Sewing Riemann surfaces together

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I shall consider compact smooth surfaces Σ with boundary, with a smooth almost complex structure (smooth on to the boundary.) Ignoring the complex structure for the moment, such a surface can always be embedded as a codimension 0 submanifold of a non-compact surface Σ^+ without boundary, so that $\partial\Sigma$ is a smoothly embedded curve in Σ^+. To see this, choose a smooth Riemannian metric on Σ, identify each boundary circle with the standard S^1 by parametrizing it proportionally to arc-length, and then identify a neighbourhood of each boundary circle with a half-open annulus $\{z \in \mathbb{C} : 1 - \epsilon < |z| \leq 1\}$ by using the geodesics perpendicular to the boundary. Of course, Σ^+ is not functorially associated to Σ, but we can extend the almost complex structure smoothly to Σ^+.

Proposition 0.1 *The almost-complex structure on Σ^+ is integrable, i.e. Σ^+ possesses a smooth holomorphic atlas.*

I shall omit the proof of this.

Proposition 0.2 *An open Riemann surface which is topologically an open annulus is isomorphic to a standard annulus $\{z \in \mathbb{C} : r < |z| < R\}$ for some $0 \leq r < R \leq \infty$.*

This is proved by applying the Riemann mapping theorem to the simply connected covering of the annulus. By applying it, in turn, to a neighbourhood of a a boundary circle C of Σ we obtain the following convenient fact.

Proposition 0.3 *A neighbourhood of a boundary circle C in Σ^+ can be identified with a neighbourhood of a simple closed curve S in $\mathbb{C}$.*

In fact the following argument shows that we can take S to be the standard $S^1 \subset \mathbb{C}$ (though with a non-standard smooth parametrization).

Proposition 0.4 *If φ is a holomorphic isomorphism from the interior U of a smooth simple closed curve S in $\mathbb{C}$ to the standard open disc, then we can extend φ to a smooth isomorphism of the closures.*

To see this, assume, without loss of generality, that $0 \in U$. Then choose a harmonic function $f : U \to \mathbb{R}$ whose boundary value on S is $z \mapsto \log |z|$. Then solve the Cauchy-Riemann equations to find $g : U \to \mathbb{R}$ so that $h = f + ig$ is holomorphic in U. Then $z \mapsto \varphi(z) = ze^{-h}$ is the desired map from U to the standard disc. We need to show that the derivative of φ does not vanish on the boundary circle. By continuity the derivative is $\mathbb{C}$-linear, and it is easy to see that it cannot vanish to finite order, but to show that it cannot vanish to infinite order at a boundary point z of U I have had to resort to the following lemma, which should be applied to $\psi = \chi \circ \varphi \circ \rho$, where ρ maps the standard unit disc into U taking 1 to z, and χ maps the standard disc to the upper half-plane taking $\varphi(z)$ to 0.

Lemma 0.5 *Let $\psi : D \to \mathbb{C}$ be a smooth map, where D is the closed disc $\{z \in \mathbb{C} : |z| \leq 1\}$. Suppose that ψ maps the interior of D into the upper half-plane, and that $\psi(1) = 0$. Then $\psi'(1) \neq 0$.*

Proof. Let $\Im\psi(\mathrm{e}^{\mathrm{i}\theta}) = a(\theta)$. We can write

$$a(\theta) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{\mathrm{i}\theta}.$$

Then

$$\psi(z) = \Re\psi(0) + \mathrm{i}\{a_0 + 2\sum_{k>0} a_k z^k\}.$$

For $0 < r < 1$ we have

$$\begin{aligned} \psi'(r) &= 2\mathrm{i}\sum_{k>0} k a_k r^{k-1} \\ &= \frac{1}{r\pi}\int a(\theta) g_r(\theta)\, d\theta, \end{aligned}$$

where

$$
\begin{aligned}
g_r(\theta) &= \sum_{k>0} k r^k \mathrm{e}^{-\mathrm{i}k\theta} \\
&= \frac{r\mathrm{e}^{-\mathrm{i}\theta}}{(1-r\mathrm{e}^{-\mathrm{i}\theta})^2}.
\end{aligned}
$$

