TOPOLOGY and its APPLICATIONS
TOPOLOGIJA i ee PRIMENENIJA
Budva 1972 (Beograd 1973) 200—212

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ON THE STABLE HOMOTOPY CATEGORY

1. HISTORY AND MOTIVATION. Many different attempts have been made to find a good framework for stable homotopy theory. The best versions so far were given in the semisimplicial setting by KAN [11], KAN and G. W. WHITEHEAD [12] and in a geometric setting by BOARDMAN [2], [3], [19]. The stable homotopy categories they construct are equivalent to each other (TIERNEY [17]). It seems that they have all good properties that one can reasonably expect, but since in both approaches a lot of technical difficulties appear it is useful to look for still other constructions of an equivalent category. I shall describe one here which is also geometric but more direct than Boardman's. It combines some of Boardman's methods with those of [15]. I got the idea from a remark in the introduction to [3]. Recently I learnt that a similar approach was used by ADAMS in his Chicago lectures in 1971. The notes of these lectures [1] have just become available.

Before writing down the definition of my category I shall try to motivate it. The first version of stable homotopy theory was the S-theory of SPANIER and J. H. C. WHITEHEAD [16]. It may be described as follows: Let $\mathcal{SW}$ be the category whose objects are pointed spaces and whose morphism sets are

$$\mathcal{SW}(A, B) = \operatorname*{colim}_{k\to\infty} \mathcal{T}op^0(\Sigma^k A, \Sigma^k B).$$

Here Σ denotes reduced suspension, $\mathcal{T}op^0(\ ,\ )$ is the set of pointed continuous maps and the colimit (= direct limit) is taken over the suspension maps

$$\mathcal{T}op^0(\Sigma^k A, \Sigma^k B) \to \mathcal{T}op^0(\Sigma^{k+1} A, \Sigma^{k+1} B)$$
$$f \mapsto \Sigma f.$$

There is an obvious notion of homotopy in $\mathcal{SW}$ which gives rise to the homotopy category $\mathcal{SW}h$. This category reflects nicely what happens in the "stable range" of ordinary homotopy theory. It is however not good enough to allow all the constructions one would like to have. For several purposes (e.g. Postnikov systems and the Adams spectral sequence) one should have

Eilenberg-Mac Lane objects. If G is an abelian group and $n \in \mathbf{Z}$, an Eilenberg--Mac Lane object of type (G, n) in $\mathcal{SW}h$ is a space B such that for all $r \in \mathbf{Z}$

$$\pi_r^S(B) \cong \begin{cases} G, & r=n \\ 0, & r \neq n. \end{cases}$$

Here π_r^S denotes the stable homotopy group which may be defined by

$$\pi_r^S(B) = \mathcal{SW}h(S^{r+q}, \Sigma^q B)$$

where q is any natural number such that $r+q \geqslant 0$. Eilenberg-Mac Lane objects are very rare in $\mathcal{SW}h$. There are none of type (G, n) if G has a direct summand which is finitely generated and not zero ([15] 8.3, 8.4).

On the other hand if $K(G, n)$ is an ordinary Eilenberg-Mac Lane space of type (G, n) we do have

$$\pi_r^S K(G, n) \cong \begin{cases} G, & r=n \\ 0, & r \neq n, \; r \leqslant 2n-2 \end{cases}$$

by the Freudenthal suspension theorem. Thus when n increases we get better and better approximations to an Eilenberg-Mac Lane object. If $K(G, n)$ is chosen to have the homotopy type of a CW-complex then there is a homotopy equivalence $K(G, n) \to \Omega K(G, n+1)$ and hence by adjointness of Σ and the loop space functor Ω a map

$$\Sigma K(G, n) \to K(G, n+1).$$

The sequence $K(G, n)$ together with these maps forms a substitute for an Eilenberg-Mac Lane object in $\mathcal{SW}h$ and serves as a model for the notion of a spectrum which was first considered by LIMA [13].

