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MORITA EQUIVALENCE FOR C*-ALGEBRAS AND W*-ALGEBRAS

Marc A. RIEFFEL
University of California, Berkeley, Calif. 94720, U.S.A.

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0. Introduction

Two rings are said to be Morita equivalent if their categories of left modules are equivalent [26, 2, 6]. This provides a useful equivalence relation on rings which is considerably coarser than that of isomorphism. In this paper we study the corresponding notion of Morita equivalence for C^*-algebras and W^*-algebras, where as their categories of left modules we take their categories of non-degenerate $*$-representations on Hilbert spaces (normal ones in the case of W^*-algebras).

In the algebraic case, Morita's basic theorem concerning Morita equivalence [26, 2, 6] gives a description of how two rings which are Morita equivalent are constructed from each other, namely that each must be the full endomorphism ring of an appropriate type of module over the other. The main theorem of the present paper (Theorem 7.9) is an analogous description of how two C^*-algebras or W^*-algebras which are Morita equivalent are constructed from each other. Specifically, if M and N are W^*-algebras, then we show that M and N are Morita equivalent if and only if each is the full algebra of "bounded" operators on a non-commutative analogue over the other of the "C^*-modules" which Kaplansky [21] defined over commutative C^*-algebras.

An important step in studying algebraic Morita equivalence is the study of functors between categories of modules which preserve certain categorical limits of the type which any equivalence must preserve. The basic theorem concerning such functors is the Eilenberg–Watts theorem [11, 38, 2] asserting that any such functor is equivalent to a functor consisting of taking tensor products with a bimodule. Similarly, a substantial portion of the present paper is devoted to studying functors between categories of modules over C^*-algebras and W^*-algebras and obtaining an analogue of the Eilenberg–Watts theorem (Theorem 5.5). This analogue states that any such functor which is continuous in a certain sense is equivalent to a functor consisting of forming a certain type of topological tensor product with one of the non-commutative analogues mentioned above of Kaplansky's "C^*-modules". This topological tensor product is essentially just the inducing process which was studied in [30] for C^*-algebras, where it was shown that the formation of Mackey's induced representations [23] is just a special case of this process, and that Mackey's imprimitivity theorem [24] can be viewed as a special case of a Morita equivalence of the kind we study here. Indeed, the present paper grew directly out of this previous work, and the results on functors obtained here will be useful in further study of induced representations.

Since Morita equivalence is an equivalence relation on C^*-algebras or W^*-algebras which is considerably coarser than that of isomorphism, it will be a useful tool in studying various aspects of these algebras, such as their classification, although such a study is not included here. The purpose of this paper is to lay the basic groundwork for the theory. Much work remains to be done in obtaining detailed understanding of Morita equivalence for special classes of algebras.

This paper is organized in the following way. Section 1 contains basic facts concerning modules over C^*-algebras and W^*-algebras. In particular, modules which are generators for the corresponding categories are studied, as these provide a tool which is important in later sections. In Section 2, categories of modules over C^*-algebras and W^*-algebras are studied. In particular, the question of how much information about an algebra can be recovered from its category of modules is considered, and this question is seen to be closely related to Takesaki's duality theorem in the representation theory of C^*-algebras [33, 3], as well as a number of other results in the literature. General methods for constructing functors between categories of modules over C^*-algebras are described in Section 3. These methods generalize the inducing process in [30]. Properties of these functors are studied in Section 4. In Section 5, similar results for modules over W^*-algebras are considered, and the existence part of our analogue of the Eilenberg–Watts theorem is proved. Section 6 is devoted to the uniqueness part of our analogue of the Eilenberg–Watts theorem. This involves the self-dual modules introduced by Paschke [27] as a generalization to the non-commutative case of the "AW^*-modules" defined by Kaplansky [21] over commutative AW^*-algebras. Morita equivalence is studied in Section 7, and the main theorem of this paper describing how W^*-algebras which are Morita equivalent are constructed from each other is proved there. Finally, in Section 8, various general facts concerning C^*-algebras and W^*-algebras which are Morita equivalent are gathered together.

I would like to thank George M. Bergman and John E. Roberts for some very helpful comments which are acknowledged more specifically later. I would also like to thank William L. Paschke for making available to me a pre-print of the paper [27] in which he introduced the self-dual modules which are used in Section 6.

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1. Modules over operator algebras

Let B be a C^*-algebra (possibly without identity element). By a (left) *Hermitian B-module* we will mean (as in [30]) the Hilbert space of a non-degenerate $*$-representation of B [8] with the corresponding action of B. We will usually use module notation when working with Hermitian modules. If V and W are Hermitian B-modules, then we will denote by $\mathrm{Hom}_B(V, W)$ the Banach space (with the operator norm) of bounded B-module homomorphisms from V to W (also known as the intertwining operators between the representations of B on V and W). For convenience we will consider the zero-dimensional Hilbert space with the zero-representation of B to be a Hermitian B-module, so that, for example, we can say that the kernel of any f in $\mathrm{Hom}_B(V, W)$ is a Hermitian B-module, even when f is injective. The collection of all Hermitian B-modules together with the corresponding spaces of intertwining operators forms a category, which we will denote by Hermod-B. (For general information about categories we refer the reader to [25].) This category will be one of the principal objects studied in this paper.

Now let N be a W^*-algebra [7, 32]. (We use the term "W^*-algebra" for the algebras considered abstractly, while we reserve the term "von Neumann algebra" for W^*-algebras which are realized as algebras of operators on specific Hilbert spaces.) By a normal representation of N on a Hilbert space V we mean a normal $*$-homomorphism [7] of N into the algebra of all bounded operators on V which carries the identity element of N to the identity operator on V. (These are what Sakai, in [32, Definition 1.16.4], calls W^*-representations.) By a *normal N-module* we will mean the Hilbert space of a normal representation of N with the corresponding action of N. As with C^*-algebras, we will consider the zero representation of N on a zero-dimensional Hilbert space to be a normal N-module. Every normal N-module is, of course, a Hermitian N-module when N is considered to be only a C^*-algebra. If V and W are normal N-modules, then we will denote by $\mathrm{Hom}_N(V, W)$ the Banach space of bounded module homomorphisms from V to W. This notation is compatible with that given above for Hermitian modules. The collection of all normal N-modules together with the corresponding spaces of intertwining operators forms a category, which we will denote by Normod-N. We remark that Normod-N is a full subcategory of Hermod-N.

Let B be a C^*-algebra and let $n(B)$ be the enveloping W^*-algebra of B [8, § 12]. Then $n(B)$ has the property that the action of B on any Hermitian B-module V extends uniquely to an action of $n(B)$ on V such that V becomes a normal $n(B)$-module [8, 12.1.5]. (In fact, as Guichardet points out on [19, p.60], the functor $B \mapsto n(B)$ is the left adjoint of the forgetful functor from the category of W^*-algebras to the category of C^*-algebras.) In this way there is established a bijection between the Hermitian B-modules and normal $n(B)$-modules. This bijection preserves the spaces of intertwining operators, and thus provides an isomorphism between the category of Hermitian B-modules and the category of normal $n(B)$-modules. Thus, if we wish to study Hermod-B, it suffices, as long as we are not considering this category to carry additional structure, to study the category Normod-$n(B)$. The converse is not true however as there are many W^*-algebras whose category of normal modules is not isomorphic to the category of Hermitian modules over any C^*-algebra. For example, a category of Hermitian modules will always contain simple modules, that is, irreducible representations, whereas this need not be true for a category of normal modules. For these reasons we will make many of our later definitions only for categories of normal modules. These definitions will then have immediate specializations to categories of Hermitian modules.

Let $\{V_i\}$ be a family of normal N-modules. Then the direct sum of this family, denoted by $\oplus V_i$, is defined to be the Hilbert space direct sum of the V_i together with the obvious coordinate-wise action of N. (To show that this action is well-defined it is necessary to use the fact that $*$-representations are norm-decreasing – see the comments after Theorem 2.11 of [29].) It is easily verified that $\oplus V_i$ is a normal N-module. A bit of care must be taken in using direct sums of normal modules since these direct sums do not satisfy the usual universal property for algebraic direct sums [25] if there is an infinite number of summands, but rather satisfy this universal property only for families of homomorphisms whose family of operator norms is square-summable. The definition of direct sums of Hermitian modules follows from that for normal modules. Categories of normal modules have the pleasant property that every submodule of a normal module is a direct summand, the complementary submodule being, of course, just its orthogonal complement.

We now investigate the subject of generators in categories of modules. These will be an important tool later in our study of functors. In general category theory, an object U is a generator [17, p.68] if the functor represented by U, namely $V \mapsto \mathrm{Hom}(U, V)$, is an embedding into the category of sets, that is, is injective on spaces of morphisms. The next proposition gives other characterizations of generators in categories of normal modules, and in fact for much of this paper it will be quite sufficient to take properties (3) or (4) below as the definition of a generator.

1.1. Proposition. *Let N be a W^*-algebra and let $U \in$* Normod-N. *Then the following conditions are equivalent*:

(1) *U is a generator for* Normod-N.

(2) *For any non-zero normal N-module V there is a non-zero element of* $\mathrm{Hom}_N(U, V)$.

(3) *Every normal N-module is isomorphic to a submodule of a (possibly infinite) direct sum of copies of U.*

(4) *Every normal N-module is a (possibly infinite) direct sum of copies of submodules of U.*

Proof. We indicate why (1) implies (2). Let V be a non-zero normal N-module, so that $\mathrm{Hom}_N(V, V)$ contains non-zero elements. Since U is assumed to be a generator, so that the functor represented by U is an embedding, it is easily seen to follow that $\mathrm{Hom}_N(U, V)$ must contain non-zero elements as desired.

We show next that (4) implies (1). Let V and W be normal N-modules. Since

$\mathrm{Hom}_N(V, W)$ is a group (under addition) and since the functor represented by U is clearly additive, it suffices to show that no non-zero element of $\mathrm{Hom}(V, W)$ is carried to the zero map by this functor, that is, that given $f \in \mathrm{Hom}_N(V, W)$ with $f \neq 0$, there exists $g \in \mathrm{Hom}_N(U, V)$ with $f \circ g \neq 0$. But, given such an f, let V' be the orthogonal complement of the kernel of f. Then V' is a non-zero submodule of V and f is injective on V'. Since we are assuming condition (4) to hold, V' is isomorphic to a direct sum of copies of non-zero submodules of U. Then f composed with the projection of U onto any of these submodules must be non-zero as desired.

It is clear that (3) and (4) are equivalent. Finally, to show that (2) implies (4) we need the following lemma, which we state for C^*-algebras, since it will be useful in that form later.

1.2. Lemma on polar decomposition. *Let C be a C^*-algebra, let V and W be Hermitian C-modules, and let $f \in \mathrm{Hom}_C(V, W)$. Then $f = p\,|f|$, where $|f| = (f^*f)^{1/2}$ (so $|f| \in \mathrm{Hom}_C(V, V)$) and p is a partial isometry in $\mathrm{Hom}_C(V, W)$ from the orthogonal complement of the kernel of $|f|$ to the closure of the range of $|f|$.*

Proof. Let X be the Hermitian C-module $V \oplus W$, and define $\tilde{f} \in \mathrm{Hom}_C(X, X)$ by $\tilde{f}(v, w) = (0, f(v))$. Then it is easily verified that the usual polar decomposition for operators ([32, Theorem 1.12] or [7, Appendix III]) when applied to $\tilde{f}$ yield this lemma. □

We return to the proof of Proposition 1.1. Let V be a non-zero normal N-module. Since we are assuming (2) to hold, we can find a non-zero $f \in \mathrm{Hom}_N(U, V)$. Then from Lemma 1.2 applied to f there is a non-zero partial isometry p in $\mathrm{Hom}_N(U, V)$. Thus V contains a non-zero submodule isomorphic to a submodule of U. If this submodule of V is not all of V, then its orthogonal complement is a non-zero submodule to which we can apply (2) and the above argument. The result now follows by an application of Zorn's lemma. □

The four equivalent conditions of Proposition 1.1 are all stated in category-theoretic terms, and so apply immediately to the category of Hermitian modules over a C^*-algebra. We now give a criterion for a module to be a generator in a category of normal modules which is not category theoretic, and which is false for C^*-algebras. This criterion, which will be important later, follows readily from well-known facts concerning von Neumann algebras, but these facts, unlike those used in the proof of Proposition 1.1 above, are not entirely elementary.

1.3. Proposition. *Let N be a W^*-algebra and let U be a normal N-module. Then U is a generator for* Normod-N *if and only if the representation of N on U is faithful (that is, injective).*

Proof. Suppose that U is a generator, and let N_0 be the kernel of the corresponding normal homomorphism of N into the algebra of bounded operators on U. Then N_0 will be in the kernel of any representation on a submodule of a direct sum of copies of U, and so in the kernel of any representation of N on any member of Normod-N. But any W^*-algebra has a faithful normal module [32, Theorem 1.16.7], and so $N_0 = \{0\}$. Thus the representation of N on U is faithful.

Conversely, suppose that the representation of N on U is faithful, and let $V \in$ Normod-N. Then on examining [7, Theorem 3, p. 53] we see that V is obtained by first taking an "ampliation", that is, a direct sum of copies of U, then an "induction", that is, a submodule of this direct sum, and then taking an isomorphism of this submodule with V. Thus condition (3) of Proposition 1.1 holds. And so this half of our proposition can be viewed as simply a reformulation of the indicated theorem in [7]. □

We now justify the comment made above that for C^*-algebras the property of a representation being faithful is not categorical, by giving an example of two C^*-algebras which have isomorphic categories of Hermitian modules for which the isomorphism does not preserve faithfulness.

1.4. Example. Let $A = c_0$, the C^*-algebra of sequences of complex numbers which converge to zero, and let B be the C^*-algebra of sequences $\{r_n\}$ of complex numbers having the property that $\lim r_n = r_1$. Both algebras can be viewed as subalgebras of the W^*-algebra l_∞ of bounded sequences. In fact, it is easily seen that the dual of each algebra is l_1, so that the double dual enveloping W^*-algebra of each algebra is just l_∞. Thus the two algebras have isomorphic categories of Hermitian modules, each isomorphic to Normod-l_∞. But let V be the subspace of l_2 consisting of sequences having zero as first term. Then V is faithful as a B-module but not as an A-module.

Actually, generators in the category of normal N-modules or Hermitian B-modules have already played an important role in representation theory, although they have not been called generators. For any C^*-algebra B its "universal" representation ([8, 2.7.6] or [32, 1.16.5]) is easily seen to be a generator for Hermod-B, while for any W^*-algebra N its universal normal representation will be a generator for Normod-N. (In particular, generators always exist.) But other generators may be useful. For example, if B is a type I C^*-algebra [8, 5.4], then the multiplicity free representation quasi-equivalent [8, 5.3.2] to the universal representation of B will be a generator.

The enveloping W^*-algebra of a C^*-algebra is usually defined using the universal representation, but the next two results show that any generator would do.

