneous and S is β-homogeneous, then TS is $(\alpha+\beta)$-homogeneous. If T is densely defined and α-homogeneous, then T^* is also α-homogeneous. If T is positive self-adjoint and α-homogeneous and $\beta \in \mathbb{R}_+$, then T^β is $(\alpha\beta)$-homogeneous (use $\rho(x)T^\beta\rho(x^{-1}) = (\rho(x)T\rho(x^{-1}))^\beta$) .

(4) If T is α-homogeneous for some $\alpha \in \mathbb{R}$, then the projection onto $N(T)^\perp$ belongs to M (since $N(T)$ is invariant under all $\rho(x)$, $x \in G$) .

(5) If a preclosed operator T is α-homogeneous, then its closure $[T]$ is also α-homogeneous.

(6) For each $\alpha \in \mathbb{R}$, $\Delta^{-\alpha}$ is α-homogeneous.

Lemma 2.1. Let T be a closed densely defined operator on $L^2(G)$ with polar decomposition $T = U|T|$. Let $\alpha \in \mathbb{R}$. Then T is α-homogeneous if and only if $U \in M$ and $|T|$ is α-homogeneous.

Proof. If T is α-homogeneous, then, by Remark (3), $|T| = (T^*T)^{\frac{1}{2}}$ is also α-homogeneous. Then for all $x \in G$ and $\xi \in D(|T|)$ we have $\rho(x)U|T|\xi = \rho(x)T\xi = \Delta^{-\alpha}(x)T\rho(x)\xi = \Delta^{-\alpha}(x)U|T|\rho(x)\xi = U\rho(x)|T|\xi$, i.e. $\rho(x)U \subseteq U\rho(x)$ on $R(|T|)$. Since the projection onto $R(|T|) = N(|T|)^{\perp}$ belongs to M, we conclude that U commutes with all $\rho(x)$; thus $U \in M$.

The "if"-part follows directly from Remarks (3) and (1). ∎

Lemma 2.2. Let T be a closed densely defined operator on $L^2(G)$, and let $\alpha \in \mathbb{C}$. Suppose that

$$\forall x \in G: \rho(x)T \subseteq \Delta^{-\alpha}(x)T\rho(x) .$$

Then

$$\forall f \in \mathcal{K}(G): \lambda'(f)T \subseteq T\lambda'(\Delta^{\alpha}f) .$$

Proof. Let $f \in \mathcal{K}(G)$ and $\xi \in D(T)$. Then for all $\eta \in D(T^*)$ we have

$$\begin{aligned}(\rho(f)T\xi|\eta) &= \int f(x)(\rho(x)T\xi|\eta)dx \\ &= \int f(x)\Delta^{-\alpha}(x)(T\rho(x)\xi|\eta)dx \\ &= \int \Delta^{-\alpha}(x)f(x)(\rho(x)\xi|T^*\eta)dx \\ &= (\rho(\Delta^{-\alpha}f)\xi|T^*\eta) .\end{aligned}$$

This shows that $\rho(\Delta^{-\alpha}f)\xi \in D(T^{**}) = D(T)$ and $T\rho(\Delta^{-\alpha}f)\xi = \rho(f)T\xi$ for all $\xi \in D(T)$, i.e.

$$\rho(f)T \subseteq T\rho(\Delta^{-\alpha}f) .$$

Hence for all $f \in \mathcal{K}(G)$ we have

$$\lambda'(f)T = \rho(\Delta^{-\frac{1}{2}}\check{f})T \subseteq T\rho(\Delta^{-\alpha}\Delta^{-\frac{1}{2}}\check{f}) = T\lambda'(\Delta^{\alpha}f) .$$ ∎

Lemma 2.3. Let T be a closed densely defined operator on $L^2(G)$, α-homogeneous for some $\alpha \in \mathbb{R}$. Let $\xi \in \mathfrak{A}_\ell$. Then for all $t \in \mathbb{R}$ we have $|T|^{it}\xi \in \mathfrak{A}_\ell$ and

$$\|\lambda(|T|^{it}\xi)\| \leq \|\lambda(\xi)\| \quad .$$

Proof. By Lemma 2.1, we have $\rho(x)|T|\rho(x^{-1}) = \Delta^{-\alpha}(x)|T|$ for all $x \in G$, whence $\rho(x)|T|^{it}\rho(x^{-1}) = \Delta^{-i\alpha t}(x)|T|^{it}$ for all $x \in G$ and all $t \in \mathbb{R}$. Then, applying the preceding lemma to $|T|^{it}$, we obtain for all $\eta \in \mathcal{K}(G)$ that

$$|T|^{it}\xi*\eta = \lambda'(\eta)|T|^{it}\xi = |T|^{it}\lambda'(\Delta^{i\alpha t}\eta)\xi = |T|^{it}\lambda(\xi)\Delta^{i\alpha t}\eta \quad ,$$

and thus

$$\||T|^{it}\xi*\eta\|_2 \leq \||T|^{it}\| \; \|\lambda(\xi)\| \; \|\Delta^{i\alpha t}\eta\|_2 \leq \|\lambda(\xi)\| \; \|\eta\|_2 \; .$$

We conclude that $|T|^{it}\xi$ is left bounded and that

$$\|\lambda(|T|^{it}\xi)\| \leq \|\lambda(\xi)\| \quad . \quad \blacksquare$$

Remark. In particular, $\Delta^{it}\xi \in \mathfrak{A}_\ell$ with $\|\lambda(\Delta^{it}\xi)\| \leq \|\lambda(\xi)\|$ for all $\xi \in \mathfrak{A}_\ell$ and $t \in \mathbb{R}$.

Our next lemma shows that α-homogeneity as defined here is equivalent to homogeneity of degree α with respect to φ_0 as defined in [2, Definition 17].

Lemma 2.4. Let $\alpha \in \mathbb{R}$, and let T be a closed densely defined operator on $L^2(G)$ with polar decomposition $T = U|T|$. Then the following conditions are equivalent:

(i) T is α-homogeneous,

(ii) $U \in M$ and $\forall y \in M' \ \forall t \in \mathbb{R}$: $\sigma_{\alpha t}^{\psi_0}(y)\ |T|^{it} = |T|^{it} y$.

Proof. By Lemma 2.1, we may assume that T is positive self-adjoi

Denote by P the projection onto $N(T)^{\perp}$. If either (i) or (ii) holds, then P is in M , and thus the subspace $P\,L^2(G)$ is invariant under all operators considered. Therefore, we may suppose that $P \in M$, and the lemma is proved when we have shown the equivalence of

(1) $$\forall x \in G:\ \rho(x)T\rho(x^{-1})P = \Delta^{-\alpha}(x)T\,P$$

and

(2) $$\forall t \in \mathbb{R}\ \forall y \in M':\ \sigma_{\alpha t}^{\psi_0}(y)P = T^{it}\,y\,T^{-it}\,P \ .$$

Now for all $x \in G$ we have

$$\sigma_{\alpha t}^{\psi_0}(\rho(x)) = \Delta^{-i\alpha t}\rho(x)\Delta^{i\alpha t} = \Delta^{i\alpha t}(x)\rho(x) \ ,$$

since

$$(\Delta^{-i\alpha t}\rho(x)\Delta^{i\alpha t}f)(z)$$

$$= \Delta^{-it}(z)\Delta^{\frac{1}{2}}(x)\Delta^{it}(zx)f(zx)$$

$$= \Delta^{-it}(x)\,(\rho(x)f)(z)$$

for all $f \in L^2(G)$ and all $x,z \in G$. Then, since M' is generated by the $\rho(x)$, the condition (2) is equivalent to

$$\forall x \in G\ \ \forall t \in \mathbb{R}:\ \Delta^{i\alpha t}(x)\rho(x)P = T^{it}\,\rho(x)T^{-it}P$$

or (changing t into $-t$)

$$\forall x \in G \quad \forall t \in \mathbb{R}: \ \rho(x)T^{it}\rho(x)P = \Delta^{-i\alpha t}(x)T^{it}P ,$$

which in turn is equivalent to (1). ∎

Now, by [2, Theorem 13] a positive self-adjoint operator on $L^2(G)$ is (-1)-homogeneous if and only if it has the form $\frac{d\varphi}{d\psi_0}$ for a (necessarily unique) normal semi-finite weight φ on M.

We define the "integral with respect to ψ_0" of a positive self-adjoint (-1)-homogeneous operator T as

$$\int T \, d\psi_0 = \varphi(1) \in [0,\infty] ,$$

where $T = \frac{d\varphi}{d\psi_0}$. If $\int T \, d\psi_0 < \infty$, i.e. if φ is a functional, we shall say that T is integrable. (These definitions agree with those given in [2, remarks following Corollary 18].)

For each $p \in [1,\infty[$, we denote by $L^p(\psi_0)$ the set of closed densely defined $(-\frac{1}{p})$-homogeneous operators T on $L^2(G)$ satisfying

$$\int |T|^p d\psi_0 < \infty .$$

(Note that $|T|^p$ is (-1)-homogeneous, so that $\int |T|^p \, d\psi_0$ is defined.) We put $L^\infty(\psi_0) = M$.

The spaces $L^p(\psi_0)$ introduced here are special cases of the spatial L^p-spaces of M. Hilsum [12]. We recall their main properties (note, however, that our notation differs from that of [12] in that we maintain throughout the distinction between operators and their closures):

If $T, S \in L^p(\psi_0)$, then $T+S$ is densely defined and preclosed, and the closure $[T+S]$ belongs to $L^p(\psi_0)$. With the obvious scalar multiplication and the sum $(T,S) \mapsto [T+S]$, $L^p(\psi_0)$ is a linear space, and even a Banach space with the norm $\|\cdot\|_p$ defined by $\|T\|_p = (\int |T|^p \, d\psi_0)^{1/p}$ if $p \in [1,\infty[$ and $\|T\|_p = \|T\|$ (operator norm) if $p = \infty$. The operation $T \mapsto T^*$ is an isometry of $L^p(\psi_0)$ onto $L^p(\psi_0)$. We denote $L^p(\psi_0)_+$ the set of positive self-adjoint operators belonging to $L^p(\psi_0)$.

By linearity, $T \mapsto \int T \, d\psi_0$ defined on $L^1(\psi_0)_+$ extends to a linear form on the whole of $L^1(\psi_0)$ satisfying $\int T^* d\psi_0 = \overline{\int T \, d\psi_0}$ and $|\int T \, d\psi_0| \leq \|T\|_1$ for all $T \in L^1(\psi_0)$.

Let $p_1, p_2, p \in [1,\infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. If $T \in L^{p_1}(\psi_0)$ and $S \in L^{p_2}(\psi_0)$, then the operator TS is densely defined and preclosed, its closure $[TS]$ belongs to $L^p(\psi_0)$, and

$$\|[TS]\|_p \leq \|T\|_{p_1} \|S\|_{p_2} .$$

In particular, if $T \in L^p(\psi_0)$ and $S \in L^q(\psi_0)$ where $\frac{1}{p} + \frac{1}{q} = 1$, then $[TS] \in L^1(\psi_0)$ and $\|[TS]\|_1 \leq \|T\|_p \|S\|_q$ (Hölder's inequalit[y]; furthermore, $\int [TS] d\psi_0 = \int [ST] d\psi_0$.

If $p \in [1,\infty[$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we identify $L^q(\psi_0)$ wit[h] the dual space of $L^p(\psi_0)$ by means of the form $(T,S) \mapsto \int [TS] d\psi_0$, $T \in L^p(\psi_0)$, $S \in L^q(\psi_0)$. In particular, $L^1(\psi_0)$ is the predual of $M = L^\infty(\psi_0)$. The space $L^2(\psi_0)$ is a Hilbert space with the inner product $(T|S)_{L^2(\psi_0)} = \int [S^*T] d\psi_0$.

Remark. Suppose that G is unimodular. Then the α-homogeneous operators for any α are simply the operators affiliated with M

and the canonical weight φ_0 on M is a trace. We claim that $\int T\, d\psi_0 = \varphi_0(T)$ for all positive self-adjoint operators T affiliated with M, where we have written $\varphi_0(T)$ for the value of $\varphi = \varphi_0(T\cdot)$ at 1 (with $\varphi_0(T\cdot)$ defined as in [17, Section 4]). To see this, recall that $\frac{d\varphi_0}{d\psi_0} = \Delta = 1$, so that using [2, Theorem 9, (2)], we have

$$T^{it} = (D\varphi : D\varphi_0)_t = \left(\frac{d\varphi}{d\psi_0}\right)^{it} \left(\frac{d\varphi_0}{d\psi_0}\right)^{-it} = \left(\frac{d\varphi}{d\psi_0}\right)^{it}$$

for all $t \in \mathbb{R}$. Thus $T = \frac{d\varphi}{d\psi_0}$ and $\int T\, d\psi_0 = \varphi(1) = \varphi_0(T)$. (When proving $T = \frac{d\varphi}{d\psi_0}$, we implicitly assumed that T is injective so that $\varphi = \varphi_0(T\cdot)$ is faithful. In the general case, denote by $Q \in M$ the projection onto $N(T)$, note that $T+Q$ is positive self-adjoint, affiliated with M, and injective, and verify that

$$T+Q = \frac{d\varphi_0((T+Q)\cdot)}{d\psi_0} = \frac{d\varphi_0(T\cdot)}{d\psi_0} + \frac{d\varphi_0(Q\cdot)}{d\psi_0} .$$

Since the supports of $\frac{d\varphi_0(T\cdot)}{d\psi_0}$ and $\frac{d\varphi_0(Q\cdot)}{d\psi_0}$ are $1-Q$ and Q, respectively, we conclude that $T = \frac{d\varphi_0(T\cdot)}{d\psi_0}$ as desired.) It follows that in this case the spaces $L^p(\psi_0)$ reduce to the ordinary $L^p(M,\varphi_0)$ (discussed in the introduction).

Returning to the general case, we now proceed to a more detailed study of the spaces $L^p(\psi_0)$. For this, we shall need the following slightly generalized version of [12, II, Proposition 2]:

Lemma 2.5. Let T be a positive self-adjoint operator on $L^2(G)$, α-homogeneous for some $\alpha \in \mathbb{R}$. Let $\xi \in \mathfrak{A}_\ell$. Then there exist $\xi_n \in \mathfrak{A}_\ell \cap \bigcap_{\beta \in \mathbb{R}_+} D(T^\beta)$, $n \in \mathbb{N}$, such that

(i) $\forall n \in \mathbb{N}: \|\lambda(\xi_n)\| \leq \|\lambda(\xi)\|$,

(ii) $\xi_n \to \xi$ as $n \to \infty$,

(iii) $T^\beta \xi_n \to T^\beta \xi$ as $n \to \infty$ whenever ξ and $\beta \in \mathbb{R}_+$ satisfy $\xi \in D(T^\beta)$.