A spectrum X is a sequence of pointed spaces X_n $(n \in \mathbf{Z})$ together with given (pointed) maps $\xi_n : \Sigma X_n \to X_{n+1}$. Spectra are going to be the objects of our stable category. The difficulties begin when we want to define the morphisms. The most naive definition is that a morphism f from X to another spectrum Y is a sequence of (pointed) maps $f_n : X_n \to Y_n$ such that

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\xi_n} & X_{n+1} \\ \Sigma f_n \downarrow & & \downarrow f_{n+1} \\ \Sigma Y_n & \xrightarrow{\eta_n} & Y_{n+1} \end{array}$$

commutes. But with this definition the category would not deserve the name "stable": It is natural to associate to any pointed space A the spectrum JA defined by

$$(JA)_n = \begin{cases} \Sigma^n A, & n \geqslant 0 \\ 0 & n < 0 \end{cases}$$

(where a space consisting only of the base point is denoted by 0). Then a morphism $JA \to JB$ in the sense defined above is nothing else but an ordinary (pointed) map $A \to B$. It should better be a "stable" map, i.e. an element of $\mathcal{SW}(A, B)$. This can be achieved by allowing that f_n is defined only for $n \geqslant n_0$ where n_0 may depend on f. Equivalently this means that a map of spectra

$X \to Y$ needs only to be defined on a subspectrum X' of X where $X'_n = X_n$ for n large enough.

This is not yet the right notion, because one would not get a satisfactory desuspension theorem (6.1, 6.2 below). For this theorem it is important that the domain of definition X' of f may still be smaller than stated above. A condition which one could use for this purpose is that for any n there is an r such that X'_{n+r} contains the image of $\Sigma^r X_n$ under the map

$$\xi_{n+r,n}: \Sigma^r X_n \xrightarrow{\Sigma^{r-1}\xi_n} \Sigma^{r-1} X_{n+1} \xrightarrow{\Sigma^{r-2}\xi_{n+1}} \cdots \xrightarrow{\xi_{n+r-1}} X_{n+r}.$$

But then there are other objections. The wedge of spectra should be a coproduct in the stable homotopy category. This would not be true for

$$\bigvee_{r=0}^{\infty} JS^r,$$

because (using the definition of maps of spectra just proposed) J would induce an embedding of $\mathcal{SWh}$ into the homotopy category of spectra and $\bigvee_{r=0}^{\infty} S^r$ is not a coproduct in $\mathcal{SWh}$ ([15] 7.6).

A good condition for the domain of definition X' of f seems to be:

1.1 *For any n and any compact (pointed) subset $K \subset X_n$ there exists a natural number r such that $\xi_{n+r,n}(\Sigma^r K) \subset X'_{n+r}$.*

One runs however into difficulties if the spaces involved are arbitrary. Hence I restrict myself to CW-complexes and give now the systematic construction of "the stable homotopy category".

2. CONSTRUCTION. From now on we take all spaces, maps, subspaces etc. to be pointed without repeating it always explicitly. A CW-complex shall have a given cell decomposition and a 0-cell as base point. The product $A \times B$ of two CW-complexes A, B is taken with the standard decomposition into product cells and with the CW-topology (compactly generated topology). The smash product $A \wedge B$ is obtained from $A \times B$ as usual by identifying the two axes to the base point and keeping all other cells. S^n denotes the n-sphere decomposed into the base point and one n-cell. The reduced suspension is defined by $\Sigma A = S^1 \wedge A$.

A *spectrum* X will now always be a sequence of CW-complexes X_n together with CW-embeddings $\xi_n: \Sigma X_n \to X_{n+1}$ $(n \in \mathbf{Z})$. This means that ξ_n maps ΣX_n isomorphically onto a subcomplex of X_{n+1}. We call X_n the *terms* and ξ_n the *structure maps* of X.

By $\mathcal{Sp}$ we denote the (preliminary) category of such spectra and "naive" maps of spectra. Thus a map $f: X \to Y$ in $\mathcal{Sp}$ is a sequence of (pointed continuous) maps $f_n: X_n \to Y_n$ such that

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\xi_n} & X_{n+1} \\ {\scriptstyle \Sigma f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma Y_n & \xrightarrow{\eta_n} & Y_{n+1} \end{array}$$

commutes (ξ_n, η_n structure maps, $n \in \mathbf{Z}$).

A spectrum X' is called a *subspectrum* of X if X'_n is a subcomplex of X_n for every n and the sequence of inclusions $i_n: X'_n \subset X_n$ is a map i in $\mathcal{S}\mathfrak{p}$. X' is called *dense* in X and i is called a *dense inclusion* if condition 1.1 is satisfied which we reformulate in the following equivalent form:

2.1 *For any $n \in \mathbf{Z}$ and any finite subcomplex K of X_n there exists a natural number r such that X'_{n+r} contains the image of $\Sigma^r K$ under*

$$\xi_{n+r,n}: \Sigma^r X_n \xrightarrow{\Sigma^{r-1}\xi_n} \Sigma^{r-1} X_{n+1} \xrightarrow{\Sigma^{r-2}\xi_{n+1}} \cdots \xrightarrow{\xi_{n+r-1}} X_{n+r}.$$