1.5. Proposition. *Any two generators for* Hermod-B *(or* Normod-N*) are quasi-equivalent. Any module quasi-equivalent to a generator is a generator. Thus the generators form an equivalence class under the relation of quasi-equivalence.*

Proof. This is an immediate consequence of [8, 5.3.1(i)] together with Proposition 1.1 above. □

1.6. Proposition. *Let B be a C^*-algebra, and let U be any generator for* Hermod-B. *Then the weak operator closure of the image of B as operators on U is isomorphic to the enveloping W^*-algebra $n(B)$ of B.*

Proof. This is an immediate consequence of the above results and [8, 5.3.1(ii)]. □

A number of well-known results concerning spatial isomorphisms of von Neumann algebras can be given pleasant reformulations in terms of generators. For example, the following is a reformulation of [7, ch. III §1, Theorem 3]:

1.7. Proposition. *Let N be a W^*-algebra and let U and U_0 be generators for* Normod-N. *If each of U and U_0 have both a separating and a cyclic vector, then U and U_0 are isomorphic.*

Among standard results in [7, ch. III] which can be reformulated in a similar way are Theorem 6 of §1, Proposition 10 of §6 and Corollary 7 of §8.

2. Categories of modules over operator algebras

The main theme of this section is to discover how much information about a C^*-algebra or W^*-algebra can be recovered from knowing just its category of Hermitian or normal modules. We do this not only because of the intrinsic interest of this question, but also because it leads to techniques which will be important in later sections.

We recall from [2, p. 56] that the center of a category is defined to be the collection of natural transformations from the identity functor on the category to itself. The following proposition is an analogue of [2, Proposition 2.1, p. 56].

2.1. Proposition. *Let N be a W^*-algebra. Then the center of* Normod-N *is an algebra which is isomorphic with the center of N.*

Proof. Any element c of the center of N is easily seen to define a natural transformation t^c of the identity functor to itself by $t^c_V(v) = cv$ for $v \in V \in$ Normod-N. It is also easily seen that the natural transformations form an algebra.

Conversely, suppose that t is a natural transformation from the identity functor to itself. Let U be a generator for Normod-N. Then by definition $t_U \in \mathrm{Hom}_N(U, U)$, and also must commute with all the elements of $\mathrm{Hom}_N(U, U)$, that is, is in the center of $\mathrm{Hom}_N(U, U)$. Since we are assuming that N is a W^*-algebra and that it is faithfully represented on U (Proposition 1.3), it follows from the von Neumann double commutant theorem [7, p. 41] that t_U corresponds to an element c in the center of N. It is then clear that t acts like c on direct sums of copies of U and on submodules thereof, and so on any element of Normod-N. □

2.2. Corollary. *Let B be a C^*-algebra. Then the center of* Hermod-B *is isomorphic to the center of $n(B)$.*

2.3. Corollary. *Let N be a W^*-algebra. Then N is a factor if and only if the center of* Normod-N *is one-dimensional.*

For any W^*-algebra N the category Normod-N has an additional piece of structure which will be of importance, namely the involution which assigns to each $f \in \mathrm{Hom}_N(V, W)$ its adjoint $f^* \in \mathrm{Hom}_N(W, V)$. The following proposition is not surprising in view of the self-duality of Hilbert spaces.

2.4. Proposition. *Let N be a W^*-algebra. Then the contravariant functor from* Normod-N *to itself which is the identity on objects and carries each morphism f to its adjoint f^* establishes a (conjugate linear) isomorphism of* Normod-N *with its dual category.*

The proof of this proposition is trivial. The definition of a dual category can be found on [25, p.33].

There is yet additional structure on Normod-N which will be of importance. We have already mentioned the fact that each space $\mathrm{Hom}_N(V, W)$ is equipped with the operator norm, with respect to which it is a Banach space. In addition, $\mathrm{Hom}_N(V, V)$ is not only a C^*-algebra for the operator norm and involution, but is in fact a von Neumann algebra, and so can be equipped with the ultra-weak operator topology. (Indeed, $\mathrm{Hom}_N(V, W)$ can also be equipped with an ultra-weak operator topology, as we will see in the next section.) In terms of this structure we can tell whether a W^*-algebra is of type I, II or III in terms of its category of normal modules.

2.5. Proposition. *Let N be a W^*-algebra. Then N is of pure type* I *(respectively type* II, *or type* III) *if and only if for every non-zero $V \in$* Normod-N *the von Neumann algebra* $\mathrm{Hom}_N(V, V)$ *is of pure type* I *(respectively type* II, *or type* III).

Proof. From the definition of the type of a W^*- algebra [32, 2.2.9] it is easily seen that any sub-W^*-algebra of N will have the same type as N. But the image N_V of N acting on V will be a W^*-subalgebra of N (as is easily seen from [32, 1.10.1, 1.16.2]), and so of the same type as N. But $\mathrm{Hom}_N(V, V)$ is just the commutant of N_V, and so is also of the same type [32, 2.9.6]. □

On the other hand, we shall see later (Corollary 8.13) that all type I_n factors for different values of n have equivalent categories of modules, so that they cannot be distinguished by their categories. We shall also see that every type II_∞ factor has associated with it at least one type II_1 factor having an equivalent category of normal modules, and conversely, so that one cannot tell whether a factor is of type II_∞ or II_1 in terms of its category of normal modules.

We can never actually recover a C^*-algebra or W^*-algebra just from its category of modules – in fact that is what makes Morita equivalence interesting. However, if there is yet additional structure present, then it may be possible to recover the algebra. We now consider one example of this. For any W^*-algebra N let H_N (or just H when there is no chance of confusion) denote the forgetful functor from Normod-N to the category of Hilbert spaces and bounded linear maps which assigns to every normal N-module its underlying Hilbert space. The next theorem shows that we can recover N from the data consisting of Normod-N together with this forgetful functor. This fact and its proof are just a slight generalization of a result (see Corollary 2.7 below) discovered by John E. Roberts in the course of his work on mathematical physics, where similar considerations are at play. (For a hint of this see the footnote on [9, p.217] and [10, Theorem 3.6].) I would like to thank him for showing me this result and for several very enlightening conversations concerning the general subject.

2.6. Theorem. *Let N be a W^*-algebra, and let H be the forgetful functor from* Normod-N *to the category of Hilbert spaces. Then the collection C of natural transformations from H to itself can be given in a natural way the structure of a W^*-algebra, and this W^*-algebra is naturally isomorphic to N.*

Proof. Let $t \in C$. Then t assigns to any $V \in$ Normod-N a bounded linear operator t_V on $H(V)$. If scalar multiples, sums, products and adjoints of elements of C are defined in terms of the corresponding operations on the associated operators, it is easily seen that C becomes a $*$-algebra. A norm can be defined on C by

$$\|t\| = \sup \{\|t_V\|: V \in \text{Normod-}N\}.$$

It is easily verified that this norm is finite, is a C^*-algebra norm, and that C is complete for this norm, so that C becomes a C^*-algebra. An analogue of the ultra-weak operator topology can be defined by means of the linear functionals $p_{V,u,w}$ defined for $V \in$ Normod-N, $v, w \in H(V)$ by

$$p_{V,u,w}(t) = \langle t_V u, w \rangle,$$

and it is easily verified that C becomes a W^*-algebra for this topology.

For any $n \in N$ we define a natural transformation t^n from H to itself by

$$(t^n)_V v = nv$$

for $v \in H(V)$, $V \in$ Normod-N. It is easily verified that the correspondence $n \mapsto t^n$ is an isometric $*$-homomorphism of N into C. We indicate now why this homomorphism is surjective. Let $t \in C$, and let U be a generator for Normod-N. Then N is faithfully represented on U (Proposition 1.3) and $\mathrm{Hom}_N(U, U)$ is just the commutant of N acting on U. Now by the definition of a natural transformation t_U must commute with every element of $\mathrm{Hom}_N(U, U)$. It follows from the von Neumann double commutant theorem [7, p.41] that there is an $n \in N$ such that n and t_U coincide as operators on U. It follows by arguments similar to those at the end of the proof of

Proposition 2.1 that t_V and n must coincide as operators on $H(V)$ for every $V \in$ Normod-N, that is, that $t = t^n$. Finally, it is easily verified that the analogue of the ultra-weak operator topology on C defined above corresponds to the weak topology of N. □

We notice from the above that the morphisms in Normod-N form a generalization of the commutant of N, while the natural transformations of the forgetful functor are like the double commutant of N.

2.7. Corollary (John E. Roberts). *Let B be a C^*-algebra and let H be the forgetful functor from* Hermod-B *to the category of Hilbert spaces. Then the collection C of natural transformations from H to itself can be given in a natural way the structure of a W^*-algebra, and this W^*-algebra is naturally isomorphic to the W^*-enveloping algebra $n(B)$ of B.*

As T. Cartier pointed out to me, the above results are closely related to the Yoneda Lemma [25, p.61], but the Yoneda Lemma is not applicable since the forgetful functor is not representable.

The above corollary is closely related to results of Takesaki [33] and Bichteler [3], and in fact categories provide a more natural setting for their results. Specifically, if B is a C^*-algebra and if K is a Hilbert space, then the set Rep(B, K) of representations of B on subspaces of K can be viewed as a category if the corresponding intertwining operators are adjoined. It is, of course, a full subcategory of Hermod-B. The forgetful functor on Rep(B, K) has as its values on objects just subspaces of K. Then the "admissible operator fields" of [33] and [3] are easily seen to correspond just to natural transformations of this forgetful functor to itself. For example, it is [3, condition (iv), p.94] which corresponds to the requirement that a natural transformation must commute with morphisms. This is seen by using the following slight generalization of [7, Proposition 3, p.4];

2.8. Proposition. *Let B be a C^*-algebra, and let $V, W \in$* Hermod-B. *Then any $f \in$* Hom$_B(V, W)$ *is a linear combination of two partial isometries in* Hom$_B(V, W)$.

Proof. According to Lemma 1.2 above, $f = p\,|f|$, where p is a partial isometry in Hom$_B(V, W)$ and $|f|$ is a positive element of Hom$_B(V, V)$. But by [7, Proposition 3, p.4], $|f|$ is a linear combination of two unitary operators in Hom$_B(V, V)$. □

Making the correspondence indicated above between admissible operator fields and natural transformations of the forgetful functor, we see that [33, Theorem 2] is just Corollary 2.7 above but with Hermod-B replaced by Rep(B, K). The proof is almost the same except that K may not be large enough to be the space of a generator for Hermod-B, and so one must work instead with a generating family of disjoint elements of Rep(B, K). But this causes no difficulties.

The above considerations are also very closely related to the "big group algebra" of Ernest [12, 13, 14], and to the enveloping algebra of a covariant system introduced by Ernest in [15]. In all these papers the "options" employed can be reinterpreted to be natural transformations of a forgetful functor to itself.

As John E. Roberts also pointed out to me, the Tannaka duality theorem can be reinterpreted in a similar way, where the "Darstellungen der Dualhalbgruppe" of [36], or the "representations" of [5, Definition 2, p.196] or the "operations" on [20, p. 75] can all be viewed as natural transformations of the forgetful functor. Undoubtedly the "operator fields" of [37] can be interpreted in a similar way, as can the maps J of [14, Remark 3.14]. Extension of these duality theorems to covariance algebras can be found in [16, Section 6]. Such generalizations of Tannaka's theorem seem to be more in the spirit of Tannaka's original theorem than those which involve Hopf algebras [22, 14, 34]. I also suspect that the above ideas are related to the work of Saavedra Rivano [31], but the situation here is not at all clear to me.

Since non-isomorphic C^*-algebras can have isomorphic enveloping W^*-algebras (Example 1.4), one would not hope, in view of Corollary 2.7, to be able to recover a C^*-algebra from the data consisting of its category of Hermitian modules together with the forgetful functor. However, one can imagine that if additional structure is added (probably of a topological nature), then one could recover the C^*-algebra itself. But it is not clear to me how to do this. Here we will content ourselves with reformulating in terms of categories the further results of Takesaki and Bichteler, in which they consider a topology on Rep(B, K). (It is not clear to me how to put a similarly useful topology on Hermod-B.)

Let B be a C^*-algebra, and let K be an infinite-dimensional Hilbert space of dimension large enough so that every cyclic representation of B can be realized on a subspace of K. As in [33] and [3] we put on the objects of the category Rep(B, K) a topology, which in [18] is called the strong topology, but which we will call the strong Fell topology. Specifically, any element of Rep(B, K) whose underlying Hilbert space is $J \subseteq K$ can be put in correspondence with the homomorphism from B to the algebra of bounded operators on J which defines the corresponding representation of B. But every operator on J can be viewed as an operator on K by defining it to be zero on the orthogonal complement of J. Thus the objects of Rep(B, K) are in bijective correspondence with the $*$-homomorphisms of B into the algebra $L(K)$ of bounded operators on K which define (possibly degenerate) $*$-representations of B on K. In this way the objects of Rep(B, K) correspond to certain functions from B to $L(K)$, and so if we equip $L(K)$ with the strong operator topology, we can equip Rep(B, K) with the corresponding topology of pointwise convergence of functions. It is this topology which we call the strong Fell topology. (As the lemma on [3, p.90] shows, we could just as well have used the weak, ultra-weak or ultra-strong operator topologies on $L(K)$.)

We would now like to define what we mean by saying that a natural transformation of the forgetful functor from Rep(B, K) to the category of Hilbert spaces is continuous. For this purpose let Sub(K) denote the category whose objects consist of the subspace of K and whose morphisms consist of the bounded operators between these subspaces. The forgetful functor H is a functor from Rep(B, K) to Sub(K), and a natural transformation from H to itself will be a map from the objects of Rep(B, K) to morphisms in Sub(K). Thus we need a topology on the set of morphisms of Sub(K). Now any morphism in Sub(K) can be viewed as a bounded operator on K by defining it to be zero on the orthogonal complement of its domain. We obtain in this way a mapping from the set of morphisms of Sub(K) onto $L(K)$, and we equip the morphisms of Sub(K) with the pre-image under this mapping of the strong operator topology on $L(K)$. Note that the resulting topology on the morphisms will not be Hausdorff. (As in the theorem on [3, p. 97] we could just as well use the weak, ultra-weak or ultra-strong operator topologies on $L(K)$.)

2.9. Definition. Let H be the forgetful functor from Rep(B, K) to Sub(K). A natural transformation from H to itself will be said to be *continuous* if it is continuous as a function from the set of objects of Rep(B, K) equipped with the strong Fell topology to the set of morphisms of Sub(K) equipped with the topology defined above.

The following is our reformulation of the main theorem of [33] and [3]:

2.10. Theorem. *Let B be a C^*-algebra, and let K be a Hilbert space of infinite dimension large enough so that every cyclic representation of B can be realized on some subspace of K. Let H be the forgetful functor from* Rep(B, K) *to* Sub(K), *and let C be the collection of natural transformations from H to itself which are continuous in the sense of the above definition. Then C forms a C^*-algebra which is in a natural way isomorphic to B.*

This theorem can be proved by making trivial modifications of the proof given in [3], so we will not include a proof here.

3. Functors between categories of Hermitian modules

Let A and B be C^*-algebras. In this section we will study some general methods for constructing functors from Hermod-B to Hermod-A.

