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Right Inverse of the Module of Approximately Finite Dimensional Factors of Type III and Approximately Finite Ergodic Principal Measured Groupoids

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Introduction

It is well known from the work of Connes [**2, 3**], Krieger [**10, 11**], and Haagerup [**6**], that the classifications of ergodic transformations up to orbit equivalence, of approximately finite dimensional (AFD) factors up to isomorphism, and of ergodic flows up to conjugacy are all equivalent problems. The proof proceeds by constructing maps from transformations to flows (the stable range or Poincaré flow construction), from transformations to factors (the group-measure-space construction) and from factors to flows (the flow of weights construction) and showing that these maps implement the desired equivalence.

Notwithstanding its beauty, the proof is unsatisfactory in at least two respects. Firstly, it is far from clear what the inverse of the various maps above (particularly those to flows) are. In fact surjectivity of the stable range map involves constructing for each flow an action of $\mathbb{Z}^2$ with the given flow as stable range and using the fact that all such actions are singly generated up to orbit equivalence. Secondly, each of the maps involved is in fact a covariant functor, transforming orbit equivalence and automorphisms to self-conjugacies, automorphisms and self conjugacies respectively; the proof fails to show whether or not these various homomorphisms are split.

Our purpose in this paper is to remedy this by constructing explicitly for each ergodic flow an AFD factor and an ergodic transformation which have the given flow as flow of weights and stable range respectively. These constructions are readily

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seen to be functorial, thus proving in particular that the map mod of [**4**] from automorphisms of AFD factors to automorphisms of the corresponding flows of weights is in fact split. Our results are also in the spirit of [**7, 8**], [**14**] and [**5**], emphasizing the necessity of adopting a functorial point of view in analyzing non-smooth classification problems.

In the first section, we construct a functorial inverse to the flow of weights construction. Section Two contains work based on unpublished results of Robert Wong [**15**] on inverting the stable range map, generalising earlier work of [**9**] on measure preserving flows. Although Wong's results are now quite old, they have never been circulated owing to his departure from mathematics and we feel they should be made available to the mathematical community.

1 Inverting the Flow of Weights and Module of an AFD Factor

First we want to construct an explicit functor from the category of ergodic flows to that of AFD factors of type III. We fix a separable abelian von Neumann algebra $\mathcal{C}$ and a $\sigma-$weakly dense separable C$^*-$algebra subalgebra B of $\mathcal{C}$. To each ergodic one parameter automorphism group θ of $\mathcal{C}$ we associate the C$^*-$algebra A generated by $\{\theta_f(a) : a \in B, f \in L^1(\mathbb{R})\}$ where $\theta_f(a) = \int_{\mathbb{R}} f(t)\theta_t(a)dt$. Then $\{A, \mathbb{R}, \theta\}$ is a separable C^*-covariant system. Set $X = \mathrm{Sp}(A)$, a compact metrizable space. The action θ of $\mathbb{R}$ on A gives rise to a topological transformation group $\{X, \mathbb{R}, F\}$ such that

$$(\theta_t(f))(x) = f(F_t^{-1}x), \quad x \in X, f \in A, t \in \mathbb{R}.$$

Let $\mathfrak{S}_*(\mathcal{C})$ be the set of all normal states on $\mathcal{C}$ and $\mathfrak{S}_*^0(\mathcal{C})$ be the set of all faithful normal states on $\mathcal{C}$. To each $\varphi \in \mathfrak{S}_*(\mathbb{C})$ there corresponds a Radon measure μ_φ on X such that $\varphi(f) = \int_X f(x)d\mu_\varphi(x), f \in A$ and if $\varphi \in \mathfrak{S}_*^0$ then $\mathcal{C}$ is canonically identified with $L^\infty(X, \mu_\varphi)$. To save notations, we will identify the measure μ_φ with $\varphi \in \mathfrak{S}_*(\mathcal{C})$. We want to find a functorial way of constructing an AFD factor $\mathcal{R}$ of type III whose flow of weights is precisely the given flow $\{\mathcal{C}, \mathbb{R}, \theta\}$. First observe that the C$^*-$algebra A, and therefore X as well, depend on θ, so it is appropriate to write them as $A(\theta)$ and $X(\theta)$. Let $\mathcal{G}(\theta) = \mathbb{R} \ltimes X(\theta)$ be the associated groupoid so that

$$\begin{cases} s(t,x) = x, r(t,x) = F_t x, & (t,x) \in \mathcal{G}(\theta); \\ (s, F_t x)(t,x) = (s+t, x), & s,t \in \mathbb{R}, x \in X(\theta). \end{cases} \tag{1.1}$$

Let $\{\mathcal{R}_{0,1}, \mathbb{R}, \alpha\}$ be a fixed AFD factor of type II$_\infty$ equipped with a trace scaling one parameter automorphism group $\{\alpha_t : t \in \mathbb{R}\}$, whose existence is guaranteed by the existence of an AFD factor of type III$_1$, and set

$$\begin{cases} \delta_\varphi(t,x) & = \frac{d\varphi \circ F_t}{d\varphi}(x), \quad \varphi \in \mathfrak{S}_*^0; \\ \theta^\varphi_{(t,x)} & = \alpha_{t+\log(\delta_\varphi(t,x))}, \quad (t,x) \in \mathcal{G}(\theta). \end{cases} \tag{1.2}$$