The stable category $\mathcal{S}$ which we propose is obtained from $\mathcal{S}\mathfrak{p}$ by inverting all dense inclusions and making $\mathcal{S}$ universal with respect to this property. More precisely: We look for a category $\mathcal{S}$ together with a functor $P: \mathcal{S}\mathfrak{p} \to \mathcal{S}$ such that:

2.2 (a) *Pi is an isomorphism in $\mathcal{S}$ for any dense inclusion i in $\mathcal{S}\mathfrak{p}$.*
(b) *If $P': \mathcal{S}\mathfrak{p} \to \mathcal{S}'$ is any functor satisfying* (a) *then there is one and only one functor F such that*

$$\begin{array}{ccc} & P & \mathcal{S} \\ \mathcal{S}\mathfrak{p} & \begin{smallmatrix}\nearrow \\ \searrow\end{smallmatrix} & \big\downarrow F \\ & P' & \mathcal{S}' \end{array}$$

commutes.

In the terminology of Gabriel-Zisman [5] p. 6.7 $\mathcal{S}$ is the category of fractions of $\mathcal{S}\mathfrak{p}$ for the class of dense inclusions. P is necessarily bijective on objects. Hence we may assume that $\mathcal{S}$ has the same objects as $\mathcal{S}\mathfrak{p}$ and P is the identity on objects. That $\mathcal{S}$ exists is clear from general considerations [5]. But here the description of $\mathcal{S}$ is particularly simple because the class of dense inclusions in $\mathcal{S}\mathfrak{p}$ admits a calculus of right fractions in the sense of [5] p. 12. The following properties are sufficient for this:

2.3 (a) *All identities of $\mathcal{S}\mathfrak{p}$ are dense inclusions.*

(b) *Any composition of dense inclusions is a dense inclusion.*

(c) *For each diagram $X \xrightarrow{f} Y \xleftarrow{j} Y'$* in $\mathcal{S}\mathfrak{p}$, *where j is a dense inclusion, there exists a commutative square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \uparrow & & \uparrow j \\ X' & \xrightarrow{f'} & Y' \end{array}$$

such that i is a dense inclusion.
(d) *Dense inclusions are monomorphisms in $\mathcal{S}\mathfrak{p}$.*

Proof. (a), (b) and (d) are obvious. In (c) one may define X'_n as the union of all subcomplexes L of X_n such that $f_n L \subset im\,(j_n: Y'_n \to Y_n)$.

Using [5] p. 12—14 it follows that a morphism $X \to Y$ in $\mathcal{S}$ may be represented by a diagram $X \supset X' \xrightarrow{f} Y$ in $\mathcal{S}\mathfrak{p}$ where X' is dense in X, and that another such diagram $X \supset X'' \xrightarrow{g} Y$ represents the same morphism in $\mathcal{S}$ if and only if there exists a dense subspectrum X''' of both X' and X'' such that f and g coincide on X'''. In other words $\mathcal{S}(X, Y)$ is the colimit of $\mathcal{S}_{\mathfrak{p}}(X', Y)$ where X' runs through the dense subspectra of X.

3. COLIMITS. We shall construct coproducts and certain pushouts in our category.

The wedge

$$X = \bigvee_{\lambda} X^{\lambda}$$

of an arbitrary family of spectra X^{λ} is formed by taking the wedge of the corresponding terms and structure maps.

3.1 Proposition. *The wedge of spectra with the canonical inclusions is a coproduct in* $\mathcal{S}_{\mathfrak{p}}$ *and in* $\mathcal{S}$.

The proof is trivial for $\mathcal{S}_{\mathfrak{p}}$. For $\mathcal{S}$ it relies on the fact that a subspectrum X' of X is dense if and only if $X' \cap X^{\lambda}$ is dense in X^{λ} for all λ.

The existence of pushouts is subject to analogous restrictions as in the category of CW-complexes. A map $f: X \to Y$ in $\mathcal{S}_{\mathfrak{p}}$ is called *skeletal* if $f_n: X_n \to Y_n$ is skeletal for each n i.e. f_n maps each skeleton of X_n into the corresponding skeleton of Y_n. f is called an *embedding* if f_n is a CW-embedding for each n. A map $X \to Y$ in $\mathcal{S}$ is called skeletal or an embedding resp. if it can be represented by a diagram $X \supset X' \xrightarrow{f} Y$ in $\mathcal{S}_{\mathfrak{p}}$ where X' is dense in X and f has the corresponding properties.