Proof. For each $n \in \mathbb{N}$, define $f_n: [0,\infty[\to \mathbb{C}$ by

$$f_n(x) = \begin{cases} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} x^{it/\sqrt{n}}\, dt & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}.$$

Since for all $x \in [0,\infty[$ we have $|f_n(x)| \leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt = 1$, the operators $f_n(T)$ are bounded. For each $n \in \mathbb{N}$, put $\xi_n = f_n(T)\xi$.

To prove that the ξ_n belong to $\mathfrak{A}_\ell$ and satisfy (i), denote by P the projection onto $N(T)^\perp$ and observe that for all $\eta \in \mathcal{K}(G)$ we have

$$\begin{aligned} f_n(T)P\xi * \eta &= \lambda'(\eta) f_n(T)P\xi \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \lambda'(\eta) T^{it/\sqrt{n}}\, \xi dt \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} T^{it/\sqrt{n}} \lambda'(\Delta^{i\alpha t/\sqrt{n}}\eta)\xi\, dt \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} T^{it/\sqrt{n}} (\xi * \Delta^{i\alpha t/\sqrt{n}}\eta)\, dt , \end{aligned}$$

where we have used Lemma 2.2. It follows that

$$\| f_n(T)P\xi*\eta\|_2 \leq \frac{1}{\sqrt{\pi}} \int e^{-t^2} \|\lambda(\xi)\| \, \|\Delta^{i\alpha t/\sqrt{n}}\eta\|_2 dt \leq \|\lambda(\xi)\| \, \|\eta\|_2 .$$

On the other hand,

$$\|(1-P)\xi*\eta\|_2 \leq \|\lambda((1-P)\xi)\| \, \|\eta\|_2 \leq \|\lambda(\xi)\| \, \|\eta\|_2 ,$$

since $P \in M$.

In all, $f_n(T)\xi = f_n(T)P\xi + (1-P)\xi$ belongs to $\mathfrak{A}_\ell$ and $\|\lambda(f_n(T)\xi)\| \leq \|\lambda(\xi)\|$.

Now, to see that $\xi_n \in D(T^\beta)$ for all $\beta \in \mathbb{R}_+$, note that

$$f_n(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} e^{(it/\sqrt{n})\log x} dt$$

$$= e^{-\frac{1}{4n}(\log x)^2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t-\frac{i}{2\sqrt{n}} \log x)^2} dt$$

$$= e^{-\frac{1}{4n}(\log x)^2}$$

for all $x>0$. Then $x \mapsto x^\beta f_n(x) = e^{(\beta \log x - \frac{1}{4n}(\log x)^2)}$ is bounded, so that $T^\beta f_n(T)$ is a bounded operator, and thus $f_n(T)\xi \in D(T^\beta)$.

Since f_n is bounded and $f_n(x) \to 1$ as $n \to \infty$ for all $x \in [0,\infty[$, we have

$$f_n(T)\zeta \to \zeta \quad \text{as} \quad n \to \infty$$

for all ζ. From this, we immediately get (ii) and (iii). Indeed, $\xi_n = f_n(T)\xi \to \xi$, and if $\xi \in D(T^\beta)$, then

$$T^\beta \xi_n = T^\beta f_n(T)\xi = f_n(T)T^\beta \xi \to T^\beta \xi . \quad \blacksquare$$

Proposition 2.1. Let T be a closed densely defined (-1)-homogeneous operator on $L^2(G)$. Then the following conditions are equivalent:

(i) $T \in L^1(\psi_0)$,

(ii) there exists a constant $C \geq 0$ such that

$\forall \xi \in \mathfrak{A}_\ell \cap D(T) \quad \forall \eta \in \mathfrak{A}_\ell : \; |(T\xi|\eta)| \leq C\|\lambda(\xi)\| \; \|\lambda(\eta)\|$,

(iii) there exists a constant $C \geq 0$ such that

$\forall \xi \in \mathfrak{A}_\ell \cap D(|T|^{\frac{1}{2}}) : \; \||T|^{\frac{1}{2}}\xi\|^2 \leq C\|\lambda(\xi)\|^2$,

(iv) there exists an approximate identity $(\xi_i)_{i\in I}$ in $\mathcal{K}(G)_+$ such that all $\xi_i \in D(|T|^{\frac{1}{2}})$ and

$$\liminf \||T|^{\frac{1}{2}}\xi_i\| < \infty \; .$$

If $T \in L^1(\psi_0)$, then $\mathfrak{A}_\ell \subseteq D(|T|^{\frac{1}{2}})$, and for any approximate identity $(\xi_i)_{i\in I}$ in $\mathcal{K}(G)_+$ we have

$$\|T\|_1 = \lim \| \; |T|^{\frac{1}{2}}\xi_i\|^2 \; .$$

Furthermore, $\|T\|_1$ is the smallest C satisfying (ii) and the smallest C satisfying (iii).

Proof. Let $T = U|T|$ be the polar decomposition of T .

First, suppose that $T \in L^1(\psi_0)$. Then $|T| \in L^1(\psi_0)_+$, and therefore $|T| = \frac{d\varphi}{d\psi_0}$ for some positive functional φ on M . Recall that $\mathfrak{A}_\ell \subseteq D(|T|^{\frac{1}{2}})$. Thus for all $\xi \in \mathfrak{A}_\ell \cap D(T)$ and $\eta \in \mathfrak{A}_\ell$ we have

$$\begin{aligned} |(T\xi|\eta)| &= |(|T|^{\frac{1}{2}}\xi \,|\, |T|^{\frac{1}{2}}U^*\eta)| \\ &= |\varphi(\lambda(\xi)\lambda(U^*\eta))| \\ &\leq \|\varphi\| \; \|\lambda(\xi)\| \; \|\lambda(U^*\eta)\| \\ &\leq \|T\|_1 \; \|\lambda(\xi)\| \; \|\lambda(\eta)\| \; , \end{aligned}$$

i.e., (ii) holds.

Next, suppose that T satisfies (ii). Then for all $\xi \in \mathfrak{A}_\ell \cap D(|T|)$ we have

$$\begin{aligned} \| \, |T|^{\frac{1}{2}}\xi \|^2 &= |(T\xi | U\xi)| \\ &\leq C\|\lambda(\xi)\| \ \ \|\lambda(U\xi)\| \\ &\leq C\|\lambda(\xi)\|^2 . \end{aligned}$$

Now if $\xi \in \mathfrak{A}_\ell \cap D(|T|^{\frac{1}{2}})$, there exist (by Lemma 2.5) $\xi_n \in \mathfrak{A}_\ell \cap D(|T|)$ such that $|T|^{\frac{1}{2}}\xi_n \to |T|^{\frac{1}{2}}\xi$ and $\|\lambda(\xi_n)\| \leq \|\lambda(\xi)\|$. Since

$$\| \, |T|^{\frac{1}{2}}\xi_n\|^2 \leq C\|\lambda(\xi_n)\|^2 \leq C\|\lambda(\xi)\|^2 ,$$

we conclude that $\| \, |T|^{\frac{1}{2}}\xi\|^2 \leq C\|\lambda(\xi)\|^2$. Thus (iii) is proved.

Now suppose that T satisfies (iii). First we show that this implies $\mathfrak{A}_\ell \subseteq D(|T|^{\frac{1}{2}})$. Let $\xi \in \mathfrak{A}_\ell$. Then by Lemma 2.5 there exist $\xi_n \in \mathfrak{A}_\ell \cap D(|T|^{\frac{1}{2}})$ such that $\xi_n \to \xi$ and $\|\lambda(\xi_n)\| \leq \|\lambda(\xi)\|$. Then for all $\eta \in D(|T|^{\frac{1}{2}})$ we have

$$\begin{aligned} |(|T|^{\frac{1}{2}}\xi_n | \eta)| &\leq \| \, |T|^{\frac{1}{2}}\xi_n\| \ \|\eta\| \\ &\leq C^{\frac{1}{2}}\|\lambda(\xi_n)\| \ \|\eta\| \\ &\leq C^{\frac{1}{2}}\|\lambda(\xi)\| \ \ \|\eta\| \end{aligned}$$

and

$$(|T|^{\frac{1}{2}}\xi_n | \eta) = (\xi_n \mid |T|^{\frac{1}{2}}\eta) \to (\xi \mid |T|^{\frac{1}{2}}\eta) .$$

We conclude that

$$\forall \eta \in D(|T|^{\frac{1}{2}}) : |(\xi \mid |T|^{\frac{1}{2}}\eta)| \leq C^{\frac{1}{2}}\|\lambda(\xi)\| \ \|\eta\| \quad .$$

Thus $\xi \in D(|T|^{\frac{1}{2}})$ as wanted.

Now, still assuming (iii), let us prove (iv). Let $(\xi_i)_{i\in I}$ be any approximate identity in $\mathcal{K}(G)_+$. Then automatically all $\xi_i \in \mathcal{K}(G) \subseteq \mathfrak{A}_\ell \subseteq D(|T|^{\frac{1}{2}})$, and $\|\lambda(\xi_i)\| \leq \|\xi_i\|_1 = 1$ so that

$$\| |T|^{\frac{1}{2}}\xi_i\|^2 \leq C\|\lambda(\xi_i)\|^2 \leq C \quad ,$$

whence $\liminf\| |T|^{\frac{1}{2}}\xi_i\| \leq C^{\frac{1}{2}} < \infty$.

Finally, suppose that T satisfies (iv) for some $(\xi_i)_{i\in I}$. Note that since $\int(\xi_i*\xi_i^*)(x)dx = 1$, $(\xi_i*\xi_i^*)_{i\in I}$ is again an approximate identity in $\mathcal{K}(G)_+$. Therefore, $\lambda(\xi_i)\lambda(\xi_i)^* = \lambda(\xi_i*\xi_i^*)$ converges strongly, and hence weakly, to 1 in M. Since all $\|\lambda(\xi_i)\lambda(\xi_i)^*\| \leq 1$, this convergence is also σ-weak, and by the σ-weak lower semicontinuity of φ, this implies

$$\begin{aligned}\varphi(1) &\leq \liminf \varphi(\lambda(\xi_i)\lambda(\xi_i)^*)\\ &= \liminf\| |T|^{\frac{1}{2}}\xi_i\|^2\\ &\leq C \liminf\|\lambda(\xi_i)\|^2\\ &\leq C < \infty \quad .\end{aligned}$$

Since $\varphi(1) = \int|T|d\psi_0 < \infty$, we have $T \in L^1(\psi_0)$, i.e. (i) holds

Note that once $\varphi(1) < \infty$ is established, φ is known to be σ-weakly lower continuous and thus

$$\varphi(1) = \lim \varphi(\lambda(\xi_i)\lambda(\xi_i)^*) = \lim\| |T|^{\frac{1}{2}}\xi_i\|^2$$

for any approximate identity $(\xi_i)_{i\in I}$, i.e.

$$\|T\|_1 = \lim\| |T|^{\frac{1}{2}}\xi_i\|^2 \ .$$

In the course of the proof we observed that $\|T\|_1$ may be used as the constant C in (ii), that every constant C satisfying (ii) also satisfies (iii), and that any C satisfying (iii) is bigger than $\lim \| |T|^{\frac{1}{2}} \xi_i \|^2$, i.e. bigger than $\|T\|_1$. This proves the remarks that end Proposition 2.1. ∎

As an immediate corollary, we have:

<u>Proposition 2.2</u>. Let T be a closed densely defined $(-\frac{1}{2})$-homogeneous operator on $L^2(G)$. Then the following conditions are equivalent:

(i) $T \in L^2(\psi_0)$,

(ii) there exists a constant $C \geq 0$ such that

$\forall \xi \in \mathfrak{A}_\ell \cap D(T): \|T\xi\| \leq C\|\lambda(\xi)\|$,

(iii) there exists an approximate identity $(\xi_i)_{i \in I}$ in $\mathcal{K}(G)_+$ such that all $\xi_i \in D(T)$ and

$$\liminf \|T\xi_i\| < \infty .$$

If $T \in L^2(\psi_0)$, then $\mathfrak{A}_\ell \subseteq D(T)$, and for any approximate identity $(\xi_i)_{i \in I}$ in $\mathcal{K}(G)_+$ we have

$$\|T\|_2 = \lim \|T\xi_i\| ;$$

furthermore, $\|T\|_2$ is the smallest constant C satisfying (ii).

We now come to the case of a general $p \in [1,\infty[$. Suppose that $T \in L^p(\psi_0)$ and $S \in L^q(\psi_0)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then by [12, II, Proposition 5,1)], we have

$$(T\xi | S\eta) = \langle [S^*T], \lambda(\xi)\lambda(\eta)^* \rangle$$

for all $\xi \in \mathfrak{A}_\ell \cap D(T)$ and $\eta \in \mathfrak{A}_\ell \cap D(S)$. (Here, $\langle\cdot,\cdot\rangle$ denotes the form giving the duality of $L^1(\psi_0)$ and M.) Using Hölder's inequality, we get

$$|(T\xi|S\eta)| \leq \|[S^*T]\|_1 \, \|\lambda(\xi)\lambda(\eta)^*\| \leq \|T\|_p \, \|S\|_q \, \|\lambda(\xi)\| \, \|\lambda(\eta)\|$$

for all such ξ and η. This kind of inequality in fact characterizes $L^p(\psi_0)$-operators among all $(-\frac{1}{p})$-homogeneous operators:

<u>Proposition 2.3</u>. Let $p \in [1,\infty]$ and define q by $\frac{1}{p}+\frac{1}{q} = 1$. Let T be a closed densely defined $(-\frac{1}{p})$-homogeneous operator on $L^2(G)$. Then the following conditions are equivalent:

(i) $T \in L^p(\psi_0)$,

(ii) there exists a constant $C \geq 0$ such that

$$\forall S \in L^q(\psi_0) \quad \forall \xi \in \mathfrak{A}_\ell \cap D(T) \quad \forall \eta \in \mathfrak{A}_\ell \cap D(S):$$
$$|(T\xi|S\eta)| \leq C\|S\|_q \, \|\lambda(\xi)\| \, \|\lambda(\eta)\| .$$

If $T \in L^p(\psi_0)$, then $\|T\|_p$ is the smallest C satisfying (ii).

<u>Proof</u>. In view of the remarks preceding this proposition, we just have to show that if T satisfies (ii) for some constant C, then $T \in L^p(\psi_0)$ and $\|T\|_p \leq C$.

