Fusion of representations of loop groups

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§1. Notations and conventions

Let G be the complexification of a compact Lie group.

If S is a compact 1-manifold then G_S denotes the group of smooth maps $S \to G$.

If Σ is a Riemann surface with boundary then G_Σ denotes the group of **holomorphic** maps $\Sigma \to G$.

We choose a "level" $k \in H^4(BG; \mathbb{Z})$. This defines for each oriented 1-manifold S a central extension $\widetilde{G}_S$ of G_S by $\mathbb{C}^\times$. There are natural maps

$$\widetilde{G}_{S_1} \times \widetilde{G}_{S_2} \to \widetilde{G}_{S_1 \amalg S_2} \quad .$$

Reversing the orientation of S reverses the central extension, and in particular there is a diagonal inclusion $G_S \subset \widetilde{G}_{S \amalg \bar{S}}$,where $\bar{S}$ denotes S with reversed orientation.

There is a basic **reciprocity theorem**: for each Riemann surface Σ the extension $\widetilde{G}_\Sigma$ of G_Σ induced by the restriction $G_\Sigma \hookrightarrow G_{\partial\Sigma}$ is canonically split, i.e. $\widetilde{G}_\Sigma = G_\Sigma \times \mathbb{C}^\times$.

§2. Aims and results

In my Swansea talk [S] I explained how, given G and k, one can associate to a Riemann surface Σ with boundary a representation $E(\Sigma)$ of $\tilde{G}_{\partial\Sigma}$ with three basic properties:

($P0$) if Σ is closed then $E(\Sigma)$ is a finite dimensional vector space;

($P1$) $E(\Sigma_1 \amalg \Sigma_2) = E(\Sigma_1) \otimes E(\Sigma_2)$, where $\otimes$ denotes a suitably completed tensor product; and

($P2$) if $\check{\Sigma}$ is formed from Σ by sewing together two components S and S' of $\partial\Sigma$ by an orientation-reversing diffeomorphism then

$$E(\check{\Sigma}) = E(\Sigma)^{G_S} \quad (= G_S\text{-invariant part of } E(\Sigma)),$$

where G_S is regarded as a subgroup of $\tilde{G}_{S \amalg S'}$ as above.

Properties ($P1$) and ($P2$) imply

($P3$) if Σ is formed by sewing Σ_1 and Σ_2 together along S then

$$E(\Sigma) = \{E(\Sigma_1) \otimes E(\Sigma_2)\}^{G_S}$$

Furthermore if $\partial\Sigma = \bar{S}_0 \amalg S_1$ then $E(\Sigma)$ defines a functor

$$E(\Sigma) : \{\text{representations of } \tilde{G}_{S_0}\} \to \{\text{representations of } \tilde{G}_{S_1}\}$$

by

$$E \mapsto \{E(\Sigma) \otimes E\}^{G_{S_0}}.$$

In fact the construction is more general. If σ is a finite set of marked points in the interior of Σ, and each $x \in \sigma$ is labelled with a finite dimensional representation V_x of G then the construction gives a representation $E(\Sigma; \sigma, V)$ of $\tilde{G}_{\partial\Sigma}$. The **Borel-Weil** theorem for loop groups is the assertion that any positive energy representation of $\widetilde{\mathcal{L}G} = \tilde{G}_{S^1}$ is of the form $E_V = E(D; \{0\}, V)$, where D is the standard disc with centre 0 and boundary S^1.

From now on we shall usually omit (σ, V) from the notation, as it plays no role in the discussion.

The properties $(P0) - (P2)$ follow directly from the definition of $E(\Sigma)$ given in [S], which will be recalled presently. My purpose in this note is to explain how the definition can be altered slightly so that the following two desirable properties hold in addition.

($P4$) $E(\Sigma)$ is independent of the complex structure of Σ, and depends only on the smooth surface together with a **rigging**, as defined below.

$(P5)$ $E(\Sigma)$ is a **positive energy** representation of $\tilde{G}_{\partial\Sigma}$.

As was explained in $[S]$, property $(P4)$ follows from $(P5)$, essentially because positive energy representations of loop groups admit an intertwining action of $Diff(S^1)$.