We then integrate the above system:

$$\begin{cases} \widetilde{\mathcal{R}} = \int_{X(\theta)}^{\oplus} \mathcal{R}_{0,1} d\varphi(x) = \mathcal{C} \bar{\otimes} \mathcal{R}_{0,1}; \\ \tau_\varphi(a) = \int_{X(\theta)} \tau(a(x))d\varphi(x), \quad a = \int_{X(\theta)}^{\oplus} a(x)d\varphi(x) \in \widetilde{\mathcal{R}}; \\ (\theta_t^\varphi(a))(F_t x) = \theta^\varphi_{(t,x)}(a(x)),\ t \in \mathbb{R},\ x \in X(\theta), \end{cases} \tag{1.3}$$

where τ means a fixed faithful semi-finite trace on $\mathcal{R}_{0,1}$ which is unique up to scalar multiples by positive numbers. It is straightforward to check that the one parameter automorphism group θ^{φ} of $\widetilde{\mathcal{R}}$ scales the trace τ_{φ} and extends the given one parameter automorphism group on the center $\mathcal{C}$ of $\widetilde{\mathcal{R}}$. Finally set

$$\mathcal{R}(\theta,\varphi) = \widetilde{\mathcal{R}} \rtimes_{\theta^{\varphi}} \mathbb{R}. \tag{1.4}$$

By the structure theorem for factors of type III, [**13**], $\mathcal{R}(\theta,\varphi)$ is a factor of type III with core $\widetilde{\mathcal{R}}$ except for the case that $\{\mathcal{C},\mathbb{R},\theta\} \cong \{L^{\infty}(\mathbb{R}),\mathbb{R},\text{translation}\}$ in which case $\mathcal{R}(\theta,\varphi) \cong \mathcal{R}_{0,1}$. Thus the flow of weights of $\mathcal{R}(\theta,\varphi)$ is conjugate to the given flow $\{\mathcal{C},\mathbb{R},\theta\}$. The construction of $\mathcal{R}(\theta,\varphi)$ is canonical except for the dependence on the state φ. So we need to look at the dependence of $\mathcal{R}(\theta,\varphi)$ on $\varphi \in \mathfrak{S}_*^0 = \mathfrak{S}_*^0(\mathcal{C})$. Choose another $\psi \in \mathfrak{S}_*^0$. With $f(x) = \left(\frac{d\psi}{d\varphi}\right)(x), x \in X(\theta)$, we have

$$\begin{cases} \delta_{\psi}(t,x) = f(F_t x)\delta_{\varphi}(x)f(x)^{-1}, & x \in X(\theta); \\ \theta^{\psi}_{(t,x)} = \alpha_{\log(f(F_t x))} \circ \theta^{\varphi}_{(t,x)} \circ \alpha_{-\log(f(x))}, & (t,x) \in \mathcal{G}(\theta). \end{cases} \tag{1.5}$$

Therefore with $\tilde{\alpha}_{(\psi,\varphi)} = \int^{\oplus}_{X(\theta)} \alpha_{(\log(f(x)))} d\varphi(x) \in \operatorname{Aut}(\widetilde{\mathcal{R}})$, we have

$$\begin{cases} \theta^{\psi}_t = \tilde{\alpha}_{(\psi,\varphi)} \circ \theta^{\varphi}_t \circ \tilde{\alpha}^{-1}_{(\psi,\varphi)}, & t \in \mathbb{R},\ \varphi,\psi \in \mathfrak{S}_*^0; \\ \tau_{\varphi} = \tau_{\psi} \circ \alpha_{(\psi,\varphi)}; & \\ \tilde{\alpha}_{(\rho,\varphi)} = \tilde{\alpha}_{(\rho,\psi)} \circ \tilde{\alpha}_{(\psi,\varphi)}, & \rho,\psi,\varphi \in \mathfrak{S}_*^0. \end{cases} \tag{1.6}$$

It is easy to check that the canonical extension $\alpha_{(\psi,\varphi)}$ of $\tilde{\alpha}_{(\psi,\varphi)}$ gives an isomorphism $\mathcal{R}(\theta,\varphi)$ onto $\mathcal{R}(\theta,\psi)$ and that the system $\{\alpha_{(\psi,\varphi)} : \psi,\varphi \in \mathfrak{S}_*^0\}$ satisfies the chain rule.