Therefore suppose that T with polar decomposition $T = U|T|$ satisfies (ii). Then also

$$\begin{aligned} |(|T|\xi|S\eta)| &= |(T\xi|U^*S\eta)| \\ &\leq C\|[U^*S]\|_q \, \|\lambda(\xi)\| \, \|\lambda(\eta)\| \\ &\leq C\|S\|_q \, \|\lambda(\xi)\| \, \|\lambda(\eta)\| \end{aligned}$$

for all S, ξ, and η chosen as in (ii). Thus we may assume that T is positive self-adjoint.

Let $S \in L^q(\psi_0)$ and $\eta \in \mathfrak{A}_\ell \cap D(T^{\frac{1}{2}}S)$. We claim that for all $\xi \in \mathfrak{A}_\ell \cap D(T^{\frac{1}{2}})$ we have

$$(1) \qquad |(T^{\frac{1}{2}}\xi | T^{\frac{1}{2}}S\eta)| \leq C\|S\|_q \, \|\lambda(\xi)\| \, \|\lambda(\eta)\| .$$

If $\xi \in \mathfrak{A}_\ell \cap D(T)$, this follows directly from the hypothesis. In case of a general $\xi \in \mathfrak{A}_\ell \cap D(T^{\frac{1}{2}})$, choose (by Lemma 2.5) $\xi_n \in \mathfrak{A}_\ell \cap D(T)$ such that $T^{\frac{1}{2}}\xi_n \to T^{\frac{1}{2}}\xi$ and $\|\lambda(\xi_n)\| \leq \|\lambda(\xi)\|$. Then (1) follows by passing to the limit.

Now since T is $(-\frac{1}{p})$-homogeneous, there exist $T_i \in L^p(\psi_0)_+$ satisfying $T_i{}^p \leq T^p$ and $\int T^p d\psi_0 = \sup \int T_i{}^p d\psi_0$. (To see this, recall that $T^p = \frac{d\varphi}{d\psi_0}$ for some normal semi-finite weight φ on M; put $T_i = \left(\frac{d\varphi_i}{d\psi_0}\right)^{1/p}$ where the φ_i are positive normal functionals such that $\varphi_i \nearrow \varphi$; then $\frac{d\varphi_i}{d\psi_0} \leq \frac{d\varphi}{d\psi_0}$ by [2, Proposition 8], and $\int T^p d\psi_0 = \varphi(1) = \sup \varphi_i(1) = \sup \int T_i{}^p d\psi_0$.)

Since the function $t \mapsto t^{1/p}$ is operator monotone on $[0,\infty[$ (by [16, Proposition 1.3.8]), we have $T_i \leq T$, i.e. $D(T_i{}^{\frac{1}{2}}) \supseteq D(T^{\frac{1}{2}})$ and

$$\forall \xi \in D(T^{\frac{1}{2}}): \|T_i{}^{\frac{1}{2}}\xi\| \leq \|T^{\frac{1}{2}}\xi\| ,$$

for each $i \in I$ (cf. also the remark following this proof).

For each i, let B_i be the bounded operator characterized by $B_i T^{\frac{1}{2}}\xi = T_i{}^{\frac{1}{2}}\xi$ for all $\xi \in D(T^{\frac{1}{2}})$ and $B_i\xi = 0$ for all $\xi \in R(T^{\frac{1}{2}})^{\perp}$. Then $\|B_i\| \leq 1$. Since $B_i T^{\frac{1}{2}} \subseteq T_i{}^{\frac{1}{2}}$, and since $T^{\frac{1}{2}}$ and $T_i{}^{\frac{1}{2}}$ are $(-\frac{1}{p})$-homogeneous, B_i is 0-homogeneous, i.e. $B_i \in M$. Put $A_i = B_i{}^*$. Then $A_i \in M$, $\|A_i\| \leq 1$, and

$$T_i{}^{\frac{1}{2}} \subseteq T^{\frac{1}{2}}A_i .$$

Using this, the fact that

$$T_i^{p-1} = T_i^{p/q} \in L^q(\psi_0) \quad \text{with} \quad \|T_i^{p-1}\|_q = \|T_i\|_p^{p-1} ,$$

and (1), we find that for all $\xi \in \mathfrak{A}_\ell \cap \bigcap_{\beta \in \mathbb{R}_+} D(T_i^{\beta})$, we have

$$\begin{aligned} \|T_i^{p/2}\xi\|^2 &= (T_i^{\frac{1}{2}}\xi | T_i^{\frac{1}{2}} T_i^{p-1}\xi) \\ &= (T^{\frac{1}{2}} A_i \xi | T^{\frac{1}{2}} A_i T_i^{p-1}\xi) \\ &\leq C\|[A_i T_i^{p-1}]\|_q \; \|\lambda(A_i\xi)\| \; \|\lambda(\xi)\| \\ &\leq C\|A_i\| \; \|T_i^{p-1}\|_q \; \|A_i\| \; \|\lambda(\xi)\|^2 \\ &= C\|T_i\|_p^{p-1} \; \|\lambda(\xi)\|^2 . \end{aligned}$$

By means of Lemma 2.5, we conclude that the estimate

$$\|T_i^{p/2}\|^2 \leq C\|T_i\|_p^{p-1} \; \|\lambda(\xi)\|^2$$

holds for all $\xi \in \mathfrak{A}_\ell \cap D(T_i^{p/2})$. Thus by Proposition 2.1,

$$\|T_i\|_p^p = \|T_i^p\|_1 \leq C\|T_i\|_p^{p-1} ,$$

i.e.

$$\|T_i\|_p \leq C .$$

Since this holds for all i , we have

$$\int T^p d\psi_0 = \sup \int T_i^p d\psi_0 \leq C^p < \infty ;$$

thus $T \in L^p(\psi_0)$ and $\|T_i\|_p \leq C$. ∎

<u>Remark</u>. We have used the fact that if a continuous function f on $[0,\infty[$ is operator monotone in the sense that $R \leq S$ implies $f(R) \leq f(S)$ for all positive bounded operators R and S , then

the same is true for all - possibly unbounded - positive self-adjoint R and S. To see this, suppose that $R \leq S$. Then for all $\varepsilon \in \mathbb{R}_+$, we have $R(1+\varepsilon R)^{-1} \leq S(1+\varepsilon S)^{-1}$ by [17, Section 4], and hence $f(R(1+\varepsilon R)^{-1}) \leq f(S(1+\varepsilon S)^{-1})$. Now if $\xi \in D(f(S)^{\frac{1}{2}})$, we have by spectral theory

$$(f(R(1+\varepsilon R)^{-1})\xi|\xi) \leq (f(S(1+\varepsilon S)^{-1})\xi|\xi)$$
$$\rightarrow \|f(S)^{\frac{1}{2}}\xi\|^2 \quad \text{as} \quad \varepsilon \rightarrow 0 .$$

Again by spectral theory, we conclude that $\xi \in D(f(R)^{\frac{1}{2}})$ and that

$$\|f(R)^{\frac{1}{2}}\xi\|^2 = \lim_{\varepsilon\to 0}(f(R(1+\varepsilon R)^{-1})\xi|\xi) \leq \|f(S)^{\frac{1}{2}}\xi\|^2 .$$

In all, we have proved that $f(R) \leq f(S)$.

Recall from [12, §1, Théorème 4, 1)], that if T_1 and T_2 belong to some $L^p(\psi_0)$, $1 \leq p < \infty$, and if $T_2 \subseteq T_1$, then $T_1 = T_2$. Actually, a stronger result holds:

<u>Lemma 2.6</u>. Let $p \in [1,\infty]$. Let $T_1 \in L^p(\psi_0)$ and let T_2 be a closed densely defined $(-\frac{1}{p})$-homogeneous operator on $L^2(G)$. Suppose that

$$T_2 \subseteq T_1 \quad \text{or} \quad T_1 \subseteq T_2 .$$

Then $T_1 = T_2$.

<u>Proof</u>. 1) First suppose that $T_2 \subseteq T_1$. If $p = \infty$, the result is well-known (a closed densely defined operator having a bounded and everywhere defined extension is equal to that extension). If

$p \in [1,\infty[$, we conclude by Proposition 2.3 that also $T_2 \in L^p(\psi$ and thus by [12, §1, Théorème 4, 1)], $T_1 = T_2$. (Alternatively, this can be proved directly, i.e. without using Proposition 2.3, by the methods of the proof of [12, §1, Théorème 4, 1)].)

If $T_1 \subseteq T_2$, apply the first part of the proof to $T_2^* \subseteq T_1^*$. ∎

A specific form of this lemma will be crucial to much of the following:

<u>Proposition 2.4</u>. Let $p \in [1,\infty]$.

1) Let T and S be closed densely defined $(-\frac{1}{p})$-homogeneou operators on $L^2(G)$ with $\mathcal{K}(G) \subseteq D(T)$ and $\mathcal{K}(G) \subseteq D(S$ Suppose that

$$\forall \xi \in \mathcal{K}(G): T\xi = S\xi .$$

Then if one of the operators, say T , belongs to $L^p(\psi_0)$, we may conclude that $T = S$.

2) If $T \in L^p(\psi_0)$ and $\mathcal{K}(G) \subseteq D(T)$, then $T = [T|_{\mathcal{K}(G)}]$

<u>Proof</u> (of both parts). Suppose that $T \in L^p(\psi_0)$. Then $T|_{\mathcal{K}(G)}$ being a restriction of a $(-\frac{1}{p})$-homogeneous operator to a right inva riant subspace, is itself $(-\frac{1}{p})$-homogeneous. Therefore also $[T|_{\mathcal{K}(G)}]$ is $(-\frac{1}{p})$-homogeneous. Since $[T|_{\mathcal{K}(G)}] \subseteq T$, we conclude by the above lemma that $T = [T|_{\mathcal{K}(G)}]$. This proves 2). - As for 1), note that $S \supseteq S|_{\mathcal{K}(G)} = T|_{\mathcal{K}(G)}$, and thus $S \supseteq [T|_{\mathcal{K}(G)}] = T$. Again we conclude $S = T$. ∎

Finally, for later reference, we summarize in a lemma some remarks of Hilsum [12]:

Lemma 2.7. Let $q \in [2,\infty[$. Let $T \in L^q(\psi_0)$. Then $\mathfrak{A}_\ell \subseteq D(T)$, and for all $\xi \in \mathfrak{A}_\ell$ we have

$$\|T\xi\| \leq \|T\|_q \|\lambda(\xi)\|^{2/q} \|\xi\|^{1-2/q} .$$

Proof. Since $|T|^{q/2} \in L^q(\psi_0)$, we have $\mathfrak{A}_\ell \subseteq D(|T|^{q/2})$. Now let $\xi \in \mathfrak{A}_\ell$. Then by spectral theory $\xi \in D(|T|)$ and

$$\begin{aligned} \| |T|\xi\|^2 &\leq (\| |T|^{q/2}\xi\|^2)^{2/q} \cdot (\|\xi\|^2)^{1-2/q} \\ &\leq (\| |T|^q\|_1 \|\lambda(\xi)\|^2)^{2/q} \cdot \|\xi\|^{2(1-2/q)} \\ &= (\|T\|_q \|\lambda(\xi)\|^{2/q} \|\xi\|^{1-2/q})^2 . \quad \blacksquare \end{aligned}$$

3. The Plancherel transformation.

Given any functions $f \in L^2(G)$ and $\xi \in L^2(G)$, the convolution product $f*\Delta^{\frac{1}{2}}\xi$ exists and belongs to $L^\infty(G)$. Thus the following definition makes sense:

Definition. Let $f \in L^2(G)$. The Plancherel transform $\mathcal{P}(f)$ of f is the operator on $L^2(G)$ given by

$$\mathcal{P}(f)\xi = f*\Delta^{\frac{1}{2}}\xi , \quad \xi \in D(\mathcal{P}(f)) ,$$

where

$$D(\mathcal{P}(f)) = \{\xi \in L^2(G) \mid f*\Delta^{\frac{1}{2}}\xi \in L^2(G)\} .$$

Theorem 3.1. (Plancherel).

1) Let $f \in L^2(G)$. Then $\mathcal{P}(f)$ belongs to $L^2(\psi_0)$, and

$$\|\mathcal{P}(f)\|_2 = \|f\|_2 .$$

2) The Plancherel transformation $\mathcal{P} : L^2(G) \to L^2(\psi_0)$ is a unitary transformation of $L^2(G)$ onto $L^2(\psi_0)$.

Proof. 1) First note that $\mathcal{P}(f)$ is $(-\frac{1}{2})$-homogeneous: for all $x,y \in G$ and $\xi \in D(\mathcal{P}(f))$, we have

$$\begin{aligned}\rho(x)(\mathcal{P}(f)\xi)(y) &= \Delta^{\frac{1}{2}}(x)(f*\Delta^{\frac{1}{2}}\xi)(yx) \\ &= \Delta^{\frac{1}{2}}(x)\int f(z)\Delta^{\frac{1}{2}}(z^{-1}yx)\xi(z^{-1}yx)dz \\ &= \Delta^{\frac{1}{2}}(x)\int f(z)\Delta^{\frac{1}{2}}(z^{-1}y)(\rho(x)\xi)(z^{-1}y)dz \\ &= \Delta^{\frac{1}{2}}(x)(f*\Delta^{\frac{1}{2}}\rho(x)\xi)(y) ,\end{aligned}$$

i.e. $\rho(x)\mathcal{P}(f) \subseteq \Delta^{\frac{1}{2}}(x)\mathcal{P}(f)\rho(x)$.

We next show that $\mathcal{P}(f)$ is closed. Suppose that $\xi_n \to \xi$ in $L^2(G)$ and $\mathcal{P}(f)\xi_n \to \eta$ in $L^2(G)$, where all the $\xi_n \in D(\mathcal{P}(f))$ Then $f*\Delta^{\frac{1}{2}}\xi_n \to f*\Delta^{\frac{1}{2}}\xi$ uniformly (by a simple case of Lemma 1.1). Since $f*\Delta^{\frac{1}{2}}\xi_n \to \eta$ in $L^2(G)$, we conclude that $\eta = f*\Delta^{\frac{1}{2}}\xi$. Thus $\xi \in D(\mathcal{P}(f))$ and $\mathcal{P}(f)\xi = \eta$, so that $\mathcal{P}(f)$ is closed.

Obviously, $\mathcal{K}(G) \subseteq D(\mathcal{P}(f))$. In all, we have shown that $\mathcal{P}(f)$ is closed, densely defined, and $(-\frac{1}{2})$-homogeneous, so that we are now in a position to apply Proposition 2.2.