Various corollaries can now be read off, including the usual version of Verlinde's "fusion rules". For in the category $\mathcal{C}_S$ of positive energy representations of $\tilde{G}_S$ there are only finitely many irreducible objects, and every object is a direct sum of irreducibles. If S consists of a single circle the irreducibles are the E_V already mentioned, where V runs through a finite set Φ of irreducible representations of G. If S has n components I shall still denote the irreducibles by E_V, where $V = (V_1, ..., V_m) \in \Phi^m = \Phi_S$, and $E_V = E_{V_1} \otimes \ldots \otimes E_{V_m}$. Then if S is part or all of $\partial\Sigma$ - say $\partial\Sigma = S \amalg S'$ - we can write

$$E(\Sigma) = \bigoplus_{V\in\Phi_S} E_V \otimes E_V(\Sigma)$$

as representations of $\tilde{G}_S \times \tilde{G}_{S'}$. By Schur's lemma $\{E_V \otimes E_W\}^{G_S} = \mathbb{C}$ or 0 according as $W = V^*$ or not, and so $(P2)$ and $(P3)$ can be reformulated as

$$\text{(P2')} \qquad E(\check{\Sigma}) = \oplus_{V\in\Phi_S} E_{V,V^*}(\Sigma)$$

$$(P3') \qquad E(\Sigma_1 \cup_S \Sigma_2) = \oplus_{V\in\Phi_S} E_V(\Sigma_1) \otimes E_{V^*}(\Sigma_2).$$

$(P6)$ Finally, if $S = \partial\Sigma$, property $(P3')$ shows that $E_V(\Sigma)$ can be identified with $E(\hat{\Sigma}; \sigma, V)$, where $\hat{\Sigma}$ is the closed surface got by sewing a disc on to each boundary circle of Σ, and σ is the set of centres of the discs.

§3. Rigged surfaces

A rigging of a smooth surface is rather analogous to a choice of a spin structure on it. Up to isomorphism each surface can be rigged in only one way, but the group of automorphisms of a rigged surface is a central extension by $\mathbb{Z}$ of the group of diffeomorphisms of the surface. The definition of rigging I shall give is not the shortest possible, but it is the one which seems to fit in best with the concept of a rigged 1-manifold.

For each oriented 1-manifold S we have the restricted Grassmannian $Gr(S)$ consisting of all closed subspaces W of $C^\infty(S)$ which are not too far away from the subspace spanned by $\{e^{in\theta}\}$ for $n \geq 0$, where $\theta : S \to \mathbb{R}/2\pi\mathbb{Z}$ is a parametrization of S. (In fact $Gr(S)$ does not depend on the choice of parametrization.) For W_1 and W_2 in $Gr(S)$ there is a canonical determinant $Det(W_1 : W_2)$ such that

$$Det(W_1 : W_2) \otimes Det(W_2 : W_3) = Det(W_1 : W_3).$$

Definition A **rigging** of S is a holomorphic line bundle L on $Gr(S)$ with a holomorphic isomorphism

$$L_{W_1} \otimes Det(W_1 : W_2) = L_{W_2}$$

over $Gr(S) \times Gr(S)$.

Example A point W_0 of $Gr(S)$ defines a rigging of S by $L_W = Det(W_0 : W)$. Hence a parametrization of S defines a rigging, and so does a Riemann surface Σ such that $\partial\Sigma = S$, for the boundary values of holomorphic functions on Σ are a subspace $Hol(\Sigma)$ of $C^\infty(\partial\Sigma)$ which belongs to $Gr(\partial\Sigma)$.

Definition For an oriented smooth surface Σ let $\mathcal{C}(\Sigma)$ denote the space of complex structures on Σ modulo diffeomorphisms of Σ which are isotopic to the identity **and** equal to the identity on $\partial\Sigma$.

The space $\mathcal{C}(\Sigma)$ is contractible.

There is a holomorphic map $\mathcal{C}(\Sigma) \to Gr(\partial\Sigma)$ defined by $\Sigma_c \mapsto Hol(\Sigma_c)$.

Definition If $\partial\Sigma$ is rigged by L then $Det(\Sigma) = Det_L(\Sigma)$ is the pull-back of L to $\mathcal{C}(\Sigma)$. A **rigging** of Σ relative to L is a choice of a universal covering space of the complement of the zero-section in $Det(\Sigma)$.

The important thing about riggings is that they can be sewn together. A rigging L of S defines a rigging of $\bar{L}$ of $\bar{S}$, and

$$Det(\Sigma_1 \cup_S \Sigma_2) = Det_L(\Sigma_1) \otimes Det_{\bar{L}}(\Sigma_2)$$

over $\mathcal{C}(\Sigma_1) \times \mathcal{C}(\Sigma_2)$ if L is a rigging of S. Thus coverings of $Det_L(\Sigma_1)$ and $Det_{\bar{L}}(\Sigma_2)$ define a covering of $Det(\Sigma_1 \cup_S \Sigma_2)$.

