Two-dimensional conformal field theories and modular functors

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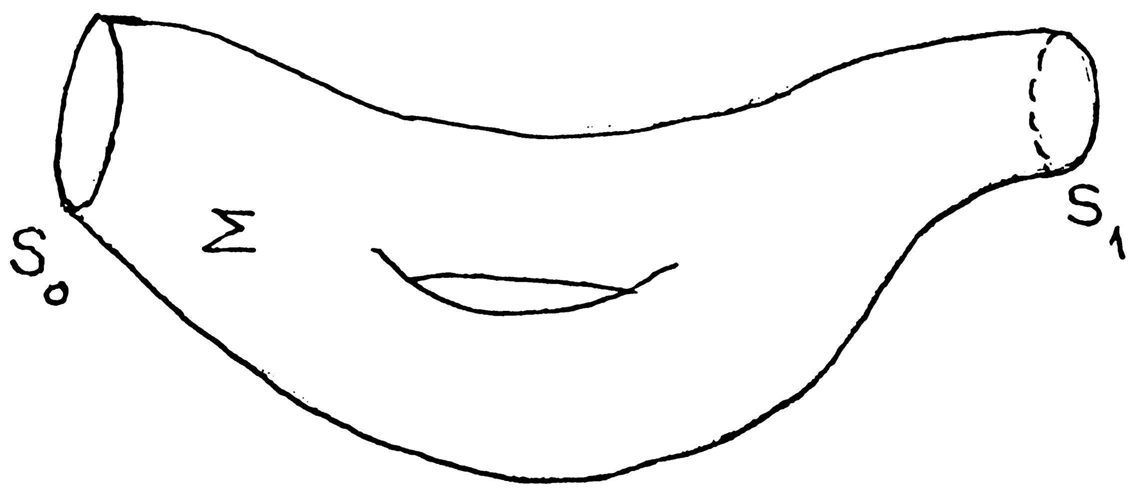
§1. The definition of a field theory

A two-dimensional conformal field theory comprises a great deal of data. The essential part is

(i) a Hilbert space $\mathcal{H}$ of states, and

(ii) an operator $U_\Sigma : \mathcal{H} \to \mathcal{H}$ for each Riemann surface Σ whose boundary consists of two circles.



Here Σ should be thought of as a two-dimensional Euclidean "space-time" interpolating between two one-dimensional "spaces" S_0 and S_1, each being a circle. $\mathcal{H}$ should be imagined as consisting of wave-functions Ψ defined on some space of "classical fields" which are functions on the circle. Schematically the operator U_Σ can be written

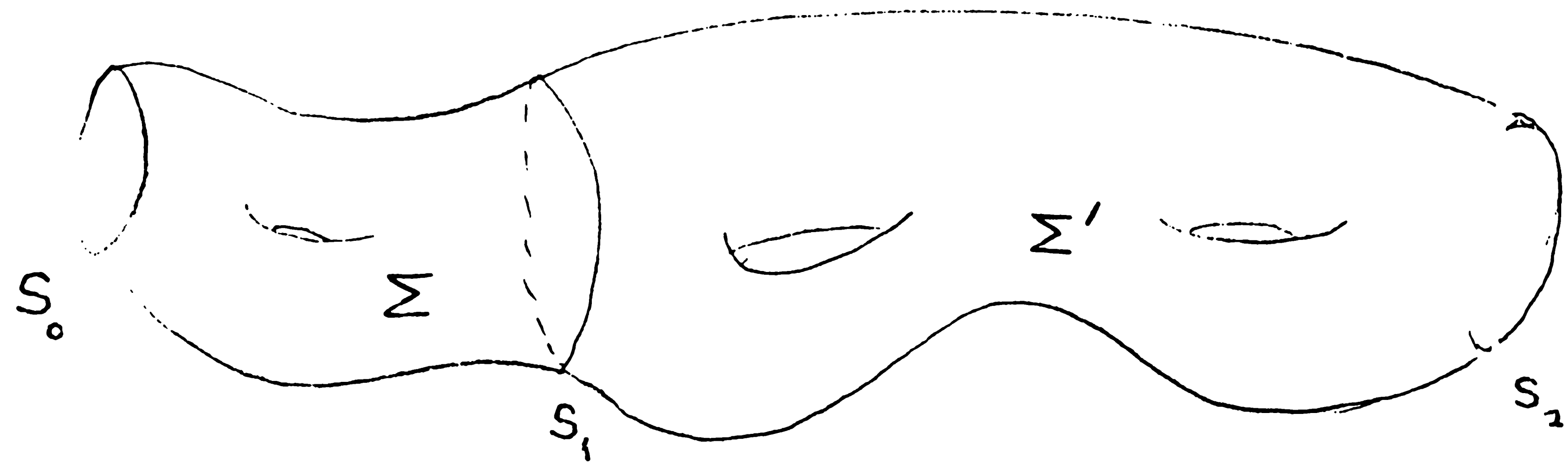
$$(U_\Sigma\Psi)(\gamma_1) = \int K\,(\gamma_0,\gamma_1)\Psi(\gamma_0)\mathcal{D}\gamma_0\ , \qquad (1.1)$$