Now $g_r(\theta) \to -\frac{1}{4}\csc^2\frac{\theta}{2}$ as $r \uparrow 1$. Because $a(\theta) = \mathcal{O}(\theta^2)$ near $\theta = 0$ we can take the limit under the integral, and find

$$\psi'(1) = -\frac{1}{4\pi}\int a(\theta)\csc^2\frac{\theta}{2}\,d\theta.$$

As $a(\theta) \geq 0$, this shows that $\psi'(1) \neq 0$.

After these preliminaries we can turn to the question of sewing surfaces together. Suppose we have Riemann surfaces Σ_1, Σ_2, and an orientation-reversing diffeomorphism $\varphi : C_1 \to C_2$, where C_i is a boundary circle of Σ_i. We form the topological space $\Sigma = \Sigma_1 \cup_\varphi \Sigma_2$, and define a subsheaf $\mathcal{O}_\Sigma$ of the sheaf of continuous complex-valued functions on Σ by $f \in \mathcal{O}_\Sigma(U)$ if each $f|\Sigma_i$ is smooth, and f is holomorphic in the interior of each Σ_i. I shall call sections of $\mathcal{O}_\Sigma$ *pseudoholomorphic* functions.

Proposition 0.6 *The ringed space $(\Sigma, \mathcal{O}_\Sigma)$ is a Riemann surface in which $C = \partial\Sigma_1 = \partial\Sigma_2$ is a smooth submanifold.*

Proof. As the question is purely local, we can assume, in the light of the preceding discussion, that Σ_1 and Σ_2 are the interior discs bounded by the unit circle $S^1 \in \mathbb{C}$, and that φ is an orientation-preserving diffeomorphism of S^1.

Let $V = C^\infty(S^1;\mathbb{C})$, let U be the closed subspace of V spanned by z^k for $k \geq 1$, and let W be the closed subspace spanned by z^k for $k \leq 1$. The subspace U belongs to the smooth restricted Grassmannian of V in the sense of [PS], and so does $\varphi^*(U)$.

Lemma 0.7 *The intersection $\varphi^*(U) \cap W$ is 1-dimensional.*

Granting the lemma, we have a pseudoholomorphic map $f : \Sigma \to S^2$, unique up to multiplication by a scalar on S^2. As f is holomorphic on each hemisphere Σ_i, the piecewise-smooth 2-form $f^*\omega$, where ω is the standard round area-form on S^2, is everywhere non-negative, and its integral is $4\pi d$, where d is the degree of f. Thus $d \geq 1$, and f is surjective. In fact $d = 1$, as $f^{-1}(\infty) = \infty \in \Sigma_2$, and f must be regular at ∞. This means that f is injective on the interiors of Σ_1 and Σ_2, and that the images of the interiors are disjoint. On the other hand Proposition 0.4 implies that $f|S$ is an immersion, and if it were not injective then the complement of $f(S)$ could not consist of two open discs. So f is bijective. Finally, f^{-1} is holomorphic, as it is continuous, and is holomorphic on the complement of $f(S)$.

Proof of Lemma 0.7.

It is enough to prove that the kernel of $p_\varphi : U \to V/W$ is 1-dimensional, where p_φ is the composite of $\varphi^* : U \to V$ with the projection $V \to V/W$. But p_φ is Fredholm, and depends continuously on φ. By deforming φ continuously to the identity we see that p_φ has index 1, and so its kernel is non-zero. But elements of the kernel correspond — as in the proof of Proposition 0.6 — to pseudoholomorphic maps $\Sigma \to S^2$, and if the kernel had dimension greater than 1 we could find a non-constant map with no pole, contradicting the positivity of the degree.