To establish the functoriality of the construction of $\mathcal{R}(\theta,\varphi)$, we need to get rid of the dependence on φ. So set

$$\mathcal{R}(\theta) = \left\{ x = \{x_{\varphi}\} \in \prod_{\varphi \in \mathfrak{S}_*^0} : x_{\psi} = \alpha_{(\psi,\varphi)}(x_{\varphi}), \quad \psi,\varphi \in \mathfrak{S}_*^0 \right\}. \tag{1.7}$$

We now move on to the question as to how the association of $\mathcal{R}(\theta)$ reacts to a conjugation of the flow θ. Let σ be an element in $\operatorname{Aut}(\mathcal{C})$ and set ${}^{\sigma}\theta_t = \sigma \circ \theta_t \circ \sigma^{-1}$, $t \in \mathbb{R}$. Then we have $\sigma(A(\theta)) = A({}^{\sigma}\theta)$. The isomorphism σ of $A(\theta)$ onto $A({}^{\sigma}\theta)$ gives rise to a homeomorphism $S : X(\theta) \mapsto X({}^{\sigma}\theta)$, so that $(\sigma(f))(x) = f(S^{-1}x), x \in X({}^{\sigma}\theta), f \in A(\theta)$. When we need to indicate the dependence of S on σ, we write $S(\sigma)$ and observe that $S(\sigma_1 \circ \sigma_2) = S(\sigma_1) \circ S(\sigma_2)$, $\sigma_1,\sigma_2 \in \operatorname{Aut}(\mathcal{C})$, i.e., S is covariant in σ. The flow ${}^{\sigma}F$ on $X({}^{\sigma}\theta)$ associated with ${}^{\sigma}\theta$ is then given by: ${}^{\sigma}F_t = S \circ F_t \circ S^{-1}, t \in \mathbb{R}$. The modules of $\mathcal{G}({}^{\sigma}\theta)$ and $\mathcal{G}(\theta)$ are linked in the following way:

$$\delta_{\psi}(t,Sx) = \delta_{\psi\circ\sigma}(t,x), \quad (t,x) \in \mathcal{G}(\theta) = \mathbb{R} \ltimes X(\theta), \psi \in \mathfrak{S}_*^0. \tag{1.8}$$

Hence

$$\theta^{\psi\circ\sigma}_{(t,x)} = ({}^{\sigma}\theta)^{\psi}_{(t,Sx)}, \quad (t,x) \in \mathcal{G}(\theta).$$

Therefore the automorphism σ of $\mathcal{C}$ extends to an automorphism $\tilde{\sigma} = \sigma \otimes \mathrm{id} \in \operatorname{Aut}(\mathcal{C} \bar{\otimes} \mathcal{R}_{0,1})$ such that

$$\begin{cases} ({}^{\sigma}\theta)^{\psi}_t = \tilde{\sigma} \circ \theta^{\psi}_t \circ \tilde{\sigma}^{-1}, t \in \mathbb{R}; & \\ \tau_{\psi\circ\sigma} = \tau_{\psi} \circ \tilde{\sigma}, & \psi \in \mathfrak{S}_*^0; \\ \tilde{\sigma} \circ \tilde{\alpha}_{(\varphi,\psi)} \circ \tilde{\sigma}^{-1} = \tilde{\alpha}_{(\varphi\circ\sigma^{-1},\psi\circ\sigma^{-1})}, & \varphi,\psi \in \mathfrak{S}_*^0. \end{cases} \tag{1.9}$$

The correspondence $\sigma \in \mathrm{Aut}(\mathcal{C}) \mapsto \tilde{\sigma} = \sigma \otimes \mathrm{id} \in \mathrm{Aut}(\mathcal{C} \bar{\otimes} \mathcal{R}_{0,1})$ obviously satisfies the chain rule:

$$\widetilde{\sigma_1 \circ \sigma_2} = \tilde{\sigma}_1 \circ \tilde{\sigma}_2, \quad \sigma_1, \sigma_2 \in \mathrm{Aut}(\mathcal{C}). \tag{1.10}$$

The conjugating map $\tilde{\sigma}$ is then canonically extended to an isomorphism, still denoted by $\tilde{\sigma}$, of $\mathcal{R}(\theta, \psi \circ \sigma)$ onto $\mathcal{R}({}^{\sigma}\theta, \psi)$ for each $\psi \in \mathfrak{S}_*^0$. The last equality in (1.9) allows us to define an isomorphism $\bar{\sigma}$ of $\mathcal{R}(\theta)$ onto $\mathcal{R}({}^{\sigma}\theta)$ by:

$$\bar{\sigma}(\{x_\varphi\}) = \{\tilde{\sigma}(x_{\varphi\circ\sigma^{-1}})\}, \quad \{x_\varphi\} \in \mathcal{R}(\theta). \tag{1.11}$$

Let $\mathrm{Erg}(\mathcal{C}, \mathbb{R})$ be the set of ergodic actions θ of $\mathbb{R}$ on a separable abelian von Neumann algebra $\mathcal{C}$ which are not conjugate to $L^\infty(\mathbb{R})$ with translation.