where

$$K_\Sigma(\gamma_0,\gamma_1) = \int_{\gamma|\partial\Sigma=(\gamma_0,\gamma_1)} e^{-S(\gamma)}\mathcal{D}\gamma\ . \qquad (1.2)$$

Here S is some conformally invariant action defined for classical fields γ on Σ, and the integral is over all fields γ whose boundary values on S_0 and S_1 are γ_0 and γ_1.

The operators U_Σ form a semigroup in the sense that when two surfaces Σ and Σ' are sewn end-to-end

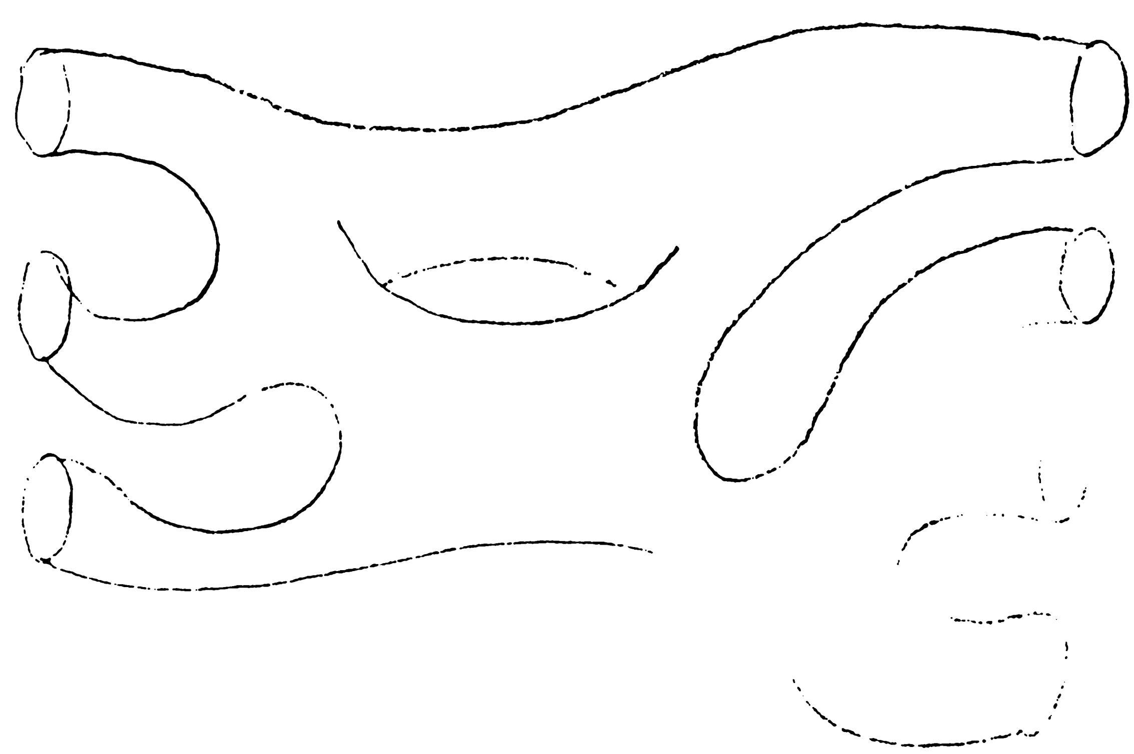


we have

$$U_{\Sigma'} \circ U_\Sigma = \lambda U_{\Sigma \cup \Sigma'} \tag{1.3}$$

for some complex number λ.

