The locality of the state space in quantum field theory

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1 Introduction

I have described elsewhere (cf. [S2] – [S5]) an attempt to axiomatize quantum field theory in a form suggested by path-integrals Wick-rotated to Riemannian space-time manifolds. In this approach, roughly speaking, a d-dimensional theory is defined as a rule which

(i) associates a complex topological vector space $\mathcal{H}_Y$ to each compact oriented Riemannian manifold Y of dimension $d-1$, functorially with respect to diffeomorphisms $Y \to Y'$; and

(ii) associates a trace-class operator $U_X : \mathcal{H}_{Y_0} \to \mathcal{H}_{Y_1}$ to each oriented Riemannian cobordism X from Y_0 to Y_1.

These data are constrained to satisfy two axioms:

(a) "concatenation", i.e.

$$U_{X' \circ X} = U_{X'} \circ U_X$$

when $X' \circ X$ is the cobordism from Y_0 to Y_2 obtained by concatenating X from Y_0 to Y_1 with X' from Y_1 to Y_2; and

(b) "tensoring", i.e. we are given associative natural isomorphisms

$$\begin{aligned} \mathcal{H}_Y \otimes \mathcal{H}_{Y'} &\xrightarrow{\cong} \mathcal{H}_{Y \sqcup Y'} \\ U_X \otimes U_{X'} &= U_{X \sqcup X'} \end{aligned}$$

when we have disjoint unions $Y \sqcup Y'$ or $X \sqcup X'$ of $(d-1)$-manifolds or cobordisms.

From the start it was clear that the structure just described is not quite rigid enough to encode what is usually meant in physics by a quantum field theory: what seemed to be lacking was some prescription of how the dependence of the state-space $\mathcal{H}_Y$ on Y is *local* with respect to Y. The aim of the present work is to propose a suitable prescription, and to justify it in the case of a free massive boson theory in an arbitrary gravitational background, and also for a class of two-dimensional conformal field theories including the WZW models. I shall say a little more about what one can hope to do with the new proposal, and where it comes from, at the end of this introduction.

The words "roughly speaking" before the definition above refer[1] to the fact that $\mathcal{H}_Y$ should really be associated, not to a compact $(d-1)$-manifold Y, but rather to a *germ* of a Riemannian d-manifold along Y, i.e.

$$\mathcal{H}_Y = \mathcal{H}_{Y,U},$$

where U is an open Riemannian d-manifold containing Y, but $\mathcal{H}_{Y,U} = \mathcal{H}_{Y,U'}$ if $Y \subset U \subset U'$. Without this modification the definition of a field theory does not really make sense, for there is no canonical smooth structure on a concatenation of cobordisms.

In practice $\mathcal{H}_{Y,U}$ will depend only on Y and a certain number of normal derivatives of the metric of U along Y. In the examples I know, $\left[\frac{d-1}{2}\right]$ normal derivatives are needed. Thus no normal information is needed for 2-dimensional theories[2], but for 4-dimensional theories we need not only the metric (or "first fundamental form") of Y but also its "second fundamental form" in U, i.e. the usual Cauchy data of general relativity. The fact that we seem to need more than one normal derivative in dimensions greater than four may perhaps be related to the difficulty of constructing interesting theories in these dimensions.

[1]One also needs to say something about the class of topological vector spaces and the notion of tensor product to be considered; but, as I have explained elsewhere (e.g. [S4],[S5]), these questions turn out to present no problems. Briefly, $\mathcal{H}_Y$ is really a *pair* $\check{\mathcal{H}}_Y, \hat{\mathcal{H}}_Y$ of topological vector spaces with an injective map $\check{\mathcal{H}}_Y \to \hat{\mathcal{H}}_Y$ with dense image, just like, for example, the smooth functions and distributions on a manifold. If $\bar{Y}$ is Y with reversed orientation, we have $(\check{\mathcal{H}}_Y)^* \cong \hat{\mathcal{H}}_{\bar{Y}}$ and $(\hat{\mathcal{H}}_Y)^* \cong \check{\mathcal{H}}_{\bar{Y}}$.

[2]This is assuming that Y is smooth. If Y is allowed to have corners we shall see in §4 that the space $\mathcal{H}_{Y,U}$ is sensitive to the angle at the corners.

The dependence of $\mathcal{H}_Y$ on the germ of the metric of U is important in connection with unitarity. The theories we shall consider are unitary. This is usually taken to mean that we have a canonical isomorphism

$$\bar{\mathcal{H}}_Y \to \mathcal{H}_{\bar{Y}}$$

from the complex-conjugate vector space of $\mathcal{H}_Y$ to the vector space for Y with reversed orientation. (As $\mathcal{H}_{\bar{Y}}$ is automatically dual to $\mathcal{H}_Y$ for any theory, this gives us a hermitian inner product on $\mathcal{H}_Y$.) But we must remember that reversing the orientation of Y really means interchanging the "in" and "out" sides of the collar U of Y, and so we do not expect $\bar{\mathcal{H}}_Y \cong \mathcal{H}_{\bar{Y}}$ unless there is an orientation-reversing reflection of U, with Y as its fixed points, which preserves whatever normal data along Y is required to define $\mathcal{H}_Y$. Thus $\mathcal{H}_Y$ will not necessarily be a Hilbert space even in a unitary theory.

In fact it is natural to enlarge somewhat the class of allowed manifold-germs (Y, U). Although U must be a Riemannian manifold, we do not need Y to be a smooth submanifold: we can permit it to be any subset of U of the form $Y = f^{-1}(0)$, where $f : U \to \mathbb{R}$ is a proper smooth map for which 0 is either a regular value (in which case Y is a submanifold) or else an *isolated* critical value. Thus Y might have the singularites of the level-set of a Morse function at a critical level, or it might be the boundary of a curvilinear d-simplex smoothly embedded in $\mathbb{R}^d$.

What does it mean to say that the vector space $\mathcal{H}_Y$ depends locally on Y? If Y is the union of two pieces Y_1 and Y_2 which are $(d-1)$-dimensional manifolds with boundary which intersect in their common boundary Z, then a too naive idea of "second quantization" might lead one to hope for naturally-defined vector spaces $\mathcal{H}_{Y_1}$ and $\mathcal{H}_{Y_2}$ such that $\mathcal{H}_Y \cong \mathcal{H}_{Y_1} \otimes \mathcal{H}_{Y_2}$. (For the Hilbert space $\mathrm{L}^2(Y)$ of L^2 functions on Y we have $\mathrm{L}^2(Y) = \mathrm{L}^2(Y_1) \oplus \mathrm{L}^2(Y_2)$, so the completed symmetric algebra $S(\mathrm{L}^2(Y))$ obeys

$$S(\mathrm{L}^2(Y)) \cong S(\mathrm{L}^2(Y_1)) \otimes S(\mathrm{L}^2(Y_2)).)$$

But that is *not* what happens even for field theories of non-interacting particles.

It is more reasonable to expect that if we enlarge Y_1 and Y_2 slightly, to manifolds Y_1^+ and Y_2^+, say, which overlap in a tubular neighbourhood of Z, then we can reconstruct $\mathcal{H}_Y$ from vector spaces $\mathcal{H}_{Y_1^+}$ and $\mathcal{H}_{Y_2^+}$ together with some gluing data which involves only the neighbourhood of Z. We shall prove that this happens in some basic examples. Roughly, we shall assign an algebra $\mathcal{A}_Z$ to Z, and shall find that $\mathcal{H}_{Y_1^+}$ and $\mathcal{H}_{Y_2^+}$ are right- and left-$\mathcal{A}_Z$-modules and that

$$\mathcal{H}_Y \cong \mathcal{H}_{Y_1^+} \otimes_{\mathcal{A}_Z} \mathcal{H}_{Y_2^+}.$$

But this is still not quite right. We shall need to adapt to our purposes a more subtle notion of tensor product introduced by Connes ([C] Chap.5) for modules over von Neumann algebras.

The general framework into which these results fit is what I have called a *3-tier* quantum field theory. Such a theory has three layers of data:

(o) to each compact oriented Riemannian $(d-2)$-manifold Z is associated a linear category $\mathcal{C}_Z$;

(i) to each $(d-1)$-dimensional Riemannian cobordism Y from Z_0 to Z_1 is associated an additive functor $\mathcal{H}_Y : \mathcal{C}_{Z_0} \to \mathcal{C}_{Z_1}$;

(ii) to each d-dimensional Riemannian cobordism X from Y to Y', where Y and Y' are cobordisms from Z_0 to Z_1, is associated a transformation of functors $U_X : \mathcal{H}_Y \to \mathcal{H}_{Y'}$.