Theorem 1.1 *The correspondences $\theta \in \mathrm{Erg}(\mathcal{C}, \mathbb{R}) \mapsto \mathcal{R}(\theta)$ and $\sigma \in \mathrm{Aut}(\mathcal{C}) \mapsto \bar{\sigma}$ define a functor from the category of ergodic flows to that of approximately finite dimensional, factors of type* III *such that the flow of weights of $\mathcal{R}(\theta)$ is conjugate to θ. In particular, if $\theta \in \mathrm{Erg}(\mathcal{C}, \mathbb{R})$ is fixed, then the map: $\sigma \in \mathrm{Aut}_\theta(\mathcal{C}) = \{\alpha \in \mathrm{Aut}(\mathcal{C}) : \alpha \circ \theta_t = \theta_t \circ \alpha\} \mapsto \bar{\sigma} \in \mathrm{Aut}(\mathcal{R}(\theta))$ is an injective homomorphism which is the right inverse of the module of $\mathcal{R}(\theta)$.*

Remark 1.2 The von Neumann algebra $\mathcal{R}(\theta)$ in Equation 1.7 is of course isomorphic to $\mathcal{R}(\theta, \varphi)$ for any $\varphi \in \mathfrak{S}_*^0$; the construction is given solely to eliminate the identifications $\alpha_{(\psi,\varphi)}$ of $\mathcal{R}(\theta, \varphi)$ and $\mathcal{R}(\theta, \psi)$ as φ and ψ vary.

It is possible to avoid this construction as follows. Fix $\varphi \in \mathfrak{S}_*^0$; it then follows from the above discussion that for any $\sigma \in \mathrm{Aut}(\mathcal{C})$, $\tilde{\sigma}$ is an isomorphism of $\mathcal{R}(\theta, \varphi\circ\sigma)$ to $\mathcal{R}({}^{\sigma}\theta, \varphi)$, so that $\tilde{\sigma} \circ \alpha_{(\varphi\circ\sigma,\varphi)} = \alpha_{(\varphi,\varphi\circ\sigma^{-1})} \circ \tilde{\sigma}$ is an isomorphism from $\mathcal{R}(\theta, \varphi)$ onto $\mathcal{R}({}^{\sigma}\theta, \varphi)$, and satisfies

$$(\widetilde{\sigma_1 \circ \sigma_2}) \circ \alpha_{(\varphi\circ(\sigma_1\circ\sigma_2),\varphi)} = \tilde{\sigma}_1 \circ \alpha_{(\varphi\circ\sigma_1,\varphi)} \circ \tilde{\sigma}_2 \circ \alpha_{(\varphi\circ\sigma_2,\varphi)} \tag{1.12}$$

for $\sigma_1 \circ \sigma_2 \in \mathrm{Aut}(\mathcal{C})$. Thus, with $\tilde{\sigma}_\varphi = \tilde{\sigma} \circ \alpha_{(\varphi\circ\sigma_2,\varphi)}$, we obtain a functor $\theta \in \mathrm{Erg}(\mathcal{C}, \mathbb{R}) \to \mathcal{R}(\theta, \varphi), \sigma \in \mathrm{Aut}(\mathcal{C}) \to \tilde{\sigma}_\varphi$.

Corollary 1.3 *If $\mathcal{R}$ is an approximately finite dimensional factor of type* III*, the short exact sequence*

$$1 \longrightarrow \overline{\mathrm{Int}}(\mathcal{R}) \longrightarrow \mathrm{Aut}(\mathcal{R}) \longrightarrow \mathrm{Aut}_\theta(\mathcal{C}) \longrightarrow 1, \tag{1.13}$$

is split, where $\{\mathcal{C}, \mathbb{R}, \theta\}$ is the flow of weights of $\mathcal{R}$.

2 Inverting the Stable Range Map

To set notation, we first recall the construction of the stable range or Poincaré flow of an ergodic action of a discrete group G on a standard measure space (X, μ). We write

$$\rho(g, x) = \log \frac{d\mu \circ g}{d\mu}(x) \tag{2.1}$$

and observe that ρ satisfies the cocycle identity

$$\rho(gh, u) = \rho(g, hx)\rho(h, x) \ \text{ a.e. in } \ x \ \text{ for } \ g, h \in G. \tag{2.2}$$

The group G acts on $(\tilde{X}, \tilde{\mu}) = (X \times \mathbb{R}, \mu \times e^{-t}dt)$ via the maps

$$(x, t) \longrightarrow (gx, t - \rho(g, x)), \ g \in G; \tag{2.3}$$

evidently, this action of G commutes with the action of $\mathbb{R}$ defined on $(\tilde{X}, \tilde{\mu})$ by

$$S_s : (x,t) \longrightarrow (x, t+s), \ s \in \mathbb{R}, \tag{2.4}$$

and hence defines an action $\{\theta_t : t \in \mathbb{R}\}$ of $\mathbb{R}$ on the algebra $\mathcal{C}_\mu$ of G-invariant elements of $L^\infty(\tilde{X}, \tilde{\mu})$. This action $\{\mathcal{C}_\mu, \theta_t, \mathbb{R}\}$, or any of its point realizations, is called the **Poincaré flow** of the original action of G on (X, μ), or the **stable range** (of the Radon-Nikodym cocycle) - see [**12**] for an equivalent discussion of terms of groupoids.