A field theory comprises more data than this, but not much more. For each surface Σ with p incoming and q outgoing boundary circles



the theory gives us an operator

$$U_\Sigma : \underbrace{H \otimes \ldots \otimes H}_{\leftarrow p \rightarrow} \rightarrow \underbrace{H \otimes \ldots \otimes H}_{\leftarrow q \rightarrow} \tag{1.4}$$

which again is described schematically by the formulae (1.1) and (1.2).

These operators satisfy rules analogous to (1.3) which are easily imagined. When all the U_Σ are given the field theory is completely described.

§2. Chiral factorization

The simplest example of a conformally invariant action is

$$S(\gamma) = \tfrac{1}{2}\int \|d\gamma\|^2 = \tfrac{1}{2}\int_\Sigma d\,\gamma \wedge {*d\gamma}\ , \tag{2.1}$$

where γ is a real-valued function on Σ. This leads to the classical field equation

$$\frac{\partial^2\gamma}{\partial t^2} - \frac{\partial^2\gamma}{\partial x^2} = 0 \tag{2.2}$$

in Minkowski space, or, in the Euclidean version we are adopting,

$$\frac{\partial^2\gamma}{\partial z \partial \bar{z}} = 0\ . \tag{2.2a}$$

It is well known that any solution of (2.2) is the sum of left- and right-moving parts:

$$\gamma(t,x) = \gamma_L(x+t) + \gamma_R(x-t)\ , \tag{2.3}$$

or, in the Euclidean version,

$$\gamma(z,\bar{z}) = \gamma_+(z) + \gamma_-(\bar{z})\ . \tag{2.3a}$$

It is natural to ask whether the state space $\mathcal{H}$ of the corresponding field theory can be factorized.

$$\mathcal{H} = \mathcal{H}^L \otimes \mathcal{H}^R\ .$$

The answer is no, for fairly elementary reasons. Even classically the decomposition (2.3) is not quite unique, as γ_L and γ_R

are determined only up to the addition of a constant. Worse still, if γ is periodic in x with period 2π then γ_L need not be periodic : the formula for γ_L at any time is

$$\gamma_L(x) = \tfrac{1}{2}\{\gamma(x) + \int_0^x \frac{\partial\gamma}{\partial t}(y)dy\} , \qquad (2.4)$$

and this is periodic only if the total momentum $p = \int_0^{2\pi} \dot{\gamma}dx$ is zero. Thus if X is the space of solutions of (2.2), and $X_p^L = X_p^R$ is the space of maps $\gamma : \mathbb{R} \to \mathbb{R}$ which satisfy

$$\gamma(\theta+2\pi) = \gamma(\theta) + \tfrac{1}{2}p ,$$

then instead of a simple product decomposition $X = X^L \times X^R$ we have

$$X = \bigcup_p (X_p^L \times X_p^R)/\mathbb{R} ,$$

which in the quantum theory should lead to a decomposition of the form

$$H = \int_p H_p^L \otimes H_p^R .$$

In fact we are interested in the case when γ takes its values not in $\mathbb{R}$ but in a circle $\mathbb{R}/\ell\mathbb{R}$ of length ℓ. Then p is quantized in units of ℓ^{-1}, and is defined only modulo ℓ. If ℓ^2 is a rational number this means there are only finitely many possibilities for p, and we expect a finite sum

$$H = \bigoplus_{p\in I} H_p^L \otimes H_p^R . \qquad (2.5)$$

If H splits in this way the next question is whether the operator U_Σ associated to a surface Σ can be written

$$U_\Sigma = \sum_p U_{\Sigma,p}^L \otimes U_{\Sigma,p}^R .$$

Once again one must expect the answer to be no, for even in the classical case a solution of (2.2a), i.e. a harmonic function, cannot usually be written as the sum of a holomorphic and an antiholomorphic function on a surface Σ which has non-trivial cohomology. In fact there is an exact sequence

$$0 \to H^0(\Sigma;\mathbb{C}) \to \mathrm{Hol}(\Sigma) \oplus \overline{\mathrm{Hol}(\Sigma)} \to \mathrm{Harm}(\Sigma) \to H^1(\Sigma;\mathbb{C}) \to 0 \ ,$$

where $f \in \mathrm{Harm}(\Sigma)$ maps to the class of the closed 1-form $*df$. (Notice that if $*df = dg$ then

$$f = \tfrac{1}{2}(f + ig) + 1(f - ig) \ ,$$

where $f \pm ig$ are holomorphic and antiholomorphic.)