As an auxiliary notion we call an embedding f in $\mathcal{S}_{\mathfrak{p}}$ *closed* if

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\xi_n} & X_{n+1} \\ \Sigma f_n \downarrow & & \downarrow f_{n+1} \\ \Sigma Y_n & \xrightarrow{\eta_n} & Y_{n+1} \end{array}$$

is a pullback for each n. Roughly speaking this means that if the suspension of a cell e of Y_n lies in (the image of) X then e itself lies in X_n.

3.2 Proposition. *Let* $Z \xleftarrow{i} X \xrightarrow{f} Y$ *be a diagram in* $\mathcal{S}_{\mathfrak{p}}$ *or* $\mathcal{S}$, *where* i *is an embedding and* f *is skeletal. In the case* $\mathcal{S}$ *the diagram has a colimit (pushout). In the case* $\mathcal{S}_{\mathfrak{p}}$ *the diagram has a colimit provided that* i *is also closed. The canonical functor* $\mathcal{S}_{\mathfrak{p}} \to \mathcal{S}$ *preserves these colimits.*

Proof. Consider first the case $\mathcal{S}_{\mathfrak{p}}$ and assume that i is closed. Then the colimit may be constructed term by term in the obvious way. In the case $\mathcal{S}$ one has to look at suitable dense subspectra of the given ones. The hypothesis "closed" is not needed in this case because any embedding $X \to Z$ in $\mathcal{S}_{\mathfrak{p}}$ can be factored into a closed embedding $X \to Z'$ and a dense inclusion $Z' \subset Z$.

Another kind of colimits will be considered in section 8.

4. SMALL SMASH PRODUCTS. Let X be a spectrum and A a CW-complex. We define the "*small smash product*" $X \wedge A$ to be the spectrum with terms $X_n \wedge A$ and structure maps

$$\Sigma (X_n \wedge A) = S^1 \wedge X_n \wedge A \xrightarrow{\xi_n \wedge id} X_{n+1} \wedge A.$$

Obviously we get a functor

$$\mathcal{S}p \times \mathcal{W}^0 \xrightarrow{\wedge} \mathcal{S}p$$

where $\mathcal{W}^0$ denotes the category of (pointed) CW-complexes and continuous maps. It induces a functor

$$\mathcal{S} \times \mathcal{W}^0 \xrightarrow{\wedge} \mathcal{S}.$$

This follows from easy general consideration about categories of fractions and from

4.1. Lemma. *If $i: X' \to X$ is a dense inclusion in $\mathcal{S}p$ then so is $i \wedge id_A$: $: X' \wedge A \to X \wedge A$ (up to a canonical isomorphism between $X' \wedge A$ and its image in $X \wedge A$).*

Proof. If K is a finite subcomplex of $X_n \wedge A$ there exists a finite sub complex L of X_n such that $K \subset L \wedge A$. By hypothesis there is an r such tha $\Sigma^r L$ is mapped into X'_{n+r} by the structure maps, hence $\Sigma^r K \subset S^r \wedge L \wedge A$ is mapped into $X'_{n+r} \wedge A = (X' \wedge A)_{n+r}$.

4.2 Proposition. *Let A be a fixed CW-complex. The endofunctors $X \mapsto X \wedge A$ of $\mathcal{S}p$ and $\mathcal{S}$ preserve coproducts and the pushouts considered in* 3.2.

Proof. Since a coproduct is just a wedge (3.1) the first assertion is trivial. For the second one looks at the explicit construction of the pushouts in 3.2.

We shall also use smash product $A \wedge X$ where the CW-complex A is the left factor. Everything is exactly the same as for $X \wedge A$ except that the structure maps are given by

$$S^1 \wedge A \wedge X_n \xrightarrow{\tau \wedge id} A \wedge S^1 \wedge X_n \xrightarrow{id \wedge \xi_n} A \wedge X_{n+1} \tag{4.3}$$

where τ interchanges the factors. $A \wedge X$ is canonically isomorphic to $X \wedge A$ by the map which interchanges the factors in each term.

In section 7 "large" smash products will be considered where both factors are spectra.

5. HOMOTOPY. Let I be the unit interval considered as an unpointed CW-complex with one 1-cell and two 0-cells. I^+ is obtained by adding an isolated basepoint. Let z be the nonbase-point of S^0 and define (pointed) maps

$$j'_\nu : S^0 \to I^+, \quad \nu = 0,1$$

$$q' : I^+ \to S^0$$

by $j'_\nu(z)=\nu$, $q'(I)=\{z\}$. This gives rise to *cylinder functors* $X \mapsto I^+ \wedge X$ in $\mathcal{S}\mathfrak{p}$ and $\mathcal{S}$ and to natural transformations

$$j_\nu : X = S^0 \wedge X \xrightarrow{j'_\nu \wedge id} I^+ \wedge X$$

$$q : I^+ \wedge X \xrightarrow{q' \wedge id} S^0 \wedge X = X.$$

Together they form homotopy systems in $\mathcal{S}\mathfrak{p}$ and $\mathcal{S}$ resp. in the sence of KAMPS [7], [8], [9].