If ν is another measure on X equivalent to μ, and

$$f(x) = \log\left(\frac{d\nu}{d\mu}(x)\right) \tag{2.5}$$

then the map $W_{\nu,\mu}$ on $X \times \mathbb{R}$

$$W_{\nu,\mu}(x,t) = (x, t + f(x)) \tag{2.6}$$

carries $\tilde{\mu}$ to $\tilde{\nu}$, commutes with the action $\{S_t : t \in \mathbb{R}\}$ in (2.4), and intertwines the action of G on $(\tilde{X}, \tilde{\mu})$ and $(\tilde{X}, \tilde{\nu})$. Consequently $W_{\nu,\mu}$ defines an isomorphism $w_{\nu,\mu} : \mathcal{C}_\mu \to \mathcal{C}_\nu$ which intertwines the two Poincaré flows. Since $w_{\lambda,\nu} \circ w_{\nu,\mu} = w_{\lambda,\mu}$, for any equivalent measures λ, ν, μ on X, the flows $\{\mathcal{C}_\mu, \theta_t, \mathbb{R}\}$ are all isomorphic in a coherent manner. In addition, if H is another countable group acting on (Y, ν) and A is an orbit equivalence from G on (X, μ) to H on (Y, ν), so that $A : X \to Y$ is a Borel isomorphism, carries G orbits to H orbits and μ to ν (or, more correctly, to a measure equivalent to ν), then the map $\tilde{A} : \tilde{X} \to \tilde{Y}$ defined by

$$\tilde{A}(x,t) = (Ax, t) \tag{2.7}$$

carries G orbits on $\tilde{X}$ to H orbits on $\tilde{Y}, \tilde{\mu}$ to $\tilde{\nu}$, and intertwines the $\mathbb{R}$-actions on $(\tilde{X}, \tilde{\mu})$ and $(\tilde{Y}, \tilde{\nu})$. This then gives rise to an isomorphism α of the flows $(\mathcal{C}_\mu, \theta_r, \mathbb{R})$ and $(\mathcal{C}_\nu, \theta_t, \mathbb{R})$ which depends covariantly on A.

In summary, we have

Theorem 2.1 *The construction of the stable range flow is a functor from the category of (ergodic) actions of discrete groups on standard measure spaces, with orbit equivalence as morphisms, to the category of (ergodic) flows on standard measure spaces with measure-class preserving conjugacies as morphisms.*

The case where $G = Z$ is of particular interest. The remainder of this section will be devoted to proving

Theorem 2.2 *There is a functor from the category of ergodic flows on standard measure spaces with measure class preserving conjugacies as morphisms to ergodic tranformations with measure class preserving conjugacies as morphisms which is a right inverse to the stable range construction.*

Note that, rather surprizingly, the morphisms between transformations are conjugacies rather than merely orbit equivalences.

The proof is somewhat analagous to that of Theorem 1.1, but carried out at the measure space level. To do so, we first need some auxilliary constructions.

Let $\mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$ be the cyclic group of order 5, and let $X = \mathbb{Z}_5^{\mathbb{N}} = \prod_{i=1}^\infty \mathbb{Z}_{5,i}$ be the cartesian product space. Let β_1, β_2, β_3 and β_4 be positive numbers such that $\log\beta_1$, $\log\beta_2$, $\log\beta_3$ and $\log\beta_4$ are rationally independent. Let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be the probability distribution on $\mathbb{Z}_5$ determined by

$$\beta_i = \alpha_{i-1}/\alpha_i, \quad i = 1, 2, 3, 4, \tag{2.8}$$

and let

$$P = \prod_{i=1}^{\infty}{}^{\otimes}\alpha \tag{2.9}$$

be the product measure on X. The *odometer* transformation T is defined as follows. For $x = \{x_i\} \in X$, let $i_0(x)$ be the first index i such that $x_i \neq 4$. Then we set

$$(Tx)_i = \begin{cases} 0, & 1 \leq i < i_0(x), \\ x_i + 1, & i = i_0(x), \\ x_i, & i > i_0(x). \end{cases} \tag{2.10}$$

In the above definition of $(Tx)_i$, $i_0(x)$ can be infinite. In this case, $x_i = 4$ for every $i \in \mathbb{N}$. Except for this isolated case, we have $x_i = (Tx)_i$ for large enough $i \in \mathbb{N}$. Since the measure P on X is non-atomic, we can exclude the point $x_0 = (4, 4, \ldots, 4, \ldots)$ and its orbit $\{T^n x_0 : n \in \mathbb{Z}\}$ from X.

Lemma 2.3 *There exist $\mathbb{Z}$-valued Borel functions $n_i(x)$, $i = 1, 2, 3, 4$, such that*

$$\frac{dP\circ T}{dP}(x) = \beta_1^{n_1(x)}\beta_2^{n_2(x)}\beta_3^{n_3(x)}\beta_4^{n_4(x)}, \quad \text{a.e.} \quad x \in X. \tag{2.11}$$

These functions n_i's do not depend on the choice of β_i's as long as the $\log\beta_i$'s are rationally independent.