Investigating the examples carefully one finds that for each surface Σ and each $p,q \in I$ there are natural finite dimensional vector spaces $V^L_{\Sigma,pq}$ and $V^R_{\Sigma,pq}$ of operators $H^L_p \to H^L_q$ (resp. $H^R_p \to H^R_q$) such that

(a) U_Σ belongs to $\bigoplus_{p,q} V^L_{\Sigma,pq} \otimes V^R_{\Sigma,pq}$, and

(b) the $V^L_{\Sigma,pq}$ are closed under composition in the sense that $V^L_{\Sigma',qr} \circ V^L_{\Sigma,pq} \subset V^L_{\Sigma\cup\Sigma',pr}$

when two surfaces Σ, Σ' are sewn together. (Similarly, of course, for V^R.)

To axiomatize the chiral fragments into which a conformal field theory breaks up one is led to introduce the concept of a modular functor, to which the remainder of this talk is devoted.

§3. Modular functors

We start with a finite set I of labels, containing a distinguished label called 1. There is an operation of "conjugation" $\alpha \mapsto \bar{\alpha}$ on I such that $\bar{1} = 1$.

A modular functor based on I is a rule which associates a finite dimensional vector space $V_{\Sigma,\alpha}$ to each Riemann surface Σ with boundary, where each boundary component of Σ is equipped with a parametrization and also a label from I. (Here $\alpha = (\alpha_1,\ldots,\alpha_k)$ is a multi-index, where $\alpha_i \in I$ is the label of the i^{th} boundary circle.) The spaces $V_{\Sigma,\alpha}$ are required to have the following four properties.

(3.1) $$V_{\Sigma_1 \amalg \Sigma_2, \alpha_1 \amalg \alpha_2} = V_{\Sigma_1,\alpha_1} \otimes V_{\Sigma_2,\alpha_2},$$

where $\Sigma_1 \amalg \Sigma_2$ denotes the disjoint union.

(3.2) If $\check{\Sigma}$ is obtained from Σ by identifying two boundary circles S_1 and S_2 (using their parametrizations) then

$$V_{\check{\Sigma},\beta} = \bigoplus_{\alpha} V_{\Sigma,\beta\alpha\bar{\alpha}},$$

where the sum is over all labellings $\beta\alpha\bar{\alpha}$ of $\partial\Sigma$ which agree with the labelling β of $\partial\check{\Sigma}$ and give conjugate labels $\alpha,\bar{\alpha}$ to S_1 and S_2.

(3.3) $$V_{D,\alpha} = \begin{cases} \mathbb{C} & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha \neq 1, \end{cases}$$

where D is the standard disc.

(3.4) $V_{\Sigma,\alpha}$ depends holomorphically on Σ, in the sense that if $\{\Sigma_t\}$ is a holomorphic family of surfaces parametrized by $t \in T$ then $\{V_{\Sigma_t,\alpha}\}$ is a holomorphic vector bundle on T.

Let us point out three immediate consequences of these axioms.

(3.5) If two surfaces Σ and Σ' are sewn end-to-end then there is a "composition" map

$$V_{\Sigma,\alpha\bar{\beta}} \otimes V_{\Sigma',\beta\gamma} \to V_{\Sigma\cup\Sigma',\alpha\gamma} .$$

(3.6) If Σ is an annulus then $V_{\Sigma,\alpha\beta} = 0$ if $\beta \neq \bar{\alpha}$, and $\dim(V_{\Sigma,\alpha\bar{\alpha}}) = 1$.

(3.7) If Σ is a torus then $V_\Sigma \cong \mathbb{C}[I]$. More precisely, for each way of cutting Σ so as to obtain an annulus we have a decomposition of V_Σ as a sum of one-dimensional spaces, one for each label.

A modular functor is a simultaneous generalization of two different concepts,

(i) a central extension of $\mathrm{Diff}(S^1)$, and

(ii) a coherent family of projective representations of the braid groups and mapping class groups.

I shall return to the first aspect in this next section. The second aspect depends on the following basic theorem about modular functors.

Theorem (3.8). If $\{\Sigma_m\}_{m\in M}$ is a holomorphic family of surfaces there is a canonical flat projective connection in the vector bundle $\{V_{\Sigma_m,\alpha}\}$ on M.

Here a projective connection means a rule which associates an isomorphism

$$p_* : V_{\Sigma_m,\alpha} \to V_{\Sigma_{m'},\alpha} ,$$

defined up to an arbitrary scalar multiple, to each smooth path p from m to m' in M. The connection is flat if p_* does not change when p is deformed smoothly leaving its ends fixed.