Let $(\xi_i)_{i\in I}$ be an approximate identity in $\mathcal{K}(G)_+$. Then

$$\mathcal{P}(f)\xi_i = f*\Delta^{\frac{1}{2}}\xi_i \to f \quad \text{in} \quad L^2(G) .$$

Thus $\|\mathcal{P}(f)\xi_i\| \to \|f\|_2$. By Proposition 2.2 we conclude that $\mathcal{P}(f) \in L^2(\psi_0)$ and that

$$\|\mathcal{P}(f)\|_2 = \|f\|_2 .$$

2) The map $\mathcal{P}$ is linear: if $f_1, f_2 \in L^2(G)$, then $[\mathcal{P}(f_1)+\mathcal{P}(f_2)]$ and $\mathcal{P}(f_1+f_2)$ obviously agree on $\mathcal{K}(G)$, and therefore by Proposition 2.4, we have

$$\mathcal{P}(f_1+f_2) = [\mathcal{P}(f_1)+\mathcal{P}(f_2)] .$$

Now, to prove that $\mathcal{P}$ is surjective, let $T \in L^2(\psi_0)$. We shall show that there exists a function $f \in L^2(G)$ such that $T = \mathcal{P}(f)$. Let $(\xi_i)_{i\in I}$ be an approximate identity in $\mathcal{K}(G)_+$. Then for all $\eta, \zeta \in \mathcal{K}(G)$ we have

$$\begin{aligned}(\eta*\Delta^{-\frac{1}{2}}\tilde{\zeta}\,|\,T\xi_i) &= (\eta\,|\,(T\xi_i)*\Delta^{\frac{1}{2}}\zeta) \\ &= (\eta\,|\,T(\xi_i*\zeta)) \\ &= (T^*\eta\,|\,\xi_i*\zeta) \\ &\qquad \to (T^*\eta\,|\,\zeta) = (\eta\,|\,T\zeta)\end{aligned}$$

(where we have used the $(-\frac{1}{2})$-homogeneity of T and the fact that $\mathcal{K}(G) \subseteq D(T^*)$ since $T^* \in L^2(\psi_0))$. Thus we can define a linear functional F on the dense subspace $\mathcal{K}(G)*\mathcal{K}(G)$ of $L^2(G)$ by

$$F(\xi) = \lim_i(\xi\,|\,T\xi_i) .$$

Since

$$|(\xi\,|\,T\xi_i)| \le \|\xi\|_2\,\|T\xi_i\|_2 \le \|\xi\|_2\,\|T\|_2\,\|\lambda(\xi_i)\| \le \|T\|_2\,\|\xi\|_2 ,$$

this functional is bounded and therefore is given by some $f \in L^2(G)$:

$$\forall \xi \in \mathcal{K}(G) * \mathcal{K}(G) : F(\xi) = (\xi | f) .$$

In particular, we have

$$(\eta | T\zeta) = F(\eta * \Delta^{-\frac{1}{2}}\tilde{\zeta}) = (\eta * \Delta^{-\frac{1}{2}}\tilde{\zeta} | f)$$

for all $\eta, \zeta \in \mathcal{K}(G)$. Since

$$(\eta * \Delta^{-\frac{1}{2}}\tilde{\zeta} | f) = (\eta | f * \Delta^{\frac{1}{2}}\zeta) = (\eta | \mathcal{P}(f)\zeta) ,$$

this implies

$$\forall \zeta \in \mathcal{K}(G) : T\zeta = \mathcal{P}(f)\zeta ,$$

and we conclude by Proposition 2.4 that $T = \mathcal{P}(f)$. ∎

<u>Proposition 3.1</u>. 1) For all $T \in M$ and all $f \in L^2(G)$, we have

$$\mathcal{P}(Tf) = [T\mathcal{P}(f)] .$$

2) For all $f \in L^2(G)$, we have

$$\mathcal{P}(Jf) = \mathcal{P}(f)^* .$$

<u>Proof</u>. 1) Let $f \in L^2(G)$ and $T \in M$. Then $[T\mathcal{P}(f)]$ and $\mathcal{P}(Tf)$ both belong to $L^2(\psi_0)$, and for all $\xi \in \mathcal{K}(G)$ we have

$$\mathcal{P}(Tf)\xi = (Tf) * \Delta^{\frac{1}{2}}\xi = T(f * \Delta^{\frac{1}{2}}\xi) = [T\mathcal{P}(f)]\xi ,$$

since T commutes with right convolution. By Proposition 2.4 we conclude that $\mathcal{P}(Tf) = [T\mathcal{P}(f)]$.

2) Let $f \in L^2(G)$. Then for all $\xi, \eta \in \mathcal{K}(G)$ we have

$$(\mathcal{P}(Jf)\xi|\eta) = (Jf*\Delta^{\frac{1}{2}}\xi|\eta)$$
$$= (Jf|\eta*\Delta^{-\frac{1}{2}}\tilde{\xi})$$
$$= (J(\eta*\Delta^{-\frac{1}{2}}\tilde{\xi})|f)$$
$$= (\xi*\Delta^{-\frac{1}{2}}\tilde{\eta}|f)$$
$$= (\xi|f*\Delta^{\frac{1}{2}}\eta) = (\xi|\mathcal{P}(f)\eta) \quad ,$$

so that $\mathcal{P}(Jf)|_{\mathcal{K}(G)} \subseteq (\mathcal{P}(f)|_{\mathcal{K}(G)})^* = [\mathcal{P}(f)|_{\mathcal{K}(G)}]^* = \mathcal{P}(f)^*$ (since $\mathcal{P}(f) = [\mathcal{P}(f)|_{\mathcal{K}(G)}]$) . We conclude by Proposition 2.4 that $\mathcal{P}(Jf) = \mathcal{P}(f)^*$. ∎

<u>Proposition 3.2</u>. Let $f \in L^2(G)$. Then $\mathcal{P}(f) \geq 0$ if and only if

$$\int f(x)(\xi*J\xi)(x)dx \geq 0$$

for all $\xi \in \mathcal{K}(G)$.

<u>Proof</u>. For all $\xi \in \mathcal{K}(G)$ we have

$$\int f(x)(\xi*J\xi)(x)dx = (f|\overline{\xi}*\Delta^{-\frac{1}{2}}\check{\xi}) = (f*\overline{\xi}|\overline{\xi}) = (\mathcal{P}(f)\overline{\xi}|\overline{\xi}) \quad .$$

Since $\mathcal{P}(f) = [\mathcal{P}(f)|_{\mathcal{K}(G)}]$, we have $\mathcal{P}(f) \geq 0$ if and only if $(\mathcal{P}(f)\eta|\eta) \geq 0$ for all $\eta \in \mathcal{K}(G)$, and the result follows. ∎

By [10, Theorem 1.21, (3)] (or, to be precise, its spatial analogue obtained by the methods of [12, §1] connecting abstract [10] and spatial [12] L^p spaces), $L^2(\psi_0)_+$ is a selfdual cone in $L^2(\psi_0)$. By Proposition 3.2 and the unitarity of $\mathcal{F}$ we conclude that

$$P_0 = \{f \in L^2(G) \mid \forall\xi \in \mathcal{K}(G): \int f(x)(\xi*J\xi)(x) \geq 0\}$$

is a selfdual cone in $L^2(G)$. Denote by P the ordinary selfdual cone in $L^2(G)$ associated with the achieved left Hilbert algebra $\mathfrak{A}_\ell \cap \mathfrak{A}_\ell{}^*$, i.e. let P be the closure in $L^2(G)$ of the set $\{\lambda(\xi)(J\xi) \mid \xi \in \mathfrak{A}_\ell \cap \mathfrak{A}_\ell{}^*\}$ (see [8, Section 1]). Since P is selfdual, we have

$$P = \{f \in L^2(G) \mid \forall \xi \in \mathfrak{A}_\ell \cap \mathfrak{A}_\ell{}^*: (f \mid \lambda(\xi)(J\xi)) \geq 0\} .$$

Thus $P \subseteq P_0$. Since P and P_0 are both selfdual, this implie that $P = P_0$. We have proved

Corollary. A function $f \in L^2(G)$ belongs to the positive selfdual cone of $L^2(G)$ if and only if

$$\forall \xi \in \mathcal{K}(G): \int f(x)(\xi * J\xi)(x)dx \geq 0 .$$

Remark. This result is similar to the characterization of the cone $P^\flat$ given in [18, p. 392] and proved in general in [9, Corollary 8]. The methods of [9] would also apply for our result. Our proof is based on the fact that $\mathcal{P}(f) = [\mathcal{P}(f)|_{\mathcal{K}(G)}]$.

Note. We have proved that $\mathcal{P} : L^2(G) \to L^2(\psi_0)$ carries the left regular representation on $L^2(G)$ into left multiplication on $L^2(\psi_0)$, takes J into $*$, and maps the positive selfdual co of $L^2(G)$ onto $L^2(\psi_0)_+$. That a unitary transformation $L^2(G) \to L^2(\psi_0)$ having these properties exists (and is unique) a follows from [8, Theorem 2.3], since both representations of M are standard (by the spatial analogue of [10, Theorem 1.21, (3)] In our approach, we have given a simple and direct definition of

We can give an explicit description of the inverse of $\mathcal{P}$:

Proposition 3.3. Let $T \in L^2(\psi_0)$, and let $(\xi_i)_{i \in I}$ be an approximate identity in $\mathcal{K}(G)_+$. Then

$$\mathcal{P}^{-1}(T) = \lim_{i \in I} T\xi_i \ .$$

Proof. Let $f = \mathcal{P}^{-1}(T)$. Then

$$T\xi_i = \mathcal{P}(f)\xi_i = f*\Delta^{\frac{1}{2}}\xi_i \to f$$

in $L^2(G)$. ∎

Remark. From Proposition 2.2 we already knew that for any approximate identity $(\xi_i)_{i \in I}$, the $\|T\xi_i\|$ tend to a limit and that this limit is independent of the choice of $(\xi_i)_{i \in I}$. Now, using that $L^2(\psi_0) = \mathcal{P}(L^2(G))$, we have proved that the same holds for the $T\xi_i$ themselves.

As a corollary, we have the following characterization of the inner product in $L^2(\psi_0)$, generalizing the formula for $\|T\|_2$ given in Proposition 2.2:

Corollary. Let $T,S \in L^2(\psi_0)$. Then

$$(T|S)_{L^2(\psi_0)} = \lim_{i \in I}(T\xi_i|S\xi_i)$$

for any approximate identity $(\xi_i)_{i \in I}$ in $\mathcal{K}(G)_+$.

Proof. Since $\mathcal{P}$ is unitary, we have

$$(T|S)_{L^2(\psi_0)} = (\mathcal{P}^{-1}(T)|\mathcal{P}^{-1}(S))_{L^2(G)} = \lim_{i \in I}(T\xi_i|S\xi_i)_{L^2(G)} \ . \ ∎$$

4. The L^p Fourier transformations.

Let $p \in [1,2]$ and define $q \in [2,\infty]$ by $\frac{1}{p} + \frac{1}{q} = 1$.

Definition. Let $f \in L^p(G)$. The L^p Fourier transform of f is the operator $\mathcal{F}_p(f)$ on $L^2(G)$ given by

$$\mathcal{F}_p(f)\xi = f*\Delta^{1/q}\xi \ , \ \xi \in D(\mathcal{F}_p(f)) \ ,$$

where $D(\mathcal{F}_p(f)) = \{\xi \in L^2(G) \mid f*\Delta^{1/q}\xi \in L^2(G)\}$.

Note that by Lemma 1.1 the convolution product $f*\Delta^{1/q}\xi$ exist and belongs to $L^r(G)$, where $r \in [2,\infty]$ is given by $\frac{1}{p} + \frac{1}{2} - \frac{1}{r} = 1$, whenever $f \in L^p(G)$ and $\xi \in L^2(G)$, so that the definition of $D(\mathcal{F}_p(f))$ makes sense.

Remark. For $p = 1$, we write $\mathcal{F}_1 = \mathcal{F}$; we have $\mathcal{F}(f)\xi = f*\xi$ and $D(\mathcal{F}(f)) = L^2(G)$, so that $\mathcal{F}(f)$ is simply $\lambda(f)$. For $p = 2$, we have $\mathcal{F}_2(f) = \mathcal{P}(f)$.

Now again let $p \in [1,2]$. Let $f \in L^p(G)$. Then the operator $\mathcal{F}_p(f)$ is closed. To see this, suppose that $\xi_i \in D(\mathcal{F}_p(f))$ converges in $L^2(G)$ to some $\xi \in L^2(G)$ and $\mathcal{F}_p(f)\xi_i$ converges in $L^2(G)$ to some $\eta \in L^2(G)$. Now by Lemma 1.1 we have $\mathcal{F}_p(f)\xi_i = f*\Delta^{1/q}\xi_i \to f*\Delta^{1/q}\xi$ in $L^r(G)$ (where $\frac{1}{p} + \frac{1}{2} - \frac{1}{r} = 1$) Therefore $f*\Delta^{1/q}\xi = \eta$, so that $f*\Delta^{1/q}\xi \in L^2(G)$, i.e. $\xi \in D(\mathcal{F}_p(f))$ and $\mathcal{F}_p(f)\xi = \eta$ as wanted.

Next we show that $\mathcal{F}_p(f)$ is $(-\frac{1}{q})$-homogeneous. For all $\xi \in D(\mathcal{F}_p(f))$ and all $x,y \in G$ we have

$$\rho(x)(\mathcal{F}_p(f)\xi)(y) = \Delta^{\frac{1}{2}}(x)(f*\Delta^{1/q}\xi)(yx)$$

$$= \Delta^{\frac{1}{2}}(x)\int f(z)\Delta^{1/q}(z^{-1}yx)\xi(z^{-1}yx)dz$$

$$= \Delta^{1/q}(x)\int f(z)\Delta^{1/q}(z^{-1}y)\Delta^{\frac{1}{2}}(x)\xi(z^{-1}yx)dz$$

$$= \Delta^{1/q}(x)\int f(z)\Delta^{1/q}(z^{-1}y)(\rho(x)\xi)(z^{-1}y)dz$$

$$= \Delta^{1/q}(x)(f*\Delta^{1/q}\rho(x)\xi)(y)$$

$$= \Delta^{1/q}(x)(\mathcal{F}_p(f)\rho(x)\xi)(y) ,$$

i.e.

$$\rho(x)\,\mathcal{F}_p(f) \subseteq \Delta^{1/q}(x)\,\mathcal{F}_p(f)\rho(x)$$

for all $x \in G$ as wanted.

Finally, note that if $\xi \in L^2(G) \cap L^s(G)$ where $s \in [1,2]$ is given by $\frac{1}{p} + \frac{1}{s} - \frac{1}{2} = 1$, then $\xi \in D(\mathcal{F}_p(f))$ by Lemma 1.1. In particular, $\mathcal{K}(G) \subseteq D(\mathcal{F}_p(f))$.