As with the earlier definition, the data are required to satisfy two axioms of concatenation and tensoring. Schematically, at least, one way to give the data is to associate an algebra $\mathcal{A}_Z$ to each $(d-2)$-manifold, and to take $\mathcal{C}_Z$ to be the category of left-$\mathcal{A}_Z$-modules; then to a cobordism Y is associated an $\mathcal{A}_{Z_1}$-$\mathcal{A}_{Z_0}$-bimodule $\mathcal{H}_Y$, which defines a functor $\mathcal{C}_{Z_0} \to \mathcal{C}_{Z_1}$ by

$$\mathcal{E} \mapsto \mathcal{H}_Y \otimes_{\mathcal{A}_{Z_0}} \mathcal{E};$$

and to a cobordism between cobordisms is associated a homomorphism of bimodules. Theories in the earlier 2-tier sense fit into the 3-tier definition by restricting to closed $(d-1)$-manifolds, which can be regarded as cobordisms from the empty $(d-2)$-manifold $\emptyset$ to itself. The tensoring axiom implies that $\mathcal{C}_\emptyset$ is the category of vector spaces, and since any additive functor $\mathcal{C}_\emptyset \to \mathcal{C}_\emptyset$ is given by tensoring with a vector space, we can identify $\mathcal{H}_Y$ with a vector space when Y is closed.

The 3-tier definition as just presented is too vague for any but heuristic purposes. How does one tensor linear categories? The only fairly well-understood example of the structure is 3-dimensional Chern-Simons theory for a compact group G (at some chosen level k). This is essentially a topological field theory, and the category $\mathcal{C}_Z$ associated to a closed 1-manifold Z is the category of positive-energy projective unitary representations (of level k) of the group of smooth maps from Z to G, which is a product of loop groups. This category is of the same size as the category of finite dimensional representations of a finite group, and so no infinite dimensional analysis is needed to handle it: there is no difficulty in interpreting the category $\mathcal{C}_{Z_1 \sqcup Z_2}$ as $\mathcal{C}_{Z_1} \otimes \mathcal{C}_{Z_2}$. The point of this paper is to treat more traditional field theories which are sensitive to the metric of space-time.

Just as the vector space $\mathcal{H}_Y$ of a 2-tier theory is actually associated to a germ (Y, U) of a d-manifold along Y, so the category $\mathcal{C}_Z$ which a 3-tier theory associates to a $(d-2)$-manifold depends on a germ (Z, U) of a Riemannian d-manifold along Z. In the version I shall describe, it depends in addition on a germ of a $(d-1)$-manifold along Z contained in U — i.e. we have a category $\mathcal{C}_{Z,V,U}$, where V is a closed $(d-1)$-dimensional submanifold of U which contains Z. It will be a category of $*$-representations of an algebra $\mathcal{A}_{Z,V,U}$ which for the moment we can think of as a von Neumann algebra.

A very interesting thing happens when we rotate the $(d-1)$-manifold V around Z inside U. If V and V' are two transversal choices then there is a natural functor $\mathcal{C}_{Z,V,U} \to \mathcal{C}_{Z,V',U}$, induced by a densely defined homomorphism of algebras from $\mathcal{A}_{Z,V,U}$ to $\mathcal{A}_{Z,V',U}$, or, better, by a bimodule for these algebras. In the case when the normal structure of Z in U is trivial, i.e. $U = Z \times \mathbb{R}^2$, and we consider the natural choices $V_\theta = Z \times L_\theta$, where L_θ is the line in $\mathbb{R}^2$ making an angle θ with the x-axis, then we get a family of algebras $\mathcal{A}_\theta$ which are all canonically isomorphic, and the morphisms from $\mathcal{A}_\theta$ to $\mathcal{A}_{\theta'}$ are precisely the *Tomita-Takesaki flow*[3] which plays a central role in Connes's theory of von Neumann algebras.

There are several kinds of applications one can hope to make of these locality results. My own main motivation has been to find a strengthening of

[3]The Tomita-Takesaki flow on a von Neumann algebra $\mathcal{A}$ is usually described as a 1-parameter group of $*$-automorphisms $\{\alpha_t : \mathcal{A} \to \mathcal{A}\}_{t \in \mathbb{R}}$. But on a dense suspace of $\mathcal{A}$ we can continue α_t analytically to purely imaginary values $t = \mathrm{i}\theta$. These unbounded automorphisms are what arise here.

the definition of a quantum field theory which will enable one to prove that — as all physicists assume as a matter of course — any infinitesimal deformation of the theory is given by a "field" in the theory. I am fairly confident that the present structure achieves this, at least for two-dimensional theories.

Another potential application is to the geometrical understanding of elliptic cohomology: here my approach is very close to that of Stolz and Teichner (cf. [ST],[S2]), and I have profited greatly from discussions of the subject with them.

A third kind of application is to the Verlinde theory of the fusion of representations of loop groups — i.e. to 3-dimensional Chern-Simons theory and its relations to the WZW model. The work of Wasserman [W] on this subject was what started me thinking along the present lines, and I am indebted to him for many helpful explanations, and in particular for bringing the paper [LRT] to my attention.

I hope to say more about all three applications elsewhere.

The plan of this paper is as follows.

Section 2 describes the state space for free bosons, and explains the nature of the problem to be solved.

Section 3 defines Connes's tensor product in a form adapted to our needs. In fact we shall not need to mention von Neumann algebras at all, though the ideas all come from the study of them.

Section 4 illustrates the use of the tensor product for "fusing" representations of loop groups, and the geometric significance of the Tomita-Takesaki flow. We begin with loop groups because the method is particularly simple and clear when applied to them. This section is essentially a reworking and extension of some of the results of Wasserman [W].

Section 5 treats the case of free massive bosons, which is our main focus of interest.

Section 6 collects some results about representations of infinite dimensional Heisenberg groups which have been used earlier. These are all in principle well-known.

2 The space $\mathcal{H}_Y$ for free massive bosons

The state-space of the theory of free massive bosons can be described very explicitly in a way that makes the question of its locality easy to understand. A classical boson field on a space-time X is a smooth map $\phi : X \to \mathbb{R}$. Its classical action is

$$S(\phi) = \frac{1}{2}\int_X (\mathrm{d}\phi * \mathrm{d}\phi + m^2\phi * \phi),$$

where m is the mass, and $*$ is the Hodge star-operator of the metric of X. The classical solutions — the critical points of S — are the fields that satisfy $(\Delta_X + m^2)\phi = 0$, where Δ_X is the Laplace operator of X (taken to be a positive operator in the Riemannian case). The Hilbert space $\mathcal{H}_Y$, for a spatial $(d-1)$-manifold Y, is the quantization of the symplectic vector space

$$\Sigma_Y = \Omega^0(Y) \oplus \Omega^{d-1}(Y),$$

which is the space of Cauchy-data for a classical solution in a neighbourhood of Y. (The Cauchy data consists of $\phi|Y$ and its normal derivative $\partial\phi/\partial n = (*\mathrm{d}\phi)|Y$.) The symplectic form is the obvious hyperbolic one coming from the duality between $\Omega^0(Y)$ and $\Omega^{d-1}(Y)$. Formally, therefore, we expect the Hilbert space $\mathcal{H}_Y$ to consist of the L^2-functions on $\Omega^0(Y)$. Furthermore, if Y is the boundary of a Riemannian manifold X, there should be a function Ψ_X in $\mathcal{H}_Y$ whose value on $f \in \Omega^0(Y)$ is the path-integral

$$\Psi_X(f) = \int_{\Phi_{X,f}} \exp\{-S(\phi)\}\mathcal{D}\phi,$$

where the integral is over the space $\Phi_{X,f}$ of all $\phi : X \to \mathbb{R}$ such that $\phi|Y = f$. Calculating formally, we find that, up to multiplication by a constant, Ψ_X is given by

$$\Psi_X(f) = \exp\{-\frac{1}{2}\langle f, A_X f\rangle\},$$

where $A_X : \Omega^0(Y) \to \Omega^{d-1}(Y)$ is the positive self-adjoint operator which associates to f the normal derivative along Y of the unique solution ϕ of the Dirichlet problem $(\Delta_X + m^2)\phi = 0$ on X with boundary value f.

For any real topological vector space V we can construct a candidate $\mathrm{L}^2_q(V)$ for the Hilbert space $\mathrm{L}^2(V)$ whenever we have a positive-definite quadratic form $q : V \to \mathbb{R}$. It is obtained by completing the vector space F_V of functions on V of the form $v \mapsto p(v)\mathrm{e}^{-\frac{1}{2}q(v)}$, where $p : V \to \mathbb{R}$ is a polynomial function which is a sum of terms of the form

$$v \mapsto q(\xi_1, v) \ldots q(\xi_k, v)$$

with $\xi_1, \ldots, \xi_k \in V$. (Here q denotes the symmetric bilinear form such that $q(v, v) = q(v)$.)

There is a positive inner product on F_V defined using the standard formulae of Gaussian integration, normalized so that the "vacuum" function $\Psi_q = \mathrm{e}^{-\frac{1}{2}q(v)}$ has norm 1. The Hilbert space completion of F_V is $\mathrm{L}^2_q(V)$. (The elements of the completion, however, cannot be interpreted as functions on V.)