§4. Representations of loop groups

A representation E of the loop group $\widetilde{\mathcal{L}G}$ has **positive energy** if it admits a positive intertwining action of the group $\mathbf{T}$ of rigid rotations of S^1. In that case E is sandwiched between canonical "minimal" and "maximal" representations

$$\check{E} \hookrightarrow E \hookrightarrow \hat{E},$$

each map being injective with dense image.

Among the representations sandwiched between $\check{E}$ and $\hat{E}$ there is a unique unitary representation E^{Hilb} of $\mathcal{L}(G_{compact})$. Usually one does not want to distinguish between representations lying between $\check{E}$ and $\hat{E}$. Nevertheless, there are two inconvenient features of the picture:

(i) the definition of $\check{E}$ and $\hat{E}$ depends on the parametrization of the circle, and $\mathrm{Diff}(S^1)$ does not act on them; while

(ii) although E^{Hilb} is independent of the parametrization of S^1 it is not acted on by the complex group $\widetilde{\mathcal{L}G}$, but only by $\widetilde{\mathcal{L}G}_{cpt}$.

I shall now sketch a way of avoiding these difficulties.

The (projective) action of $\mathrm{Diff}(S^1)$ on E^{Hilb} extends to a holomorphic action of a semigroup $\mathcal{A}$ whose elements are "annuli", i.e. Riemann surfaces diffeomorphic to $S^1 \times [0,1]$ with parametrized boundary circles. Each annulus A acts on E^{Hilb} by a trace-class operator. If E^{Hilb} is decomposed as $\bigoplus E_n$, where rotation by θ acts as $e^{in\theta}$ on E_n, then the annulus

$$A_q = \{z \in \mathbf{C} : |q| \le |z| \le 1\},$$

with its ends parametrized by $\{e^{i\theta}\}$ and $\{qe^{i\theta}\}$, acts on E_n as q^n.

Our strategy is to replace $\check{E}$ and $\hat{E}$ by the spaces

$$E' = \bigcup_A A.E^{Hilb}$$

and

$$E'' = \{\xi \in \hat{E} : A\xi \in E^{Hilb} \quad \text{for all } A\}.$$

(Here A runs through $\mathcal{A}$.) These spaces have natural topologies which make them mutually antidual. If we write $\xi = \Sigma \xi_n$ with $\xi_n \in E_n$, then

$$\xi \in E' \iff R^n \|\xi_n\| \quad \text{is bounded for some} \quad R > 1, \quad \text{and}$$

$$\xi \in E'' \iff R^{-n} \|\xi_n\| \quad \text{is bounded for all} \quad R > 1.$$

(The norm here is that of E^{Hilb}.)

Because E^{Hilb} does not depend on the parametrization of S^1 the same is true of E' and E''. It is not hard to see that $\widetilde{\mathcal{L}G}$ also acts on them. For our purposes it is best to formulate this assertion in another way, as follows.

The objects we are interested in are representations of the semi-direct product $\mathrm{Diff}(S^1)\tilde{\times}\mathcal{L}G$, but just as we passed from the group $\mathrm{Diff}(S^1)$ to the semigroup $\mathcal{A}$ so we can pass from $\mathrm{Diff}(S^1)\tilde{\times}\mathcal{L}G$ to a semigroup $\mathcal{B}$. An element of $\mathcal{B}$ is an annulus $A \in \mathcal{A}$ together with a holomorphic G-bundle B on A which is trivialized over ∂A. (Two pairs $(A, B), (A', B')$ are identified if they are isomorphic by a map which respects the boundary trivialization.) There is a forgetful homomorphism $\mathcal{B} \to \mathcal{A}$, and $\mathcal{A}$ can also be regarded as a sub-semigroup of $\mathcal{B}$ by equipping each annulus with the trivial G-bundle.

The objects that we want can now be described very simply: they are projective representations of the semigroup $\mathcal{B}$ by trace-class operators.