In all, we have proved that for all $f \in L^p(G)$, $\mathcal{F}_p(f)$ is closed, densely defined, and $(-\frac{1}{q})$-homogeneous. We shall see, using the criterion from Proposition 2.3, that actually $\mathcal{F}_p(f) \in L^q(\psi_0)$. The proof is based on interpolation from the special cases

$$\mathcal{F} : L^1(G) \to L^\infty(\psi_0)$$

and

$$\mathcal{P} : L^2(G) \to L^2(\psi_0) .$$

First we restrict our attention to $f \in \mathcal{K}(G)$.

Lemma 4.1. Let $p \in [1,2]$. Denote by A the closed strip $\{\alpha \in \mathbb{C} \mid \frac{1}{2} \le \mathrm{Re}\,\alpha \le 1\}$. Let $f \in \mathcal{K}(G)$ and $\xi \in \mathfrak{A}_\ell$. Then:

(i) for each $\alpha \in A$, the convolution product

$$\xi_\alpha = sg(f)|f|^{p\alpha} * \Delta^{1-\alpha}\xi$$

exists, and $\xi_\alpha \in L^2(G)$;

(ii) the function

$$\alpha \mapsto \xi_\alpha \ , \ \alpha \in A ,$$

with values in $L^2(G)$ is bounded;

(iii) for each $\eta \in L^2(G)$, the scalar function

$$\alpha \mapsto (\xi_\alpha|\eta) \ , \ \alpha \in A ,$$

is continuous on A and analytic in the interior of A

Proof. Write $g = \Delta^{-1/p}\check{f}$. Then

$$\forall \alpha \in A: \ sg(f)|f|^{p\alpha} = \Delta^{-\alpha}(sg(g)|g|^{p\alpha})^{\vee} .$$

Note that g as well as all $sg(g)|g|^{p\alpha}$, $\alpha \in A$, belong to $\mathcal{K}(G)$.

For each $\eta \in \mathcal{K}(G)$, we define

(1) $$H_\eta(\alpha) = \int \xi(x) \ (sg(g)|g|^{p\alpha} * \Delta^{1-\alpha}\eta)(x)dx \quad , \alpha \in A$$

i.e.

(2) $$H_\eta(\alpha) = \iint \xi(x)(sg(g)|g|^{p\alpha})(y)\Delta^{1-\alpha}(y^{-1}x)\eta(y^{-1}x)dy\,dx ,$$

(later we shall recognize $H_\eta(\alpha)$ as simply $(\xi_\alpha|\overline{\eta})$).

Note that

$$\forall\alpha \in A: \| |sg(g)|g|^{p\alpha}| * |\Delta^{1-\alpha}\eta| \|_2$$

(3) $$\leq \| |g|^{p \operatorname{Re} \alpha} \|_1 \, \| \Delta^{1-\operatorname{Re} \alpha} |\eta| \|_2$$

$$\leq K < \infty$$

where K is a constant independent of $\alpha \in A$. In particular, this allows us to apply Fubini's theorem to the double integral (2). We find

$$H_\eta(\alpha) = \iint \xi(x)(sg(g)|g|^{p\alpha})(y^{-1})\Delta^{1-\alpha}(yx)\eta(yx)\Delta^{-1}(y)dy\, dx$$

$$= \iint \xi(y^{-1}x)(sg(g)|g|^{p\alpha})(y^{-1})\Delta^{1-\alpha}(x)\eta(x)\Delta^{-1}(y)dx\, dy$$

$$= \iint (sg(f)|f|^{p\alpha})(y)\Delta^{1-\alpha}(y^{-1}x)\xi(y^{-1}x)\eta(x)dy\, dx \; ;$$

it also follows that the convolution integral

$$\xi_\alpha(x) = \int (sg(f)|f|^{p\alpha})(y)\Delta^{1-\alpha}(y^{-1}x)\xi(y^{-1}x)dy$$

exists, so that we can write

$$H_\eta(\alpha) = \int \xi_\alpha(x)\eta(x)dx \quad .$$

Now we shall prove that there exists a constant $C \geq 0$ independent of α such that

(4) $$\forall\eta \in \mathcal{K}(G): \left|\int \xi_\alpha(x)\eta(x)dx\right| \leq C\|\eta\|_2 \; .$$

This will imply that each ξ_α, $\alpha \in A$, is in $L^2(G)$ with $\|\xi_\alpha\|_2 \leq C$, i.e. (i) and (ii) will be proved.

Let us prove (4). Without loss of generality, we may assume that $\|f\|_p = 1$. We want to show then that

(5) $$\forall \eta \in \mathcal{K}(G): \ |H_\eta(\alpha)| \leq (\|\lambda(\xi)\| + \|\xi\|_2)\|\eta\|_2 .$$

To do this, we shall apply the Phragmen-Lindelöf principle [24, p 93].

Fix $\eta \in \mathcal{K}(G)$. By (2), H_η is continuous on A and analytic in the interior of A (the integrand in (2) can be majorized by an integrable function that is independent of α) . Furthermore, H_η is bounded (use (3) and (1)). Finally, we shall estimate H_η on the boundaries of A .

Let $t \in \mathbb{R}$. Then $\Delta^{-it}\xi \in \mathfrak{A}_\ell$ and $\|\lambda(\Delta^{-it}\xi)\| \leq \|\lambda(\xi)\|$. Now

$$\mathcal{P}(sg(f)|f|^{p(\frac{1}{2}+it)})(\Delta^{-it}\xi)$$

$$= sg(f)|f|^{p(\frac{1}{2}+it)} * \Delta^{1-(\frac{1}{2}+it)}\xi = \xi_{\frac{1}{2}+it} ,$$

so that $\xi_{\frac{1}{2}+it} \in L^2(G)$ with

$$\|\xi_{\frac{1}{2}+it}\|_2 \leq \|\mathcal{P}(sg(f)|f|^{p(\frac{1}{2}+it)})\|_2 \ \|\lambda(\Delta^{-it}\xi)\|$$

$$\leq \|sg(f)|f|^{p(\frac{1}{2}+it)}\|_2 \ \|\lambda(\xi)\|$$

$$= \||f|^{p/2}\|_2 \ \|\lambda(\xi)\|$$

$$= \|\lambda(\xi)\|$$

(where we have used Proposition 2.2, the fact that $\mathcal{P}$ is unitai and the hypothesis $\|f\|_p = 1$) . Similarly,

$$\mathcal{F}(sg(f)|f|^{p(1+it)})(\Delta^{-it}\xi)$$

$$= sg(f)|f|^{p(1+it)} * \Delta^{1-(1+it)}\xi = \xi_{1+it} ,$$

so that $\xi_{1+it} \in L^2(G)$ with

$$\|\xi_{1+it}\|_2 \leq \|\mathcal{F}(sg(f)|f|^{p(1+it)})\|_\infty \|\Delta^{-it}\xi\|_2$$

$$\leq \|sg(f)|f|^{p(1+it)}\|_1 \|\xi\|_2$$

$$= \||f|^p\|_1 \|\xi\|_2$$

$$= \|\xi\|_2$$

(where we have used that $\mathcal{F} : L^1(G) \to L^\infty(\psi_0)$ is norm-decreasing).

It follows that

$$\forall t \in \mathbb{R}: \quad |H_\eta(\tfrac{1}{2}+it)| = |\int \xi_{\frac{1}{2}+it}(x)\eta(x)dx|$$

$$\leq \|\xi_{\frac{1}{2}+it}\|_2 \|\eta\|_2 \leq \|\lambda(\xi)\| \|\eta\|_2$$

and

$$\forall t \in \mathbb{R}: \quad |H_\eta(1+it)| = |\int \xi_{1+it}(x)\eta(x)dx|$$

$$\leq \|\xi_{1+it}\|_2 \|\eta\|_2 \leq \|\xi\|_2 \|\eta\|_2 .$$

Then by the Phragmen-Lindelöf principle, we have established (5) and thus (i) and (ii).

Finally, (iii) is easy. Indeed, since $\alpha \mapsto \xi_\alpha$ is bounded, each $\alpha \mapsto (\xi_\alpha|\eta)$, where $\eta \in L^2(G)$, can be uniformly approximated by functions $\alpha \mapsto (\xi_\alpha|\zeta)$ with $\zeta \in \mathcal{K}(G)$, so we just have to prove (iii) in the case of $\eta \in \mathcal{K}(G)$. This is already done since $(\xi_\alpha|\eta) = H_{\bar{\eta}}(\alpha)$. ∎

Lemma 4.2. Let $p \in [1,2]$. Let $f \in \mathcal{K}(G)$ and $S \in L^p(\psi_0)$. Then for all $\xi \in \mathfrak{A}_\ell$ and $\eta \in \mathfrak{A}_\ell \cap D(S)$ we have

$$|(\mathcal{F}_p(f)\xi \mid S\eta)| \leq \|f\|_p \|S\|_p \|\lambda(\xi)\| \|\lambda(\eta)\| .$$

Note that $\xi \in D(\mathcal{F}_p(f))$ by Lemma 4.1.

Proof. We may assume that $\|f\|_p = 1$ and $\|S\|_p = 1$. Furthermore by Lemma 2.5, we need only consider $\eta \in \mathfrak{A}_\ell \cap D(|S|^p)$.

Let $\xi \in \mathfrak{A}_\ell$ and $\eta \in \mathfrak{A}_\ell \cap D(|S|^p)$. For each α in the closed strip $A = \{\alpha \in \mathbb{C} \mid \frac{1}{2} \leq \mathrm{Re}\,\alpha \leq 1\}$, put $\xi_\alpha = \mathrm{sg}(f)|f|^{p\alpha} * \Delta^{1-\alpha}\xi$ as in Lemma 4.1. Note that for all $\alpha \in A$ we have (by spectral theory) $\eta \in D(U|S|^{p\alpha})$ and

$$\|U|S|^{p\alpha}\eta\|_2^2 \leq \| |S|^{p/2}\eta\|_2^2 + \| |S|^p\eta\|_2^2 ,$$

where $S = U|S|$ is the polar decomposition of S. For each $\alpha \in A$, put

$$\eta_\alpha = U|S|^{p\alpha}\eta .$$

Then the function $\alpha \mapsto \eta_\alpha$ with values in $L^2(G)$ is bounded on A. Furthermore, by [22, 9.15], it is continuous on A and analytic in the interior of A.

Now for each $\alpha \in A$, let

$$H(\alpha) = (\xi_\alpha \mid \eta_{\overline{\alpha}}) .$$

Then obviously H is bounded on A (by Lemma 4.1 (ii), $\alpha \mapsto \xi_\alpha$ is bounded). Furthermore, H is continuous on A. To see this, note that

$$\forall\alpha,\ \alpha_0 \in A:\ (\xi_\alpha|\eta_{\bar\alpha}) - (\xi_{\alpha_0}|\eta_{\bar\alpha_0}) = (\xi_\alpha|\eta_{\bar\alpha}-\eta_{\bar\alpha_0}) + (\xi_\alpha-\xi_{\alpha_0}|\eta_{\bar\alpha_0})\ ;$$

the continuity follows since $\alpha \mapsto \xi_\alpha$ is bounded and weakly continuous (Lemma 4.1 (iii)). Finally, we claim that H is analytic in the interior of A. First note that for each $\zeta \in L^2(G)$ the function $\alpha \mapsto (\zeta|\eta_{\bar\alpha})$, being equal to $\alpha \mapsto \overline{(\eta_{\bar\alpha}|\zeta)}$, is analytic. Next, recall that $\alpha \mapsto \xi_\alpha$ is actually analytic as a function with values is $L^2(G)$ (by Lemma 4.1 (iii) and [19, Theorem 3.31]). Then, writing

$$\frac{(\xi_\alpha|\eta_{\bar\alpha})-(\xi_{\alpha_0}|\eta_{\bar\alpha_0})}{\alpha-\alpha_0} = \left(\frac{1}{\alpha-\alpha_0}(\xi_\alpha-\xi_{\alpha_0})\,|\,\eta_{\bar\alpha}\right) + \frac{(\xi_{\alpha_0}|\eta_{\bar\alpha})-(\xi_{\alpha_0}|\eta_{\bar\alpha_0})}{\alpha-\alpha_0}\ ,$$

we find that H has a derivative at each point α_0 in the interior of A.

Now suppose that

(1) $\forall t \in \mathbb{R}:\ |H(\tfrac{1}{2}+it)| \leq \|\lambda(\xi)\|\ \|\lambda(\eta)\|$

and

(2) $\forall t \in \mathbb{R}:\ |H(1+it)| \leq \|\lambda(\xi)\|\ \|\lambda(\eta)\|$.

Then by the Phragmen-Lindelöf principle [24, p. 93] we infer that

$$\forall\alpha \in A:\ |H(\alpha)| \leq \|\lambda(\xi)\|\ \|\lambda(\eta)\|\ ;$$

in particular,

$$|(\mathcal{F}_p(f)\xi|S\eta)| \leq \|\lambda(\xi)\|\ \|\lambda(\eta)\|$$

as desired, since

$$H(\tfrac{1}{p}) = (f*\Delta^{1-1/p}\xi|U|S|\eta) = (\mathcal{F}_p(f)|S\eta)\ .$$

So we just have to prove (1) and (2).

Since $S \in L^p(\psi_0)$ with $\|S\|_p = 1$ we have

(3) $U|S|^{p/2} \in L^2(\psi_0)$ with $\|U|S|^{p/2}\|_2 = 1$

and

(4) $U|S|^p \in L^1(\psi_0)$ with $\|U|S|^p\|_1 = 1$.

Now let $t \in \mathbb{R}$. Then by Lemma 2.3, we have

(5) $|S|^{-pit}\eta \in \mathfrak{A}_\ell$ with $\|\lambda(|S|^{-pit}\eta)\| \leq \|\lambda(\eta)\|$.