5.1. Proposition. *The homotopy systems just defined satisfy all extension conditions* $DNE(n, \nu, k)$ (*cf.* [7] (2.6), [8] (3.3) *or* [9] (3.2)).

Proof. One constructs the extensions in $I^+ \wedge I^+ \wedge \dots \wedge I^+ = (I^n)^+$ where they are trivial and forms then the smash product with a spectrum.

Hence for $\mathcal{S}\mathfrak{p}$ and $\mathcal{S}$ we get as corollaries all the results which KAMPS proved for categories with homotopy system satisfying the extension conditions. In particular we have homotopy notions which are equivalence relations and give rise to the homotopy (quotient) categories $\mathcal{S}\mathfrak{p}\mathfrak{h}$ and $\mathcal{S}\mathfrak{h}$.

5.2 Proposition. *The categories* $\mathcal{S}\mathfrak{p}$ *and* $\mathcal{S}$ *have double mapping cylinders for all skeletal maps.*

Proof. Let $Y \xleftarrow{f} X \xrightarrow{f'} Y'$ be skeletal maps. Then by 3.2 the pushout

$$\begin{array}{ccc} X \vee X & \xrightarrow{<j_0,\, j_1>} & I^+ \wedge X \\ f \vee f' \downarrow & & \downarrow \\ Y \vee Y' & \longrightarrow & Z_{(f,\, f')} \end{array}$$

exists (because $<j_0, j_1>$ is a closed embedding). $Z_{(f, f')}$ is the double mapping cylinder.

By specializing we get:

(a) the *mapping cylinder* $Z_f = Z_{(f,\, id_X)}$,

(b) the *mapping cone* $C_f = Z_{(f,\, 0)}$, where $0 : X \to 0$ denotes the unique map into the zero spectrum,

(c) the *suspension* $\Sigma X = C_0 = Z_{(0,\, 0)}$.

It is obvious from the definitions that ΣX is canonically isomorphic to $S^1 \wedge X$.

5.3 Proposition. *The cylinder functor in* $\mathcal{S}\mathfrak{p}$ *and* $\mathcal{S}$ *preserves all colimits considered so far, namely*

coproducts,
pushouts as in 3.2,
(double) mapping cylinders and mapping cones of skeletal maps,
suspensions.

This is a corollary of 4.2 and the proof of 5.2.

The results of 5.1 and 5.3 suffice to carry out the standard construction of a cofibre sequence

$$X \xrightarrow{f} Y \to C_f \to \Sigma X \to \Sigma Y \to \cdots$$

for all skeletal maps (**[7]** section 7, **[10]** section 5). The restriction to skeletal maps is not essential bacause we have.

5.4 Proposition. (a) *If the spectrum X is bounded below (i.e there exists $n_0 \in \mathbf{Z}$ such that $X_n = 0$ for $n < n_0$) then any $f \in \mathcal{S}\mathfrak{p}(X, Y)$ is homotopic to a skeletal map.*
(b) *Any map in $\mathcal{S}$ is homotopic to a skeletal map.*

Proof. (a) Using the relative skeletal (cellular) approximation theorem for *CW*--complexes we deform f_n into a skeletal map by induction over n making the deformation compatible with the structure maps.

(b) follows from (a) because any map in $\mathcal{S}$ may be represented by a diagram $X \supset X' \xrightarrow{f} Y$ in $\mathcal{S}\mathfrak{p}$ where X' is dense in X and X' is bounded below.

6. DESUSPENSION. To justify the name "stable" for our category we have to show that the suspension functor is invertible up to equivalence. To see this we introduce a new kind of suspension, the translation suspension, and compare it to the other one which, for distinction, shall be called geometric suspension.

The translation suspension Σ_t is simply defined by shifting the indices one step. Thus the spectrum $\Sigma_t X$ has the terms $(\Sigma_t X)_n = X_{n+1}$ and the structure maps $(\Sigma_t \xi)_n = \xi_{n+1}$.

6.1 Theorem. *The geometric suspension Σ and the translation suspension Σ_t are naturally equivalent as endofunctors of $\mathcal{S}\mathfrak{h}$.*

6.2 Corollary. *The geometric suspension $\Sigma : \mathcal{S}\mathfrak{h} \to \mathcal{S}\mathfrak{h}$ is invertible up to natural equivalence.*

Proof. Σ_t is obviously invertible.