The Heisenberg group generated by affine translations in V and multiplications by functions of the form $v \mapsto \mathrm{e}^{\mathrm{i}q(\xi,v)}$ acts on $\mathrm{L}^2_q(V)$ by an irreducible unitary representation (cf. [S1]). The vector Ψ_q is a smooth vector for this representation, i.e. the map $\mathrm{Heis}(V \oplus V) \to \mathrm{L}^2_q(V)$ given by $g \mapsto g\Psi_q$ is smooth, and even extends to a holomorphic map from the complexification $\mathrm{Heis}(V_\mathbb{C} \oplus V_\mathbb{C})$ to $\mathrm{L}^2_q(V)$. Because Ψ_q is a smooth vector we can act on it with arbitrary monomials in the elements of the complexified Lie algebra of $\mathrm{Heis}(V \oplus V)$. A crucial fact for us will be that, as a representation of $\mathrm{Heis}(V_\mathbb{C} \oplus V_\mathbb{C})$, the Hilbert space $\mathrm{L}^2_q(V)$ is characterized up to canonical isomorphism by the existence of the smooth cyclic vector Ψ_q which is annihilated by the action of the elements $v \oplus \mathrm{i}v$ of the complexified Lie algebra for all $v \in V$.

The important question for us is how $\mathrm{L}^2_q(V)$ depends on q. This is answered by a classical theorem [G].

Theorem 2.1 *There is a canonical isomorphism* $\mathrm{L}^2_q(V) \to \mathrm{L}^2_{q'}(V)$*, compatible with the action of the Heisenberg group, precisely when* $q^{-1} \circ q'$*, regarded as a linear map* $V \to V$*, differs from the identity by a trace-class operator. The isomorphism is not quite an isometry: it multiplies all inner products by* $\det(q^{-1} \circ q')$.

For a compact oriented $(d-1)$-manifold Y we shall define the state-space $\mathcal{H}_Y$ of bosonic field theory essentially as $\mathrm{L}^2_q(V)$ with $V = \Omega^0(Y)$. We must choose a quadratic form q. If Y is contained in a closed manifold X which it divides into two pieces X_1 and X_2, then a particular choice q_X is provided by the operator[4]

$$A_X = \frac{1}{2}(A_{X_1} + A_{X_2}),$$

where $A_{X_i} : \Omega^0(Y) \to \Omega^{d-1}(Y)$ are the positive isomorphisms described above. The resulting space $\mathrm{L}^2_{q_X}$, which is an irreducible representation of the Heisenberg group $\mathcal{G}_Y = \mathrm{Heis}(\Sigma_Y)$ of the symplectic vector space Σ_Y, is spanned by elements

$$\alpha_1 \ldots, \alpha_k \Psi_X,$$

where $\alpha_i \in \Omega^{d-1}(Y)$, and Ψ_X is the Gaussian function defined by q_X. The inner product

$$\langle\, \alpha_1 \ldots \alpha_k \Psi_X,\ \alpha_{k+1} \ldots \alpha_m \Psi_X \,\rangle$$

is a sum of products of terms $\langle \alpha_i, B_X \alpha_j \rangle$, where $B_X = A_X^{-1}$ is a pseudo-differential operator of order -1 on Y. This means that

$$\langle \alpha_i, B_X \alpha_j \rangle = \int_{Y \times Y} \alpha_i(x) k_X(x,y) \alpha_j(y)\ dx\, dy,$$

where $k_X(x,y)$ is an integral kernel on $Y \times Y$ which is a smooth function outside the diagonal, but is singular on the diagonal. The germ of k_X along the diagonal is determined up to the addition of smooth functions by the complete symbol of the pseudo-differential operator $A_{X_1} + A_{X_2}$, which in turn can be calculated locally from the Riemannian metric of Y and its transverse normal derivatives in X of all orders — the formulae are given in [H]. This has two consequences.

(i) Up to rescaling its inner product, the Hilbert space $\mathrm{L}^2_{q_X}(\Omega^0(Y))$ depends only on the *germ* of X along Y.

(ii) We do not need to assume that Y is a boundary: any Y sits inside its own collar neighbourhood X, and divides it into two parts X_1 and X_2. We can still define isomorphisms $A_{X_i} : \Omega^0(Y) \to \Omega^{d-1}(Y)$ by $\phi \mapsto (*\mathrm{d}f_i)|Y$, where f_i is the solution of $(\Delta_X + m^2) f_i = 0$ on X_i with boundary values ϕ on Y and 0 at the outside end of X_i.

An important point arises here. Unless $A_{X_1}^{-1} \circ A_{X_2} - 1$ is of trace class, the functions Ψ_{X_1} and Ψ_{X_2} on $\Omega^0(Y)$ do *not* lie in the Hilbert space $\mathcal{H}_Y$: instead

[4]The operator $A_{X_1} + A_{X_2}$ is sometimes called the "Neumann jump operator".

they lie in distinct spaces of functions F_{X_i} generated by the Gaussian functions defined by A_{X_i}. These spaces should be completed to a pair of nuclear topological vector spaces which are in duality; but they are not isomorphic to each other, and not Hilbert spaces. For the purposes of this paper we shall not need to pursue this point. It will be enough to confine ourselves to the case when there is a nice reflection across Y, and $\mathcal{H}_Y$ is the Hilbert space we have just defined.

To construct bosonic field theory we still need to normalize the inner product of $\mathcal{H}_Y$. The positive operators A_{X_i} are of the kind[5] which have ζ-function determinants $\det_\zeta(A_{X_i})$, and these have the property

$$\det_\zeta(A') = \det(A^{-1}A')\det_\zeta(A),$$

when $A^{-1}A'$ is of the form $1 +$ (trace-class) and so has a "naive" determinant. We can therefore renormalize the inner product in $\mathrm{L}^2_{q_X}(\Omega^0(Y))$ so that

$$\|\Psi_{X_i}\|^2 = \det_\zeta(A_{X_i})^{-1/2},$$

and then $\mathcal{H}_Y$ is well-defined just by the choice of the germ of X along Y. I shall refer to [S5] for the proof that in this way we do indeed get a quantum field theory in the sense of my definition.

We can now see clearly to what extent $\mathcal{H}_Y$ is local in Y. We constructed $\mathcal{H}_Y$ by completing the symmetric algebra $S(\Omega^{d-1}(Y))$ using an inner product derived from the bilinear form defined by B_X on $\Omega^{d-1}(Y)$. Suppose now that $Y = Y_1 \cup Y_2$, as envisaged in the introduction. If we could decompose the completion of $\Omega^{d-1}(Y)$ as the sum of two orthogonal pieces associated to Y_1 and Y_2 — in the way that we can write $\mathrm{L}^2(Y) = \mathrm{L}^2(Y_1) \oplus \mathrm{L}^2(Y_2)$ — then we could factorize $\mathcal{H}_Y$ as a tensor product, just as

$$S(\mathrm{L}^2(Y)) \cong S(\mathrm{L}^2(Y_1)) \otimes S(\mathrm{L}^2(Y_2)).$$

But the operator B_X is not a differential operator, and so its distributional kernel k_X cannot be supported exactly on the diagonal in $Y \times Y$. We can, however, change it by an operator with a smooth kernel without changing $\mathcal{H}_Y$, to an operator whose kernel vanishes outside an ϵ-neighbourhood of the diagonal, for arbitrarily small ϵ. For the resulting inner product on $\Omega^{d-1}(Y)$ the completion nearly — but not quite — decomposes as the sum of the

[5] See [S5] for a fuller discussion of this.

completions of $\Omega_0^{d-1}(Y_1)$ and $\Omega_0^{d-1}(Y_2)$, where $\Omega_0^{d-1}(Y_i)$ denotes the elements whose support stays a distance $\geq \epsilon/2$ from the boundary of Y_i.

Let us restate this in the traditional language of "field operators". We write the action of $\alpha \in \Omega^{d-1}(Y)$ on $\mathcal{H}_Y$ as

$$\int \alpha(y)\phi_y \, \mathrm{d}y,$$

where $y \mapsto \phi_y$ is an "operator-valued distribution" on Y, and, similarly, we write the action of $f \in \Omega^0(Y)$ as

$$\int f(y) \, \dot{\phi}_y \, \mathrm{d}y,$$

where $y \mapsto \dot{\phi}_y$ is an operator-valued distributional $(d-1)$-form on Y. Then if k_B is the distributional kernel of an operator $B : \Omega^{d-1}(Y) \to \Omega^0(Y)$ suitably close to A_X^{-1} there is a corresponding Gaussian element $\Psi_B \in \mathcal{H}_B$ such that

$$\langle \, \Psi_B \, , \, \phi_y \phi_{y'} \, \Psi_B \rangle \; = \; k_B(y, y').$$

The "vectors" $\phi_{y_1}\phi_{y_2} \ldots \phi_{y_k}\Psi_B$ — which actually make sense only when "smeared" in $y_1, \ldots, y_k$ — then span $\mathcal{H}_Y$, and by choosing k_B with support in an ϵ-neighbourhood of the diagonal we can assume that $\phi_{y_1}\phi_{y_2} \ldots \phi_{y_k}\Psi_B$ is orthogonal to $\phi_{y'_1}\phi_{y'_2} \ldots \phi_{y'_m}\Psi_B$ if the distance from y_i to y'_j is greater than ϵ for all i, j. We should beware, however, that

$$\langle \, \Psi_B \, , \, \dot{\phi}_y \dot{\phi}_{y'} \; \Psi_B \rangle \; = \; \tilde{k}_B(y, y'),$$

where $\tilde{k}_B$ is the kernel of B^{-1}, and we cannot assume that k_B and $\tilde{k}_B$ are *simultaneously* localized in a neighbourhood of the diagonal. That would contradict the following version of the Reeh-Schlieder theorem. To state it, recall that $\mathcal{H}_Y$ has a canonical dense subspace $\check{\mathcal{H}}_Y$ which consists of the images of all operators $U_{X_1} : \mathcal{H}_{Y_1} \to \mathcal{H}_Y$, where X_1 is a half-collar of Y, regarded as a cobordism from Y_1 to Y. The vector Ψ_B chosen above will belong to $\mathcal{H}_Y$.