Using this, Proposition 2.2, the estimate $\|\xi_{\frac{1}{2}+it}\|_2 \leq \|\lambda(\xi)\|$ give in the proof of Lemma 4.1, and (3), we get

$$\begin{aligned}|H(\tfrac{1}{2}+it)| &= |(\xi_{\frac{1}{2}+it}|U|S|^{p/2}|S|^{-pit}\eta)| \\ &\leq \|\xi_{\frac{1}{2}+it}\|_2 \, \|U|S|^{p/2}|S|^{-pit}\eta\|_2 \\ &\leq \|\lambda(\xi)\| \, \|U|S|^{p/2}\|_2 \, \|\lambda(|S|^{-pit}\eta)\|_2 \\ &\leq \|\lambda(\xi)\| \, \|\lambda(\eta)\| ,\end{aligned}$$

i.e., (1) is proved. To prove (2), note that

$$\begin{aligned}\xi_{1+it} &= sg(f)|f|^{p(1+it)} * \Delta^{1-(1+it)}\xi \\ &= \lambda(sg(f)|f|^{p(1+it)})\Delta^{-it}\xi \in \mathfrak{A}_\ell\end{aligned}$$

and

$$\begin{aligned}\|\lambda(\xi_{1+it})\| &\leq \|\lambda(sg(f)|f|^{p(1+it)})\| \, \|\lambda(\Delta^{-it}\xi)\| \\ &\leq \|sg(f)|f|^{p(1+it)}\|_1 \, \|\lambda(\xi)\| \\ &\leq \|\lambda(\xi)\|\end{aligned}$$

since $\|sg(f)|f|^{p(1+it)}\|_1 = \||f|^p\|_1 = 1$. Using this together with (5), Proposition 2.1, and (4), we find

$$|H(1+it)| = |(\xi_{1+it}|U|S|^p|S|^{-pit}\eta)|$$

$$\leq \|\lambda(\xi_{1+it})\| \; \|U|S|^p\|_1 \; \|\lambda(|S|^{-pit}\eta)\|$$

$$\leq \|\lambda(\xi)\| \; \|\lambda(\eta)\| \quad ,$$

so that (2) is proved. ∎

In the formulation of the following theorem we include the case $p = 2$. Note however that the proof is based on the results for this special case (they were used for the preceding lemmas).

<u>Theorem 4.1</u>. (Hausdorff-Young). Let $p \in]1,2]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

1) Let $f \in L^p(G)$. Then $\mathcal{F}_p(f) \in L^q(\psi_0)$ and

$$\| \mathcal{F}_p(f)\|_q \leq \|f\|_p \; .$$

2) The mapping

$$\mathcal{F}_p \colon L^p(G) \to L^q(\psi_0)$$

is linear, norm-decreasing, injective, and has dense range.

3) For all $h \in L^1(G)$ and $f \in L^p(G)$, we have

$$\mathcal{F}_p(h*f) = [\lambda(h)\, \mathcal{F}_p(f)] \; .$$

4) For all $f \in L^p(G)$, we have

$$\mathcal{F}_p(J_p f) = \mathcal{F}_p(f)^* \; .$$

<u>Proof</u>. 1) First suppose that $f \in \mathcal{K}(G)$. Then, using Proposition 2.3, we conclude from Lemma 4.2 that $\mathcal{F}_p(f) \in L^q(\psi_0)$ with

$\| \mathcal{F}_p(f)\|_q \leq \|f\|_p$. Thus we have defined a norm-decreasing mappin

$$\mathcal{F}_p|_{\mathcal{K}(G)} : L^p(G) \to L^q(\psi_0) .$$

Furthermore $\mathcal{F}_p|_{\mathcal{K}(G)}$ is linear: for all $f_1, f_2 \in \mathcal{K}(G)$ and all $\xi \in \mathcal{K}(G)$ we have

$$(f_1+f_2)*\Delta^{1/q}\xi = f_1*\Delta^{1/q}\xi + f_2*\Delta^{1/q}\xi$$

so that $\mathcal{F}_p(f_1+f_2) = [\mathcal{F}_p(f_1)+\mathcal{F}_p(f_2)]$ by Proposition 2.4.

Now $\mathcal{F}_p|_{\mathcal{K}(G)}$ extends by continuity to a norm-decreasing linear mapping

$$\mathcal{F}_p' : L^p(G) \to L^q(\psi_0) .$$

We claim that for all $f \in L^p(G)$, we have

$$\mathcal{F}_p'(f) = \mathcal{F}_p(f) .$$

This will prove 1).

Let $f \in L^p(G)$. Then $\mathcal{F}_p'(f) \in L^q(\psi_0)$ and $\mathcal{K}(G) \subseteq D(\mathcal{F}_p'(f))$ by Lemma 2.7. On the other hand, by the remarks at tl beginning of this section, $\mathcal{F}_p(f)$ is closed, densely defined, and $(-\frac{1}{q})$-homogeneous, and $\mathcal{K}(G) \subseteq D(\mathcal{F}_p(f))$. Thus by Lemma 2.· to conclude that $\mathcal{F}_p'(f) = \mathcal{F}_p(f)$ we just have to show that

$$\forall \xi \in \mathcal{K}(G) : \mathcal{F}_p'(f)\xi = \mathcal{F}_p(f)\xi .$$

Now, take $f_n \in \mathcal{K}(G)$ such that $f_n \to f$ in $L^p(G)$. Then for all $\xi \in \mathcal{K}(G)$, we have

$$\begin{aligned}\mathcal{F}_p(f_n)\xi &= f_n*\Delta^{1/q}\xi \\ &\to f*\Delta^{1/q}\xi = \mathcal{F}_p(f)\xi \quad \text{in } L^p(G) .\end{aligned}$$

On the other hand, since $\mathcal{F}_p'$ is continuous,

$$\mathcal{F}_p(f_n)\xi = \mathcal{F}_p'(f_n)\xi \to \mathcal{F}_p'(f)\xi \quad \text{in} \quad L^2(G)$$

by Lemma 2.7. We conclude that $\mathcal{F}_p(f)\xi = \mathcal{F}_p'(f)\xi$ as desired. Thus 1) is proved.

2) By the proof of 1), we just have to show that $\mathcal{F}_p$ is injective and has dense range. The injectivity is evident: if $\mathcal{F}_p(f) = 0$ for some $f \in L^p(G)$, then $f*\Delta^{1/q}\xi = 0$ for all $\xi \in \mathcal{K}(G)$, and thus $f = 0$. That $\mathcal{F}_p(L^p(G))$ is dense will be proved later.

3) For all $h \in L^1(G)$, $f \in L^p(G)$, and $\xi \in \mathcal{K}(G)$ we have

$$h*(f*\Delta^{1/q}\xi) = (h*f)*\Delta^{1/q}\xi$$

(in $L^p(G)$). Thus by Proposition 2.4,

$$[\lambda(h)\ \mathcal{F}_p(f)] = \mathcal{F}_p(h*f) .$$

4) Let $f \in \mathcal{K}(G)$. Then for $\xi, \eta \in \mathcal{K}(G)$ we have

$$\begin{aligned}
(\mathcal{F}_p(J_pf)\xi|\eta) &= (J_pf*\Delta^{1/q}\xi|\eta) \\
&= (\Delta^{1/q}\xi|\Delta^{-1}(J_pf)^{\sim}*\eta) \\
&= (\xi|\Delta^{1/q}(\Delta^{-1}\Delta^{1/p}f*\eta)) \\
&= (\xi|f*\Delta^{1/q}\eta) \\
&= (\xi|\ \mathcal{F}_p(f)\eta) ,
\end{aligned}$$

so that $\mathcal{F}_p(J_pf)|\mathcal{K}(G) \subseteq (\mathcal{F}_p(f)|\mathcal{K}(G))^*$. By Proposition 2.4, we conclude that

$$\mathcal{F}_p(J_p f) = \mathcal{F}_p(f)^* .$$

By the continuity of J_p, $\mathcal{F}_p$, and $*$, this holds for all $f \in L^p(G)$.

Finally, let us show that $\mathcal{F}_p(L^p(G))$ is dense in $L^q(\psi_0)$. By the duality between $L^q(\psi_0)$ and $L^p(\psi_0)$, this is equivalen to proving that if $T \in L^p(\psi_0)$ satisfies $\int [\mathcal{F}_p(f)T]d\psi_0 = 0$ fc all $f \in L^p(G)$, then $T = 0$.

Suppose that $T \in L^p(\psi_0)$ is such that

$$\forall f \in L^p(G): \int [\mathcal{F}_p(f)T]d\psi_0 = 0 .$$

Let $f \in L^p(G)$. Then for all $h \in L^1(G)$ we have

$$\int [\mathcal{F}_p(h*f)T]d\psi_0 = 0 .$$

Alternatively stated, since $[\mathcal{F}_p(h*f)T] = [[\lambda(h)\mathcal{F}_p(f)]T] = [\lambda(h)[\mathcal{F}_p(f)T]]$, we have

$$\forall h \in L^1(G): \int [\lambda(h)[\mathcal{F}_p(f)T]]d\psi_0 = 0 .$$

We conclude that the normal functional on M defined by $[\mathcal{F}_p(f)T] \in L^1(\psi_0)$ is 0, so that

$$[\mathcal{F}_p(f)T] = 0 .$$

Changing f into $J_p f$ and using 4) this gives

$$\forall f \in L^p(G): [\mathcal{F}_p(f)^*T] = 0 .$$

Now let $\xi \in D(T)$. Then using [12, II, Proposition 5, 1)], we find that

$$\forall f,\eta \in \mathcal{K}(G): (T\xi \mid f*\Delta^{1/q}\eta)$$

$$= (T\xi \mid \mathcal{F}_p(f)\eta)$$

$$= \langle [\mathcal{F}_p(f)*T], \lambda(\xi)\lambda(\cdot)^*\rangle = 0 .$$

Thus $T\xi = 0$. This proves that $T = 0$ as wanted. ∎

Proposition 4.1. Let $p \in [1,2]$. Let $f \in L^p(G)$. Then $\mathcal{F}_p(f) \geq 0$ if and only if

$$\forall \xi \in \mathcal{K}(G): \int f(x)(\xi * J_p\xi)(x)dx \geq 0 .$$

Proof. We have

$$(\mathcal{F}_p(f)\xi \mid \xi) = \int (f*\Delta^{1/p}\xi)(x)\overline{\xi(x)}dx$$

$$= \int f(x)(\overline{\xi}*\Delta^{-1/p}\check{\xi})(x)dx$$

for all $\xi \in \mathcal{K}(G)$. The result follows by changing ξ into $\overline{\xi}$ and recalling that $\mathcal{F}_p(f) = [\mathcal{F}_p(f)|_{\mathcal{K}(G)}]$. ∎

The L^p Fourier transformations are well-behaved with respect to convolution as the following proposition shows. The result generalizes 3) of the theorem.

Proposition 4.2. Let $p_1,p_2,p \in [1,2]$ such that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} = 1$. Define $q_1 \in [2,\infty]$ by $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Let $f_1 \in L^{p_1}(G)$ and $f_2 \in L^{p_2}(G)$. Then

$$\mathcal{F}_p(f_1 * \Delta^{1/q_1} f_2) = [\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)] .$$

Proof. By Lemma 1.1, we have $f_1 * \Delta^{1/q_1} f_2 \in L^p(G)$, and $(f_1,f_2) \mapsto \mathcal{F}_p(f_1 * \Delta^{1/q_1} f_2)$ maps $L^{p_1}(G) \times L^{p_2}(G)$ continuously into $L^q(\psi_0)$ (where $\frac{1}{p} + \frac{1}{q} = 1$). Also $[\mathcal{F}_{p_1}(f_1)\,\mathcal{F}_{p_2}(f_2)]$ is continuous as a function of $(f_1,f_2) \in L^{p_1}(G) \times L^{p_2}(G)$ with values in $L^q(\psi_0)$. Thus we need only prove the statement for $f_1, f_2 \in \mathcal{K}(G)$. Since

$$(f_1 * \Delta^{1/q_1} f_2) * \Delta^{1/q}\xi = f_1 * \Delta^{1/q_1}(f_2 * \Delta^{1/q_2}\xi)$$

(where $\frac{1}{p_2} + \frac{1}{q_2} = 1$) for all $f_1, f_2, \xi \in \mathcal{K}(G)$, the result follows by Proposition 2.4 as usual. ∎

We conclude this section by the following characterization of the image of $L^p(G)$ under $\mathcal{F}_p$:

Proposition 4.3. Let $p \in]1,2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $T \in L^q(\psi_0)$

1) If $T = \mathcal{F}_p(f)$ for some $f \in L^p(G)$, then for any approximate identity $(\xi_i)_{i \in I}$ in $\mathcal{K}(G)_+$ we have

$$T\xi_i \to f \text{ in } L^p(G).$$

In particular, $\lim \|T\xi_i\|_p = \|f\|_p < \infty$.

2) Conversely, suppose that for some approximate identity $(\xi_i)_{i \in I}$ in $\mathcal{K}(G)_+$ we have $T\xi_i \in L^p(G)$ for all $i \in$ and

$$\liminf \|T\xi_i\|_p < \infty .$$

Then $T \in \mathcal{F}_p(L^p(G))$.

<u>Proof</u>. The first part is obvious since $T\xi_i = f*\Delta^{1/q}\xi_i \to f$ in $L^p(G)$ and therefore $\|T\xi_i\|_p \to \|f\|_p$. Now suppose that the hypothesis of 2) holds for some $(\xi_i)_{i\in I}$. We then proceed as in the proof of the surjectivity of $\mathcal{P}$ (Theorem 3.1). For all $\eta,\zeta \in \mathcal{K}(G)$ we have

$$\begin{aligned}(\eta*\Delta^{-1/q}\tilde{\zeta}\,|\,T\xi_i) &= (\eta\,|\,(T\xi_i)*\Delta^{1/q}\zeta)\\ &= (\eta\,|\,T(\xi_i*\zeta))\\ &= (T^*\eta\,|\,\xi_i*\zeta)\\ &\to (T^*\eta\,|\,\zeta) = (\eta\,|\,T\zeta)\ .\end{aligned}$$

Thus we can define a linear functional F on $\mathcal{K}(G)*\mathcal{K}(G)$ by

$$F(\xi) = \lim_i \int \xi(x)\,\overline{(T\xi_i)(x)}\,dx\ .$$

Since

$$\left|\int \xi(x)\,\overline{(T\xi_i)(x)}\,dx\right| \leq \|\xi\|_q\ \|T\xi_i\|_p$$

we have

$$|F(\xi)| \leq (\liminf \|T\xi_i\|_p)\cdot\|\xi\|_q\ .$$

Now since $\mathcal{K}(G)*\mathcal{K}(G)$ is dense in $L^q(G)$, F extends to a bounded functional on $L^q(G)$ and therefore is given by some $\overline{f} \in L^p(G)$:

$$F(\xi) = \int \xi(x)\,\overline{f(x)}\,dx\ .$$

In particular,

$$(\eta|T\zeta) = F(\eta*\Delta^{-1/q}\tilde{\zeta}) = \int(\eta*\Delta^{-1/q}\tilde{\zeta})(x)\overline{f(x)}dx$$

for all $\eta,\zeta \in \mathcal{K}(G)$. Since

$$\int(\eta*\Delta^{-1/q}\tilde{\zeta})(x)\overline{f(x)}dx = \int\eta(x)\overline{(f*\Delta^{1/q}\zeta)(x)}dx = (\eta|\ \mathcal{F}_p(f)\zeta) \ ,$$

this implies that

$$\forall\zeta \in \mathcal{K}(G): T\zeta = \mathcal{F}_p(f)\zeta \ ,$$

and we conclude by Proposition 2.4 that $T = \mathcal{F}_p(f)$. ∎

<u>Remark</u>. For $p = 1$, part 2) of the above proposition fails. (For a counter-example, take $T = \lambda(x)$, $x \in G$.)