6.3 Corollary. *$\mathcal{S}\mathfrak{h}$ is an additive category with arbitrary coproducts.*

Proof. Up to equivalence every object of $\mathcal{S}\mathfrak{h}$ is of the form ΣX and every morphism of the form Σf (6.2). From the usual cogroup structure on S^1 one gets a cogroup structure on $\Sigma X = S^1 \wedge X$ and hence a group structure on the morphism sets of $\mathcal{S}\mathfrak{h}$ which is compatible with composition. The coproduct in $\mathcal{S}$ (3.1) is also a coproduct in $\mathcal{S}\mathfrak{h}$, because the cylinder functor preserves coproducts (5.3).

We may turn $\mathcal{S}\mathfrak{h}$ into a graded category $\mathcal{S}\mathfrak{h}$ by defining

$$\mathcal{S}\mathfrak{h}_p(X, Y) = \mathcal{S}\mathfrak{h}(\Sigma_t^p X, Y).$$

For any skeletal map f we consider the cofibre sequence

$$X \xrightarrow{f} Y \to C_f \to \Sigma X$$

and compose the last map with the natural equivalence $\Sigma X \to \Sigma_t X$ of 6.1, obtaining a "triangle"

$$X \to Y \to C_f \to X$$

in which the last map has degree -1.

6.4 Theorem. *$\mathcal{S}h_*$ together with the class of triangles isomorphic to some triangle of the form just described is a triangulated category.*

The name "triangulated" is taken from VERDIER [18] (compare also HELLER [6]). Almost the same concept was called "stable" in [14] and [15]. There are minor differences in conventions about signs and there is an additional axiom in [18] not contained in [14] or [15], the octahedral axiom. What we mean here is that all the axioms are satisfied including the octahedral axiom. We use however the sign conventions of [15]. The proof of the axioms is rather straightforward from the properties of the cofibre sequence given in [7] section 7. One may also take the proof of [15] Satz 3.5 and adapt it to the present situation with slight changes.

Proof of Theorem 6.1. Let $\Sigma_t' X$ be the spectrum with terms $(\Sigma_t')X_n = \Sigma X_n$ and structure maps $(\Sigma_t'\xi)_n = \Sigma\xi_n$. Then the maps

$$(\Sigma_t' X)_n = \Sigma X_n \xrightarrow{\xi_n} X_{n+1} = (\Sigma_t X)_n$$

fit together forming an embedding $\xi\colon \Sigma_t' X \to \Sigma_t X$ in $\mathcal{S}p$ The image is a dense subspectrum of $\Sigma_t X$, hence ξ is an isomorphism in $\mathcal{S}$. At first sight one could think that $\Sigma_t' X$ equals ΣX (which would finish the proof), but this is not so. The structure maps of ΣX are

$$(\Sigma\xi)_n\colon S^1 \wedge S^1 \wedge X_n \xrightarrow{\tau \wedge id} S^1 \wedge S^1 \wedge X_n \xrightarrow{id \wedge \xi_n} S^1 \wedge X_{n+1}$$

where τ interchanges the two factors S^1 (4.3). They are not even homotopic to the structure maps

$$(\Sigma_t'\xi)_n = \Sigma\xi_n\colon S^1 \wedge S^1 \wedge X_n \xrightarrow{id \wedge \xi_n} S^1 \wedge X_{n+1}$$

of $\Sigma_t' X$. To get around this difficulty we consider the reflection σ of S^1 (given by $\sigma(t) = 1 - t$ if $S^1 = I/\{0, 1\}$) and define $\Sigma_- X$ to be the spectrum with terms $(\Sigma_- X)_n = \Sigma X_n$ and structure maps

$$(\Sigma_-\xi)_n\colon S^1 \wedge S^1 \wedge X_n \xrightarrow{\tau \wedge id} S^1 \wedge S^1 \wedge X_n \xrightarrow{\sigma \wedge \xi_n} S^1 \wedge X_{n+1}.$$

Now we have an isomorphism

$$k\colon \Sigma X \to \Sigma_- X \text{ in } \mathcal{S}p$$

given by

$$k_n = \sigma^n \wedge id\colon S^1 \wedge X_n \to S^1 \wedge X_n,$$

and on the other hand $(\Sigma_-\xi)_n$ is homotopic to $(\Sigma_t'\xi)_n$. One can show that this latter fact implies that $\Sigma_- X$ is isomorphic to $\Sigma_t' X$ in $\mathcal{S}h$. Actually we have to prove a little more, namely that there is a natural isomorphism.