Proposition 2.2 *If U is an open subset of Y, and $\Psi \in \check{\mathcal{H}}$, then the Hilbert space $\mathcal{H}_Y$ is spanned by the orbit of Ψ under the subgroup $\mathcal{G}_U$ of the Heisenberg group $\mathcal{G}_Y$ consisting of elements with support in U.*

Equivalently, $\mathcal{H}_Y$ is spanned by the "vectors"

$$\psi_{y_1}\psi_{y_2}\ldots\psi_{y_k}\Psi$$

with $y_1,\ldots,y_k \in U$, where each ψ_{y_i} is either ϕ_{y_i} or $\dot{\phi}_{y_i}$.

Proof The essential point (see [S3], [S5]) is that the vector-valued distribution $y \mapsto \phi_y\Psi$ is the boundary-value of an actual smooth function $x \mapsto \phi_x\Psi = \Phi(x)$ which is defined in the interior of the cobordism X_1 and satisfies the classical field equation $(\Delta_X + m^2)\Phi = 0$. Suppose that there is some non-zero $\eta \in \mathcal{H}_Y$ such that

$$\langle \eta\ ,\ \psi_{y_1}\psi_{y_2}\ldots\psi_{y_k}\Psi\rangle = 0$$

for all $y_1,\ldots,y_k \in U$. Thinking of the left-hand side as a distributional "function" of y_k, first when $\psi_{y_k} = \phi_{y_k}$ and then when $\psi_{y_k} = \dot{\phi}_{y_k}$, the previous remark shows that for y_k in a small neighbourhood of any point of $U - \{y_1,\ldots,y_{k-1}\}$ the two distributions are the boundary-value and the boundary-normal-derivative of a solution of the classical field equation which is smooth in the interior of X_1. Because the classical equation is elliptic, a solution vanishes in all of X_1 if its Cauchy data vanish in any open subset of the boundary. It follows that the inner-product above vanishes for all $y_k \in Y - \{y_1,\ldots,y_{k-1}\}$. Applying the same argument to the other y_i, we find that the inner-product vanishes for all sets of distinct points $y_1,\ldots,y_k \in Y$, which is a contradiction. ♠

In the next section we shall describe the theory of Connes which will enable us in §5 to refine the preceding discussion to prove a precise locality theorem.

3 The Connes tensor product

In understanding the Connes tensor product the following example seems to me very helpful.

Let E be a smooth complex vector bundle equipped with a hermitian inner product on a compact manifold M. We can form a Hilbert space $\mathrm{L}^2(E)$ by completing the space of smooth 1/2-density sections of E — i.e. of smooth sections of $E \otimes \omega^{1/2}$, where ω is the volume line bundle on M — in its natural inner product. This Hilbert space is a module over the algebra

$\mathcal{A} = \mathrm{C}^\infty(M)$ of smooth functions on M. Now let F be another such vector bundle on M. We should like a notion of tensor product of $\mathcal{A}$-modules which produces $\mathrm{L}^2(E \otimes F)$ from the $\mathcal{A}$-modules $\mathrm{L}^2(E)$ and $\mathrm{L}^2(F)$.

There are two different reasons why there is no natural bilinear map $\mathrm{L}^2(E) \times \mathrm{L}^2(F) \to \mathrm{L}^2(E \otimes F)$. First, the product of two L^2 functions belongs to L^1 rather than to L^2. But, more fundamentally, the product of two 1/2-densities is a density and not a 1/2-density. Now the line bundle $\omega^{1/2}$ can be encoded as the $\mathcal{A}$-module $\mathcal{H} = \mathrm{L}^2(M)$. Furthermore, it is clear[6] that the space of $\mathcal{A}$-module homomorphisms from $\mathcal{H}$ to $\mathrm{L}^2(E)$ is the space $\mathrm{L}^\infty(E)$, and similarly for $\mathrm{L}^2(F)$. There is a natural linear map

$$\mathrm{L}^\infty(E) \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \mathrm{L}^\infty(F) \to \mathrm{L}^2(E \otimes F),$$

given by pointwise multiplication, where the tensor product on the left is the usual algebraic tensor product. The right-hand side can now be obtained by completing the left-hand side with respect to the inner product defined by

$$\langle e_1 \otimes \lambda_1 \otimes f_1,\ e_2 \otimes \lambda_2 \otimes f_2 \rangle \ = \ \langle \lambda_1,\ \langle e_1, e_2 \rangle \langle f_1, f_2 \rangle \lambda_2 \rangle,$$

where on the right $\langle e_1, e_2 \rangle$ and $\langle f_1, f_2 \rangle$ denote the pointwise inner products, which, being L^∞ functions on M, act as operators on $\mathcal{H}$. (To see that the right-hand side is indeed the completion of the left, it is enough to observe that the map preserves the inner product, and that its image is dense.)

The important thing to bear in mind is that whereas for the usual tensor product of modules we have

$$(\text{sections of } E) \ \otimes \ (\text{sections of } F) \ = \ (\text{sections of } E \otimes F),$$

for the Connes tensor product we have

$$(\text{sections of } E) \ \otimes_{\text{Connes}} \ (\text{sections of } F) \ = \ (\text{sections of } E \otimes F \otimes \omega^{-1/2}).$$

We can now give a general definition. We shall consider $*$-algebras $\mathcal{A}$, i.e. ones with a complex-antilinear involution $a \mapsto a^*$ such that $(ab)^* = b^* a^*$. The left- and right-$\mathcal{A}$-modules we shall consider will always be Hilbert spaces on which $\mathcal{A}$ acts by a $*$-homomorphism into the algebra of bounded operators.

Now suppose that $\mathcal{A}_L$ and $\mathcal{A}_R$ are $*$-algebras, and that we are given an $\mathcal{A}_L$-$\mathcal{A}_R$-bimodule $\mathcal{H}$ — i.e. $\mathcal{H}$ is a left $\mathcal{A}_L$-module and a right $\mathcal{A}_R$-module,

[6] because the operators on $\mathcal{H}$ which commute with all multiplications by smooth functions are the multiplications by L^∞ functions, and (as E and F are summands of trivial bundles) the $\mathcal{A}$-modules $\mathrm{L}^2(E)$ and $\mathrm{L}^2(F)$ are summands in finite sums of copies of $\mathcal{H}$.

and the left- and right-actions commute. Thus $\mathcal{A}_R$ is contained in $\mathrm{End}_{\mathcal{A}_L}(\mathcal{H})$, the algebra of bounded endomorphisms of $\mathcal{H}$ which commute with $\mathcal{A}_L$. We shall assume that this inclusion is dense for the topology of pointwise convergence, which, by von Neumann's double-commutant theorem, is equivalent to assuming that the algebras $\mathrm{End}_{\mathcal{A}_L}(\mathcal{H})$ and $\mathrm{End}_{\mathcal{A}_R}(\mathcal{H})$ are each other's commutants in the algebra $\mathrm{End}(\mathcal{H})$.