5. The L^p Fourier cotransformations.

<u>Definition</u>. Let $p \in [1,2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. For each $T \in L^p(\psi$ denote by $\overline{\mathcal{F}}_p(T)$ the unique function in $L^q(G)$ such that

$$\int h(x)\overline{\mathcal{F}}_p(T)(x)dx = \int[\mathcal{F}_p(h)T]d\psi_0$$

for all $h \in L^p(G)$ (or just $h \in \mathcal{K}(G)$, or $h \in \mathcal{K}(G)*\mathcal{K}(G)$) . The mapping

$$\overline{\mathcal{F}}_p: L^p(\psi_0) \to L^q(G)$$

thus defined will be called the L^p Fourier cotransformation. For $p = 1$, we write $\overline{\mathcal{F}} = \overline{\mathcal{F}}_1$.

Note that if $1 < p \leq 2$, then $\overline{\mathcal{F}}_p$ is simply the transpose of $\mathcal{F}_p : L^p(G) \to L^q(\psi_0)$ when we identify the dual spaces of $L^p(G)$ and $L^q(\psi_0)$ with $L^q(G)$ and $L^p(\psi_0)$, respectively.

The mapping $\overline{\mathcal{F}}$ takes an element $T \in L^1(\psi_0)$ into the unique function $\varphi \in A(G)$ that defines the same element of M_* as T does; in particular,

$$\overline{\mathcal{F}}\left(\frac{d\varphi}{d\psi_0}\right) = \varphi$$

for all $\varphi \in (M_*)^+ \simeq A(G)_+$.

In view of these remarks, we obviously have

Theorem 5.1.

1) Let $p \in]1,2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\overline{\mathcal{F}}_p : L^p(\psi_0) \to L^q(G)$$

is linear, norm-decreasing, injective, and has dense range.

2) The mapping

$$\overline{\mathcal{F}} : L^1(\psi_0) \to A(G)$$

is an isometry of $L^1(\psi_0)$ onto $A(G)$.

Remark. With our definition of the cotransformations, $\overline{\mathcal{F}}_2$ is not exactly the inverse of $\mathcal{P}$; they are related by the formula

$$\forall T \in L^2(\psi_0) : \overline{\mathcal{F}}_2(T) = \overline{\mathcal{P}^{-1}(T^*)}$$

(since for all $h \in L^2(G)$ we have $\int h(x)\overline{\mathcal{F}}_2(T)(x)dx =$

$\int [\mathcal{F}_2(h)T]d\psi_0 = (\mathcal{F}_2(h) | T^*)_{L^2(\psi_0)} = (h | \mathcal{P}^{-1}(T^*))_{L^2(G)} =$

$\int h(x)\overline{\mathcal{P}^{-1}(T^*)(x)dx)}$. It follows that $\overline{\mathcal{F}}_2: L^2(\psi_0) \to L^2(G)$ is unitary.

The classical Hausdorff-Young theorem [24, p. 101] has a second part, stating that with each $c \in \ell_p(\mathbb{Z})$, $1 \leq p \leq 2$, we can associate a function $f \in L^q(\mathbb{T})$ with $\|f\|_q \leq \|c\|_p$, such that c is the sequence of Fourier coefficients of f . Theorem 5.1 is a generalization of this result. Indeed, let $T \in L^p(\psi_0)$ and put $g = \Delta^{-1/q}\,\overline{\mathcal{F}}_p(T)^{\vee}$. Then $g \in L^q(G)$ and $\|g\|_q = \|\overline{\mathcal{F}}_p(T)\|_q \leq \|T\|_p$, and we shall see that T is close to being the "L^q Fourier transform" of g in the sense that $T\xi = g*\Delta^{1/p}\xi$ for certai ξ (note that we do not in general define L^q Fourier transforms $q \geq 2$) .

<u>Proposition 5.1</u>. Let $p \in [1,2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $T \in L^p(\psi_0)$, we have

$$\overline{\mathcal{F}}_p(T^*) = J_q(\overline{\mathcal{F}}_p(T)) .$$

<u>Proof</u>. For all $h \in L^p(G)$ we have

$$\begin{aligned}\int h(x)\overline{\mathcal{F}}_p(T^*)(x)dx &= \int [\mathcal{F}_p(h)T^*]d\psi_0 \\ &= \overline{\int [T\,\mathcal{F}_p(h)^*]d\psi_0} = \overline{\int [T\,\mathcal{F}_p(J_p h)]d\psi_0} \\ &= \overline{\int \mathcal{F}_p(T)(x)\Delta^{-1/p}(x)\overline{h(x^{-1})}dx} \\ &= \int \Delta^{-1/q}(x)\,\overline{\mathcal{F}_p(T)(x^{-1})}h(x)dx .\end{aligned}$$ ∎

emma 5.1. Let $h,k \in \mathcal{K}(G)$ and put $\varphi = h*\tilde{k}$. Then $\lambda(\varphi)\Delta] \in L^1(\psi_0)$ and

$$\int[\lambda(\varphi)\Delta]d\psi_0 = \varphi(e) .$$

roof. Since

$$\lambda(\varphi)\Delta = \lambda(h)\lambda(\tilde{k})\Delta^{\frac{1}{2}}\Delta^{\frac{1}{2}}$$

$$\subseteq \lambda(h)\Delta^{\frac{1}{2}}\lambda(\Delta^{-\frac{1}{2}}\tilde{k})\Delta^{\frac{1}{2}} \subseteq \mathcal{P}(h)\ \mathcal{P}(k)^* ,$$

the closure $[\lambda(\varphi)\Delta]$ exists and $[\lambda(\varphi)\Delta] \subseteq [\mathcal{P}(h)\ \mathcal{P}(k)^*]$. One easily checks that for all $x \in G$ we have $\rho(x)\lambda(\varphi)\Delta \subseteq \Delta(x)\lambda(\varphi)\Delta\rho(x)$, i.e. that $\lambda(\varphi)\Delta$ is (-1)-homogeneous. Then also $[\lambda(\varphi)\Delta]$ is (-1)-homogeneous, and we conclude by Proposition 2.4 that $[\lambda(\varphi)\Delta] = [\mathcal{P}(h)\ \mathcal{P}(k)^*]$, so that $[\lambda(\varphi)\Delta] \in L^1(\psi_0)$ and

$$\int[\lambda(\varphi)\Delta]d\psi_0 = (\mathcal{P}(h) \mid \mathcal{P}(k))_{L^2(\psi_0)}$$

$$= \int h(x)\overline{k(x)}dx = (h*\tilde{k})(e) = \varphi(e) . \blacksquare$$

Suppose that $f_1 \in L^{p_1}(G)$ and $f_2 \in L^{p_2}(G)$, where $p_1, p_2 \in [1,2]$. In Proposition 4.2, a formula relating $f_1*\Delta^{1/q_1}f_2$ and $[\mathcal{F}_{p_1}(f_1)\ \mathcal{F}_{p_2}(f_2)]$ was given in the case where $\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{3}{2}$ (under this assumption, $p \in [1,2]$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} = 1$ exists). The following proposition takes care of the case where $\frac{1}{p_1} + \frac{1}{p_2} \leq \frac{3}{2}$.

Proposition 5.2. Let $p_1, p_2 \in [1,2]$ and $q \in [2,\infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{q} = 1$. Let $f_1 \in L^{p_1}(G)$ and $f_2 \in L^{p_2}(G)$. Then

$$\overline{\mathcal{F}}_p([\mathcal{F}_{p_1}(f_1)\,\mathcal{F}_{p_2}(f_2)]) = \Delta^{-1/q}(f_1 * \Delta^{1/q_1} f_2)^{\vee},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$.

Proof. Both expressions exist, belong to $L^q(G)$, and are continuous as functions of $(f_1, f_2) \in L^{p_1}(G) \times L^{p_2}(G)$. Thus we need only prove the formula for $f_1, f_2 \in \mathcal{K}(G)$. In this case, for all $h \in \mathcal{K}(G)$ and $\xi \in \mathcal{K}(G)$ we have

$$h * \Delta^{1/q}(f_1 * \Delta^{1/q_1}(f_2 * \Delta^{1/q_2}\xi)) = h * \Delta^{1/q}(f_1 * \Delta^{1/q_1} f_2) * \Delta\xi ,$$

where $\frac{1}{p_2} + \frac{1}{q_2} = 1$. We conclude by Proposition 2.4 that

$$\forall h \in \mathcal{K}(G) : [\mathcal{F}_p(h)[\mathcal{F}_{p_1}(f_1)\,\mathcal{F}_{p_2}(f_2)]] = [\lambda(h * \Delta^{1/q} f)\Delta] ,$$

where we have written $f = f_1 * \Delta^{1/q_1} f_2$. Using this and Lemma 5.1, we find

$$\begin{aligned}
\forall h \in \mathcal{K}(G) : \int [\mathcal{F}_p(h)[\mathcal{F}_{p_1}(f_1)\,\mathcal{F}_{p_2}(f_2)]]\,d\psi_0 \\
&= \int [\lambda(h * \Delta^{1/q} f)\Delta]\,d\psi_0 \\
&= (h * \Delta^{1/q} f)(e) \\
&= \int h(x)\Delta^{1/q}(x^{-1}) f(x^{-1})\,dx .
\end{aligned}$$

We conclude that

$$\overline{\mathcal{F}}_p([\mathcal{F}_{p_1}(f_1)\,\mathcal{F}_{p_2}(f_2)]) = \Delta^{-1/q}\check{f} ,$$

as desired. ∎

Corollary. Let $f,g \in L^2(G)$. Then

$$f*\tilde{g} = \overline{\mathcal{F}}([\mathcal{P}(\overline{g})\,\mathcal{P}(\overline{f})^*]) \ .$$

Proof. Letting $p_1 = p_2 = 2$ and $q = \infty$ in Proposition 5.2, we obtain

$$\overline{\mathcal{F}}([\mathcal{P}(\overline{g})\,\mathcal{P}(\overline{f})^*]) = \overline{\mathcal{F}}([\mathcal{F}_2(\overline{g})\,\mathcal{F}_2(J\overline{f})])$$

$$= (\overline{g}*\Delta^{\frac{1}{2}}J\overline{f})^{\vee} = f*\tilde{g} \ . \ \blacksquare$$

Remark. Since $A(G) = \overline{\mathcal{F}}(L^1(\psi_0))$ and since every $T \in L^1(\psi_0)$ can be written $T = [RS^*]$ where $R,S \in L^2(\psi_0) = \mathcal{P}(L^2(G))$ (just put $R = U|T|^{\frac{1}{2}}$ and $S^* = |T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of T) , we have reproved the fact [6, Théorème, p. 218] that $A(G) = \{f*\tilde{g} \mid f,g \in L^2(G)\}$. It also follows that $\|\varphi\|_{A(G)} \leq \|f\|_2 \|g\|_2$ whenever $\varphi = f*\tilde{g}$, $f,g \in L^2(G)$ (since $\|[\mathcal{P}(\overline{g})\,\mathcal{P}(\overline{f})^*]\|_1 \leq \|\mathcal{P}(\overline{g})\|_2\,\|\mathcal{P}(\overline{f})\|_2$), and that, given $\varphi \in A(G)$, there exist $f,g \in L^2(G)$ with $\varphi = f*\tilde{g}$ such that $\|\varphi\|_{A(G)} = \|f\|_2 \|g\|_2$ (use that $\|T\|_1 = \|U|T|^{\frac{1}{2}}\|_2\,\||T|^{\frac{1}{2}}\|_2$ for $T \in L^1(\psi_0)$).

Propositon 5.3. Let $p \in [1,2]$ and $q_1,q_2 \in [2,\infty]$ such that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p}$. Let $T \in L^{q_1}(\psi_0)$ and $S \in L^{q_2}(\psi_0)$. Then

$$(T\xi|S\eta) = \int \overline{\mathcal{F}}_p([S^*T])(x)\,(\xi*J_p\eta)(x)\,dx$$

for all $\xi,\eta \in \mathcal{K}(G)$.

<u>Proof</u>. By Lemma 2.7, the left hand side of the equation to be proved is a continuous function of T and S. The same is true of the right hand side. Therefore it is enough to prove the statement for T and S belonging to the (dense) sets $\mathcal{F}_{p_1}(\mathcal{K}(G))$ and $\mathcal{F}_{p_2}(\mathcal{K}(G))$ (where, as usual, $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $\frac{1}{p_2} + \frac{1}{q_2} = 1$).

Now suppose that $T = \mathcal{F}_{p_1}(h)$ and $S = \mathcal{F}_{p_2}(k)$ where $h,k \in \mathcal{K}(G)$. Then

$$
\begin{aligned}
(T\xi|S\eta) &= (h*\Delta^{1/q_1}\xi|k*\Delta^{1/q_2}\eta) \\
&= (\Delta^{1/q_1}\xi*\Delta^{-1/q_2}\tilde{\eta}|\Delta^{-1}\tilde{h}*k) \\
&= (\xi*\Delta^{-1/q_1-1/q_2}\tilde{\eta}|\Delta^{-1/p_1}\tilde{h}*\Delta^{-1/q_1}k) \\
&= \int (\xi*J_p\eta)(x)(\Delta^{-1/p_1}\check{h}*\Delta^{-1/q_1}\bar{k})(x)\,dx .
\end{aligned}
$$

Since

$$
\begin{aligned}
\overline{\mathcal{F}}_p([S^*T]) &= \overline{\mathcal{F}}_p([\mathcal{F}_{p_2}(J_{p_2}k)\,\mathcal{F}_{p_1}(h)]) \\
&= \Delta^{-1/q}(J_{p_2}k*\Delta^{1/q_2}h)^{\vee} \\
&= \Delta^{-1+1/p}\,\Delta^{-1/q_2}\check{h}*\Delta^{-1+1/p}\,\Delta^{1/p_2}\bar{k} \\
&= \Delta^{-1/p_1}\check{h}*\Delta^{-1/q_1}\bar{k}
\end{aligned}
$$

we have proved the formula. ∎

<u>Proposition 5.4</u>. Let $p \in [1,2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $T \in L^p(\psi_0)$ with polar decomposition $T = U|T|$. Put $g = \Delta^{-1/q}\mathcal{F}_p(T)^{\vee}$. Then

$$
(|T|^{\frac{1}{2}}\xi|\,|T|^{\frac{1}{2}}U^*\eta) = \int (g*\Delta^{1/p}\xi)(x)\overline{\eta(x)}\,dx
$$

for all $\xi,\eta \in \mathcal{K}(G)$.