For this we choose a specific (pointed) homotopy

$$\rho^t\colon S^1 \wedge S^1 \to S^1 \wedge S^1,\quad t \in I$$

such that $\rho^0 = (\sigma \wedge id)\tau$ and $\rho^1 = id$. Then

$$S^1 \wedge S^1 \wedge X_n \xrightarrow{\rho^t \wedge id} S^1 \wedge S^1 \wedge X_n \xrightarrow{id \wedge \xi_n} S^1 \wedge X_{n+1}$$

is a homotopy from $(\Sigma_- \xi)_n$ to $(\Sigma'_t \xi)_n$. Let MX be the spectrum with terms $(MX)_n = S^1 \wedge X_n \wedge I^+$ and structure maps

$$(M\xi)_n : S^1 \wedge S^1 \wedge X_n \wedge I^+ \to S^1 \wedge X_{n+1} \wedge I^+$$

$$(s_1, s_2, x, t) \mapsto ((id \wedge \xi_n)(\rho^t(s_1, s_2), x), t).$$

Define embeddings in $\mathcal{S}p$

$$i_0 : \Sigma_- X \to MX, \quad i_1 : \Sigma'_t X \to MX$$

by $i_{\nu, n}(s, x) = (s, x, \nu)$, $s \in S^1$, $x \in X_n$, $\nu = 0,1$. Obviously M may be considered as an endofunctor of $\mathcal{S}p$ and i_ν as a natural transformation. $i_{\nu, n}$ is a homotopy equivalence for each single n. By Lemma 6.5 below, i_ν is an isomorphism in $\mathcal{S}ph$ provided that the spectrum X is bounded below (compare 5.4). This last restriction is irrelevant as soon as we pass to $\mathcal{S}h$, because any spectrum X has a dense subspectrum X' which is bounded below and X' is isomorphic to X in $\mathcal{S}$. Altogether we have

$$\Sigma X \underset{\mathcal{S}p}{\xrightarrow{k}} \Sigma_- X \underset{\mathcal{S}ph}{\xrightarrow{i_0}} MX \underset{\mathcal{S}ph}{\xleftarrow{i_1}} \Sigma'_t X \underset{\mathcal{S}}{\xrightarrow{\xi}} \Sigma_t X.$$

Each arrow is a natural equivalence in the category indicated (if we assume X to be bounded below) and induces a natural equivalence in $\mathcal{S}h$ (without restriction).

6.5. Lemma. *Let $f : Y \to Y$ be a map in $\mathcal{S}p$ such that $f_n : X_n \to Y_n$ is a homotopy equivalence for each n and X, Y are bounded below. Then f is a homotopy equivalence in $\mathcal{S}p$, i.e. an isomorphism in $\mathcal{S}ph$.*

Proof. We shall show that for any spectrum Z the map

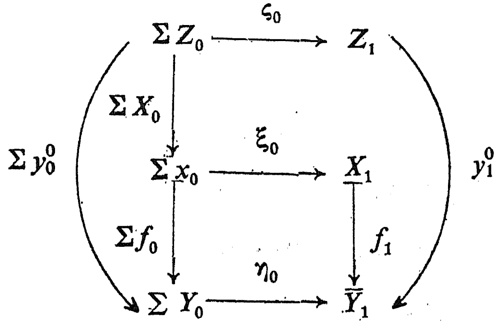
$$f_* : \mathcal{S}ph\,(Z, X) \to \mathcal{S}ph\,(Z, Y)$$

induced by f is a surjection. Taking $Z = Y$ this gives a map $g \in \mathcal{S}p\,(Y, X)$ such that fg is homotopic to id_Y in $\mathcal{S}p$. Since g_n is a homotopy inverse of f_n we may apply everything to g instead of f obtaining $h \in \mathcal{S}p\,(X, Y)$ such that $gh \simeq id_X$. This shows that g and hence f are isomorphisms in $\mathcal{S}ph$.

To prove f_* surjective we assume $X_n = Y_n = Z_n = 0$ for $n < 0$. This is no loss of generality and it simplifies the notation a little. Let $y \in \mathcal{S}p\,(Z, Y)$

Since f_0 is a homotopy equivalence there is a map $x_0 : Z_0 \to X_0$ such that $y_0 \simeq f_0\, x_0$. Let $\varphi_0^0 : Z_0 \wedge I^+ \to Y_0$ be a homotopy deforming y_0 into $f_0\, x_0$. Using the homotopy extension property for $\zeta_n : \Sigma Z_n \to Z_{n+1}$ we extend φ_0^0 to a homotopy $\varphi^0 : Z \wedge I^+ \to Y$ deforming y into some $y^0 \in \mathcal{S}p\,(Z, Y)$.