One general method for constructing functors from Hermod-B to Hermod-A was introduced in [30], in terms of what were called there Hermitian B-rigged A-modules (these being a generalization to the non-commutative case of the "C^*-modules" which Kaplansky [21] introduced for commutative C^*-algebras). For our present purposes it will be useful to change slightly the definition of these objects (by no longer requiring the range of the B-valued inner product to span a dense submanifold of B, and by requiring completeness).For this reason we will include the definitions here. Contrary to the case in [30], there seems to be no advantage here in considering pre-C^*-algebras, so we make our definitions only for C^*-algebras.

3.1. Definition. Let B be a C^*-algebra. By a (right) *pre-B-rigged space* we mean a vector space, X, over the complex numbers on which B acts by means of linear transformations in such a way that X is a right B-module (in the algebraic sense), and on which there is defined a B-valued pre-inner-product, that is, a B-valued sesquilinear form $\langle\ ,\ \rangle_B$ conjugate linear in the first variable, such that

(1) $\langle x, x\rangle_B \geqslant 0$ for all $x \in X$,

(2) $(\langle x, y\rangle_B)^* = \langle y, x\rangle_B$ for all $x,y \in X$,

and having the further property that

(3) $\langle x, yb\rangle_B = \langle x, y\rangle_B\, b$ for all $x,y \in X$, $b \in B$.

It is easily seen that if we factor a pre-B-rigged space by the subspace of the elements x for which $\langle x, x\rangle_B = 0$, the quotient becomes in a natural way a pre-B-rigged space having the additional property that its inner product is definite, that is, $\langle x, x\rangle_B > 0$ for all non-zero x in X. On a pre-B-rigged space with definite inner product we can define a norm $\|\ \|$ by setting

$$\|x\| = \|\langle x, x\rangle_B\|^{1/2} \tag{3.2}$$

for $x \in X$. (See [30, Proposition 2.10] for the verification that this is indeed a norm.) From now on we will always view a pre-B-rigged space with definite inner product as being equipped with this norm. Then the completion of X with respect to this norm is easily seen to become again a pre-B-rigged space. (This matter is discussed around [27, 2.5].)

3.3. Definition. Let B be a C^*-algebra. By a *B-rigged space* we will mean a pre-B-rigged space, X, satisfying the following additional conditions:

(1) If $\langle x, x\rangle_B = 0$, then $x = 0$, for all $x \in X$.

(2) X is complete for the norm defined in (3.2).

Viewing a B-rigged space as a generalization of an ordinary Hilbert space, we can define what we mean by bounded operators on a B-rigged space, as was done in [30, Definition 2.3] and following [27, 2.5]. That these two definitions are equivalent is shown in [27, Theorem 2.8]. For completeness we include here a definition of bounded operators which lumps together the definitions from [30] and [27].

3.4. Definition. Let X be a B-rigged space. By a *bounded operator* on X we mean a linear operator, T, from X to itself which satisfies either of the equivalent conditions

(1) for some constant k_T we have

$$\langle Tx, Tx\rangle_B \leqslant k_T\langle x, x\rangle_B \quad \text{for all } x \in X\,;$$

(1$'$) T is continuous with respect to the norm on X;

and the condition

(2) there is a linear operator, T^*, on X, satisfying conditions (1) and (1$'$) above, such that

$$\langle Tx, y\rangle_B = \langle x, T^*y\rangle_B \quad \text{for all } x,y \in X.$$

It is easily seen that any bounded operator on a B-rigged space X will automatically commute with the action of B on X (because it has an adjoint).

We will denote by $L(X)$ (or $L_B(X)$ if there is a chance of confusion) the set of all bounded operators on X. Then it is easily verified that with the operator norm $L(X)$ is a C^*-algebra ([30, Proposition 2.12] or comments after [27, 2.5]).

3.5. Definition. Let A and B be C^*-algebras. By a *Hermitian B-rigged A-module* we mean a B-rigged space, X, which is a left A-module by means of a $*$-homomorphism of A into $L(X)$, and which is non-degenerate as an A-module in the sense that AX is dense in X with respect to the norm on X.

This definition differs from [30, Definition 4.19] only in that we do not require here that the range of the B-valued inner product on X span a dense submanifold of B.

In [30, Section 5] it was shown how to use a Hermitian B-rigged A-module, X, to construct a functor from Hermod-B to Hermod-A. We recall the definition here. If $V \in$ Hermod-B, then we can form the algebraic tensor product $X \otimes_B V$, and equip it with an ordinary pre-inner-product which is defined on elementary tensors by

$$\langle x \otimes v, x' \otimes v' \rangle = \langle\langle x', x\rangle_B v, v'\rangle_V .$$

Completing the quotient of $X \otimes_B V$ by the subspace of vectors of length zero, we obtain an ordinary Hilbert space, on which A acts (by $a(x \otimes v) = ax \otimes v$) to give a $*$-representation of A. We will denote the corresponding Hermitian A-module by ${}^A V$ (or ${}^A_X V$ if there is a chance of confusion concerning which Hermitian B-rigged A-module is being used). The only difference between the situation here and that in [30] is that, because we no longer require the range of the B-valued inner product on B to span a dense subset of B, we can no longer conclude that ${}^A V$ will not be the zero-dimensional A-module. In fact, it is easily seen that whenever the kernel of the representation of B on V contains the range of the B-valued inner product, then indeed ${}^A V$ will be the zero-dimensional A-module. However, as long as we consider the zero-dimensional A-module to be a non-degenerate A-module, we can still assert that ${}^A V$ will be a non-degenerate A-module for all $V \in$ Hermod-B. Actually, it is easily seen that if the above construction is applied even to a degenerate B-module, it will nevertheless produce a non-degenerate A-module, because of the fact that we are assuming that X is non-degenerate. This remark will be useful shortly.

The above construction defines a functor if for $V, W \in$ Hermod-B and $f \in \mathrm{Hom}_B(V, W)$ we define ${}^A f \in \mathrm{Hom}_A({}^A V, {}^A W)$ on elementary tensors by ${}^A f(x \otimes v) = x \otimes f(v)$.

3.6. Definition. For any Hermitian B-module V the Hermitian A-module ${}^A V$ will be called the *Hermitian A-module obtained by inducing V up to A via X.* We will call the corresponding functor the *inducing functor determined by X.*

These inducing functors have some fairly nice properties. For example, they preserve weak containment of representations, as can be seen by arguments similar to those in the proof of [30, Proposition 6.26], and they can be seen to preserve direct integrals when sense can be made of this.

We would now like to generalize the above construction to obtain a wider class of functors. As before, let A and B be C^*-algebras, and now let $n(B)$ be the W^*-enveloping algebra of B. Let D be any C^*-subalgebra of $n(B)$, and let X be a Hermitian D-rigged A-module. Then we can use X to define a functor from Hermod-B to Hermod-A as follows. For any $V \in$ Hermod-B, we can view V as a normal $n(B)$-module, and we can then restrict the action to D, obtaining a (possibly degenerate) D-module. We can then apply the construction described above using X to obtain an A-module which will be non-degenerate for the reason mentioned at the end of the next to last paragraph preceding 3.6. In this way we obtain a functor from Hermod-B to Hermod-A. This process is slightly round-about, and we will see shortly that it can be given a neater formulation. But first we will consider some examples.

3.7. Example. Let $A = B = C([0, 1])$, the algebra of continuous functions on the unit interval. The dual B' of B consists of the Borel measures on $[0, 1]$, and so it is clear that the algebra of bounded Borel functions on $[0, 1]$ can be viewed as a subalgebra of $B'' = n(B)$. Let D be the subalgebra of bounded functions on $[0, 1]$ which have value zero except at a countable set of points, and let $X = D$. Let A and D act on X by pointwise multiplication, and define a D-valued inner product on X by letting $\langle x, y\rangle_D$ be the pointwise product of y with the complex conjugate of x. Note that for $x, y \in D$ the element $\langle x, y\rangle_D$, viewed as a linear functional on B', is zero on any purely continuous measure. Then it is easily seen that the functor determined by X as described above will assign to every Hermitian B-module its atomic part, that is, the sum of the irreducible modules which it contains. A similar construction can be carried out for any C^*-algebra. This functor preserves neither weak containment nor direct integrals.

3.8. Example. Let $A = B = C([0, 1])$ as above, and let $x = C([0, 1])$ with the evident pointwise action of A. Let D be the subalgebra of $B'' = n(B)$ which is the range of the map p of $C([0, 1])$ into B'' which assigns to each $f \in C([0, 1])$ the linear function $p(f)$ on B' defined by

$$p(f)(m) = \int f \, \mathrm{d}m_c ,$$

where m_c is the continuous part of the measure m. Let D, viewed as $C([0, 1])$, act on X in the evident pointwise way, and define a D-valued inner product on X by

$$\langle f, g\rangle_D = p(\bar{f} g) \quad \text{for } f, g \in X .$$

Then it is easily seen that the functor determined by X assigns to every Hermitian B-module its continuous part, that is, the complement of its atomic part. A similar construction can be carried out for any C^*-algebra.

3.9. Example. Let $A = B = C([0, 1])$ as in the two examples above, and let D be the algebra of bounded Borel functions on $[0, 1]$, viewed as a subalgebra of $n(B)$ as described in Example 3.7. Let X be as in Example 3.7 with pointwise action of A and D. Let Y be D with D-valued inner product defined in the usual way. It is easily seen that the functor determined by Y is naturally equivalent to the identity functor on Hermod-B (a similar situation occurs in the paragraph before [30, Theorem 6.23]).

Let $Z = X \oplus Y$, viewed as a Hermitian D-rigged A-module in the obvious way. Then the functor determined by Z doubles the atomic part of any Hermitian B-module while keeping the continuous part fixed (up to equivalence). This functor preserves weak containment but not direct integrals.

3.10. Example. Let $A = B$ be the B of Example 1.4 and let $D = n(B) = l_\infty$. Let X be the A of Example 1.4, with B and D acting by pointwise multiplication on X, and with the obvious D-valued inner product on X. Then it is easily seen that the functor determined by X is naturally equivalent to the identity functor. But this is also true if we let $X = D$. Thus non-isomorphic Hermitian rigged modules, in our present generalized sense, can define equivalent functors, contrary to the purely algebraic case (see [2, Theorem 2.3]). However, we will see later that we can recover uniqueness by putting further conditions on the Hermitian rigged modules which are used (see Section 6).

3.11. Example. Let S and T be compact Hausdorff spaces. Let $B = C(S)$, $A = C(T)$, and let D be the algebra of bounded Borel functions on S, identified with a subalgebra of $n(B) = B''$ in the usual way. Let $X = D$. Suppose we are given a Borel measurable mapping q from S to T. Define an action of A on X by

$$(h \cdot f)(s) = h(q(s)) f(s)$$

for $h \in A$, $f \in X$, $s \in S$. Then X becomes a Hermitian D-rigged A-module, and the corresponding functor maps irreducible B-modules according to the mapping q and other modules according to their direct integral decompositions into irreducibles. In particular, this functor will preserve direct integrals, but it will not preserve weak containment unless q is continuous.

3.12. Example. Let $A = B = C([0, 1])$, and let D and X be defined as in Example 3.7, except for the action of A. Let q be a (possibly non-measurable) permutation of the points of $[0, 1]$, and let A act on X by setting

$$(h \cdot f)(s) = h(q(s)) f(s) .$$

Then the corresponding functor will send all continuous modules to the zero module, while it will permute the irreducible modules (and so the atomic modules) according to q. In particular, such a functor can carry a measurable field of irreducible representations to a field which is not measurable.

3.13. Example. Let $A = B = C([0, 1])$. Notice that the D's of Examples 3.9 and 3.12 are orthogonal subalgebras of $n(B)$, so that the direct sum of these two algebras can be viewed in a natural way as a subalgebra of $n(B)$. Let X be the direct sum of the X's from Examples 3.9 and 3.12. Then the corresponding functor carries each purely continuous Hermitian module to an equivalent module, while it permutes the atomic representations according to the permutation q of Example 3.12. In particular, this functor is an equivalence of Hermod-B with itself which need not preserve weak containment or direct integrals and can carry a measurable field of modules to a non-measurable field.

We will now reformulate the above general construction in a way which is more elegant but which is somewhat more difficult to work with in specific examples such as those given above. Specifically, let A and B be C^*-algebras, D a subalgebra of $n(B)$ and X a Hermitian D-rigged A-module. Then we can form the algebraic tensor product $X \otimes_D n(B)$ and define on it an $n(B)$-valued sesquilinear form by

$$\langle x \otimes n, x' \otimes n' \rangle_{n(B)} = \langle \langle x', x \rangle_D n, n' \rangle_{n(B)} = n^* \langle x, x' \rangle_D n' .$$

This is just a special case of the construction used in [30, Theorem 5.9] or in [27, Section 4], except that $n(B)$ may be degenerate as a D-module. Nevertheless the sesquilinear form can be shown to be non-negative, either by imitating the proof indicated for [30, Theorem 5.9] (and splitting the degenerate D-modules which occur into their non-degenerate part plus null part) or by using [27, Proposition 6.1].

3.14. Proposition. *Let A and B be C^*-algebras, let D be a C^*-subalgebra of $n(B)$ and let X be a Hermitian D-rigged A-module. Let Y be the Hermitian $n(B)$-rigged A-module obtained by equipping $X \otimes_D n(B)$ with the pre-inner-product defined above, by factoring by the elements of length zero, and completing. Then the functor F_Y initially defined on* Hermod-$n(B)$ *but restricted to* Normod-$n(B)$ *(which equals* Hermod-B*) is naturally equivalent to the functor from* Hermod-B *to* Hermod-A *constructed from X by the method described just before Example* 3.7.

Proof. Let $V \in$ Hermod-B, and view V as a normal $n(B)$-module which can be restricted to D. Then the natural map t_V from $Y \otimes_{n(B)} V$ to $X \otimes_D V$ defined on elementary tensors by

$$t_V((x \otimes n) \otimes v) = x \otimes nv$$

is easily verified to provide the required natural unitary equivalence. □

In view of this result we may from now on restrict our attention to Hermitian $n(B)$-rigged A-modules and the functors which they define from Hermod-B to Hermod-A, though the earlier construction is still useful for considering specific examples.

We remark that even by so restricting the rigged modules we consider to those defined over $n(B)$ (in other words, by requiring the algebra D of the earlier construction to be $n(B)$ itself), it is still possible for two non-isomorphic Hermitian $n(B)$-rigged A-modules to define equivalent functors, as is shown by Example 3.10.

4. Properties of functors

With the eventual aim of characterizing the functors which are defined by Hermitian $n(B)$-rigged A-modules, we now study the properties which these functors possess. Throughout this paper we will always assume that the functors considered are linear with respect to the linear structures on the Hom spaces.

As was mentioned earlier, Hermod-B carries a natural involution, that is, a conjugate linear contravariant functor of period two which is the identity on objects, namely the functor which takes each morphism to its adjoint.

4.1. Definition. Let C and D be categories whose spaces of morphisms carry the structure of complex vector spaces, and let C and D each have an involution, denoted by *. Then a linear functor F from C to D will be said to be a *$*$-functor* if

$$F(f^*) = F(f)^*$$

for every morphism f in C.