<u>oof</u>. Put $q_1 = q_2 = 2p$. Then $|T|^{\frac{1}{2}} \in L^{q_1}(\psi_0)$ and $|^{\frac{1}{2}}U^* \in L^{q_2}(\psi_0)$, and by Proposition 5.3 we get

$$(|T|^{\frac{1}{2}}\xi \,|\, |T|^{\frac{1}{2}}U^*\eta)$$

$$= \int \overline{\mathcal{F}_p(T)}(x)(\xi * J_p\eta)(x)\,dx$$

$$= \int \overline{\mathcal{F}_p(T)}(x^{-1})(\Delta^{1/p}\overline{\eta} * \overset{\vee}{\xi})(x^{-1})\Delta^{-1}(x)\,dx$$

$$= \int g(x)(\overline{\eta} * \Delta^{-1/p}\overset{\vee}{\xi})(x)\,dx$$

$$= \int (g * \Delta^{1/p}\xi)(x)\overline{\eta(x)}\,dx \; . \quad ∎$$

<u>roposition 5.5</u>. Let $p \in [1,2]$ and $T \in L^p(\psi_0)$. Put $g = \Delta^{-1/q}\,\mathcal{F}_p(T)^{\vee}$. Let $\xi \in \mathcal{K}(G)$. Then $\xi \in D(T)$ if and only f $g * \Delta^{1/p}\xi \in L^2(G)$, and if this is the case, we have

$$T\xi = g * \Delta^{1/p}\xi \; .$$

<u>roof</u>. First suppose that $\xi \in D(T)$. Then for all $\eta \in \mathcal{K}(G)$ we have

$$\int (T\xi)(x)\overline{\eta(x)}\,dx = (T\xi \,|\, \eta)$$

$$= (|T|^{\frac{1}{2}}\xi \,|\, |T|^{\frac{1}{2}}\,U^*\eta)$$

$$= \int (g * \Delta^{1/p}\xi)(x)\overline{\eta(x)}\,dx \; .$$

Hence $g * \Delta^{1/p}\xi = T\xi$ and thus $g * \Delta^{1/p}\xi \in L^2(G)$.

Conversely, if $g * \Delta^{1/p}\xi \in L^2(G)$, then

$$|(|T|^{\frac{1}{2}}\xi \mid |T|^{\frac{1}{2}}U^*\eta)|$$

$$= |\int (g*\Delta^{1/p}\xi)(x)\overline{\eta(x)}\,dx|$$

$$\leq \|g*\Delta^{1/p}\xi\|_2 \|\eta\|_2$$

for all $\eta \in \mathcal{K}(G)$. We conclude that $|T|^{\frac{1}{2}}\xi \in D([|T|^{\frac{1}{2}}U^*|_{\mathcal{K}(G)}]^*)$. Now $[|T|^{\frac{1}{2}}U^*|_{\mathcal{K}(G)}]^* = [|T|^{\frac{1}{2}}U^*]^* = U|T|^{\frac{1}{2}}$, so that $|T|^{\frac{1}{2}}\xi \in D(U|T|^{\frac{1}{2}})$, whence $\xi \in D(T)$. ∎

Theorem 5.2. Let $p \in [1,2]$ and $T \in L^p(\psi_0)$. Put $g = \Delta^{-1/q}\overline{\mathcal{F}}_p(T)^\vee$. Suppose that $g \in L^2(G)$. Then T is the closure of the operator

$$\xi \mapsto g*\Delta^{1/p}\xi \quad , \quad \xi \in \mathcal{K}(G) \quad .$$

Proof. When $g \in L^2(G)$, we have $g*\Delta^{1/p}\xi \in L^2(G)$ for all $\xi \in \mathcal{K}(G)$. Thus, by Proposition 5.5, $\mathcal{K}(G) \subseteq D(T)$, and $T\xi = g*\Delta^{1/p}\xi$ for all $\xi \in \mathcal{K}(G)$. Since $T = [T|_{\mathcal{K}(G)}]$ by Proposition 2.4, the theorem is proved. ∎

As a corollary, we have

Theorem 5.3. (Fourier inversion). Let $p \in [1,2]$ and $\frac{1}{p} + \frac{1}{q} = 1$

1) Let $T \in L^p(\psi_0)$. Put $g = \Delta^{-1/q}\overline{\mathcal{F}}_p(T)^\vee$. If $g \in L^r(G)$ for some $r \in [1,2]$, then $\mathcal{F}_r(g)\Delta^{1/r-1/q}$ is closable, and

$$T = [\mathcal{F}_r(g)\Delta^{1/r-1/q}] \quad .$$

2) Let $f \in L^p(G)$. If for some $r \in [1,2]$, the closure $S = [\mathcal{F}_p(f)\Delta^{1/r-1/q}]$ exists and belongs to $L^r(\psi_0)$, then

$$f = \Delta^{-1/s}\, \overline{\mathcal{F}}_r(S)^{\vee}$$

where $\frac{1}{r} + \frac{1}{s} = 1$.

roof. 1) Since $g \in L^r(G) \cap L^q(G)$, we also have $g \in L^2(G)$. hen by Theorem 5.2 we have

$$T\xi = g*\Delta^{1/p}\xi = g*\Delta^{1/s}\Delta^{-1+1/r+1/p}\xi = \mathcal{F}_r(g)\Delta^{1/r-1/q}\xi$$

or all $\xi \in \mathcal{K}(G)$. Thus $T|_{\mathcal{K}(G)} \subseteq \mathcal{F}_r(g)\Delta^{1/r-1/q}$. As is asily seen $\mathcal{F}_r(g)\Delta^{1/r-1/q}$ is $(-\frac{1}{p})$-homogeneous. It is also clos-ble, since its adjoint is densely defined (indeed, $(\mathcal{F}_r(g)\Delta^{1/r-1/q})^* \subseteq (T|_{\mathcal{K}(G)})^* = T^*$ so that $(\mathcal{F}_r(g)\Delta^{1/r-1/q})^* = T^*)$. We conclude that $T = [\mathcal{F}_r(g)\Delta^{1/r-1/q}]$ (since $T \subseteq [\mathcal{F}_r(g)\Delta^{1/r-1/q}])$.

2) For all $\xi \in \mathcal{K}(G)$, we have $\xi \in D(S)$ and by Proposition 5.5,

$$f*\Delta^{1/r}\xi = \mathcal{F}_p(f)\Delta^{1/r-1/q}\xi = S\xi = \Delta^{-1/s}\, \overline{\mathcal{F}}_r(S)^{\vee}*\xi \ .$$

The result follows. ∎

Putting $p = r = 1$ in the first part of Theorem 5.2 and recall-ing that $\overline{\mathcal{F}}\left(\frac{d\varphi}{d\psi_0}\right) = \varphi$ for $\varphi \in A(G)_+$ we obtain

Corollary. Let $\varphi \in A(G)_+$. If $\check{\varphi} \in L^1(G)$, then

$$\frac{d\varphi}{d\psi_0} = [\lambda(\check{\varphi})\Delta] \ .$$

Finally we shall give some results on positive operators $T \in L^p(\psi_0)$ valid without any restriction on $\mathcal{F}_p(T)$.

Note that for all $f \in L^q(G)$ and $\xi, \eta \in \mathcal{K}(G)$ we have

$$\int f(x)(\xi * J_p\eta)(x)dx$$

$$= \iint f(x)\xi(y)\Delta^{-1/p}(y^{-1}x)\tilde{\eta}(y^{-1}x)dy\ dx$$

$$= \iint f(yx)\xi(y)\Delta^{-1/p}(x)\tilde{\eta}(x)dx\ dy$$

$$= \iint f(yx^{-1})\xi(y)\Delta^{1/q}(x)\overline{\eta(x)}dx\ dy\ .$$

Proposition 5.6. Let $p \in [1,2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $T \in L^p(\psi_0)_+$. Put $f = \overline{\mathcal{F}}_p(T)$. Let

$$q(\xi) = \int f(x)(\xi * J_p\xi)(x)dx$$

$$= \iint f(yx^{-1})\Delta^{1/q}(x)\xi(y)\overline{\xi(x)}dy\ dx$$

for all $\xi \in \mathcal{K}(G)$. Then q is a closable positive quadratic form, and the positive self-adjoint operator associated with its closure is T.

Proof. By (the proof of) Proposition 5.4, we have

$$(T^{\frac{1}{2}}\xi | T^{\frac{1}{2}}\xi) = \int f(x)(\xi * J_p\xi)(x)dx = q(\xi)$$

for all $\xi \in \mathcal{K}(G)$, and $T^{\frac{1}{2}} = [T^{\frac{1}{2}}|_{\mathcal{K}(G)}]$. Thus q is a closable positive quadratic form with closure corresponding to T. ∎

Corollary. Let $\varphi \in A(G)_+$. Then $\frac{d\varphi}{d\psi_0}$ is the positive self-adjoint operator associated with the closure of the positive quadratic form given by

$$q(\xi) = \int \varphi(x)(\xi*\xi^*)(x)dx$$
$$= \iint \varphi(yx^{-1})\xi(y)\overline{\xi(x)}dy\ dx$$

for all $\xi \in \mathcal{K}(G)$.

Remark. This result also follows directly from the definition of $\frac{d\varphi}{d\psi_0}$. Indeed,

$$\left\| \left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}} \xi \right\|^2 = \varphi(\lambda(\xi)\lambda(\xi)^*) = \int \varphi(x)(\xi*\xi^*)(x)dx$$

for all $\xi \in \mathcal{K}(G)$, and we have $\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}} = \left[\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}} \Big| \mathcal{K}(G)\right]$ by Proposition 2.4 (or, alternatively, by an application of [9, Theorem] together with the fact that $\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}} = \left[\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}} \Big| \mathfrak{a}_l\right]$) .

Actually, the property of defining closable quadratic forms on $\mathcal{K}(G)$ characterizes $A(G)_+$-functions among all positive definite continuous functions. The precise statement is as follows:

Theorem 5.4. Let φ be a positive definite continuous function. Define q on $\mathcal{K}(G)$ by

$$q(\xi) = \int \varphi(x)(\xi*\xi^*)(x)dx$$
$$= \iint \varphi(yx^{-1})\xi(y)\overline{\xi(x)}dy\ dx\ ,\quad \xi \in \mathcal{K}(G)\ .$$

Then q is a positive quadratic form on $\mathcal{K}(G)$, and q is closable if and only if $\varphi \in A(G)$.

Proof. That q is a quadratic form is obvious, and since φ is positive definite, q is positive.

Now suppose that q is closable. Denote by T the positive self-adjoint operator associated with its closure; then T is characterized by the properties $\mathcal{K}(G) \subseteq D(T^{\frac{1}{2}})$, $T^{\frac{1}{2}} = [T^{\frac{1}{2}}|_{\mathcal{K}(G)}]$, and

$$\forall \xi \in \mathcal{K}(G):\ \| T^{\frac{1}{2}}\xi \|^2 = q(\xi) .$$

Let us show that T is (-1)-homogeneous. Let $x \in G$. Then $T_x = \Delta^{-1}(x)\rho(x)T\rho(x^{-1})$ is positive self-adjoint and $T_x^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}(x)\rho(x)T^{\frac{1}{2}}\rho(x^{-1})$. Therefore $\mathcal{K}(G) \subseteq D(T_x^{\frac{1}{2}})$ and $T_x^{\frac{1}{2}} = [T_x^{\frac{1}{2}}|_{\mathcal{K}(G)}]$. Furthermore, for all $\xi \in \mathcal{K}(G)$ we have

$$\begin{aligned} \| T_x^{\frac{1}{2}}\xi \|^2 &= \| \Delta^{-\frac{1}{2}}(x)\rho(x)T^{\frac{1}{2}}\rho(x^{-1})\xi \|^2 \\ &= \Delta^{-1}(x)\ \| T^{\frac{1}{2}}\rho(x^{-1})\xi \|^2 \\ &= \Delta^{-1}(x) q(\rho(x^{-1})\xi) \\ &= \Delta^{-1}(x) \iint \varphi(yz^{-1}) (\rho(x^{-1})\xi)(y) \overline{(\rho(x^{-1})\xi)(z)}\, dy\, dz \\ &= \iint \Delta^{-1}(x)\varphi(yz^{-1})\Delta^{\frac{1}{2}}(x^{-1})\xi(yx^{-1})\Delta^{\frac{1}{2}}(x^{-1})\overline{\xi(zx^{-1})}\, dy\, dz \\ &= \Delta^{-1}(x) \iint \varphi(yxz^{-1})\xi(y)\overline{\xi(zx^{-1})}\, dy\, dz \\ &= \iint \varphi(yz^{-1})\xi(y)\overline{\xi(z)}\, dz\, dy \\ &= q(\xi) . \end{aligned}$$

We conclude from the characterization of T that $T_x = T$, so that

$$\forall x \in G: \Delta^{-1}(x)\rho(x)T\rho(x^{-1}) = T ,$$

ie. T is (-1)-homogeneous.

Now let $(\xi_i)_{i\in I}$ be an approximate identity in $\mathcal{K}(G)_+$. Then

$$\|T^{\frac{1}{2}}\xi_i\|^2 = q(\xi_i)$$
$$= \int \varphi(x)(\xi_i * \xi_i^*)(x)\,dx$$
$$\leq \sup\left\{ |\varphi(x)| \;\middle|\; x \in \operatorname{supp}(\xi_i * \xi_i^*) \right\} \cdot \|\xi_i * \xi_i^*\|_1$$
$$\leq \sup\left\{ |\varphi(x)| \;\middle|\; x \in \operatorname{supp}(\xi_i * \xi_i^*) \right\} .$$

ince φ is continuous and the supports of the $\xi_i * \xi_i^*$ tend to e} , we get

$$\liminf_{i\in I} \|T^{\frac{1}{2}}\xi_i\|^2 \leq \varphi(e) .$$

y Proposition 2.1, this shows that $T \in L^1(\psi_0)$.

Put $\varphi_1 = \overline{\mathcal{F}}(T) \in A(G)$. Then

$$\forall \xi \in \mathcal{K}(G): \int \varphi_1(x)(\xi * \xi^*)(x)\,dx = \|T^{\frac{1}{2}}\xi\|^2 = q(\xi)$$
$$= \int \varphi(x)(\xi * \xi^*)(x)\,dx .$$

e conclude that $\varphi = \varphi_1$ and thus $\varphi \in A(G)$. ∎

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