Let us be a little more explicit about the projectiveness. The universal central extension of $\mathrm{Diff}(S^1)$ has centre $\mathbf{R} \oplus \mathbf{Z}$, and it is best to keep the two factors separate. The $\mathbf{Z}$ factor is $\pi_1(Diff(S^1) = \pi_1(\mathcal{A})$. We deal with it by replacing $\mathcal{A}$ and $\mathcal{B}$ by the obvious $\mathbf{Z}$-fold coverings, which we shall do from now on without changing the notation. (The new $\mathcal{A}$ is the same as the space $\mathcal{C}_{S^1\times[0,1]}$ of §2. above.) The chosen level k defines not only a central extension $\widetilde{\mathcal{L}G}$ but also an extension $\tilde{\mathcal{B}}$ of $\mathcal{B}$ by $\mathbf{C}^\times$ which acts on all positive energy representations of level k. (The central $\mathbf{C}^\times$ acts in the obvious way.)

§5. The old definition.

For a Riemann surface Σ with non-empty boundary the space $E(\Sigma;\sigma,V)$ was defined as the space of holomorphic maps $f:\tilde{G}_{\partial\Sigma}\to V=\bigotimes_{x\in\sigma}V_x$ which are equivariant with respect to $\tilde{G}_\Sigma$ in the sense that

$$f(hg)=hf(g)$$

where $h\in\tilde{G}_\Sigma=G_\Sigma\times\mathbb{C}^\times$ acts on V via the restriction to the points $x\in\sigma$ (and $\mathbb{C}^\times$ acts naturally on V). I shall now denote this space $F(\Sigma;\sigma,V)$ to distinguish it from the modification to be introduced presently. Thus $F(\Sigma;\sigma,V)$ is the representation of $\tilde{G}_{\partial\Sigma}$ holomorphically induced from the representation V of $\tilde{G}_\Sigma$.

It is better to state the definition in terms of the moduli space $\mathcal{M}(\Sigma)$ of holomorphic G-bundles on Σ which are trivialized on $\partial\Sigma$. Because all holomorphic bundles on Σ are trivial $\mathcal{M}(\Sigma)$ is simply the homogeneous space $G_{\partial\Sigma}/G_\Sigma=\tilde{G}_{\partial\Sigma}/\tilde{G}_\Sigma$. The data (σ,V) define a holomorphic vector bundle $L_{\sigma,V}$ on $\mathcal{M}(\Sigma)$ with fibre V, and $F(\Sigma;\sigma,V)$ is the space of holomorphic sections of $L_{\sigma,V}$. Property (P2) is now just the fact that $\mathcal{M}(\check{\Sigma})=\mathcal{M}(\Sigma)/G_S$.

If $\partial\Sigma$ is empty one can define $F(\Sigma)$ as $F(\Sigma_0)^{G_D}$, where Σ_0 is obtained from Σ by deleting the interior of a small disc D (disjoint from σ). This does not depend on the choice of D, for if Σ_1 is obtained by deleting another, disjoint, disc D' from Σ, and $\Sigma_{01}=\Sigma_0\cap\Sigma_1$, then $F(\Sigma_0)^{G_D}=F(\Sigma_{01})^{G_{D\amalg D'}}=F(\Sigma_1)^{G_D}$ by property (P3).

The description of $F(\Sigma)$ in terms of the moduli space $\mathcal{M}(\Sigma)$ is valid even when $\partial\Sigma=\emptyset$, but in that case $\mathcal{M}(\Sigma)$ is normally defined by considering only **semistable** bundles on Σ. We shall return to this point in an appendix, as it is irrelevant for our purposes.

From the definition we are using it is clear that $F(\Sigma)$ is finite dimensional when Σ is closed, for we can write $F(\Sigma)=F(D)^{G_{\Sigma_0}}$, and there is a compact algebraic variety (a union of Bruhat cells) in $\mathcal{M}(D)=G_S/G_D$ which maps surjectively on to $G_{\Sigma_0}\backslash G_S/G_D$.