Examples

(i) Let M be the upper half-plane in $\mathbb{C}$. Consider the family of tori

$$\Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$$

for $\tau \in M$. Because M is simply connected the projective connection identifies all the V_{Σ_τ} canonically as projective spaces. On the other hand $PSL_2(\mathbb{Z})$ acts on M, and for each $g \in PSL_2(\mathbb{Z})$ we have an isomorphism $\Sigma_\tau \cong \Sigma_{g\tau}$ and hence an isomorphism $V_{\Sigma_\tau} \cong V_{\Sigma g\tau}$. So $PSL_2(\mathbb{Z})$ acts projectively on V_{Σ_τ}.

(ii) For some small number ε let M be the subspace of $\mathbb{C}^k$ consisting of all $(z_1,\ldots,z_k)$ such that $|z_i| < 1 - \varepsilon$ and $|z_i - z_j| > 2\varepsilon$. The fundamental group of M is the coloured braid group CBr_k on k strings. For each $m \in M$ let Σ_m be the surface got by removing k open discs of radius ε with centres $z_1,\ldots,z_k$ from the closed unit disc D. Then for any modular functor and any labels $\alpha_1,\ldots,\alpha_k$ the group CBr_k acts projectively on $V_{\Sigma,\alpha}$. Furthermore, the surface Σ_m does not depend on the order of the z_i, so if $\alpha_1 = \ldots = \alpha_k$ the braid group Br_k acts on $V_{\Sigma,\alpha}$.

In his talk in this volume Witten has pointed out a remarkable consequence of the basic theorem (3.8).

Corollary (3.9) For any modular functor the projective space of V_Σ is naturally associated to the smooth surface Σ without any choice of a complex structure on it.

Proof: The space J_Σ of complex structures on Σ (not identifying structures which are related by diffeomorphisms of Σ) is contractible, so, just as in example (i) above, we can apply (3.8) to the family $\{V_{\Sigma_m}\}$ parametrized by $m \in J_\Sigma$.

We can, however, be a little more precise. We shall see in the next section that a modular functor has a central charge $c \in \mathbb{R}$. If $c = 0$ there is a flat connection, and not just a projective connection, in the bundle $\{V_{\Sigma_m}\}$, and so the vector space V_{Σ_m} depends only on Σ. Now the simplest modular functor is the determinant line Det_Σ, which needs no labelling set. Its central charge is 1. For any modular functor the vector space

$$\tilde{V}_\Sigma = V_{\Sigma_m} \otimes (\mathrm{Det}\ \Sigma_m)^{\otimes(-c)}$$

is therefore independent of the complex structure on Σ. It does, however, depend on the choice of the non-integral power of the determinant line bundle on the contractible space J_Σ. To choose the power it is enough to choose a universal covering space $\tilde{P}_\Sigma$ of the total space

$$P_\Sigma = \bigcup_m (\mathrm{Det}_{\Sigma_m} - \{0\}) \ . \ .$$

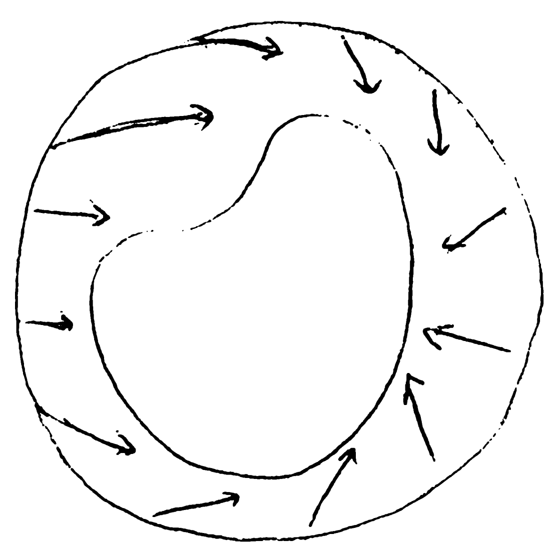
The space P_Σ is functorially associated to Σ, and has the homotopy type of a circle. Thus $\tilde{V}_\Sigma$ is nearly, but not quite, functorially associated to Σ. If $\phi : \Sigma_0 \to \Sigma_1$ is a diffeomorphism, then to get an isomorphism $\tilde{V}_{\Sigma_0} \to \tilde{V}_{\Sigma_1}$ we must choose a map $\tilde{\phi}_* : \tilde{P}_{\Sigma_0} \to \tilde{P}_{\Sigma_1}$ covering the map $\phi_* : P_{\Sigma_0} \to P_{\Sigma_1}$ induced by ϕ. For a given ϕ the choices of $\tilde{\phi}_*$ differ by integers, and so a central extension of $\mathrm{Diff}(\Sigma)$ by $\mathbb{Z}$ acts naturally on $\tilde{V}_\Sigma$.