4.2. Proposition. *Let A and B be C^*-algebras, let X be a Hermitian $n(B)$-rigged A-module, and let F_X be the corresponding functor from* Hermod-B *to* Hermod-A *(equipped with their natural involutions). Then F_X is a $*$-functor.*

The proof consists of a straightforward computation.

We now show that $*$-functors between categories of Hermitian modules are norm decreasing, in analogy with the well-known fact for $*$-homomorphisms between C^*-algebras.

4.3. Proposition. *Let A and B be C^*-algebras, and let F be a $*$-functor from* Hermod-B *to* Hermod-A. *Then $\| F(f) \| \leq \| f \|$ for every morphism f in* Hermod-B.

Proof. Let $f \in \mathrm{Hom}_B(V, W)$ for $V, W \in$ Hermod-B. Now $\mathrm{Hom}_B(V, V)$ and $\mathrm{Hom}_A(F(V), F(V))$ are both von Neumann algebras, and F restricted to $\mathrm{Hom}_B(V, V)$ is a $*$-homomorphism, and so is norm decreasing [7, p. 8]. But $f^*f \in \mathrm{Hom}_B(V, V)$, so that

$$\| F(f) \|^2 = \| F(f)^* F(f) \| = \| F(f^*f) \| \leq \| f^*f \| = \| f \|^2 . \square$$

We now consider a somewhat less obvious property of the functors defined by Hermitian rigged modules. If $V \in$ Hermod-B, then $\mathrm{Hom}_B(V, V)$ is not only a Banach space, but in fact a von Neumann algebra, carrying in particular the ultra-weak operator topology. More generally, given $V, W \in$ Hermod-B, the space $\mathrm{Hom}_B(V, W)$, while not an algebra, still carries a natural analogue of the ultra-weak operator topology, which, following the usage in [32] for W^*-algebras, we will call simply the weak topology. We will also have brief use for the analogue of the ultra-strong operator topology on $\mathrm{Hom}_B(V, W)$.

4.4. Definition. Let $V, W \in$ Hermod-B. By the *weak topology* on $\mathrm{Hom}_B(V, W)$ we will mean the topology defined by the linear functionals of the form

$$f \mapsto \sum \langle f(v_i), w_i \rangle ,$$

where $\{v_i\}$ and $\{w_i\}$ are sequences of elements from V and W respectively such that

$$\sum \| v_i \|^2 < \infty, \qquad \sum \| w_i \|^2 < \infty .$$

By the *ultra-strong operator topology* on $\mathrm{Hom}_B(V, W)$ we will mean the topology defined by the seminorms of the form

$$f \mapsto (\sum \| f(v_i) \|^2)^{1/2} ,$$

where $\{v_i\}$ is a sequence of elements from V such that $\Sigma \| v_i \|^2 < \infty$.

From [28, Theorem 1.4] it follows immediately that $\mathrm{Hom}_B(V, W)$ is a dual Banach space, with the weak-$*$ topology corresponding to the weak topology just defined, in analogy with the well-known situation for von Neumann algebras.

We have the following analogue of well-known facts for von Neumann algebras, principally [7, Lemma 2, p. 35]:

4.5. Proposition. *Let $V, W \in$* Hermod-B. *Then the ultra-strong operator topology on* $\mathrm{Hom}_B(V, W)$ *is stronger than the weak topology. The ultra-strongly continuous linear functionals on* $\mathrm{Hom}_B(V, W)$ *are exactly the linear functionals used in Definition* 4.4 *to define the weak topology. A net $\{f_k\}$ of elements of* $\mathrm{Hom}_B(V, W)$ *converges ultra-strongly to* 0 *if and only if the net $\{f_k^* f_k\}$ converges to* 0 *in the weak topology of* $\mathrm{Hom}_B(V, V)$.

The proof is obtained by making minor modifications to the proofs of the corresponding facts for von Neumann algebras.

In view of Definition 4.4 it now makes sense to ask whether a mapping between spaces of homomorphisms is normal, that is, continuous for the weak topology, and in particular we can ask this of the mappings between spaces of homomorphisms defined by a functor.

4.6. Definition. Let A and B be C^*-algebras, and let F be a $*$-functor from Hermod-B to Hermod-A (or between full subcategories thereof). Then we will say that F is *normal* if for any $V, W \in$ Hermod-B (or its full subcategory) the mapping from $\mathrm{Hom}_B(V, W)$ to $\mathrm{Hom}_A(F(V), F(W))$ defined by F is normal, that is, continuous for the weak topologies.

The main full subcategory which will interest us (in the next section) is the category of normal modules over a W^*-algebra.

Actually, the next result shows that in order for a $*$-functor to be normal it suffices for it to be normal on the von Neumann algebras of form $\mathrm{Hom}_B(V, V)$.

4.7. Proposition. *Let F be a $*$-functor from* Hermod-B *to* Hermod-A. *If for every $V \in$* Hermod-B *the homomorphism from* $\mathrm{Hom}_B(V, V)$ *to* $\mathrm{Hom}_A(F(V), F(V))$ *defined by F is normal, then F is normal.*

Proof. Let $V, W \in$ Hermod-B. We must show that the map from $\mathrm{Hom}_B(V, W)$ to $\mathrm{Hom}_A(F(V), F(W))$ defined by F is normal. We show first that it is continuous for the ultra-strong operator topologies. Let $\{f_k\}$ be a net of elements of $\mathrm{Hom}_B(V, W)$ which converges ultra-strongly to 0. Then by the last part of Proposition 4.5 the net $\{f_k^* f_k\}$ in $\mathrm{Hom}_B(V, V)$ converges weakly to 0. Since F is assumed normal on $\mathrm{Hom}_B(V, V)$, it follows that the net $\{F(f_k^* f_k)\}$ in $\mathrm{Hom}_A(F(A), F(A))$ converges weakly to 0. But F is a $*$-functor, and so $F(f_k^* f_k) = F(f_k)^* F(f_k)$. Then again by the last part of Proposition 4.5 it follows that the net $\{F(f_k)\}$ in $\mathrm{Hom}_A(F(V), F(W))$ converges ultra-strongly to 0. Thus F is ultra-strongly continuous. But by using the first two parts of Proposition 4.5 it is easily seen that any linear map between spaces of homomorphisms which is ultra-strongly continuous is also weakly continuous. □

4.8. Theorem. *Let A and B be C^*-algebras, let X be a Hermitian $n(B)$-rigged A-module, and let F_X be the corresponding functor from* Hermod-B *to* Hermod-A. *Then F_X is normal.*

Proof. According to Proposition 4.7 it suffices to show that for every $V \in$ Hermod-B the $*$-homomorphism from $\mathrm{Hom}_B(V, V)$ to $\mathrm{Hom}_A(F_X(V), F_X(V))$ defined by F_X is normal. Now let $x \otimes v, x' \otimes v' \in F_X(V)$ be given, and let $f \in \mathrm{Hom}_B(V, V)$. Then

$$\langle F_X(f)(x \otimes v), x' \otimes v' \rangle = \langle f(v), \langle x, x' \rangle_{n(B)} v' \rangle ,$$

and so it is clear that the composition of the homomorphism defined by F_X with the linear functional defined by two elementary tensors is weakly continuous. This will then also be true for any two finite tensors. But the finite tensors are dense in $F_X(V)$, and a routine argument using [7, Theorem 1, p. 38] shows from this that the homomorphism defined by F_X is normal as desired. □

We now examine an important property of normal $*$-functors.

4.9. Proposition. *Let A and B be C^*-algebras, and let F be a normal $*$-functor from* Hermod-B *to* Hermod-A. *Then F preserves (possibly infinite) Hilbert space direct sums. That is, if $\{V_i\}$ is a family of elements of* Hermod-B *and if $V \cong \oplus V_i$, then $F(V) \cong \oplus F(V_i)$.*

Proof. For each k let p_k denote the canonical mapping of $\oplus V_i$ onto V_k. Then p_k^* is the canonical inclusion of V_k into $\oplus V_i$, $p_k p_k^*$ is the identity map on V_k, $p_k^* p_k$ is the projection onto $p_k^*(V_k)$, and $p_k p_j^* = 0$ if $k \neq j$. Then $F(p_k) F(p_k)^*$ will be the identity on $F(V_k)$, $F(p_k)^* F(p_k)$ will be a projection in $F(\oplus V_i)$, and $F(p_k) F(p_j)^* = 0$ if $k \neq j$. It follows that the $F(p_k)^*$ define a natural injection of $\oplus F(V_i)$ into $F(\oplus V_i)$. We must use the normality of F to show that this injection is also surjective.

Now for any finite subset N of the index set for the V_i, the operator

$$p_N = \sum \{p_k^* p_k : k \in N\}$$

is the projection of $\oplus V_i$ onto the subspace corresponding to $\oplus \{V_k : k \in N\}$. Then it is easily seen that the net $\{p_N\}$ converges weakly to the identity operator on $\oplus V_i$. Since we are assuming that F is normal, it follows that $\{F(p_N)\}$ converges weakly to the identity operator on $F(\oplus V_i)$. From this it follows easily that the injection of $\oplus F(V_i)$ into $F(\oplus V_i)$ is surjective. □

We remark that since in Hermod-B every short exact sequence splits, it follows that any functor preserves cokernels. Thus normal functors, in that they also preserve direct sums, are a natural analogue of the right-continuous functors defined on [2, p. 58]. Indeed, one of the main results of this paper will be an analogue for normal functors of the Eilenberg–Watts theorem for right-continuous functors ([11, 38] or [2, p. 58]).

We now give an example of a $*$-functor which is not normal.

4.10. Example. We begin by considering a method for constructing functors from the category, Hilbert, of Hilbert spaces (Hermitian C-modules) to itself. This construction, which deserves further study, was suggested to me by George M. Bergman. Let $K \in$ Hilbert, let $L(K) = \mathrm{Hom}(K, K)$, and let p be any state (normal or not) on $L(K)$. We use p to construct a functor F_p from Hilbert to itself. Given $V \in$ Hilbert, define a pre-inner-product on $\mathrm{Hom}(K, V)$ by

$$\langle f, g \rangle = p(g^* f) ,$$

and let $F_p(V)$ be the corresponding Hilbert space obtained by factoring by the vectors of length zero and completing. Given $V, W \in$ Hilbert and $h \in \mathrm{Hom}(V, W)$, let $\tilde{h}$ denote the map from $\mathrm{Hom}(K, V)$ to $\mathrm{Hom}(K, W)$ defined by $\tilde{h}(f) = h \circ f$ for $f \in \mathrm{Hom}(K, V)$. Then it is easily verified that $\tilde{h}$ is continuous for the pre-inner-products defined above, and so defines a continuous operator $F_p(\tilde{h})$ from $F_p(V)$ to $F_p(W)$. It is easily seen that F_p, defined in this way, is a $*$-functor.

Now let K be an infinite-dimensional Hilbert space, and let $L_c(K)$ denote the algebra of compact operators on K, so that $L_c(K)$ is a two-sided ideal in $L(K)$. Let p be a state of $L(K)$ which is zero on $L_c(C)$ (so that p is not a normal state). Then F_p is not normal, for it is easily seen that if $h \in \mathrm{Hom}(V, W)$ and if h is compact, then $F_p(h) = 0$. But the identity operator on any Hilbert space is the weak limit of compact operators.

5. Normal modules and the Eilenberg–Watts theorem

We have seen that if B is any C^*-algebra, then Hermod-B is isomorphic to Normod-$n(B)$. Thus any functor from Hermod-B to Hermod-A can equally well be viewed as a functor from Normod-$n(B)$ to Normod-$n(A)$. We are thus led naturally to make a general study of functors between categories of normal modules over W^*-algebras.

Let M and N be W^*-algebras. Since Normod-N is a full subcategory of Hermod-N, we know what is meant by a normal $*$-functor from Normod-N to Normod-M. In view of the results of Section 3 we would expect to construct such functors in terms of some kind of M-N-bimodules with N-valued inner product. We will show now that this is in fact the case, the main difference from the previous section being that we must here ensure that the range of the functor consists of elements of Normod-M and not just Hermod-M. In what follows we will, as before, refer to the ultra-weak operator topology on W^*-algebras as just the weak topology.

5.1. Definition. Let M and N be W^*-algebras. By a *normal N-rigged M-module* we mean a Hermitian N-rigged M-module, X, which has the added property that for every $x, y \in X$ the linear map $m \mapsto \langle x, my \rangle_N$ from M to N is normal, that is, weakly continuous.

5.2. Theorem. *Let M and N be W^*-algebras, and let X be a normal N-rigged M-module. For any $V \in$ Normod-N let $F_X(V)$ be the Hermitian M-module obtained by inducing V up to M via X. Then $F_X(V)$ is in fact in* Normod-M. *In this way we obtain a normal $*$-functor F_X from* Normod-N *to* Normod-M.

Proof. Consider two elementary tensors $x \otimes v$ and $x' \otimes v'$ in $F_X(V)$. Then the functional

$$m \mapsto \langle m(x \otimes v), x' \otimes v' \rangle = \langle \langle x', mx \rangle_N v, v' \rangle$$

is weakly continuous because of the normality of X and V. But the elementary tensors span a dense subspace of $F_X(V)$, and from this it is easily seen that $F_X(V)$ is normal.

Thus the inducing functor from Hermod-N to Hermod-M defined by X as in Section 3 carries the full subcategory Normod-N into the full subcategory Normod-M.

Since it is normal as a functor from Hermod-N (Theorem 4.8), it follows immediately that its restriction to Normod-N is also a normal $*$-functor. □

Note that we will use the symbol F_X to denote both the functor from Hermod-N and its restriction to Normod-N, but this should not cause any confusion.

We would like to prove conversely that every normal $*$-functor from Normod-N to Normod-M is of the form F_X for some normal N-rigged M-module X. To do this we must somehow produce such an X from any normal $*$-functor. As motivation for how to do this, we now consider how, given a normal N-rigged M-module X, we can recover X from F_X. Now for any $V \in$ Normod-N and any $x \in X$ we can define a bounded linear operator t_V^x from V to $F_X(V) = {}^M V$ by

$$t_V^x(v) = x \otimes v .$$

In general t_V^x will not respect any actions of M or N. However, if $W \in$ Normod-N and if $f \in \mathrm{Hom}_N(V, W)$, then it is easily seen that

$$t_W^x(f(v)) = F_X(f)(t_V^x(v))$$

for any $v \in V$. This says that, for fixed x, the family of maps t_V^x as V ranges over Normod-N forms a natural transformation from the forgetful functor H_N from Normod-N to Hilbert (the category of Hilbert spaces), to the forgetful functor H_M from Normod-M composed with F_X, that is, a natural transformation from H_N to $H_M \circ F_X$. Thus we can hope to recover X as the collection of all natural transformations from H_N to $H_M \circ F_X$. We will see later (Section 6) that this can often enough be done, but not always, since we have already seen that non-isomorphic X's can define equivalent functors (Example 3.15 with A taken to be $L(K)$).

In view of the above considerations it is appropriate to develop some tools for handling natural transformations. Note that the category Hilbert can be considered to be Normod-**C**, and that the definition of a normal N-module says that the functor H_N is a normal functor. Similarly $H_M \circ F_X$ will be a normal $*$-functor. Thus it is appropriate to study natural transformations between pairs of normal $*$-functors from Normod-N to Normod-M, where in our first applications we will take M to be **C**. But the following results will also be useful when we study the uniqueness of X in the next section.