We now come to the crucial point, why $F(\Sigma)$ is a positive-enery representation of $\tilde{G}_{\partial\Sigma}$. Let us begin with the case when Σ is the standard annulus A_q. By adding caps with trivial bundles a point of $\mathcal{M}(A)$ defines a bundle on the Riemann sphere, and so a point of $\mathcal{M}(A)$ can be described as a bundle on S^2 with given trivializations of its restrictions to $|z|\le|q|$ and $|z|\ge 1$. Generically the bundle on S^2 is trivial, so a dense open subset $\mathcal{M}^0(A)$ is isomorphic to $(G_D\times G_{\bar{D}})/G$, compatibly with the actions of $G_D\times G_{\bar{D}}\subset G_{\partial A}$. The line bundle L is trivial over $\mathcal{M}^0(A)$, so the space $F(A)$ is a subspace of the space of holomorphic functions on $G_D\times G_{\bar{D}}$. There is an obvious action of the rotation group $\mathbb{T}\times\mathbb{T}$ of the two boundary circles of A on the holomorphic functions, and it has positive energy, and intertwines correctly with the action of the subgroup $G_D\times G_{\bar{D}}$ of $G_{\partial A}$. Unfortunately this is not enough to ensure that $F(A)$ is a positive energy representation. In the next section we shall see that all the same the argument is effectively correct. Meanwhile let us assume that $F(A)$ has positive energy.

If $F(A)$ has positive energy then so does $F(\Sigma)$ for any Riemann surface Σ. For by property (P3) we can write $F(\Sigma) = \{F(\Sigma_1) \otimes F(A)\}^{G_S}$ where $\Sigma = \Sigma_1 \cup_S A$. (I assume for simplicity that $\partial\Sigma$ consists of a single circle.) But the positiveness of $F(A)$ allows us to apply the decomposition theory of $[PS]$, which shows that $F(A)$, as a representation of $\tilde{G}_{\partial A} = \tilde{G}_S \times \tilde{G}_{\partial\Sigma}$, can be densely embedded in a finite sum of representations $\hat{E}_V \otimes \hat{E}_W$. Then

$$\begin{aligned} F(\Sigma) &= \bigoplus \{F(\Sigma_1) \otimes \check{E}_V\}^{G_S} \otimes \check{E}_W \\ &= \bigoplus \{F(\Sigma_1) \otimes F(D; 0, V)\}^{G_S} \otimes \check{E}_W \\ &= \bigoplus F_V(\Sigma_1) \otimes \check{E}_W, \end{aligned}$$

which is of positive energy because $F_V(\Sigma_1)$ is finite dimensional.

§6. The positive energy property for $F(A)$ and the Peter-Weyl theorem for a loop group

This is the crucial part of the whole discussion.

Let A be the annulus $\{z : a \leq |z| \leq 1\}$ in the Riemann sphere Σ. Write $\partial A = \bar{S}_a \amalg S$, and $\Sigma = D_a \cup_{S_a} A \cup_S D_\infty$. We can think of the loop group G_S as a dense subspace of $\mathcal{M}(A) = (G_{S_a} \times G_S)/G_A$ by the map $g \mapsto (1, g)$, and so $F(A)$ is a dense subspace of the space of holomorphic sections of the level k line bundle on G_S. We are going to prove that

$$F(A) \cong \bigoplus_{V \in \Phi} E_{V^*} \otimes E_V \qquad (1)$$

as representations of $\tilde{G}_{S_a} \times \tilde{G}_S$. This is a version of the Peter-Weyl theorem.

It is useful to think of an element of $\mathcal{M}(A)$ as a bundle on the sphere Σ trivialized over D_a and D_∞. By forgetting each part of the trivialization in turn we get a map

$$\mathcal{M}(A) \to \mathcal{M}(D) \times \mathcal{M}(\tilde{D}),$$

where $D = D_a \cup A$ and $\tilde{D} = A \cup D_\infty$. This map is not quite injective, but we can make it so by introducing the slightly larger spaces $\mathcal{M}_*(D), \mathcal{M}_*(\tilde{D})$ of bundles equipped with a trivialization of the fibres at the centres of the discs. Then the map

$$\mathcal{M}(A) \to \{\mathcal{M}_*(D) \times \mathcal{M}_*(\tilde{D})\}/G \qquad (2)$$

is injective. (Its image is the dense open subset where the two bundles on Σ are isomorphic.)

The right-hand side of (2) will be denoted by $\mathcal{M}(A_\infty)$, as it is the moduli space of bundles on an "infinitely long annulus" A_∞, which is interpreted as the singular surface got by attaching D to $\tilde{D}$ at their centres. The map (2) is **not** equivariant with respect to $G_{\partial A}$. Nevertheless the striking fact is that the spaces $F(A)$ and $F(A_\infty)$ of holomorphic sections of the standard line bundles on $\mathcal{M}(A)$ and $\mathcal{M}(A_\infty)$ **are** isomorphic as representations of $\tilde{G}_{\partial A}$. The representation $F(A_\infty)$ obviously has positive energy and satisfies (1). On the other hand, if we know that $F(A)$ has positive energy then it is isomorphic to $F(A_\infty)$ by property (P6).