We can now define the purely algebraic tensor product

$$(\text{right } \mathcal{A}_R\text{-modules}) \ \times \ (\text{left } \mathcal{A}_L\text{-modules}) \quad \longrightarrow \quad (\text{vector spaces})$$

by

$$(\ \mathcal{E}\ ,\ \mathcal{F}\ ) \ \mapsto \ \ \mathrm{Hom}_{\mathcal{A}_R}(\mathcal{H};\mathcal{E}) \otimes_{\mathcal{A}_L} \mathcal{H} \otimes_{\mathcal{A}_R} \mathrm{Hom}_{\mathcal{A}_L}(\mathcal{H};\mathcal{F}).$$

(Notice here that the *left* action of $\mathcal{A}_L$ on $\mathcal{H}$ makes $\mathrm{Hom}_{\mathcal{A}_R}(\mathcal{H};\mathcal{E})$ into a *right* $\mathcal{A}_L$-module, and similarly for $\mathrm{Hom}_{\mathcal{A}_L}(\mathcal{H};\mathcal{F})$.) This algebraic tensor product can be completed as a Hilbert space by giving it the inner product

$$\langle e_1 \otimes \lambda_1 \otimes f_1,\ e_2 \otimes \lambda_2 \otimes f_2 \rangle \ = \ \langle \lambda_1,\ (e_1^* e_2)\lambda_2(f_1^* f_2)\rangle,$$

where $e_1^* e_2 \in \mathrm{End}_{\mathcal{A}_R}(\mathcal{H})$ and $f_1^* f_2 \in \mathrm{End}_{\mathcal{A}_L}(\mathcal{H})$ are thought of as lying in completions of $\mathcal{A}_L$ and $\mathcal{A}_R$ respectively which act on the left and right of the bimodule $\mathcal{H}$. It seems reasonable to denote this completed tensor product by

$$\mathcal{E} \otimes_{\mathcal{H}} \mathcal{F}.$$

It may perhaps be unhelpful to write the operators $e_1^* e_2$ and $f_1^* f_2$ on opposite sides of λ_2 in the formula for the inner product, as they are, after all, just endomorphisms of $\mathcal{H}$. But the crucial thing is that they must *commute* for the formula for the inner product to be well-defined and positive. In other words, though we need no properties of the algebras $\mathcal{A}_L$ and $\mathcal{A}_R$, the one "von-Neumann-like" feature we need is this commuting of $\mathrm{End}_{\mathcal{A}_L}(\mathcal{H})$ and $\mathrm{End}_{\mathcal{A}_R}(\mathcal{H})$.

In the applications — and automatically in Connes's theory — the gluing bimodule $\mathcal{H}$ will contain vectors Ω such that $a \mapsto a\Omega$ and $a \mapsto \Omega a$ are dense embeddings $\mathcal{A}_L \to \mathcal{H}$ and $\mathcal{A}_R \to \mathcal{H}$, and $a \mapsto \tilde{a}$, where $a\Omega = \Omega\tilde{a}$, is a densely defined "unbounded" homomorphism of algebras from $\mathcal{A}_L$ to $\mathcal{A}_R$.

In Connes's work, the algebras are always assumed to be von Neumann algebras, and then the bimodule $\mathcal{H}$ is canonically determined by the algebra.

But, conversely, giving $\mathcal{H}$ defines a specific way of completing the algebra to get a von Neumann algebra, and I have found it clearer to leave the algebras uncompleted. Indeed, although I have expressed the preceding discussion in the language of algebras, in our applications no algebras will actually enter. Instead we shall be concerned with unitary representations of groups, say $\mathcal{G}_L$ and $\mathcal{G}_R$. We need a Hilbert space $\mathcal{H}$ with a unitary action of $\mathcal{G}_L$ on the left, and a commuting unitary action of $\mathcal{G}_R$ on the right. (I shall call this structure a "bi-representation".) The crucial point is that $\mathrm{End}_{\mathcal{G}_L}(\mathcal{H})$ and $\mathrm{End}_{\mathcal{G}_R}(\mathcal{H})$ commute. We can then use $\mathcal{H}$ to define a tensor product $\mathcal{E} \otimes_{\mathcal{H}} \mathcal{F}$ for Hilbert spaces $\mathcal{E}$ and $\mathcal{F}$ with right- and left-actions of $\mathcal{G}_L$ and $\mathcal{G}_R$ respectively.

Example

Let $\mathcal{G}_L = \mathcal{G}_R = \mathcal{G}$ be any group, and let $\{\mathcal{P}_\alpha\}$ be any family of irreducible unitary representations of $\mathcal{G}$. The bi-representation $\mathcal{H} = \bigoplus \mathcal{P}_\alpha \otimes \mathcal{P}_\alpha^*$ of $\mathcal{G}$ has the property that $\mathrm{End}_{\mathcal{G}_L}(\mathcal{H})$ and $\mathrm{End}_{\mathcal{G}_R}(\mathcal{H})$ commute, and so we can use it as a gluing bimodule to form the Connes tensor product $\mathcal{E} \otimes_{\mathcal{H}} \mathcal{F}$ of any right- and left- unitary representations $\mathcal{E}$ and $\mathcal{F}$ of $\mathcal{G}$.

Let $\mathcal{F}_\alpha = \mathrm{Hom}_{\mathcal{G}_L}(\mathcal{P}_\alpha; \mathcal{F})$. This has an inner-product defined by

$$f_1^* f_2 \;=\; \langle f_1,\; f_2\,\rangle\, 1_{\mathcal{P}_\alpha},$$

and for any choice of a unit vector $\xi \in \mathcal{P}_\alpha$ it can be identified with a sub-Hilbert-space of $\mathcal{F}$ by $f \mapsto f(\xi)$. (In fact, the canonical embedding

$$\bigoplus \mathcal{P}_\alpha \otimes \mathcal{F}_\alpha \to \mathcal{F}$$

is an isometry.) We define $\mathcal{E}_\alpha = \mathrm{Hom}_{\mathcal{G}_R}(\mathcal{P}_\alpha^*; \mathcal{E})$ similarly. Then

Proposition 3.1 *We have a canonical isomorphism of Hilbert spaces*

$$\mathcal{E} \otimes_{\mathcal{H}} \mathcal{F} \;\cong\; \bigoplus \mathcal{E}_\alpha \otimes \mathcal{F}_\alpha.$$

Proof We can embed $\mathcal{P}_\alpha \otimes \mathcal{F}_\alpha$ in $\check{\mathcal{F}} = \mathrm{Hom}_{\mathcal{G}_L}(\mathcal{H}; \mathcal{F})$ by $\xi \otimes f \mapsto f_\xi$, where

$$f_\xi\Big(\sum \xi_\beta \otimes \eta_\eta\Big) \;=\; \langle \eta_\alpha, \xi\rangle\; \xi_\alpha.$$

Similarly $\mathcal{E}_\alpha \otimes \mathcal{P}_\alpha^* \hookrightarrow \check{\mathcal{E}} = \mathrm{Hom}_{G_R}(\mathcal{H}; \mathcal{E})$.

To prove 3.1 we have only to check that the composition

$$\mathcal{E}_\alpha \otimes \mathcal{P}_\alpha^* \otimes \mathcal{H} \otimes \mathcal{P}_\alpha \otimes \mathcal{F}_\alpha \to \check{\mathcal{E}} \otimes \mathcal{H} \otimes \check{\mathcal{F}} \to \mathcal{E} \otimes_{\mathcal{H}} \mathcal{F}$$

is an isometry, which is straightforward. ♠

4 Loop groups and Tomita-Takesaki theory

We must begin by recalling the main facts about representations of loop groups.

Let us fix a compact Lie group G, and for any compact oriented 1-manifold S let G_S denote the group of piecewise-smooth maps $S \to G$. This is of a finite product of copies of the usual loop group $\mathcal{L}G$, except that we have chosen to work with piecewise-smooth loops rather than smooth ones, which makes no difference to the theory. A choice of a "level" k — which, properly speaking, is an element of the cohomology group $H^4(BG;\mathbb{Z})$ — defines a central extension of G_S by the circle-group $\mathbb{T}$. We are interested in projective unitary representations of G_S of the chosen level k, i.e. representations of the central extension in which the circle $\mathbb{T}$ acts by scalar multiplication. Among these we restrict ourselves to the class of so-called "positive energy" representations. The basic results about representations of this class are

(i) any representation is a sum of irreducible representations, and, up to isomorphism, there are only finitely many different irreducible representations;

(ii) there is a *canonical* projective unitary action of the group Diff(S) of orientation-preserving diffeomorphisms of S on each representation, intertwining in the natural way with the action of G_S;

(iii) there is a *canonical* dense subspace in each representation on which the action of G_S extends to a holomorphic action of the complexified group $G_S^{\mathbb{C}}$;

(iv) if S is the boundary of a connected Riemann surface Σ then the group $G_\Sigma^{\mathbb{C}}$ of holomorphic maps from Σ to $G^{\mathbb{C}}$ is a subgroup of $G_S^{\mathbb{C}}$, and the central extension of $G_S^{\mathbb{C}}$ is *canonically* split over $G_\Sigma^{\mathbb{C}}$, which can therefore be regarded as a subgroup of the central extension;

(v) if S is a circle then among the irreducible representations of G_S there is precisely one, called the *basic* representation, which contains a ray fixed by $G_D^{\mathbb{C}}$, where D is a disc with conformal structure whose boundary is S.

This is all described and proved in [PS], among other places.