5.3. Proposition. *Let M and N be W^*-algebras, and let F and G be normal $*$-functors from* Normod-N *to* Normod-M. *Let s and t be two natural transformations from F to G. If $s_U = t_U$ for some generator U for* Normod-N, *then $s = t$.*

Proof. Let $V \in$ Normod-N, and assume first that there is an isometric isomorphism f of V onto a submodule of U, so that $f^* f$ is the identity operator on V. Then

$$G(f)\, s_V = s_U F(f) = t_U F(f) = G(f)\, t_V .$$

Multiplying on the left by $G(f^*)$ and using the fact that $G(f^*)\, G(f)$ is the identity operator on $G(V)$, we find that $s_V = t_V$.

Suppose now that V is an arbitrary element of Normod-N. Then V is the direct sum of copies of submodules of U (Proposition 1.1), on which we have just seen that s and t agree. From the fact that normal $*$-functors preserve direct sums (Proposition 4.9) it follows easily that s and t agree on V. □

In preparation for the next result we note that if $V \in$ Normod-N, then $\mathrm{Hom}_N(V, V)$, which we will denote by $c(N)$ (or $c_V(N)$ if there is a chance of confusion), is the commutant of the action of N on V, and is a von Neumann algebra. As a result, V (or more precisely $H_N(V)$) can also be viewed as a normal $c(N)$-module. Then if F is a normal $*$-functor from Normod-N to Normod-M, F will define a normal homomorphism from $c(N) = \mathrm{Hom}_N(V, V)$ into $\mathrm{Hom}_M(F(V), F(V))$, and via this homomorphism $F(V)$ also can be viewed as a normal $c(N)$-module.

5.4. Proposition. *Let M and N be W^*-algebras, and let F and G be normal $*$-functors from* Normod-N *to* Normod-M. *Let U be a generator for* Normod-N, *and let $c(N) = \mathrm{Hom}_N(U, U)$, so that $F(U)$ and $G(U)$ can be viewed as normal $c(N)$-modules (as well as normal M-modules). Then the assignment to any natural transformation t from F to G of the linear transformation t_U from $F(U)$ to $G(U)$ establishes a bijection between the natural transformations from F to G and the linear transformations from $F(U)$ to $G(U)$ which commute both with the actions of M and with the actions of $c(N)$ defined by F and G respectively. Under this bijection natural equivalences correspond to invertible transformations.*

Proof. If t is a natural transformation from F to G, then t_U must be a morphism in Normod-M, and so is a linear transformation which commutes with the action of M. But by the definition of a natural transformation, t_U must also commute with the action of $c(N)$. Thus t_U commutes with the actions of both M and $c(N)$ as required. Furthermore, the mapping $t \mapsto t_U$ is injective by Proposition 5.3. We must show that this mapping is surjective.

Let T be a linear transformation from $F(U)$ to $G(U)$ which commutes with the actions of both M and $c(N)$. We wish to extend T to a natural transformation from F to G. Let $V \in$ Normod-N, and assume first that there is an isometric isomorphism f of V onto a submodule of U, so that $f^* f$ is the identity operator on V. Define t_V by

$$t_V = G(f^*)\, TF(f)\,.$$

Since T commutes with the action of M, it is clear that $t_V \in \mathrm{Hom}_M(F(V), G(V))$. We must show that the definition of t_V does not depend on the choice of f. Let g be another isometric isomorphism of V into U. Then $fg^* \in \mathrm{Hom}_N(U, U) = c(N)$. Since T is assumed to commute with the actions of $c(N)$ defined by F and G, it follows that

$$TF(fg^*) = G(fg^*)\, T\,.$$

Furthermore $fg^* g = f$. Then a straightforward calculation using these results shows that

$$t_V = G(g^*)\, TF(g)\,,$$

so that t_V does not depend on the choice of f.

Suppose now that $V, W \in$ Normod-N and that there are isometric isomorphisms f and g of V and W respectively with submodules of U. Then for any $h \in \mathrm{Hom}_N(V, W)$ we have $ghf^* \in \mathrm{Hom}_N(U, U) = c(N)$, so that

$$TF(ghf^*) = G(ghf^*)\, T\,.$$

Then a straightforward calculation shows that

$$t_W F(h) = G(h)\, t_V\,,$$

which is the characteristic property of natural transformations.

If V is an arbitrary element of Normod-N, the definition of t_V and the verification that it is a natural transformation now follow in a routine way by decomposing V into the direct sum of modules which are isomorphic to submodules of U, and by using the fact that normal $*$-functors commute with direct sums (Proposition 4.9).

Finally, if T is invertible, then its inverse can be extended by the above process to a natural transformation, say r, from G to F, and $(r \circ t)_U = r_U \circ t_U$ will be the identity of $F(U)$. Thus $r \circ t$ agrees on U with the identity natural transformation from F to itself. It follows from Proposition 5.3 that $r \circ t$ is the identity natural transformation. In a similar way it is seen that $t \circ r$ is the identity natural transformation from G to itself. Thus t is a natural equivalence as desired. □

Suppose now that F is a normal $*$-functor from Normod-N to Normod-M, and that we wish to find an X such that F is naturally equivalent to F_X. In the discussion preceding Proposition 5.3 it was suggested that we take as X the collection of all natural transformations from H_N to $H_M \circ F$. But in view of Proposition 5.4 this is equivalent to choosing a generator U and then taking all linear transformations from $H_N(U)$ to $H_M(F(U))$ which commute with the action of $c(N) = \mathrm{Hom}_N(U, U)$. (Note that since H_M and H_N have values in Hilbert, the M of Proposition 5.4 is just $\mathbf{C}$, and so plays little role.) For simplicity of notation we will omit the symbols H_N and H_M, so that we can write $X = \mathrm{Hom}_{c(N)}(U, F(U))$. We are now in a position to prove one of the main theorems of this paper, namely, the analogue of the Eilenberg–Watts theorem ([11, 38] or [2, p. 58]).

5.5. Theorem. *Let M and N be W^*-algebras, and let F be a normal $*$-functor from* Normod-N *to* Normod-M. *Then there is a normal N-rigged M-module X such that F is naturally equivalent to F_X. In fact, if U is any generator for* Normod-N, *and if $c(N) = \mathrm{Hom}_N(U, U)$, then we can take X to be*

$$X = \mathrm{Hom}_{c(N)}(U, F(U))\,.$$

Proof. Let U be a generator for Normod-N and let X be defined as indicated. We must first show in what way X is a normal N-rigged M-module. Now since the action of N on U commutes with that of $c(N)$, X becomes a right N-module if an action is defined by $(xn)(u) = x(n(u))$ for $x \in X$, $n \in N$, $u \in U$. Since U is a generator for Normod-N, N is faithfully represented on U (Proposition 1.3), and it follows from von Neumann's double commutant theorem [7, p. 41] that N can be identified with $\mathrm{Hom}_{c(N)}(U, U)$. Now if $x, y \in X$, then $x^* y \in \mathrm{Hom}_{c(N)}(U, U)$, and so, from the above, $x^* y$ can be viewed as an element of N. Accordingly we define an N-valued inner product on X by

$$\langle x, y \rangle_N = x^* y$$

for $x, y \in X$. It is easily verified that in this way X becomes an N-rigged space. (This is a special case of [30, Example 4.26].)

Now $F(U)$ is an M-module, and so we can define an action of M on X by

$$(mx)(u) = m(x(u))$$

for $m \in M$, $x \in X$, $u \in U$. It is easily verified that with this action X becomes a Hermitian N-rigged M-module. Furthermore, for any $x, y \in X$ the map

$$m \mapsto \langle x, my \rangle_N = x^* my$$

from M to N is clearly continuous for the weak topologies on M and N. Thus X is in fact a normal N-rigged M-module.

We can now use X to define a normal $*$-functor F_X from Normod-N to Normod-M as in Theorem 5.2. We would like to show that F is naturally equivalent to F_X. Since F and F_X are both normal $*$-functors, it suffices, in view of Proposition 5.4, to show that there is an invertible transformation from $F_X(U)$ to $F(U)$ which commutes with the actions of both M and $c(N)$. Accordingly, define a bilinear map from $X \times U$ into $F(U)$ by $(x, u) \mapsto x(u)$. This map is clearly N-balanced, and so defines a linear map T from $X \otimes_N U$ into $F(U)$ satisfying $T(x \otimes u) = x(u)$. A simple calculation then shows that T is isometric, and so extends to an isometry of $F_X(U)$ into $F(U)$. Two more simple calculations show that T commutes with the actions of both M and $c(N)$. Thus what remains to be shown is that T is surjective.

Now it is clear that the representation of $c(N)$ on U is faithful, and so according to Proposition 1.3, U is also a generator for Normod-$c(N)$. Let W denote the range of T in $F(U)$, and view $F(U)$ as a $c(N)$-module. Since T is a $c(N)$ homomorphism, W is a $c(N)$-submodule of $F(U)$, and so its orthogonal complement $W^\perp$ is also. Suppose that $W^\perp$ is not zero. Then it follows from Proposition 1.1 that there is a non-zero element z in $\mathrm{Hom}_{c(N)}(U, W^\perp)$. We can view z as an element of X whose range is in $W^\perp$. Choose $u \in U$ such that $z(u) \neq 0$. Then $T(z \otimes u) = z(u) \in W^\perp$, which contradicts the fact that W is the range of T. Thus T must be surjective. □

6. Uniqueness and self-dual modules

We have seen that non-isomorphic rigged modules can define equivalent functors (Example 3.10). In this section we shall see that if we impose an additional condition on the rigged modules considered, then the correspondence between normal $*$-functors and rigged modules becomes essentially bijective. We will motivate this additional condition from a different direction. Let A and B be C^*-algebras, and let X be a Hermitian $n(B)$-rigged A-module as in Section 3. Then X defines a functor from Hermod-B to Hermod-A, and so from Normod-$n(B)$ to Normod-$n(A)$. But according to Theorem 5.5 this functor should be defined by a normal $n(B)$-rigged $n(A)$-module. It is natural to ask whether the action of A on X can be extended to an action of $n(A)$, in which case X would become the desired normal $n(B)$-rigged $n(A)$-module. We will see shortly that it is not always possible to so extend the action of A. But, in view of the universal property of $n(A)$, one sufficient condition for extending the action of A is that the algebra $L(X)$ of all bounded operators on X be a W^*-algebra. The following example shows that in general $L(X)$ is not a W^*-algebra even when X is defined over a W^*-algebra.

6.1. Example. Let $N = l^\infty$, the W^*-algebra of bounded sequences. Let c_0 be the ideal of l^∞ consisting of sequences converging to zero, viewed as an N-rigged space in the usual way. Let $X = c_0 \oplus l^\infty$, viewed as an N-rigged space in the obvious way. Then it is easily seen that $L(X)$ consists of the 2×2 matrices whose diagonal elements come from l^∞ but whose off-diagonal elements come from c_0, and this algebra is not a W^*-algebra. Furthermore, if A is the algebra of 2×2 matrices all of whose entries come from c_0, acting on X in the obvious way, then it is easily seen that this action of A on X does not extend to an action of $n(A)$.

It is thus natural to look for conditions on an N-rigged space which will ensure that its algebra of bounded operators is a W^*-algebra. Now such conditions have already been found by Paschke [27], namely, that the N-rigged space should be self-dual in a sense analogous to that for ordinary Hilbert spaces. We will see that such self-dual spaces are also the ones which give uniqueness in our analogue of the Eilenberg–Watts theorem. For the convenience of the reader we recall Paschke's results here, stated in our terminology.

6.2. Definition. Let B be a C^*-algebra, and let X be a B-rigged space. Then X is said to be *self-dual* if every continuous (for the usual norm on X) B-module homomorphism R from X into B (with B acting on itself on the right) is of the form

$$R(x) = \langle y, x \rangle_B$$

for some $y \in X$.

6.3. Proposition ([27, Proposition 3.10]). *Let N be a W^*-algebra and let X be a self-dual N-rigged space. Then $L(X)$ is a W^*-algebra. In fact, if N_* denotes the pre-*

dual of N, then the functionals

$$T \mapsto p(\langle y, Tx\rangle_N)$$

for $p \in N_*$, $x,y \in X$, $T \in L(X)$, *span a norm-dense submanifold of the predual of* $L(X)$.

In addition, Paschke showed that any N-rigged space can be enlarged to form a self-dual N-rigged space [27, Theorem 3.2], the enlargement being in a vague sense an analogue of the weak operator closure of an operator algebra. The developments in this section will as a by-product lead to a different (and perhaps simpler) proof of this result (Proposition 6.10).

We remark that in order for $L(X)$ to be a W^*-algebra, it is not necessary for X to be defined over a W^*-algebra or for X to be self-dual, as can be seen by considering either the algebra of compact operators on a Hilbert space as a rigged space over itself, or c_0 as a rigged space over itself, or, more generally, any C^*-algebra B which is an ideal in $n(B)$.

Paschke also indicated the importance of "polar decompositions" for the study of self-dual rigged spaces. We shall not need the next result (except in the form of Lemma 1.2 – but see the comments after Proposition 6.12), but we state it here to provide a backdrop for our use of "polar decompositions" in the following results.

6.4. Proposition ([27, Proposition 3.11]). *Let N be a* W^**-algebra and let X be a self-dual N-rigged space. Then each element x of X can be written uniquely in the form* $y|x|$, *where* $|x| = (\langle x, x\rangle_N)^{1/2}$ *and y is an element of X such that* $\langle y, y\rangle_N$ *is the range projection of* $|x|$, *that is, the smallest projection e in N such that* $e|x| = |x|$.

We will now exhibit a class of self-dual spaces which will be of great importance in this section.

6.5. Theorem. *Let C be a* C^**-algebra and let V and W be Hermitian C-modules. Let* $N = \mathrm{Hom}_C(V, V)$ *and* $X = \mathrm{Hom}_C(V, W)$, *and let N act on the right on X by composition of operators. On X define an N-valued inner product by* $\langle x, y\rangle_N = x^*y$ *for* $x,y \in X$. *Then X is a self-dual N-rigged space.*

Proof. It is easily seen that X is an N-rigged space. Furthermore, Lemma 1.1 shows that X has polar decompositions in the sense of Proposition 6.4. We remark that if y is an element of a B-rigged space such that $\langle y, y\rangle_B$ is a projection, then $y = y\langle y, y\rangle_B$, as is seen immediately from calculating the B-inner-product of $y - y\langle y, y\rangle_B$ with itself. To continue the proof we shall need to take orthogonal expansions:

6.6. Definition. Let M be a W^*-algebra, let Y be an M-rigged space, and let $\{y_i\}$ be a family of elements of Y. We will say that $\{y_i\}$ is an *orthonormal* family of elements of Y if

(1) $\langle y_i, y_j\rangle_M = 0$ if $i \neq j$;

(2) $\langle y_i, y_i\rangle_M$ is a projection in M for each i.