Proof Fix an $x \in X$. Let $A_k = (x_1, x_2, \ldots, x_k) \times \prod_{k+1}^{\infty}\mathbb{Z}_5$ be the cylinder set determined by the first k coordinates of x. We then have

$$P(A_k) = \prod_{j=1}^{k}\alpha_{x_j}, \quad P(TA_k) = \prod_{j=1}^{k}\alpha_{(Tx)_j}.$$

For each j, we have

$$\frac{\alpha_{x_j}}{\alpha_{(Tx)_j}} = \begin{cases} 1 & \text{if} \quad j > i_0(x); \\ \beta_{(Tx)_j} & \text{if} \quad j = i_0(x); \\ \beta_1\beta_2\beta_3\beta_4 & \text{if} \quad j < i_0(x). \end{cases}$$

Therefore, there exist $\mathbb{Z}$-valued functions $n_1(x), \ldots, n_4(x)$ such that

$$\frac{P(TA_k)}{P(A_k)} = \prod_{i=1}^{4}\beta_i^{n_i(x)} \qquad \text{for large} \quad k.$$

Then we have

$$P(TA_k) = P(A_k)\prod_{i=1}^{4}\beta_i^{n_i(x)} = \int_{A_k}\prod_{i=1}^{4}\beta_i^{n_i(x)}\,dP(x).$$

Since these A's are closed under finite intersections and generate the Borel field of X, we conclude that T leaves the measure P quasi-invariant, and that

$$\frac{dP \circ T}{dP}(x) = \prod_{i=1}^{4} \beta_i^{n_i(x)} .$$

In the above proof, the functions $n_i(x)$, $1 \le i \le 4$, do not depend on the choice of β_i's, so that they are intrinsic to the odometer. □

We now let H be the group of all finite permutations on $\mathbb{N}$, i.e. the group of permutations on $\mathbb{N}$ which change only a finite number of elements of $\mathbb{N}$. We then let H act on X in the canonical way, i.e., if $g \in H$, then $(gx)_i = x_{g^{-1}(i)}$ for $x = \{x_i\} \in X$. We observe that $H \subset [T]$, the full group of T, and H leaves the measure P invariant.

Lemma 2.4 *Let $x \in X$. If k is a positive integer such that $T^k(x)$ is a finite permutation of x, then we have*

$$\sum_{j=0}^{k-1} n_i(T^j(x)) = 0 \quad \text{for} \quad i = 1, 2, 3, 4 . \tag{2.12}$$

Proof Choose $h \in H$ with $h(x) = T^k(x)$. For $j = 0, 1, \ldots, k$, $T^j x$ differs from x in only a finite number of coordinates, so that there exists $n \in \mathbb{N}$ such that $(Tx)_j = x_j$ for every $j \ge n$. Set

$$A_0 = (x_1, x_2, \ldots, x_n) \times \prod_{n+1}^{\infty} \mathbb{Z}_5 ,$$

$$A_j = T^j(A_0), \qquad j = 0, 1, \ldots, k .$$

Then we have

$$\frac{P(A_{j+1})}{P(A_j)} = \prod_{i=1}^{4} \beta_i^{n_i(T^j(x))} , \qquad j = 0, 1, \ldots, k-1 .$$

Since $P(A_k) = P(h(A_0)) = P(A_0)$, we have

$$1 = \frac{P(A_k)}{P(A_0)} = \prod_{j=0}^{k-1} \frac{P(A_{j+1})}{P(A_j)} = \prod_{i=1}^{4} \beta_i^{\sum_{j=0}^{k-1} n_i(T^j(x))} .$$

Since β_i's are multiplicatively independent we obtain the formula (2.12). □

Let $\{\Omega, \mu, F_t\}$ be a properly ergodic flow on the standard measure space $\{\Omega, \mu\}$. Set

$$p(t, \omega) = \frac{d\mu \circ F_t}{d\mu}(\omega), \qquad (t, \omega) \in \mathbb{R} \times \Omega . \tag{2.13}$$

We further introduce notations:

$$\begin{cases} a(x) = n_1(x) \log \beta_1 + n_2(x) \log \beta_2, \\ b(x) = n_3(x) \log \beta_3 + n_4(x) \log \beta_4, \quad x \in X . \end{cases} \tag{2.14}$$

We now prove a result which is the key step in proving Theorem 2.2.