A good example of the use of the Connes tensor product in the representation theory of loop groups is explained in the thesis of H. Postuma [P]. If Σ is a connected Riemann surface with boundary which is a cobordism from a 1-manifold S_0 to a 1-manifold S_1, and we consider the group $G^{\mathbb{C}}_{\Sigma}$ of holomorphic maps $\Sigma \to G^{\mathbb{C}}$ as a subgroup of $G^{\mathbb{C}}_{S_0} \times G^{\mathbb{C}}_{S_1}$, then it is the graph of a densely-defined homomorphism α_{Σ} from $G^{\mathbb{C}}_{S_0}$ to $G^{\mathbb{C}}_{S_1}$. Using α_{Σ} we can associate to a unitary representation $\mathcal{F}$ of G_{S_1} a highly non-unitary action of a dense subgroup of $G^{\mathbb{C}}_{S_0}$ on a dense subspace of $\mathcal{F}$. The surprising thing is that this action automatically comes from a canonical *unitary* representation of G_{S_0}, which one can call $\alpha^*_{\Sigma}\mathcal{F}$. The reason is that there is a canonical bi-representation $\mathcal{H}_{\Sigma}$ of (G_{S_0}, G_{S_1}) associated to Σ which allows us to define

$$\alpha^*_{\Sigma}\mathcal{F} \;=\; \mathcal{H}_{\Sigma} \otimes_{\mathcal{H}_{S_1}} \mathcal{F},$$

where $\mathcal{H}_{S_1}$ is an appropriate (G_{S_1}, G_{S_1}) bi-representation.

In fact the isomorphism-class of the bi-representation $\mathcal{H}_{\Sigma}$, and hence the equivalence-class of the functor

$$\alpha^*_{\Sigma} : \{\text{unitary representations of } G_{S_1}\} \to \{\text{unitary representations of } G_{S_0}\}$$

is independent of the complex structure of Σ.

One important example of this functor is when Σ is an annulus. Then the functor is equivalent to the identity, i.e. for any representation $\mathcal{H}$ and any annulus Σ there is a contraction operator $U_{\Sigma} : \mathcal{H} \to \mathcal{H}$ such that

$$\alpha_{\Sigma}(\gamma) \circ U_{\Sigma} = U_{\Sigma} \circ \gamma$$

for every $\gamma \in G^{\mathbb{C}}_{S_0}$. This is an extension of the property (ii) above of positive energy energy representations: it is explained in [S3] how the semigroup of annuli Σ, with a chosen diffeomorphism between their incoming and outgoing boundary circles, can be regarded as part of the (non-existent) complexification of the group of diffeomorphisms of the circle, a diffeomorphism being regarded as corresponding to an annulus which has shrunk to a circle. We shall need below to know that U_{Σ} is defined even in the degenerate case when Σ is the region of the complex plane bounded by two simple closed curves, one contained in the closed disc bounded by the other, but coinciding with it for part of its length.

The second important example is when Σ is a pair of pants, with two incoming boundary circles and one outgoing. Then α^*_{Σ} is an operation called *fusion* which combines two representations of a loop group to produce a representation of the same level.

We shall give a simple proof of these facts for abelian G in section 6, but they are not directly relevant to the theme of this paper.

Our interest here is in the group G_I of smooth maps $I \to G$, where I is an oriented 1-manifold *with boundary*, and the maps are required to map the boundary points to the identity element of G. One fact about groups of this type is that if we decompose a circle S into intervals $I_1, \ldots, I_k$ which meet only at their end-points, and think of $G_{I_1} \times \ldots \times G_{I_k}$ as a subgroup of G_S, then a positive energy irreducible representation of G_S remains irreducible when restricted to $G_{I_1} \times \ldots \times G_{I_k}$. This is because the product group is dense in G_S for the coarsest topology for which the representation of G_S is continuous: a proof can be found in [W], but the result follows easily from the particular case when $G = \mathbb{T}$, for which the proof is given in section 5 below.

The same density argument proves that distinct irreducible representations of G_S remain non-isomorphic when restricted to the product.

The situation is completely different when we restrict an irreducible representation $\mathcal{H}$ of G_S to a subgroup G_I, where I is a proper subinterval of S. As a representation of G_I the representation $\mathcal{H}$ is of *type III*, i.e. it contains no irreducible subrepresentation at all, and any non-zero subrepresentation is isomorphic to $\mathcal{H}$ itself. Although this is a fundamental part of the picture, we shall not, strictly speaking, make use of it in what follows.

In the following discussion it will be convenient to denote the concatenation of two oriented intervals I, I' by $I \circ I'$. (For this to make sense, we assume that all our intervals are equipped with germs of parametrizations at their ends.) I shall also write $I\circ$ for the circle obtained by attaching the ends of I to each other.

Let $\mathcal{H}$ denote the bi-representation of $G_I \times G_{I'}$ which is the restriction of the basic representation of the loop group $G_{I \circ I' \circ}$. Wasserman[W] has proved — and in any case we shall give a simple proof for abelian G below — that $\mathrm{End}_{G_I}(\mathcal{H})$ and $\mathrm{End}_{G_{I'}}(\mathcal{H})$ commute, so that we can use $\mathcal{H}$ for constructing Connes tensor products.

Suppose now that we have an interval $I_L \circ I \circ I_R$ partitioned into three subintervals. Let $\mathcal{R}$ denote the category of projective unitary representations of $G_{I_L} \times G_I \times G_{I_R}$ which are the restrictions of positive energy representations of G_S, where S is the circle $I_L \circ I \circ I_R \circ$. We shall think of the objects of $\mathcal{R}$ as bi-representations of $G_{I_L} \times G_{I_R}$ with an additional action of G_I. If we first ignore the G_I-action, and regard the objects simply as bi-representations, then there is a tensoring operation $(\mathcal{E}, \mathcal{F}) \mapsto \mathcal{E} \otimes_{\mathcal{H}} \mathcal{F}$, with $\mathcal{H}$ as in the previous paragraph. The bi-representation $\mathcal{E} \otimes_{\mathcal{H}} \mathcal{F}$ has an additional intertwining action of the group $G_{I \circ I}$, which can be identified with G_I by choosing a diffeomorphism between I and $I \circ I$. We have

Proposition 4.1 *When composed with a diffeomorphism $I \to I \circ I$ the Connes tensor product defines a functor*

$$\mathcal{R} \times \mathcal{R} \to \mathcal{R}.$$

Proof A representation $\mathcal{E}$ of a loop group G_S is of positive energy if and only if it is generated by the subspace $\mathcal{E}_0$ of fixed vectors of the subgroup $G^{\mathbb{C}}_{D,z}$ consisting of holomorphic maps $D \to G^{\mathbb{C}}$ which take the value 1 at a point z in the interior of the disc D. The space $\mathcal{E}_0$ is a representation of the finite dimensional group G. As already mentioned, the action of $\mathrm{Diff}(S)$ on $\mathcal{E}$ extends to an action of the semigroup of annuli Σ by operators U_Σ. We need to know the following strengthening of this fact, which applies when D is a disc bounded by S and z is a point in the interior of the annulus Σ obtained from D by removing the interior of a smaller disc D_0.

Lemma 4.2 *In the situation just described, if $\mathcal{H}_{S_0}$ is the basic representation of the loop group G_{S_0}, where S_0 is the boundary of D_0, there is a continuous bilinear map*

$$U_{\Sigma,z} : \mathcal{E}_0 \times \mathcal{H}_{S_0} \to \mathcal{E}$$

which intertwines with the action of the group $G^{\mathbb{C}}_{\Sigma}$, which acts on $\mathcal{E}_0$ by evaluation at z, and on $\mathcal{H}_{S_0}$ and $\mathcal{E}$ by restriction to S_0 and S.

Furthermore, the same holds in the degenerate case when the circles S_0 and S_1 have an interval in common, providing z is in the interior of the region between them.

This lemma is due, essentially, to Tsuchiya and Kanie [TK], and is well known in conformal field theory; a proof will be given in [S5], but we shall not include it here, as it is not our main concern in this paper.

Granting the lemma, the proof of 4.1 is very simple: we have only to show that the representation $\mathcal{E} \otimes_{\mathcal{H}} \mathcal{F}$ of $G_{I \circ I}$ is generated by vectors fixed by $G^{\mathbb{C}}_{D,z}$, where D is a disc with boundary $S = I_L \circ I \circ I \circ I_R \circ$, and z is a point in the interior of D. Let $s_0 \in S$ be the point at which the two copies of I are joined together, and let $s_\infty \in S$ be the point of intersection of I_R and I_L. Let I'_L be a smooth curve inside S from s_∞ to s_0 which fits together with $I \circ I_R$ to form a smooth simple closed curve S_L in D, and let I'_R be a similar curve from s_0 to s_∞ forming a smooth curve $S_R = I_L \circ I \circ I'_R \circ$. Finally, let S_0 be the smooth closed curve $I'_L \circ I'_R \circ$. We can choose points z_L, z_R in the regions between S_0 and S_L and between S_0 and S_R respectively. We can assume that $\mathcal{E}$ is a representation of G_{S_L} in which $\mathcal{E}_0$ is the subspace fixed by G_{D_L, z_L}, and *matatis mutandis* for $\mathcal{F}$. The lemma tells us that for any $\xi \in \mathcal{E}_0$ there is a continuous linear map $e_\xi : \mathcal{H}_{S_0} \to \mathcal{E}$ which takes the vacuum vector $\Omega \in \mathcal{H}_{S_0}$ to ξ. Similarly for any $\eta \in \mathcal{F}_0$ we have $f_\eta : \mathcal{H}_{S_0} \to \mathcal{F}$ taking Ω to η. As a representation of $G_{I \circ I}$ the space $\mathcal{E} \otimes_{\mathcal{H}_{S_0}} \mathcal{F}$ is plainly spanned by the vectors