6.7. Lemma (Gram–Schmidt process). *Let M be a* W^**-algebra, and let Y be an M-rigged space which has polar decompositions in the sense of Proposition 6.4. Let* $\{x_1, ..., x_k\}$ *be a finite family of elements of X. Then there is an orthonormal family* $\{y_1, ..., y_k\}$ *of elements of Y and a family* $\{m_i^j : 1 \leqslant i,j \leqslant k\}$ *of elements of M such that*

$$x_i = \sum \{y_i m_i^j : 1 \leqslant j \leqslant k\} \quad \textit{for each } i.$$

Proof. Let the polar decomposition of x_1 be $y_1 m_1$, and let

$$z_2 = x_2 - y_1\langle y_1, x_2\rangle_M .$$

Then a routine calculation shows that $\langle z_2, y_1\rangle_M = 0$. Let the polar decomposition of z_2 be $y_2 m_2$. Let $p = \langle y_2, y_2\rangle_M$ so that p is the range projection of m_2. Then

$$\langle y_1, z_2\rangle_M = \langle y_1, y_2\rangle_M\, pm_2 .$$

Since the left-hand side is zero, it follows that

$$0 = \langle y_1, y_2\rangle_M p = \langle y_1, y_2 p\rangle_M = \langle y_1, y_2\rangle_M$$

by the comment before Definition 6.6. Thus y_1 and y_2 are orthogonal. The rest of the proof continues in the same way to imitate the proof of the usual Gram–Schmidt process. □

We now complete the proof of Theorem 6.5. Let R be a continuous N-module homomorphism of X into N. We wish to find an element y_R of X such that $R(x) = \langle y_R, x\rangle_N$ for all $x \in X$. We will do this by first finding y_R^*, which will be an element of $\mathrm{Hom}_C(W, V)$. We begin by defining y_R^* on elements of W of the form $\Sigma x_i(v_i)$, where $\{x_i\}$ and $\{v_i\}$ are finite families of elements of X and V. Specifically, if we knew that y_R existed, then we would have

$$y_R^*(\sum x_i(v_i)) = \sum \langle y_R, x\rangle_N (v_i) = \sum R(x_i)\, v_i .$$

Accordingly, we define y_R^* by

$$y_R^*(\sum x_i(v_i)) = \sum R(x_i)\, v_i .$$

We must show that y_R^* is well defined. For this it clearly suffices to show that if $\Sigma x_i(v_i) = 0$, then $\Sigma R(x_i)\, v_i = 0$. Now, since X has the polar decomposition property by Lemma 1.1, we can apply Lemma 6.7 to find an orthonormal family $\{y_i\}$ of elements of X and a family $\{n_i^j\}$ of elements of N such that $x_i = \Sigma y_j n_i^j$ for each i. Let $z = \Sigma y_j R(y_j)^*$. Then a straightforward calculation, using the fact that $y_j\langle y_j, y_j\rangle_N = y_j$ for each j and that R is an N-module homomorphism, shows that

$$R(x_i) = \langle z, x_i\rangle_N$$

for each i. Then, since $\Sigma x_i(v_i) = 0$, we have

$$(6.8) \qquad \sum R(x_i)\, v_i = \sum \langle z, x_i \rangle_N\, v_i = z^*(\Sigma x_i(v_i)) = 0\,,$$

as desired.

We now show that y_R^*, defined so far only on the linear submanifold XV of W, is bounded there. Note that y_R^* is clearly linear. Now from the calculations above it is clearly seen that any element of XV is of the form $\Sigma y_i(v_i)$, where $\{y_i\}$ is an orthonormal family of elements of X, and $\{v_i\}$ is some family of elements of V. As above, set $z = \Sigma y_i(R(y_i))^*$, so that as in the derivation of (6.8) above we have

$$y_R^*(\sum y_i(x_i)) = \sum R(y_i)\, v_i = z^*(\sum y_i(v_i))\,.$$

Then

$$\| y_R^*(\sum y_i(v_i)) \| \leqslant \| z^* \| \, \| \sum y_i(v_i) \|\,.$$

But straightforward calculations show that

$$\| z^* \|^2 = \| z^* z \| = \| \sum R(y_i) \langle y_i, y_i \rangle_N R(y_i)^* \| = \| R(z) \| \leqslant \| R \| \, \| z \| = \| R \| \, \| z^* \|\,,$$

so that $\| z^* \| \leqslant \| R \|$, and $\| y_R^* \| \leqslant \| R \|$.

We can now extend y_R^* by continuity to the closure of XV, and define y_R^* to be zero on the orthogonal complement of XV. Then for any $x \in X$ and $v \in V$ we have

$$R(x)(v) = y_R^*(x(v)) = \langle y_R, x \rangle_N (v)\,,$$

so that $R(x) = \langle y_R, x \rangle_N$ as desired.

Finally, we must show that y_R commutes with the action of C. Now XV and its complement are both invariant under C, and so, since y_R^* is zero on the complement, it suffices to show that y_R^* commutes with the action of C on the submanifold XV. But this is shown by a simple calculation. It follows that y_R itself commutes with the action of C. □

We would now like to show that any two self-dual normal N-rigged M-modules which define equivalent functors from Normod-N to Normod-M are isomorphic (contrary to what happens if we drop the condition of self-duality).

6.9. Theorem. *Let M and N be W^*-algebras, and let F be a normal $*$-functor from* Normod-N *to* Normod-M. *Then the space Y of natural transformations from H_N to $H_M \circ F$ has a natural structure as a self-dual normal N-rigged M-module. Furthermore, if X is a self-dual normal N-rigged M-module and F_X is equivalent to F, then X is naturally isomorphic to Y. In particular, any two self-dual normal N-rigged M-modules which define equivalent functors are isomorphic. Thus there is a bijection between equivalence classes of normal $*$-functors from* Normod-N *to* Normod-M *and isomorphism classes of self-dual normal N-rigged M-modules.*

Proof. The elements of Y form a vector space under pointwise operations. Furthermore, Y forms an M-N-bimodule with actions defined by

$$(mt)_V(v) = m(t_V(v)), \qquad (tn)_V(v) = t_V(nv)\,.$$

(For simplicity of notation we omit the forgetful functors H_N and H_M here, and whenever convenient later.) Let U be a generator for Normod-N, and let $c(N) = \mathrm{Hom}_N(U, U)$, so that $\mathrm{Hom}_{c(N)}(U, F(U))$ is a normal N-rigged M-module as shown in the proof of Theorem 5.5. Now, by Proposition 5.4, there is a bijection between Y and $\mathrm{Hom}_{c(N)}(U, F(U))$, and this bijection is easily seen to preserve the actions of M and N on these two spaces. Using this bijection we can transfer the N-valued inner product on $\mathrm{Hom}_{c(N)}(U, F(U))$ to Y, so that Y becomes a normal N-rigged M-module. We show that this inner product on Y does not depend on the choice of generator. Suppose that U_0 is another generator for Normod-N, and assume at first that there is an isomorphism f of U onto a submodule of U_0. Because U and U_0 are generators, the map $n \mapsto f^* nf$ is an isomorphism of N acting on U_0 onto N acting on U. But for $s, t \in Y$, it is easily seen from the properties of natural transformations that

$$s_U^* t_U = f^* s_{U_0}^* t_{U_0} f\,,$$

so that $s_U^* t_U$ and $s_{U_0}^* t_{U_0}$ represent the same element of N. If U_0 does not contain a copy of U, then we can compare both U and U_0 with the generator $U \oplus U_0$ to obtain the general case. Finally, since $\mathrm{Hom}_{c(N)}(U, F(U))$ is a self-dual module by Theorem 6.5, it follows that Y is a self-dual normal N-rigged M-module.

Suppose now that X is a self-dual normal N-rigged M-module, and that F_X is equivalent to F, so that Y can be viewed as the space of natural transformations from H_N to $H_M \circ F_X$. We show that the mapping of X into Y defined by assigning to $x \in X$ the natural transformation t^x defined by $t^x(v) = x \otimes v$ (as in the discussion after Theorem 5.2) is an isomorphism of X with Y. Now this mapping is easily seen to be an injective isometric M-N-bimodule homomorphism. The only issue is whether it is surjective.

Let $t \in Y$. We wish to find $z \in X$ such that $t = t^z$. Let U be a generator for Normod-N, and let $c(N) = \mathrm{Hom}_N(U, U)$. Then for any $x \in X$, $(t^x)_U$ and t_U are both in $\mathrm{Hom}_{c(N)}(U, F_X(U))$, so that $t_U^*(t^x)_U \in \mathrm{Hom}_{c(N)}(U, U)$, and so can be viewed as an element of N by the double commutant theorem. We thus have a map R from X into N defined by

$$R(x) = t_U^*(t^x)_U\,.$$

It is easily verified that R is an N-module homomorphism (with N acting on itself on the right), and that R is bounded (by $\| t_U^* \|$). Since X is assumed to be self-dual, it follows that there is an element z of X such that $R(x) = \langle z, x \rangle_N$ for all $x \in X$, that is,

$$t_U^*(t^x)_U = \langle z, x \rangle_N\,.$$

But for any $x \in X$ and $u, u' \in U$, we then have

$$\langle t_U(u), x \otimes u' \rangle = \langle u, t_U^*(t^x)_U(u') \rangle = \langle u, \langle z, x \rangle_N u' \rangle = \langle z \otimes u, x \otimes u' \rangle .$$

It follows that $t_U = (t^z)_U$. From Proposition 5.3 it then follows that $t = t^z$.

We have thus shown that the correspondence from isomorphism classes of self-dual normal N-rigged M-modules to equivalence classes of normal $*$-functors is injective. But this correspondence is also seen to be surjective by Theorem 5.5 if we note that the X constructed there is self-dual by Theorem 6.5. □

We now indicate how the above results can be used to give a new proof of Paschke's theorem [27, 3.2] that any N-rigged space can be embedded in a self-dual N-rigged space. In fact, we show this for any normal N-rigged M-module (Paschke's case being that in which $M = \mathbf{C}$). Let Z be a normal N-rigged M-module. Then we can form the functor F_Z and apply Theorem 5.5 to form $X = \mathrm{Hom}_{c(N)}(U, F_Z(U))$, so that F_Z is equivalent to F_X. This suggests that there is an embedding of Z into X, and this is in fact the case, namely, the mapping $z \mapsto x_z$, where $x_z(u) = z \otimes u$ for all $u \in U$. This mapping is easily seen to be an M-N-bimodule homomorphism of Z into X which preserves the N-valued inner products. Now Paschke showed that any self-dual N-rigged space Y is a dual Banach space, with the functionals $y \mapsto p(\langle y', y \rangle_N)$ spanning a norm-dense submanifold of the pre-dual of Y as y' ranges over Y and p ranges over the pre-dual M_* of M [27, Proposition 3.8]. For $X = \mathrm{Hom}_{c(N)}(U, F_Z(U))$ this is particularly apparent (and follows from [28, Theorem 1.4]). Furthermore, it is easily seen that the image of Z in X is dense for the weak-$*$ topology. Finally, as in Paschke's theorem, there is a natural conjugate isomorphism of the N-dual $\mathrm{Hom}_N(Z, N)$ of Z onto X given by assigning to any $h \in \mathrm{Hom}_N(Z, N)$ the element x_h in X defined by setting

$$x_h^*(z \otimes u) = h(z)\, u .$$

We have thus sketched the proof of all but the last statement of the following result, part of which generalizes [27, Theorem 3.2]:

6.10. Proposition. *Any normal N-rigged M-module Z can be embedded as a weak-$*$ dense subspace of a self-dual normal N-rigged M-module X which is conjugate isomorphic to the N-dual of Z. As X we can take* $\mathrm{Hom}_{c(N)}(U, F_Z(U))$, *where U is any generator of* Normod-*U. Also, X is unique up to isomorphism. The functors F_Z and F_X will be naturally equivalent.*

The last statement of the above proposition follows from:

6.11. Proposition. *Let X be a self-dual normal N-rigged M-module and let Z be a sub-M-N-bimodule of X, so that X can be viewed as a normal N-rigged M-module. Assume further that Z is dense in X for the weak-$*$ topology on X (defined above). Then the functors F_X and F_Z are equivalent.*

Proof. For $V \in$ Normod-N define a natural transformation t from F_Z to F_X by

$$t_V(z \otimes v) = z \otimes v$$

for $z \in Z$, $v \in V$. It is easily verified that t_V is an isometry. We must use the density of Z to show that t_V is surjective. If it is not, then there is a $w \in F_X(V)$, $w \neq 0$, which is orthogonal to the range of t_V. Since $w \neq 0$, there must exist $y \in X$, $v \in V$ such that $\langle y \otimes v, w \rangle \neq 0$. Then the functional h on X defined by $h(x) = \langle x \otimes v, w \rangle$ is zero on Z but not on X. But h is easily seen to be in the pre-dual of X, and so Z could not be weak-$*$ dense in X. □

From the considerations leading to Proposition 6.10 we also get a representation for self-dual normal N-rigged M-modules:

6.12. Proposition. *Let X be any self-dual normal N-rigged M-module. Let U be a generator for* Normod-*N, and let* $c(N) = \mathrm{Hom}_N(U, U)$. *Then there is a normal M-module W which is also a normal c(N)-module with the two actions commuting (namely, $F_X(U)$) such that X is isomorphic to* $\mathrm{Hom}_{c(N)}(U, W)$, *with N-valued inner product defined by*

$$\langle x, y \rangle_N = x^* y .$$

This representation immediately yields such results as Paschke's proposition on polar decompositions [27, 3.11] (by applying Lemma 1.2).

7. Morita equivalence and its characterization

In this section we will study normal $*$-functors which establish an equivalence between the categories Normod-N and Normod-M. Such functors will, of course, be faithful (that is, injective on morphisms), and so we begin by considering when the functor defined by a normal N-rigged M-module is faithful.

7.1. Proposition. *Let M and N be W^*-algebras and let X be a normal N-rigged M-module. Let N_0 be the weak closure of the span of the range of the inner product on X, so that N_0 is a two-sided ideal in N. Then F_X is faithful if and only if $N_0 = N$. In particular, if $V \in$* Normod-*N then $F_X(V) = \{0\}$ if and only if $N_0 V = \{0\}$.*

Proof. It is easily seen that if $N_0 V = \{0\}$, then $F_X(V) = \{0\}$. Furthermore, if $N_0 \neq N$, then, since N_0 is weakly closed, it is generated by a central idempotent [32, 1.10.5] and so has a complementary weakly closed two-sided ideal. If V is any normal module over this complementary ideal, we can view V as a normal N-module, and we see that $F_X(V) = \{0\}$. Thus F_X is not faithful if $N_0 \neq N$.

Suppose conversely that $V, W \in$ Normod-N, that $f \in \mathrm{Hom}_N(V, W)$ and $f \neq 0$, but that $F_X(f) = 0$. We will show that it follows that $N_0 \neq N$.