Theorem 2.5 *Let $\{T, X, P\}$ and $\{F_t, \Omega, \mu\}$ be as above. Then the flow $\{F_t, \Omega, \mu\}$ is the Poincaré flow of the following ergodic single transformation S on the space $\{X \times \Omega \times \mathbb{R}, P \otimes \mu \otimes e^{-s}\, ds\}$:*

$$S(x, \omega, r) = (Tx, F_{a(x)}(\omega), r + \log p(a(x), \omega) + b(x)) . \tag{2.15}$$

Proof Let $Y = X \times \Omega \times \mathbb{R}$ and $d\nu = dP \otimes d\mu \otimes e^{-s}\, ds$. To find out the Poincaré flow of S, we set

$$\begin{cases} \tilde{S}(y, t) = \left(Sy, t + \log \dfrac{d\nu \circ S}{d\nu}(y)\right), (y, t) \in Y \times \mathbb{R} = \tilde{Y}, \\ d\tilde{\nu}(y, t) = d\nu \otimes e^{-t}\, dt . \end{cases} \tag{2.16}$$

Let A be a cylinder subset of X:

$$A = (x_1, x_2, \ldots, x_n) \times \prod_{n+1}^{\infty} \mathbb{Z}_5 ,$$

such that some x_j, $1 \le j \le n$, is less than 4, so that $n_i(x)$, $1 \le i \le 4$, are constant on A. Let $a = a(x)$ and $b = b(x)$ for all $x \in A$. Set

$$\begin{cases} Z_{ab}(\omega, r) = (F_a\omega, r + \log p(a, \omega) + b), (\omega, r) \in \Omega \times \mathbb{R}, \\ d\tilde{\mu} = d\mu \otimes e^{-r}\, dr . \end{cases} \tag{2.17}$$

It then follows that Z_{ab} is the composition of a measure preserving transformation and the translation which scales the measure by e^{-b}. Thus Z_{ab} scales the measure $\tilde{\mu}$ by the factor e^{-b}. For any Borel set $C \subset \tilde{\Omega} = \Omega \times \mathbb{R}$, we have

$$\begin{aligned} \nu(S(A \times C)) &= \int_Y \chi_{S(A\times C)}(y)\, d\nu(y) = \int_Y \chi_{A\times C}(S^{-1}(y))\, d\nu(y) \\ &= \iint_{X\times\tilde{\Omega}} \chi_{A\times C}(F_{-a}(\omega), r + \log p(-a, \omega) - b)\, d\nu(x, \omega, r) \\ &= \int_X \chi_{T(A)}(x)\, dP(x) \int_{\tilde{\Omega}} \chi_{Z_{ab}(C)}(\omega, r)\, d\tilde{\mu}(\omega, r) \\ &= P(T(A))\tilde{\mu}(Z_{ab}(C)) = e^{a+b} P(A) e^{-b}\, \tilde{\mu}(C) \\ &= e^a P(A)\tilde{\mu}(C) = \int_{A\times C} e^{a(x)}\, d\nu(y) . \end{aligned}$$

Since the collection of such $A \times C$ is closed under finite intersections and generates the σ-field of Borel sets in $Y = X \times \tilde{\Omega}$, we conclude that

$$\frac{d\nu \circ S}{d\nu}(y) = e^{a(x)}, \quad (x, \tilde{\omega}) = y \in Y = X \times \tilde{\Omega} . \tag{2.18}$$

Hence we have

$$\tilde{S}(x, \omega, r, t) = (S(x, \omega, r), t + a(x)), (x, \omega, r, t) \in \tilde{Y} . \tag{2.19}$$

For each $h \in H$, let $k(h,x)$, or simply $k(x)$, be the integer valued function such that $hx = T^{k(x)}x,\ x \in X$. For abbreviation, set

$$\begin{cases} s(k,x) = \sum\limits_{j=0}^{k} a(T^j x) \qquad k \geq 0\,, \\ t(k,x) = \sum\limits_{j=0}^{k} b(T^j x)\,. \end{cases}$$

If $k(x) = k(h,x) \geq 0$, then we have

$$\tilde{S}^{k(x)}(x,\omega,r,t) = (S^{k(x)}(x,\omega,r), t + s(k(x)-1,x))\,.$$

By Lemma 2.4, we know $s(k(x)-1,x) = 0$ and $t(k(x)-1,x) = 0$, so that

$$\tilde{S}^{k(x)}(x,\omega,r,t) = (S^{k(x)}(x,\omega,r), t)\,.$$

Furthermore, we have

$$S^{k(x)}(x,\omega,r) = (T^{k(x)}x, \omega, r + \log p(s(k(x)-1,\omega)) + t(k(x)-1,x)) = (hx,\omega,r)\,.$$

If $k(x) = k(h,x) < 0$, then we consider h^{-1} applied to $h(x)$ to conclude that

$$\tilde{S}^{k(x)}(x,\omega,r,t) = (hx,\omega,r,t)\,.$$

Therefore, the full group $[\tilde{S}]$ of $\tilde{S}$ on $\tilde{Y}$ contains the subgroup $H \times \mathrm{id}$ on $X \times \tilde{\Omega} \times \mathbb{R} = \tilde{Y}$. Hence any $\tilde{S}$-invariant measurable subset of $\tilde{Y}$ is of the form $X \times B$ for some measurable subset $B \subset \Omega \times \mathbb{R} \times \mathbb{R}$, which is invariant under the transformation:

$$(\omega,r,t) \mapsto (F_{a(x)}(\omega), r + \log p(a(x),\omega) + b(x), t + a(x))$$

for all $x \in X$.