Witten's striking discovery is that $\tilde{V}_\Sigma$, which arose in the study of two-dimensional conformal field theories, is in fact the state space of what he calls a <u>topological</u> field theory in three dimensions.

§4. <u>The semigroup of annuli</u>

Let $\mathcal{A}$ denote the set of isomorphism classes of Riemann surfaces which are topologically annuli, and are equipped with parametrizations of their boundaries. The set $\mathcal{A}$ is a <u>semi-group</u> under the operation of sewing.

Very roughly speaking, an element A of $\mathcal{A}$ is got by exponentiating an inward-pointing vector field defined along S^1:



Such vector fields form a cone in the complexification of the Lie algebra of $\mathrm{Diff}(S^1)$. The group $\mathrm{Diff}(S^1)$ does not possess a complexification, but the semigroup $\mathcal{A}$ plays the role of a subsemigroup of the non-existent complexification. The relation between $\mathrm{Diff}(S^1)$ and $\mathcal{A}$ is the same as that between the unitary group U_n and the contraction semigroup

$$GL_n^{<1}(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) : \|A\| < 1\}$$

contained in its complexification. Like $GL_n^{<1}(\mathbb{C})$ the semigroup $\mathcal{A}$ is a bounded complex domain, and $\mathrm{Diff}(S^1)$ is part of its boundary. The "positive energy" representations of $\mathrm{Diff}(S^1)$ are characterized as those that are boundary values of holomorphic representations of $\mathcal{A}$.

We can now explain the sense in which a modular functor generalizes the notion of a central extension of $\mathrm{Diff}(S^1)$. Indeed for each label $\alpha \in I$ a modular functor defines a central extension $\tilde{\mathcal{A}}_\alpha$ of $\mathcal{A}$ by $\mathbb{C}^\times$: an element of $\tilde{\mathcal{A}}_{\dot{\alpha}}$ is a pair (A,λ), with $A \in \mathcal{A}$ and $\lambda \in V_{A_{\bar{\alpha}\alpha}}$. Composition is defined using (3.5) above. On the boundary of $\mathcal{A}$ the extension $\tilde{\mathcal{A}}_\alpha$ gives rise to a central extension of $\mathrm{Diff}(S^1)$, which in turn determines $\tilde{\mathcal{A}}_\alpha$ completely.

Let us recall that a central extension of $\mathrm{Diff}(S^1)$ is determined by a pair (c,h), where $c \in \mathbb{R}$ is called the <u>central charge</u> and $h \in \mathbb{R}/\mathbb{Z}$ the <u>spin</u>.

§5. The proof of the basic theorem

The idea of the proof of theorem (3.8) can be explained quite simply. For simplicity let us consider the case of a family $\{\Sigma_m\}$ of surfaces for which $\partial\Sigma_m$ consists of a single circle. Let $\hat{\Sigma}_m$ be the closed surface got by sewing a disc on to $\partial\Sigma_m$. It is a basic fact about the complex structures on a surface that when m' is sufficiently near m the surface $\Sigma_{m'}$ can be holomorphically embedded in $\hat{\Sigma}_m$. We can therefore find annuli A and A' such that

$$\Sigma_m \cup A = \Sigma_{m'} \cup A' .$$

So

$$V_{\Sigma_m} \otimes V_A = V_{\Sigma_{m'}} \otimes V_{A'} \ .$$

Because $\dim V_A = \dim V_{A'} = 1$ this gives us an identification of V_{Σ_m} and $V_{\Sigma_{m'}}$ up to a scalar factor, and hence a projective connection in the family $\{V_{\Sigma_m}\}$.