$$e_\xi \otimes \Omega \otimes f_\eta.$$

To go further we need a slight extension of the usual loop-group theory. Positive energy representations of $\mathcal{L}G$ are characterized by the fact that they are generated by "lowest-weight vectors", i.e. vectors which are left fixed[7] by the subgroup $G^{\mathbb{C}}_{D,0}$ of $\mathcal{L}G^{\mathbb{C}}$ consisting of boundary-values of maps $\gamma : D \to G^{\mathbb{C}}$ which are holomorphic in the interior of the disc $D = \{z \in \mathbb{C} : |z| \leq 1\}$ and satisfy[8] $\gamma(0) = 1$. If S is any smooth simple closed curve in $\mathbb{C}$ then positive energy representations of G_S have the same property when D is replaced by the region D_S bounded by the curve S in $\mathbb{C}$, for there is a smooth bijection $D \to D_S$, holomorphic in the interior of D, which maps the standard circle diffeomorphically to S. But if the simple closed curve S is only *piecewise* smooth then the bijection $f : D \to D_S$ given by the Riemann mapping theorem will *not* map S^1 smoothly to S — think of the map $z \mapsto (z-1)^a$, with $0 < a < 2$, which maps the circle to a curve which is smooth except for one corner where the tangent vector changes direction by the angle $(a-1)\pi$. There is a generalization of the notion of "positive energy" associated to the curve S, obtained by modifying the definition of a lowest-weight vector to

[7]This makes sense, because the central extension of $\mathcal{L}G^{\mathbb{C}}$ is canonically split over $G^{\mathbb{C}}_D$, which can therefore be regarded as a subgroup of the extension.

[8]Among the level k irreducible representations there is precisely one, called the *basic representation*, which contains a ray fixed by the larger group $G^{\mathbb{C}}_D$ of boundary-values of holomorphic maps γ with no condition on $\gamma(0)$.

mean one fixed by the group $G^{\mathbb{C}}_{D_S,z}$ of boundary-values of maps holomorphic in the interior of D_S (and vanishing at a chosen point z in the interior). The resulting class of representations of G_S is different from the class of positive energy representations defined by the structure of S as an abstract piecewise-smooth manifold: it "sees" the corners in S.

To show that the corners actually are detected, it is as usual enough to consider the case $G = \mathbb{T}$ at level 1, when the basic irreducible representation of $\mathcal{L}\mathbb{T}$ can be realized on the fermionic Fock space constructed from the Hilbert space $\mathcal{K}$ of L^2 half-densities on the circle by choosing a polarization $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ in the sense described in [PS]. Introducing corners in the circle amounts to reparametrizing it by a homeomorphism ϕ which is smooth except at the corners, where the derivative tends to infinity. This changes the orthogonal projection $\Pi : \mathcal{K} \to \mathcal{K}_+$ to $T_\phi \Pi T_\phi^{-1}$, where T_ϕ is the unitary action of ϕ on $\mathcal{K}$. The change in Π is described by the formula (6.8.3) of [PS] — given there, of course, for a smooth reparametrization. For a corner with angle α, the reparametrization looks locally like $x \mapsto x^{1-\alpha/\pi}$, and we find from the formula that the projection changes by an integral operator defined by a kernel function whose values on the diagonal behave like $1/|x|$ near the corners. This operator is therefore not of trace-class, which means that the new representation is inequivalent to the old one.

For a pair of oriented intervals I and I', and a choice of angle α in the range $-\pi < \alpha < \pi$, let us now consider the unitary representation[9] $\mathcal{H}_\alpha$ of $G_I \times G_{I'}$ obtained in the following way. We choose a positively-oriented simple closed curve S_α in $\mathbb{C}$ which is smooth except at one point s_0 at which the direction of tangent vector makes a jump to the *right* through the angle α. Then we identify I and I' smoothly with the intervals of S_α from s_∞ to s_0 and s_0 to s_∞ respectively, where s_∞ is an arbitrary point of S_α with $s_0 \neq s_\infty$. This identifies $G_I \times G_{I'}$ with a subgroup of G_{S_α}, and we take $\mathcal{H}_\alpha$ to be the restriction of the basic representation of G_{S_α} at the level we are interested in. We have made some choices here, but the discussion above tells us that $\mathcal{H}_\alpha$ is well-defined up to isomorphism. It contains a vector Ω_α which is left fixed by the group of pairs (γ, γ') in $G^{\mathbb{C}}_I \times G^{\mathbb{C}}_{I'}$ which fit together to form the boundary values of a holomorphic map from D_α to $G^{\mathbb{C}}$, where D_α is the region of the

[9] We must be careful not to think of $\mathcal{H}_\alpha$ as the state space associated to S_α by a 2-dimensional field theory. The state space for S_α would not be a Hilbert space: its dual would be the space associated to $S_{-\alpha}$. The inner product in $\mathcal{H}_\alpha$ is connected with path-integrals on the singular Riemann surface got by doubling D_α.

complex plane bounded by the curve S_α. If we write the actions of G_I and $G_{I'}$ on $\mathcal{H}_\alpha$ as left- and right-actions to keep them separate — this may or may not be a good idea — then we shall have the commutation property

$$\gamma\Omega_\alpha = \Omega_\alpha(\gamma')^{-1}$$

for such pairs (γ, γ'). A theorem of Reeh-Schlieder type — with exactly the same proof as that given in the preceding section — shows that the vector Ω_α generates $\mathcal{H}_\alpha$ under the action of either G_I or $G_{I'}$.

When $\alpha = 0$ we shall write $\mathcal{H}$ for $\mathcal{H}_\alpha$. In this case we know that $\mathrm{End}_{G_I}(\mathcal{H})$ and $\mathrm{End}_{G_{I'}}(\mathcal{H})$ commute: this is proved by Wasserman [W], and in any case we shall give a simple proof for abelian G below. We can therefore use $\mathcal{H}$ to define a Connes tensor product.

Proposition 4.1 *For any angles $\alpha, \beta > 0$ with $\alpha + \beta < \pi$ we have*

$$\mathcal{H}_\alpha \otimes_{\mathcal{H}} \mathcal{H}_\beta \cong \mathcal{H}_{\alpha+\beta}.$$

Proof

Let us assume, for the sake of clarity, that α and β are positive, and that $\alpha + \beta < \pi$. Let us assume that the regions D_α and D_β are contained in $D = D_0$, with the points s_0, s_∞ of the three curves coinciding, and the right-hand side I' of S_α coinciding with the right-hand side I' of S, and the left-hand side I of S_β coinciding with the left-hand side I of S. Thus we have a picture with four distinct arcs joining s_0 to s_∞, which we can call $I, I_\alpha, I'_\beta, I'$. The two inside arcs I_α and I'_β fit together to bound a region which we can take to be $D_{\alpha+\beta}$. By restricting elements of the group $\mathcal{G}$ of holomorphic maps $D \to G^{\mathbb{C}}$ to any of the arcs we obtain a dense subgroup of any of $G_I, G_{I_\alpha}, \ldots$.

The first step is to see that there is an isometry $e_\alpha : \mathcal{H} \to \mathcal{H}_\alpha$ of Hilbert spaces which commutes with the action of $G_{I'}$ and takes Ω to Ω_α. This is true because the orbits of these cyclic vectors span the respective Hilbert spaces, and so it is enough to check that

$$\langle \Omega \, , \, \gamma\Omega \rangle \; = \; \langle \Omega_\alpha \, , \, \gamma\Omega_\alpha \rangle$$

for all $\gamma \in G_{I'}$. When G is the circle group $\mathbb{T}$, or a torus, this is a simple calculation using the theory of Heisenberg groups, and I shall postpone it to §6. By the standard devissage techniques of loop group theory (cf. [PS] Chap. 10) it then follows for the basic representation of U_n, and then for any G for any level that can be pulled back from U_n by a homomorphism $G \to U_n$. I shall not pursue the discussion here. Using the commutation relation for the cyclic vectors given above, we find that

$$\gamma e_\alpha = e_\alpha \gamma$$

where γ denotes on the left and right the element of $G_{I_\alpha}^{\mathbb{C}}$ or $G_I^{\mathbb{C}}$ obtained by restricting an element γ of the group $\mathcal{G}$ of holomorphic maps from the disc.

Of course there is a similar homomorphism $e_\beta : \mathcal{H} \to \mathcal{H}_\beta$ intertwining with G_I.