7.2. **Lemma.** *Let M and N be W^*-algebras and let F be a normal $*$-functor from* Normod-N *to* Normod-M. *Let $V, W \in$* Normod-N *and let $f \in \mathrm{Hom}_N(V, W)$. Let the polar decomposition of f be $f = p|f|$. Then the polar decomposition of $F(f)$ is $F(f) = F(p)f(|f|)$.*

Proof. It is easily seen that $F(|f|) \geqslant 0$ and that its square is equal to $|F(f)|^2$, so that $|F(f)| = F(|f|)$. Now $F(pp^*)$ and $F(p^*p)$ are both projections, and so $F(p)$ is a partial isometry, which must be at least as big as the partial isometry in the polar decomposition of $F(f)$, since $F(p)\,|F(f)| = F(f)$. Now p^*p is the support projection of $|f|$. It follows that p^*p is the weak limit of projections which are dominated by positive scalar multiples of $|f|$. Since F is normal, it follows that $F(p^*p)$ is the weak limit of certain projections dominated by positive scalar multiples of $|F(f)|$. Putting these facts together it follows that $F(p)$ must be exactly the partial isometry in the polar decomposition of $F(f)$. □

Returning to the proof of Proposition 7.1, we let $f = p|f|$ be the polar decomposition of the f for which $F_X(f) = 0$. It follows from the above lemma that $F_X(p)$ is the partial isometry in the polar decomposition of 0, so that $F_X(p) = 0$. Let V_1 be the range of p^*p. Since p^*p acts as the identity on V_1 and $F_X(p^*p) = 0$, it follows that $F_X(V_1) = \{0\}$. In particular, $x \otimes v = 0$ as an element of $F_X(V_1)$ for all $x \in X$, $v \in V_1$, and so

$$0 = \langle x \otimes v, x' \otimes v' \rangle = \langle \langle x', x \rangle_N v, v' \rangle$$

for all $x, x' \in X$, $v, v' \in V_1$. It follows that $N_0 V_1 = \{0\}$. Since $V_1 \neq \{0\}$ (because $p \neq 0$), it follows that $N_0 \neq N$. □

Functors which establish an equivalence of categories will also be full (that is, surjective on morphisms between objects in their range).

7.3. **Proposition.** *Let M and N be W^*-algebras, and let F be a $*$-functor from* Normod-N *to* Normod-M. *If F is faithful and full, then F is automatically normal.*

Proof. If $V \in$ Normod-N, then the map defined by F from $\mathrm{Hom}_N(V, V)$ to $\mathrm{Hom}_M(F(V), F(V))$ will be a $*$-isomorphism of von Neumann algebras since F is faithful and full. But any $*$-isomorphism of von Neumann algebras is automatically normal [7, Corollary 1, 54].) It follows from Proposition 4.7 that F is normal. □

7.4. **Definition.** Let M and N be W^*-algebras. We shall say that M and N are *Morita equivalent* if there is an equivalence of Normod-N with Normod-M implemented by $*$-functors (which will automatically be normal).

We shall now see how W^*-algebras which are Morita equivalent are constructed from each other. Let M and N be W^*-algebras, and let F be a $*$-functor from Normod-N to Normod-M, which establishes an equivalence of these categories. Let

U be a generator for Normod-N, let $c(N) = \mathrm{Hom}_N(U, U)$, and let $X = \mathrm{Hom}_{c(N)}(U, F(U))$, so that F is equivalent to F_X as in Theorem 5.5. Since F is faithful, the weak closure of the range of the inner product on X must be all of N by Proposition 7.1.

Now since F establishes an equivalence, it follows that $F(U)$ will be a generator for Normod-M. It follows from Proposition 1.3 that M is faithfully represented on $F(U)$. Furthermore, since F is faithful and full, it establishes an isomorphism of $c(N) = \mathrm{Hom}_N(U, U)$ with $\mathrm{Hom}_M(F(U), F(U))$, so that $c(N)$ can be viewed as the commutant of M acting on $F(U)$, and M in turn can be viewed as the commutant of $c(N)$ acting on $F(U)$. From this fact we can equip X with an additional piece of structure, namely, an M-valued inner product for the left action of M on X. Specifically, for $x, y \in X$ we see that $xy^* \in \mathrm{Hom}_{c(N)}(F(U), F(U))$, and so can be viewed as an element of M. We set

$$\langle x, y \rangle_M = xy^* .$$

Then it is easily seen that X becomes a left M-rigged space [30, Definition 2.8]. Furthermore, the M- and N-valued inner products are readily seen to satisfy the relation

$$\langle x, y \rangle_M z = x \langle y, z \rangle_N$$

for all $x, y, z \in X$. Also for fixed $x, y \in X$ the map

$$n \mapsto \langle xn, y \rangle_M = xny^*$$

from N to M is clearly normal, so that X becomes a normal left M-rigged right N-module for the obvious definition. Finally, we show that the weakly closed two-sided ideal spanned by the range of the M-valued inner product is all of M. If this were not the case, then the complementary two-sided ideal would be generated by a non-zero projection, say p, which would have the property that $p\langle x, y \rangle_M = 0$ for all $x, y \in X$. Let $W = p(F(U))$. Since M can be viewed as the commutant of $c(N)$, W will be $c(N)$-invariant. Then, since $c(N)$ is faithfully represented on U so that U is a generator for Normod-$c(N)$ by Proposition 1.3, it follows from Proposition 1.1 that if $W \neq \{0\}$, then there is a non-zero element $x \in \mathrm{Hom}_{c(N)}(U, W)$, which can be viewed as an element of X. Then $\langle x, x \rangle_M = xx^*$ can be viewed as an element of $\mathrm{Hom}_{c(N)}(W, W)$. But p acts as the identity on W, so that we would have

$$p\langle x, x \rangle_M = pxx^* \neq 0 ,$$

contrary to assumption. Thus $p(F(U)) = \{0\}$. But $F(U)$ is a generator for Normod-M and so M is faithfully represented on $F(U)$ by Proposition 1.3. Thus $p = 0$ as desired.

This leads us to make the following definition, in analogy with [30, Definition 6.10] and [2, Definition 3.2]:

7.5. **Definition.** Let M and N be W^*-algebras. By an *M-N-equivalence bimodule* we mean an M-N-bimodule X which is equipped with M- and N-valued inner products with respect to which X is a normal N-rigged M-module and a normal left M-rigged right N-module such that

(1) $\langle x, y\rangle_M z = x\langle y, z\rangle_N$ for all $x,y,z \in X$;

(2) the range of $\langle\ ,\ \rangle_M$ spans a weakly dense subset of M, and the range of $\langle\ ,\ \rangle_N$ spans a weakly dense subset of N.

We shall say that the equivalence bimodule X is *self-dual* if it is self-dual both as an N-rigged space and as an M-rigged space.

7.6. Proposition. *Let M and N be W^*-algebras and let X be an M-N-equivalence bimodule. Then, if we view X as an N-rigged space with corresponding algebra $L_N(X)$ of bounded operators, the homomorphism defining the action of M on X is an isomorphism of M onto $L_N(X)$. In particular, $L_N(X)$ is a W^*-algebra. Similarly, viewing X as a left M-rigged space, the antihomomorphism defining the action of N on X is an anti-isomorphism of N onto $L_M(X)$, so that $L_M(X)$ is a W^*-algebra.*

Proof. Let M_0 be the (unclosed) two-sided ideal in M spanned by the range of the M-valued inner product. By assumption, M_0 is weakly dense in M. Suppose there is an element m of M such that $mX = 0$. Then

$$m\langle x, y\rangle_M = \langle mx, y\rangle_M = 0$$

for all $x, y \in X$. Since M_0 is weakly dense, it follows that $m = 0$, so that the homomorphism of M into $L_N(X)$ is injective. Now by the Kaplansky density theorem [7, p. 43] there is a net $\{m_k\}$ of elements of M_0 of norm one which converges weakly to the identity element of M. Let $T \in L_N(X)$. Then

$$T\langle x, y\rangle_M z = Tx\langle y, z\rangle_N = \langle Tx, y\rangle_M z .$$

Thus M_0 is in fact an ideal in $L_N(X)$. In particular, $Tm_k \in M_0$ for each k. Since $\| Tm_k \| \leqslant \| T \|$, and since any closed ball in M is compact for the weak topology (this being the weak-$*$ topology from the pre-dual), the net $\{Tm_k\}$ will have a subnet $\{Tm_j\}$ weakly convergent to an element m_1 of M. Then for $x,y \in X$ we have

$$\langle m_1 x, y\rangle_N = \lim\langle Tm_j x, y\rangle_N = \lim\langle m_j x, T^* y\rangle_N = \langle x, T^* y\rangle_N = \langle Tx, y\rangle_N ,$$

where the limits are in the weak topology on N. Thus $m_1 x = Tx$ for $x \in X$, so that the homomorphism of M into $L_N(X)$ is surjective, and so bijective. A similar argument applies to the antihomomorphism of N into $L_M(X)$. □

The above result should be compared with [2, Theorem 3.4, part 5].

7.7. Proposition. *Let X be an M-N-equivalence bimodule. Then the following conditions are equivalent*:

(1) *X is self-dual.*

(2) *X is self-dual as an N-rigged space.*

(3) *X is self-dual as a left M-rigged space.*

Proof. By definition, condition (1) implies conditions (2) and (3). Suppose now that condition (2) holds. Then, by Proposition 6.12, X is of the form $\mathrm{Hom}_{c(N)}(U, W)$ for U a generator of Normod-N and $W = F_X(U)$. By Proposition 7.6, M coincides with $L_N(X)$. Note that every element of $\mathrm{Hom}_{c(N)}(W, W)$ defines in an obvious way an element of $L_N(X)$, so that we obtain a homomorphism of $\mathrm{Hom}_{c(N)}(W, W)$ into $L_N(X)$. Now for $x, y, z \in X$ we have

$$\langle x, y\rangle_M z = x\langle y, z\rangle_N = xy^* z ,$$

from which it follows that

$$\langle x, y\rangle_M = xy^* ,$$

which is also an element of $\mathrm{Hom}_{c(N)}(W, W)$. Since these elements are by assumption weakly dense in M, it follows that M can be viewed as a weakly closed two-sided ideal of $\mathrm{Hom}_{c(N)}(W, W)$. But X is self-dual as a left-rigged space over $\mathrm{Hom}_{c(N)}(W, W)$ by the left-sided version of Theorem 6.5. It follows easily that X is self-dual over M, so that condition (3), and hence condition (1), is satisfied. A similar argument applies if we start with condition (3). □

Let X be an M-N-equivalence bimodule. Then exactly as in [30, Definition 6.17] we can form $\widetilde{X}$, the space X but with conjugate operations of M, N and the complex numbers (that is, $n\widetilde{x} = (xn^*)^\sim$, $\langle \widetilde{x}, \widetilde{y}\rangle_N = \langle x, y\rangle_N$, etc.). We remark that if X is self-dual, and so of the form $\mathrm{Hom}_{c(N)}(U, W)$ as in Proposition 6.11, then there is a natural identification of $\widetilde{X}$ with $\mathrm{Hom}_{c(N)}(W, U)$. At any rate, it is easily verified that:

7.8. Proposition. *If X is an M-N-equivalence bimodule, then $\widetilde{X}$ is an N-M-equivalence bimodule.*

We now come to the main theorem of this paper, which is the analogue of Morita's theorem for the algebraic case [26, 2, 6].

7.9. Theorem. *Let M and N be W^*-algebras. Suppose that there exists an M-N-equivalence bimodule X. Then M and N are Morita equivalent, with an equivalence between* Normod-N *and* Normod-M *being implemented by the functors F_X and $F_{\widetilde{X}}$. Conversely, if M and N are Morita equivalent, then there exists an M-N-equivalence bimodule X which can be chosen to be self-dual, and which is such that F_X and $F_{\widetilde{X}}$ are naturally equivalent to the functors establishing the Morita equivalence. We obtain in this way a bijection between equivalence classes of equivalences from* Normod-N *to* Normod-M *and isomorphism classes of self-dual M-N-equivalence bimodules.*

Proof. The second part of the proof follows from the discussion following Definition 7.4 which motivated our definition of an M-N-equivalence bimodule in Definition 7.5. The only detail which is not clear from that discussion is that $F_{\widetilde{X}}$ is naturally equivalent to the functor from Normod-M to Normod-N establishing the Morita equivalence. But this will follow from the first part of the theorem and the easily verified fact that the "inverses" of two naturally equivalent equivalence are equivalent.

The proof of the first part of the theorem is very similar to the proof of [30, Theorem 6.23]. We must show that $F_X \circ F_{\tilde{X}}$ and $F_{\tilde{X}} \circ F_X$ are each naturally equivalent to the appropriate identity functor. Now [30, Proposition 6.21] is immediately applicable. (The fact that we are no longer assuming that the range of the N-valued inner product is *norm* dense makes no difference.) Thus the functor $F_{\tilde{X}} \circ F_X$ is naturally equivalent to the functor F_Y, where Y is the pre-Hermitian N-rigged N-module $\tilde{X} \otimes_M X$. But in analogy with [30, Lemma 6.22] the map $\tilde{x} \otimes y \to \langle x, y \rangle_N$ is a pre-equivalence [30, Definition 5.6] of Y with the norm closed span N_0 of the range of the N-valued inner product on X, where N_0 is viewed as a Hermitian N-rigged N-module by virtue of the fact that it is a two-sided ideal in N. Then the functor F_Y is easily seen to be equivalent to the functor F_{N_0} (as in [30, Lemma 5.7]). But N_0 is weak-$*$ dense in N by assumption, and so F_{N_0} is equivalent to F_N by Proposition 6.11. But we have seen that F_N is equivalent to the identity functor. Thus $F_{\tilde{X}} \circ F_X$ is naturally equivalent to the identity functor. A similar argument works for $F_X \circ F_{\tilde{X}}$.

The uniqueness up to isomorphism of the self-dual M-N-equivalence bimodule corresponding to an equivalence class of equivalences follows from the corresponding part of Theorem 6.9. □

7.10. Corollary. *Let N be a W^*-algebra and let X be a self-dual N-rigged space the range of whose inner product spans a weakly dense ideal in N. Then N and $L(X)$ are Morita equivalent. Conversely, if M and N are W^*-algebras which are Morita equivalent, then there is a self-dual N-rigged space X the range of whose inner product spans a weakly dense ideal in N such that M is isomorphic to $L(X)$.*

Proof. On X, viewed as a left $L(X)$-module, define an $L(X)$-valued inner product by

$$\langle x, y \rangle_{L(X)} z = x \langle y, z \rangle_N$$

for $x, y, z \in X$. Then, as in [30, Proposition 6.2, 6.3] it is easily verified that X becomes an $L(X)$-N-equivalence bimodule. The converse follows immediately from Theorem 7.9 and Proposition 7.6. □

7.11. Corollary. *Let K be a Hilbert space, and let $L(K)$ denote the von Neumann algebra of all bounded operators on K. Then $L(K)$ is Morita equivalent to the one-dimensional W^*-algebra,* **C**.

Proof. View K as a self-dual **C**-rigged space. □

8. Properties of Morita equivalence

In this section we gather together various general facts about operator algebras which are Morita equivalent.

8.1. Proposition. *Let M and N be W^*-algebras which are Morita equivalent. Then the centers of M and N are isomorphic.*

Proof. This follows immediately from Proposition 2.1. □

8.2. Corollary. *Two commutative W^*-algebras are Morita equivalent if and only if they are isomorphic.*

The next result is an analogue of part of [2, Theorem 3.5(5)].