To see that the connection is both well-defined and flat it is best to consider the universal case, when the parameter space M is the space of all complex structures on the given smooth surface Σ. (Here two structures are identified if they are related by a diffeomorphism of Σ which is the identity on $\partial\Sigma$.) What the preceding argument really gives us is a projective action of the Lie algebra $V = \mathrm{Vect}(S^1)$ on the total space of the bundle $\{V_{\Sigma_m}\}$ which covers the natural action of V on M. (V acts on M by reparametrizing $\partial\Sigma_m$.) The fact that nearby surfaces differ by annuli translates into the fact that the tangent space to M at m is $\mathrm{Vect}_{\mathbb{C}}(S^1)/\mathrm{Vect}(\Sigma_m)$, where $\mathrm{Vect}(\Sigma_m)$ denotes the holomorphic vector fields on Σ_m (which move Σ_m inside $\hat{\Sigma}_m$ without changing the structure). The action of $\mathrm{Vect}_{\mathbb{C}}(S^1)$ defines a connection in the bundle $\{V_{\Sigma_m}\}$ because $\mathrm{Vect}(\Sigma_m)$ acts trivially on V_{Σ_m}. (In fact the algebra $\mathrm{Vect}(\Sigma_m)$ has no non-trivial finite dimensional projective representations.) The connection is automatically flat because it comes from a Lie algebra action of $\mathrm{Vect}(S^1)$.

§6. Verlinde's algebra

Verlinde has introduced a very elegant way of encoding the dimensions of the spaces $V_{\Sigma,\alpha}$ for any modular functor V.

Let P be a disc with two holes, i.e. with three boundary circles. For any lables $\alpha, \beta, \gamma \in I$ let $n_{\alpha\beta\gamma} = \dim V_{P,\overline{\alpha\beta}\gamma}$.

Then we can define a multiplication on the free abelian group $\mathbb{Z}[I]$ on the set of labels by the formula

$$[\alpha].[\beta] = \sum_{\gamma \in I} n_{\alpha\beta\gamma}[\gamma] .$$

This clearly makes $\mathbb{Z}[I]$ into a commutative ring.

Because any surface Σ can be cut into copies of P, (together perhaps with discs and annuli) a knowledge of the algebra $\mathbb{Z}[I]$ enables one to calculate the dimension of $V_{\Sigma,\alpha}$ in all cases.

§7. Representations of loop groups and modular functors

The basic examples of modular functors arise from representations of loop groups. Let us recall the main points of the representation theory.

If G is a simple complex Lie group (i.e. the complexification of a simple compact group) then the group LG of smooth loops in G has an important class of representations called <u>positive energy</u> representations. These are realized on vector spaces E on which there is an energy operator $H : E \to E$ satisfying

$$\frac{d}{d\alpha} U_{\gamma_\alpha} = i[H,U_{\gamma_\alpha}] .$$

Here U_γ denotes the action of $\gamma \in LG$ on E, and γ_α is "γ rotated by α", i.e. $\gamma_\alpha(\theta) = \gamma(\theta-\alpha)$. The operator H has positive integral eigenvalues, and each eigenspace

$$E_k = \{\xi \in E : H\xi = k\xi\}$$

is finite dimensional .

The positive energy representations are <u>projective</u> : they are actually representations of a canonical central extension $\tilde{LG}$ of LG by $\mathbb{C}^\times$. An element of u of the central subgroup $\mathbb{C}^\times$ acts on E by multiplication by u^k, where k is a positive integer called the <u>level</u> of the representation.

The representation E is completely determined by its level k and its lowest energy part E_0, which is a representation of G and is irreducible if E is irreducible. In fact E can be reconstructed from E_0 in the following way. Let G_D denote the group of holomorphic maps from the disc D to G. Then G_D is a subgroup of LG over which the extension $\tilde{LG}$ is canonically split:

$$\tilde{G}_D = \mathbb{C}^\times \times G_D \ ;$$

and the map

$$E \to \mathrm{Hol}_{\tilde{G}_D}(\tilde{LG};E_0) \tag{7.1}$$

is a quasi-isomorphism. Here the right-hand side denotes the holomorphic maps $f : \tilde{LG} \to E_0$ which satisfy $f(\gamma\eta^{-1}) = \eta f(\gamma)$ for $\eta \in \tilde{G}_D$, where $\tilde{G}_D = \mathbb{C}^\times \times G_D$ acts on E_0 by

$$(z,\phi).\xi = z^k\phi(0)\xi \ .$$

The map (7.1) takes $\xi \in E$ to f_ξ, where $f_\xi(\gamma) = \mathrm{pr}(\gamma\xi)$, and $\mathrm{pr} : E \to E_0$ is the projection. A <u>quasi-isomorphism</u> means an injective $\tilde{LG}$-equivariant map with dense image.