Using these elements in the definition of the Connes tensor product , we see that $\mathcal{H}_\alpha \otimes_{\mathcal{H}} \mathcal{H}_\beta$ is spanned by the elements

$$\gamma e_\alpha \otimes \Omega \otimes e_\beta \gamma',$$

where $\gamma \in G_{I_\alpha}^{\mathbb{C}}$ and $\gamma' \in G_{I'_\beta}^{\mathbb{C}}$ are each the restrictions of (different) elements of $\mathcal{G}$. Thus it is a cyclic representation of $G_{I_\alpha} \times G_{I'_\beta}$ with the cyclic vector $e_\alpha \otimes \Omega \otimes e_\beta$. Furthermore, the cyclic vector is fixed by the pairs (γ, γ') where both γ and γ' are restrictions of the same element of $\mathcal{G}$. Because of the density of $G_I \times G_{I'}$ in $G_{S_{\alpha+\beta}}$ already discussed, this is enough to characterize the left-hand side of 4.1 as the representation $\mathcal{H}_{\alpha+\beta}$.

5 The gluing theorem for free massive bosons

We shall begin by associating a symplectic vector space Σ_Y to each compact oriented Riemannian $(d-1)$-manifold Y with boundary. We shall assume that Y is a submanifold of a Riemannian d-manifold X with boundary, and that the boundary of Y is contained in the boundary of X, which Y meets transversally. As we shall also assume that the neighbourhood of Y in X admits a "sufficiently isometric" reflection across Y, we may as well assume that Y meets the boundary of X at right angles. We allow the boundary of X to have corners away from Y. We shall assume that Y divides X into two pieces X_1 and X_2.

Let $\Omega_0^0(Y)$ and $\Omega_0^{d-1}(Y)$ denote the smooth functions and forms on Y which vanish on ∂Y. These spaces are in duality, and we can define the symplectic vector space $\Sigma_Y = \Omega_0^0(Y) \oplus \Omega_0^{d-1}(Y)$.

For each $f \in \Omega^0_0(Y)$ we can solve the Dirichlet problem for the operator $\Delta_X + m^2$ first in X_1 and then in X_2, taking as boundary value the function f extended by zero to the remainder of ∂X_1 or ∂X_2. Assigning to f the normal derivatives of the solution gives us positive operators

$$A_{X_i} : \Omega^0_0(Y) \to \Omega^{d-1}_0(Y),$$

just as in §2, and we can use the Neumann jump operator

$$A_X = \frac{1}{2}(A_{X_1} + A_{X_2})$$

as we did there to define the Hilbert space $\mathcal{H}_Y = \mathrm{L}^2(\Omega^0_0(Y))$. It depends only on the germ of X along Y, and is an irreducible representation of the Heisenberg group formed from Σ_Y. It contains a vector Ψ_B for each $B : \Omega^{d-1}_0(Y) \to \Omega^0(Y)$ sufficiently close to A_X^{-1}.

Now suppose that Y is divided into two submanifolds Y_1 and Y_2 by a codimension 1 submanifold Z contained in the interior of Y. Let $\mathcal{G}_{Y_1}$ and $\mathcal{G}_{Y_2}$ denote the subgroups of $\mathrm{Heis}(\Sigma_Y)$ formed by elements with supports in the interiors of Y_1 and Y_2 respectively. As the symplectic structure of Σ_Y is completely local , it is clear that every element of $\mathcal{G}_{Y_1}$ commutes with every element of $\mathcal{G}_{Y_2}$. We shall need two basic lemmas about this situation.

Lemma 5.1 *The Hilbert space $\mathcal{H}_Y$ is irreducible as a representation of $\mathcal{G}_{Y_1} \times \mathcal{G}_{Y_2}$.*

Lemma 5.2 *The algebras $End_{\mathcal{G}_{Y_1}}(\mathcal{H}_Y)$ and $End_{\mathcal{G}_{Y_2}}(\mathcal{H}_Y)$ commute.*

The proofs of these will be postponed to section 6.

The next step is to introduce a tubular neighbourhood U of Z in Y. I shall assume it is a compact smooth manifold with boundary, and is precisely the ϵ-neighbourhood of Z in Y. Let us write $U_1 = U \cap Y_1$ and $U_2 = Y \cap Y_2$; and let $\mathcal{G}_{U_1}$ and $\mathcal{G}_{U_2}$ be the subgroups of $\mathrm{Heis}(\Sigma_Y)$ consisting of elements with supports in U_1 and U_2. We can apply Lemma 5.2 with Y replaced by U to see that $\mathcal{H}_U$ is a representation of $\mathcal{G}_{U_1} \times \mathcal{G}_{U_2}$ — a "bimodule" — which we can use to form the Connes tensor product of a representation of $\mathcal{G}_{U_1}$ and a representation of $\mathcal{G}_{U_2}$. The representations we want to use are $\mathcal{H}_{Y_1^+}$ and $\mathcal{H}_{Y_2^+}$, where $Y_1^+ = Y_1 \cup U_2$ and $Y_2^+ = Y_2 \cup U_1$.

Proposition 5.3 *There is a canonical isomorphism*

$$\mathcal{H}_{Y_1^+} \otimes_{\mathcal{H}_U} \mathcal{H}_{Y_2^+} \to \mathcal{H}_Y.$$

Proof

6 Heisenberg groups

Let V be a real locally convex topological vector space with a continuous nondegenerate skew bilinear form $S : V \times V \to \mathbb{R}$.

The Heisenberg group $\mathrm{Heis}(V)$ is the central extension of the additive group V by the circle $\mathbb{T}$ with cocycle $\mathrm{e}^{-\mathrm{i}S}$, i.e. $\mathrm{Heis}(V)$ contains elements U_v for $v \in V$ such that

$$U_{v_1}U_{v_2} = \mathrm{e}^{-\mathrm{i}S(v_1,v_2)}U_{v_1+v_2}.$$

Let $\mathcal{H}$ be a unitary representation of $\mathrm{Heis}(V)$ on which $\mathbb{T}$ acts by multiplication. We write $U_v = \exp \mathrm{i}A_v$, where A_v is an unbounded self-adjoint operator in $\mathcal{H}$.

Definition 6.1 *A vector $\xi \in \mathcal{H}$ is* Gaussian *(of type q) if*

$$\langle \xi, U_v\xi \rangle = \mathrm{e}^{-\frac{1}{2}q(v)},$$

where $q : v \to \mathbb{R}$ is a positive quadratic form.

It will appear presently that a Gaussian vector ξ is necessarily smooth. Assuming that, we can define a real bilinear form $F : V \times V \to \mathbb{C}$ by

$$F(v_1, v_2) \;=\; \langle A_{v_1}\xi,\, A_{v_2}\xi \rangle.$$

We have

$$F(v_1, v_2) \;=\; B(v_1, v_2) \;+\; \mathrm{i}S(v_1, v_2),$$

where $B : V \times V \to \mathbb{R}$ is the positive bilinear symmetric form corresponding to q, and S is the skew form of V.

Lemma 6.2

$$|S(v_1, v_2)| \;\le\; q(v_1)^{\frac{1}{2}}q(v_2)^{\frac{1}{2}},$$

with equality if and only if $(A_{v_1} + \mathrm{i}\lambda A_{v_2})\xi = 0$ *for some* $\lambda \in \mathbb{R}$.

We shall now show that for any quadratic form q satisfying the condition of 6.2 there is a unitary representation $\mathcal{H}$ of Heis(V) containing a Gaussian vector ξ of type q.

The simplest case is when the symplectic vector V has a compatible positive structure σ, i.e. a symplectic map $\sigma : V \to V$ such that

(i) $\sigma^2 = -1$, and

(ii) $q(v) = B(v, v) = S(\sigma(v), v) > 0$ for all non-zero $v \in V$.

It is very well-known (cf. [S1]) that if V is regarded as a complex vector space by means of σ, and is completed to a Hilbert space W using the sesquilinear form $B + \mathrm{i}S$, that the group Heis(V) has an irreducible unitary representation on the symmetric Fock space $S(W)$, and the vacuum vector $1 \in S^0(W)$ is Gaussian of type q.

The case of a general quadratic form q satisfying 6.2 reduces to this simple case in view of

Lemma 6.3 *Given* (V, S, q) *satisfying 6.2, there is a canonical embedding* $(V, S, q) \hookrightarrow (V_1, S_1, q_1)$*, where the symplectic and quadratic forms* S_1 *and* q_1 *extend* S *and* q*, and* V_1 *has a compatible complex structure* $\sigma : V_1 \to V_1$ *such that* $q_1(v) = S_1(\sigma(v), v)$.

Proof First consider the complexification $V_{\mathbb{C}} = V \oplus V$ with the real bilinear forms S_1 and B_1 defined by the matrices

$$S_1 = \begin{pmatrix} S & -B \\ B & S \end{pmatrix} \qquad B_1 = \begin{pmatrix} B & S \\ -S & B \end{pmatrix},$$

where B is the real bilinear form corresponding to q. The condition 6.2 is equivalent to the (semi-definite) positivity of B_1. The desired symplectic space V_1 is defined as $V_{\mathbb{C}}/K$, where $K = V_{\mathbb{C}}^{\perp}$ is the radical of the form B_1. (Notice that $\sigma(K) \subset K$.) ♠

The irreducible representation of Heis(V_1) defined by σ restricts to a representation of Heis(V) in which the vacuum vector is cyclic and Gaussian of type q. If $V_1 \neq V$, however, the representation of Heis(V) is far from irreducible: in our application it will be of type III.

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