In X, the following four sets all have nonzero measure:

$$\begin{aligned} &\{x \in X : a(x) = \log\beta_1 \quad \text{and} \quad b(x) = 0\}; \\ &\{x \in X : a(x) = \log\beta_2 \quad \text{and} \quad b(x) = 0\}; \\ &\{x \in X : a(x) = 0 \quad \text{and} \quad b(x) = \log\beta_3\}; \\ &\{x \in X : a(x) = 0 \quad \text{and} \quad b(x) = \log\beta_4\}\,. \end{aligned}$$

Therefore, the above set B must be invariant under the transformations:

$$(\omega,r,t) \mapsto (F_a\omega, r + \log p(a,\omega) + b, t + a)$$

for all $a \in G_1$ and $b \in G_2$, where G_1 and G_2 are the subgroups of $\mathbb{R}$ generated by $\log\beta_1$, $\log\beta_2$ and $\log\beta_3$, $\log\beta_4$ respectively. Since these subgroups are both dense in $\mathbb{R}$, there exists a measurable subset $C \subset \Omega \times \mathbb{R}$ such that

$$B = \{(\omega,r,t) \in \Omega \times \mathbb{R} \times \mathbb{R} : (\omega,t) \in C\}$$

and C is invariant under the transformations:

$$R_a : (\omega,t) \mapsto (F_a\omega, t + a), \quad a \in G_1\,.$$

The density of G_1 in $\mathbb{R}$ implies that C is invariant under the flow

$$R_s : (\omega,t) \mapsto (F_s\omega, t + s), \quad s \in \mathbb{R}\,.$$

We now define

$$W(\omega,t) = (F_t\omega, t)\,.$$

Then we have

$$WR_sW^{-1}(\omega, t) = (\omega, t+s)\,.$$

Therefore $W(C)$ is invariant under id$\times$ translation on $\Omega \times \mathbb{R}$. Hence, with $\tilde{W}(x, \omega, r, t) = (x, F_t\omega, r, t)$, we have

$$\tilde{W} L^\infty(\tilde{Y})^{\tilde{S}} \tilde{W}^{-1} = \mathbb{C} \otimes L^\infty(\Omega, \mu) \otimes \mathbb{C} \otimes \mathbb{C}\,,$$

where we view $\tilde{W}$ as a unitary on $L^2(\tilde{Y})$. It now follows that the associated flow of S is conjugate to F under $\tilde{W}$. □

To complete the proof of Theorem 2.2, we need to consider the effects of conjugacies of flows. So consider ergodic flows $\{F_t, \Omega, \mu\}$ and $\{G_t, \Gamma, \nu\}$ and suppose $\theta : (\Omega, \mu) \to (\Gamma, \nu)$ is a measure space isomorphism with $\theta \circ F_t = G_t \circ \theta$ for all t. Let

$$(\tilde{\Omega}, \tilde{\mu}) = (X \times \Omega \times \mathbb{R}, \quad P \times \mu \times e^{-t}dt)$$

and analogously for $(\tilde{\Gamma}, \tilde{\nu})$; also let S_F, S_G be the transformations on $(\tilde{\Omega}, \tilde{\mu})$ and $(\tilde{\Gamma}, \tilde{\nu})$ as in Equation 2.15, and define $\tilde{\theta} : \tilde{\Omega} \to \tilde{\Gamma}$ by

$$\tilde{\theta}(x, w, r) = (x, \theta w, r + \log \frac{d\nu \circ \theta}{d\mu}(w)). \tag{2.20}$$

Lemma 2.6 *With notation as above, we have*
(a) $p_F(t, w) \frac{d\nu\circ\theta}{d\mu}(F_t w) = p_G(t, \theta w) \frac{d\nu\circ\theta}{d\mu}(w)$ *a.e. for each* t*;*

(b) $\tilde{\theta} S_F = S_G \tilde{\theta}$*;*

(c) $\tilde{\theta}$ *depends covariantly on* θ*.*

Proof
(a) follows routinely from the Radon-Nikodym Theorem

(b) We calculate

$$\begin{aligned}\tilde{\theta} \circ S_F(x, w, r) = (Tx, \theta F_{a(x)}(w), r + b(x) &+ \log p_F(a(x), w) \\ &+ \log \frac{d\nu \circ \theta}{d\mu}(F_{a(x)}(w))),\end{aligned}$$

while

$$\begin{aligned}S_G \tilde{\theta}(x, w, r) = (Tx, G_{a(x)}\theta(w), r + b(x) &+ \log p_G(a(x), \theta(w)) \\ &+ \log \frac{d\nu \circ \theta}{d\mu}(w)).\end{aligned}$$

The equality of these two expressions follows from a) and the assumption that $\theta \circ F_t = G_t \circ \theta$ for all t.

(c) is routine, and left to the reader. □

Remark 2.7 With Lemma 2.6, the proof of Theorem 2.2 is complete. We also note that composing Theorem 2.2 with the group measure space construction, which is known to be covariantly functorial for orbit equivalences, yields an alternate proof of Corollary 1.2.

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