Let us restate what we want in representation-theoretic terms. We start with the "regular representations" $\Gamma(G_{\partial A})$ of all holomorphic sections of the line bundle on $G_{\partial A}$, which is a representation of $\tilde{G}_{\partial A} \times \tilde{G}_{\partial A}$. The space $F(A)$ is its G_A-invariant part, where G_A is embedded in the right-hand copy of $\tilde{G}_{\partial A}$. We want to deform G_A to $G_{A_\infty} = \{(g_0, g_\infty) \in G_D \times G_{\tilde{D}} : g_0(0) = g_\infty(\infty)\}$, and see that the representation of $\tilde{G}_{\partial A}$ does not change.

There is an exactly analogous phenomenon in a familiar finite dimensional situation. Let us try to understand the Peter-Weyl decomposition

$$Hol(G) = \bigoplus V \otimes V$$

of the space of holomorphic functions on $G = SL_2(\mathbb{C})$ from the point of view of the Borel-Weil theorem. We think of $Hol(G)$ as $Hol((G \times G)/G)$, where G is embedded diagonally. Then let $G_\lambda \subset G \times G$ be the graph of the automorphism $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & \lambda b \\ \lambda^{-1} c & d \end{pmatrix}$. As the subgroups G_λ are conjugate in $G \times G$ the spaces $\mathrm{Hol}((G \times G)/G_\lambda)$ are clearly the same as representations of $G \times G$. But as $\lambda \to \infty$ the group G_λ tends to the subgroup $\tilde{B}$ of $B_- \times B_+ \subset G \times G$ consisting of all pairs

$$\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \right).$$

Now $(G \times G)/\tilde{B}$ is a $\mathbb{C}^\times$-bundle over $(G/B_+) \times (G/B_-)$, and the Borel-Weil theorem tells us that $\mathrm{Hol}((G \times G)/\tilde{B})$ is $\bigoplus V \otimes V$.

We now come to the proof. It is enough to show that $F(A)$ has positive energy as a representation of G_S.

There is a fibration

$$N_a \to \widetilde{\mathcal{M}(A)} \to \widetilde{\mathcal{M}_*(D)},$$

where $N_a = \{g \in G_{D_a} : g(0) = 1\}$. As $\widetilde{\mathcal{M}_*(D)} = \tilde{G}_S/N$, where $N = \{g \in G_D : g(0) = 1\}$, this means that

$$F(A) = Hol_N(\tilde{G}_S; Hol(N_a)).$$

We think of N as acting on $\tilde{G}_S$ on the left, and then $\tilde{G}_S$ acts on itself on the right.

§7. The new definition

To obtain the desired spaces $E(\Sigma)$ we must modify $F(\Sigma)$ in two ways. First, we must pass to a dense subspace, just as we had to reduce the "maximal" representation $\check{E}$ of a loop group to E'' in order to make $Diff(S^1)$ act. Then we must tensor the space with a suitable fractional power of the determinant line of Σ in order to eliminate the dependence on the complex structure of Σ. It is to have canonical fractional powers of the determinant line bundle over $\mathcal{C}(\Sigma)$ that one requires the surface Σ to be rigged.

For each smooth surface Σ we introduce the moduli space $\mathcal{B}(\Sigma)$ of pairs (Σ_c, B), where Σ_c is a complex structure on Σ and B is a holomorphic G-bundle on Σ_c trivialized over $\partial\Sigma$. Two such pairs are identified if they are isomorphic by a map which is the identity over $\partial\Sigma$ and is isotopic to the identity. There is a forgetful map $\mathcal{B}(\Sigma) \to \mathcal{C}(\Sigma)$, and its fibre at Σ_c is $\mathcal{M}(\Sigma_c)$.

For any 1-manifold S the space $\mathcal{B}(S \times [0,1])$ is a semigroup. It will be denoted by $\mathcal{B}_S$. If $S = S^1$ it is the semigroup $\mathcal{B}$ of §4. For any surface Σ the semigroup $\mathcal{B}_{\partial\Sigma}$ acts on $\mathcal{B}(\Sigma)$. The rigging of Σ gives us a natural holomorphic $\mathbb{C}^\times$-bundle $\widetilde{\mathcal{B}(\Sigma)}$ on which the canonical extension $\tilde{\mathcal{B}}_{\partial\Sigma}$ acts.