8.3. Corollary. *Let M and N be W^*-algebras which are Morita equivalent. Then there is an isomorphism between their lattice of weakly closed two-sided ideals.*

Proof. The weakly closed two-sided ideals of M correspond to the projections in the center of M, and similarly for N. □

We now describe how this isomorphism is implemented by a self-dual M-N-equivalence bimodule, in analogy with [2, Theorem 3.5(5)].

8.4. Proposition. *Let M and N be W^*-algebras, and let X be a self-dual M-N-equivalence bimodule. Then there is an isomorphism of the lattice of weakly closed two-sided ideals of N (and M) and the lattice of weak-$*$ closed M-N-submodules of X. If J is such an ideal of N, generated by the central idempotent e, then this isomorphism is given in one direction by assigning the subspace $X_J = Xe$ to J. In the other direction this isomorphism is given by assigning to a weak-$*$ closed M-N-submodule Y the weakly closed two-sided ideal I_Y generated by $\langle Y, Y \rangle_N$.*

Proof. We must show that $X_{I_Y} = Y$ and $I_{X_J} = J$. To show the first equality, let e be the central projection generating I_Y. Then it is easily seen that $y = ye$ for $y \in Y$, so that $Y \subseteq Xe$. The opposite inclusion is shown by making precise the symbolic calculation

$$Xe = X \langle Y, Y \rangle_N = \langle X, Y \rangle_M Y \subseteq Y.$$

Conversely, given J, generated by e, we have

$$\langle X_J, X_J \rangle_N = \langle Xe, Xe \rangle_N = e \langle X, X \rangle_N e \subseteq J,$$

so that $I_{X_J} \subseteq J$. But since $\langle X, X \rangle_N$ is weakly dense in N by assumption, the opposite inclusion is shown by making precise the symbolic calculation

$$J = eNe = e \langle X, X \rangle_N e = \langle Xe, Xe \rangle_N \subseteq I_{X_J}. \square$$

We remark that a similar result is true for C^*-algebras which are related by an imprimitivity bimodule [30, Definition 6.10]. This fact is closely related to Mackey's normal subgroup analysis [24], as will be shown elsewhere.

We now investigate how Morita equivalence is related to forming tensor products of W^*-algebras (over the complex numbers, as on [7, p. 24] or [32, Definition 1.22.10] – for an interesting different kind of tensor product see [19]).

8.5. Proposition. *Let M, M_1, N, N_1 be W^*-algebras. Let X be a normal N-rigged M-module and Y a normal N_1-rigged M_1-module. Then the algebraic tensor product $X \otimes Y$ over the complex numbers, with $N \otimes N_1$-valued inner product defined by*

$$\langle x \otimes y, x' \otimes y' \rangle_{N \otimes N_1} = \langle x, x' \rangle_N \otimes \langle y, y' \rangle_{N_1}$$

and completed for the usual norm, is a normal $N \otimes N_1$-rigged $M \otimes M_1$-bimodule. If X and Y are in fact equivalence bimodules and if an $M \otimes M_1$-valued inner product is defined by

$$\langle x \otimes y, x' \otimes y' \rangle_{M \otimes M_1} = \langle x, x' \rangle_M \otimes \langle y, y' \rangle_{M_1},$$

then $X \otimes Y$ (completed) becomes an equivalence bimodule.

Proof. We have not found any way to prove that the indicated inner products are positive except to imitate the proof for tensor products of ordinary Hilbert spaces, which involves expressing vectors as linear combinations of orthonormal vectors. In our setting this means that we must embed X and Y in self-dual modules (Proposition 6.10) so that by Proposition 6.12 we can apply Lemma 1.2 and Lemma 6.7 to express elements of X and Y in terms of orthonormal families of elements. The proof of positivity is then carried out pretty much as for ordinary Hilbert spaces.

The rest of the proof of this proposition is carried out by routine calculations. □

8.6. Corollary. *Let N be a W^*-algebra, let K be a Hilbert space, and let $L(K)$ be the von Neumann algebra of all bounded operators on K. Then $N \otimes L(K)$ is Morita equivalent to N.*

Proof. View K as an $L(K)$-**C**-equivalence bimodule as in Corollary 7.11, and N as an N-N-equivalence bimodule. □

8.7. Corollary. *Every type* II_1 *factor is Morita equivalent to a type* II_∞ *factor. Every type* II_∞ *factor is Morita equivalent to a type* II_1 *factor.*

Proof. If N is a type II_1 factor and if K is an infinite-dimensional Hilbert space, then $N \otimes L(K)$ is a type II_∞ factor [32, Theorem 2.6.6] to which N is Morita equivalent by Corollary 8.6. Conversely, if M is a type II_∞ factor, then, as in [7, Exercise 5, p. 242], M is isomorphic to $N \otimes L(K)$ for some type II_1 factor N and Hilbert space K. □

Whether the type II_1 factor to which a type II_∞ factor is Morita equivalent is unique up to isomorphism is not clear, since it is still unknown whether a type II_1

factor is isomorphic to the algebra of 2×2 matrices over itself (which is an algebra to which it is clearly Morita equivalent). Notice also that Proposition 2.5 shows that a type I factor cannot be Morita equivalent to a type II or type III factor, nor can a type II factor be equivalent to a type III factor.

8.8. Corollary. *Let N be a W^*-algebra, and let V and W be two generators for* Normod-N. *Let N'_V and N'_W denote the commutants of the actions of N on V and W. Then N'_V and N'_W are Morita equivalent.*

Proof. The representations of N on V and W are quasi-equivalent [8, 5.3] and so [8, 5.3.1(iv)] there are Hilbert spaces H and K such that $V \otimes H$ and $W \otimes K$ are isomorphic as normal N-modules. But the commutant of N acting on $V \otimes H$ is $N'_V \otimes L(H)$ [7, p. 24], which, by Corollary 8.5, is Morita equivalent to N'_V. Similarly, $N'_W \otimes L(K)$ is Morita equivalent to N'_W. But $N'_V \otimes L(H)$ is isomorphic to $N'_W \otimes L(K)$. □

8.9. Proposition. *If M and N are W^*-algebras which are Morita equivalent, then their opposite algebras are Morita equivalent.*

Proof. Let $\widetilde{M}$ and $\widetilde{N}$ denote the opposite algebras of M and N and let X be an M-N-equivalence bimodule. Then X can be viewed as an $\widetilde{N}$-$\widetilde{M}$-equivalence bimodule by setting $\tilde{n}x = xn$, $\langle x, y \rangle_{\widetilde{M}} = (\langle x, y \rangle_M)^\sim$, etc. □

Whether a W^*-algebra is always Morita equivalent to its opposite algebra is not clear. This is related to the unsettled question of whether a W^*-algebra is always isomorphic to its opposite algebra.

8.10. Theorem. *A W^*-algebra is of type* I *if and only if it is Morita equivalent to a commutative W^*-algebra. In fact, any type* I *W^*-algebra is Morita equivalent to its center.*

Proof. If a W^*-algebra M is Morita equivalent to a commutative W^*-algebra N, then Normod-M is equivalent to Normod-N, and so by Proposition 2.5, M must be of type I. Suppose conversely that M is a type I W^*-algebra. Then by [7, Theorem 1, p. 123] there is an Abelian projection e in M whose central support is the identity element of M. Then eMe is a commutative W^*-algebra which by [7, corollary to Proposition 2, p. 18] coincides with eZe, where Z is the center of M. Now if p is any projection in Z, then $epe \neq 0$ since e has the identity of M as central support. It follows that the surjective homomorphism $z \mapsto eze$ of Z onto eZe is injective and so is an isomorphism of W^*-algebras. Now MeM (the weak closure thereof) is a two-sided ideal in M, so generated by some central projection which must dominate e, and so which must be the identity of M. Thus $MeM = M$. Finally, let $X = Me$, and define M and eZe-valued inner products on X by

$$\langle me, m_1 e \rangle_M = mem_1^*, \qquad \langle me, m_1 e \rangle_{eZe} = em^* m_1 e.$$

Then it is clear that X is an M-eZe-equivalence bimodule, so that M is Morita equivalent to eZe, and hence to Z. □

8.11. Corollary. *Two type* I W^**-algebras are Morita equivalent if and only if their centers are isomorphic.*

8.12. Corollary. *Any two type* I *factors are Morita equivalent.*

8.13. Corollary. *Any type* I W^**-algebra is Morita equivalent to its opposite algebra.*

8.14. Corollary. *Any type* I *von Neumann algebra is Morita equivalent to its commutant.*

Let M and N be two type I W^*-algebras which are Morita equivalent, so that their centers are isomorphic. Then we can realize both of their centers as $L^\infty(S, \mu)$ for an appropriate measure space [7, 32], and, in the separable case, we can form direct integral decompositions of M and N over this measure space [7, 32]. Thus we see that the way in which M and N differ is exactly in that their "full matrix algebras", that is, type I factors, over each point of S can have different dimensions. Conversely, it is clear that two W^*-algebras constructed by taking direct integrals of type I factors of possibly differing dimensions but over the same measure space will have isomorphic centers and so will be Morita equivalent. Thus at least in the separable type I case we obtain a good picture of how W^*-algebras which are Morita equivalent are related.

The above considerations are all quite elementary. But by invoking Tomita theory we can obtain a quite precise description of Morita equivalence for general W^*-algebras in terms of traditional concepts.

8.15. Theorem.[‡] *Let M and N be W^*-algebras. Then M and N are Morita equivalent if and only if there is a generator V for* Normod-N *such that M is isomorphic to the opposite algebra of the commutant of N acting on V.*

Proof. Let V be a generator for Normod-N, and let N' denote the commutant of N acting on V. By [35, Theorem 12.2] N is isomorphic to the left algebra $L(A)$ of some modular Hilbert algebra A. By the commutation theorem for left Hilbert algebras [35, Theorem 4.1], the commutant of $L(A)$ is $R(A)$, the right algebra of A, and $L(A)$ is anti-isomorphic to $R(A)$. Thus $L(A)$ is isomorphic, hence Morita equivalent, to the opposite algebra of $R(A)$. Since the representations of N on both V and the Hilbert space U of A are faithful, and thus are generators for Normod-N, we can apply Corollary 8.8 and Proposition 8.9 to conclude that N is Morita equivalent to the opposite algebra of N'.

Suppose now that M is isomorphic to the opposite algebra of N'. It follows that M and N are Morita equivalent. Conversely suppose that M and N are Morita equivalent. Choose U as above, so that $R(A)$ is anti-isomorphic to N. Then the arguments in the second paragraph after Definition 7.4 show that, if we adopt the notation used there, $F(U)$ is a faithful normal $c(N)$-module whose commutant is isomorphic to M. But $c(N) = R(A)$ so that N is anti-isomorphic to $c(N)$. In other words, N is isomorphic to the opposite of the commutant of M acting on V. Passing to the real dual of V we obtain a normal N-module such that M is isomorphic to the opposite of the commutant of N. □

‡ Alain Connes and Masamichi Takesaki have pointed out that for factors on separable Hilbert spaces, Morita equivalence turns out to be the same as the notion of genus introduced by Murray and von Neumann [26a, Section 3]. Alain Connes has also pointed out that two W^*-algebras are Morita equivalent if and only if they have faithful representations with isomorphic commutants.

8.16. Corollary. *Let M and N be type* III *von Neumann algebras on separable Hilbert spaces. If M and N are Morita equivalent, then they are isomorphic.*

Proof. Let V be a generator for Normod-N such that M is isomorphic to the opposite of the commutant of N on V. Let U be the Hilbert space of a modular Hilbert algebra of which N is isomorphic to the left algebra. We view U as a normal N-module, whose commutant is isomorphic to the opposite of N. Since both M and the opposite of N are countably decomposable and of type III, it follows from [7, Corollary 8, p. 301] that U and V are isomorphic as N-modules. Then their commutants, which are the opposites of N and M, are isomorphic, so N and M are isomorphic. □

The above results indicate that for W^*-algebras, Morita equivalence is not a fundamentally new concept. For type I algebras it does provide a pleasant point of view. In fact it suggests that perhaps a neater way to *define* type I algebras is simply to say that they are the ones which are Morita equivalent to commutative ones. On the other hand, in the type II case the fact that Morita equivalence seems to be weaker than isomorphism may possibly prove useful since isomorphism seems so intractable there – witness the questions of whether a type II_1 factor is isomorphic to 2×2 matrices over itself. It might, for example, be possible to obtain at least a partial classification of Morita equivalence classes of type II factors. A number of the invariants which have already been introduced for factors will probably turn out to be, in fact, invariants for Morita equivalence classes of factors.

8.17. Definition. Let A and B be C^*-algebras. We say that A and B are *Morita equivalent* if there is an equivalence of Hermod-B with Hermod-A which is implemented by $*$-functors (which will automatically be normal).

8.18. Proposition. *The C^*-algebras A and B are Morita equivalent if and only if their enveloping W^*-algebras $n(A)$ and $n(B)$ are Morita equivalent.*

It follows that the earlier results of this section are immediately applicable to C^*-algebras, although they do not, in general, have nice formulations in terms of the C^*-algebras themselves rather than the corresponding W^*-enveloping algebras.

The detailed study of Morita equivalence for special classes of C^*-algebras must await another time. We include here only the following result.

8.19. Proposition. *Two commutative separable C^*-algebras are Morita equivalent if and only if their spectra are Borel isomorphic.*

Proof. Any two commutative C^*-algebras (separable or not) which have Borel isomorphic spectra will be Morita equivalent, as can be seen from Example 3.11. Conversely, let A and B be two separable commutative C^*-algebras which are Morita equivalent. Then there will be a bijection between their equivalence classes of irreducible representations, and so between their spectra. Thus their spectra have the same cardinality. Since A and B are separable, their spectra are separable locally compact metric spaces, hence polonais spaces [4, p. 122], and so their Borel structures are those of standard Borel spaces [1, Proposition 2.3]. But any two standard Borel spaces of the same cardinality are isomorphic [1, Proposition 2.7]. □

We remark that if two commutative C^*-algebras A and B are Morita equivalent, so that $n(A)$ is isomorphic to $n(B)$, then the pre-duals of $n(A)$ and $n(B)$ must also be isomorphic. (The pre-dual of any W^*-algebra is unique – see [32, p. 29].) Thus, if X and Y are the spectra of A and B, and $M(X)$ and $M(Y)$ denote the spaces of regular Borel measures on X and Y respectively, there will be an isometric order-preserving isomorphism of $M(X)$ onto $M(Y)$. One can conclude from this also that X and Y have the same cardinality. But it is not clear how much more one can conclude in general about X and Y. Certainly Example 3.13 shows that the bijection of X with Y corresponding to a Morita equivalence can be very badly behaved.

The above considerations indicate that for C^*-algebras the notion of Morita equivalence as defined in Definition 8.16 is probably too weak to be very interesting. Rather, it will be strengthened forms of Morita equivalence for C^*-algebras which will provide the more interesting and useful tools – for example, Morita equivalences which preserve direct integrals, or weak containment, as do those defined by an imprimitivity bimodule [30, Proposition 6.26]. Such strengthened forms of Morita equivalence will play an important role in, among other things, the classification of C^*-algebras, but discussion of this matter must await another time.

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