We can now describe the basic modular functor. There are only a finite number of irreducible representations E of $\tilde{LG}$ of a given level k. Let us denote them by $\{E^\alpha\}_{\alpha\in I}$. (They correspond to the irreducible representations E_0 of G whose highest weights λ satisfy $||\lambda||^2 \le 2k$.)

If Σ is a Riemann surface with m boundary circles, let G_Σ denote the group of holomorphic maps $\Sigma \to G$. Such maps are determined by their boundary values, so G_Σ is a subgroup of $(LG)^m$. If we label the i^{th} boundary circle with a representation E^{α_i} then G_Σ acts projectively on

$$E^\alpha = E^{\alpha_1} \otimes \ldots \otimes E^{\alpha_m} .$$

The induced projective multiplier, however, turns out to be trivial, so it makes sense to define $V_{\Sigma,\alpha}$ as the part of E^α which is fixed under G_Σ.

Theorem (7.2). $(\Sigma,\alpha) \mapsto V_{\Sigma,\alpha}$ is a modular functor.

The essential property to check is the sewing axiom (3.2). This depends on a version of the Peter-Weyl theorem which holds for loop groups.

Recall that for a group such as G the Peter-Weyl theorem is the quasi-isomorphism

$$\bigoplus_V \bar{V} \otimes V \to \mathrm{Hol}(G) , \qquad (7.3)$$

where $\mathrm{Hol}(G)$ denotes the holomorphic functions on G, V runs through the irreducible representations of G, and the map assigns to $\bar{\nu}_1 \otimes \nu_2$ the matrix element $g \mapsto \langle \nu_1, g\nu_2 \rangle$. The map (7.3) is compatible with the left and right actions of G.

For a loop group LG with its central extension $\widetilde{LG}$ the corresponding assertion is that there is a quasi-isomorphism

$$\bigoplus_\alpha \bar{E}^\alpha \otimes E^\alpha \cong \mathrm{Hol}_k(\widetilde{LG}) , \qquad (7.4)$$

where $\mathrm{Hol}_k(\widetilde{LG})$ denotes the holomorphic functions $f : \widetilde{LG} \to \mathbb{C}$ such that $f(u\gamma) = u^k f(\gamma)$ for $u \in \mathbb{C}^\times$, and the sum is over the irreducible representations of level k. The meaning of the "quasi-isomorphism" is clarified by observing that both sides of (7.4) are bigraded, and finite dimensional in each bidegree, and the quasi-isomorphism is an isomorphism in each bidegree.

Returning to the sewing property (3.2), suppose that the surface $\check{\Sigma}$, with non-empty boundary labelled by a multi-index β, is formed by sewing together the edges S_1 and S_2 of Σ. Then G_Σ acts transitively on LG by

$$(g,\gamma) \mapsto g_1 \gamma g_2^{-1} ,$$

where $g_i = g|S_i$. The isotropy group of $1 \in LG$ is clearly $G_{\check{\Sigma}}$, so $LG = G_\Sigma/G_{\check{\Sigma}} = \tilde{G}_\Sigma/\tilde{G}_{\check{\Sigma}}$. Then

$$\begin{aligned} V_{\check{\Sigma},\beta} &= (E^\beta)^{G_{\check{\Sigma}}} \\ &= \mathrm{Hol}(\tilde{LG};E^\beta)^{\tilde{G}_\Sigma} \\ &= \{\mathrm{Hol}_k(\tilde{LG}) \otimes E^\beta\}^{G_\Sigma} \\ &= \{\bigoplus_\alpha \bar{E}^\alpha \otimes E^\alpha \otimes E^\beta\}^{G_\Sigma} \\ &= \bigoplus_\alpha V_{\Sigma,\beta\bar{\alpha}\alpha} . \end{aligned}$$

This argument is not quite complete, for the spaces

$$\mathrm{Hol}_k(\tilde{LG};E^\beta) \sim \mathrm{Hol}_k(\tilde{LG}) \otimes E^\beta \sim \bigoplus_\alpha \bar{E}^\alpha \otimes E^\alpha \otimes E^\beta$$

are quasi-isomorphic but not isomorphic. One must show that their G^Σ-invariant subspaces (which are finite dimensional) are actually isomorphic.

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