To define the representation $E(\Sigma)$ of $\tilde{\mathcal{B}}_{\partial\Sigma}$ let us choose an embedding of the standard disc D in Σ, writing $\Sigma = \Sigma_0 \cup_S D$. Then $\partial\Sigma_0 = \partial\Sigma \amalg S$, and it is convenient to think of $\mathcal{B}(\Sigma_0)$ as having a left-action of $\mathcal{B} = \mathcal{B}_S$ and a right-action of $\mathcal{B}_{\partial\Sigma}$. We define

$$E(\Sigma) = Hol_{\tilde{\mathcal{B}}}(\widetilde{\mathcal{B}(\Sigma_0)}; H),$$

where H is the basic representation of $\tilde{\mathcal{B}}$ (i.e. the Hilbert space in the representation $\check{E}_{\mathbb{C}}$), and $Hol_{\tilde{\mathcal{B}}}$ denotes the space of holomorphic maps $f : \widetilde{\mathcal{B}(\Sigma_0)} \to H$ such that $f(PQ) = Pf(Q)$ for $P \in \tilde{\mathcal{B}}$ and $Q \in \widetilde{\mathcal{B}(E_0)}$. The semigroup $\tilde{\mathcal{B}}_{\partial\Sigma}$ acts on $E(\Sigma)$ by $(Pf)(Q) = f(QP)$. The groups $G_{\partial\Sigma}$ and $Diff(\partial\Sigma)$ also act (projectively) on $E(\Sigma)$, as they act on $\mathcal{B}(\Sigma_0)$.

Let us first observe that, after tensoring with a line, $E(\Sigma)$ injects into the previously defined space $F(\Sigma)$, compatibly with the action of $\tilde{G}_{\partial\Sigma}$. For a given complex structure on Σ the moduli space $\mathcal{M}(\Sigma_0)$ is a subspace of $\mathcal{B}(\Sigma_0)$, and the relevant line bundles on these spaces differ by a power of $Det(\Sigma)$. Moreover $H \subset Hol_{\tilde{G}_D}(\tilde{G}_S; \mathbb{C})$, so there is a restriction map

$$\begin{aligned} Hol_{\tilde{\mathcal{B}}}(\widetilde{\mathcal{B}(\Sigma_0)}; H) &\to Hol_{\tilde{G}_S}(\widetilde{\mathcal{M}(\Sigma_0)}; Hol_{\tilde{G}_D}(\tilde{G}_S; \mathbb{C})) \\ &= Hol_{\tilde{G}_D}(\widetilde{\mathcal{M}(\Sigma_0)}; \mathbb{C}) \\ &= F(\Sigma). \end{aligned}$$

The map is injective - i.e. a $\mathcal{B}$-equivariant map on $\mathcal{B}(\Sigma_0)$ is determined by its restriction to $\mathcal{M}(\Sigma_0)$ - because one can get from one complex structure on Σ_0 to any other by successively adding on and taking off annuli along the boundary circle S.

The new functor E has the properties (P2) and (P3) for the same reasons as were given for F. Thus

$$\mathcal{B}(\Sigma_1 \cup_S \Sigma_2) \cong \{\mathcal{B}(\Sigma_1) \times \mathcal{B}(\Sigma_2)\}/\mathcal{B}$$

in the sense that a $\mathcal{B}$-invariant holomorphic function on $\mathcal{B}(\Sigma_1) \times \mathcal{B}(\Sigma_2)$ is the same as a holomorphic function on $\mathcal{B}(\Sigma_1 \cup_S \Sigma_2)$. The fact that $E(\Sigma)$ does not depend on the choice of the disc D is also as before.

Let us calculate the space $E(D)$, where D is a disc. It is $Hol_{\tilde{\mathcal{B}}}(\tilde{\mathcal{B}}; H)$. If $\mathcal{B}$ had an identity element this would be simply H, but as it does not we conclude that an element of $E(D)$ is a family $\{\xi_A\}$ of elements of H indexed by annuli $A \in \mathcal{A}$ and such that $A_1\xi_{A_2} = \xi_{A_1A_2}$. This is the same as an element ξ of $\check{E}_{\mathbf{C}} = F(D)$ such that $A\xi \in H$ for all annuli A, i.e. $E(D) = E''_{\mathbf{C}}$ in the notation of §4.