Next we look at the commutative diagram



in which we would like to fill in $x_1: Z_1 \to X_1$. We consider Z_1, X_1, Y_1 as spaces under $\sum Z_0$ by the maps ζ_0, $\xi_0(\sum x_0)$ and $\eta_0(\sum y_0^0) = f_1 \xi_0 (\sum x_0)$ resp. Then f_1 is a map under $\sum Z_0$ and an ordinary homotopy equivalence. Since ζ_0 is a cofibration it follows that f_1 induces a bijection of homotopy classes $Z_1 \to X_1$ rel. $\sum Z_0$ into homotopy classes $Z_1 \to Y_1$ rel $\sum Z_0$ (**[4]** (10.5) Satz, p. 165). So we find a map $x_1: Z_1 \to X_1$ and a homotopy $\varphi_1^1: Z_1 \wedge I^+ \to Y_1$ deforming y_1^0 into $f_1\, x_1$. We extend this to a homotopy $\varphi^1: Z \wedge I^+ \to Y$ deforming y^0 into some $y^1 \in \mathcal{S}p$ (Z, Y) such that $\varphi_0^1: Z_0 \wedge I^+ \to Y_0$ is constant with respect to the homotopy parameter.

Continuing in this way we recursively construct

$$x_n: Z_n \to X_n$$
$$y^n \in \mathcal{S}p\,(Z, Y)$$
$$\varphi^n \in \mathcal{S}p\,(Z \wedge I^+, Y)$$

such that

$f_n x_n = y_n^n$

φ^n deforms y^{n-1} into y^n

φ_k^n is constant with respect to the homotopy parameter if $k < n$.

These maps fit together giving $x \in \mathcal{S}p\,(Z, X)$ and $\varphi \in \mathcal{S}p\,(Z \wedge [-1, \infty]^+, Y)$ deforming y into fx, where

$$\varphi_k(z, t) = \begin{cases} \varphi_k^n(z, t), & n-1 \leq t \leq n \\ f_k x_k(z), & t = \infty \end{cases}$$

Hence $f_*[x] = [y]$.

7. LARGE SMASH PRODUCTS. In section 4 we have defined smash products between spectra and CW-complexes. It is important to extend this to smash products of two spectra.

7.1. Theorem. *There is a functor, the "large smash product",*

$$\mathcal{S}b \times \mathcal{S}b \xrightarrow{\wedge} \mathcal{S}b$$

with the following properties:

(a) *It is associative and commutative up to natural equivalence.*
(b) *The diagram*

$$\begin{array}{ccccc} \mathcal{S}b \times \mathcal{W}^0 b & \xrightarrow{id \times J} & \mathcal{S}b \times \mathcal{S}b & \xrightarrow{J \times id} & \mathcal{W}^0 b \times \mathcal{S}b \\ & \searrow^{\wedge} & \downarrow{\wedge} & \swarrow^{\wedge} & \\ & & \mathcal{S}b & & \end{array}$$

(which relates small and large smash products) commutes up to natural equivalence. This implies easily that JS^0 is a unit for the large smash product up to natural equivalence.

(c) *All the natural equivalences in* (a) *and* (b) *can be chosen to be coherent.*

The proof is too long to be included here. For Boardman's stable category there is a proof in **[3]** Chapt. II § 7. The proof in **[19]** has a gap in the proof of Lemma 9.6 p. 154. A very direct proof of 7.1 for precisely the present situation is given by ADAMS **[1]** §4. My own proof is different from Adams' but follows Boardman's ideas: For the purpose of this proof I pass from ordinary spectra to a new kind where the indexing set **Z** is replaced by the set $\mathfrak{E}$ of finite dimensional subspaces of a euclidean vector space of countable dimension. The structure maps have the form

$$S_{B,A} \wedge X_A \to X_B,$$

where $A, B \in \mathfrak{E}$, $A \subset B$ and $S_{B,A}$ is the one point compactification of the orthogonal complement of A in B. Details will appear elsewhere.

8. COMPARISON TO BOARDMAN'S CATEGORY. BOARDMAN defines first the category $\mathcal{F}_s$ of finite spectra by "stabilizing" the category $\mathcal{F}$ of finite CW-complexes (cf. **[3]**, **[17]**, **[19]**). It is canonically isomorphic to the full subcategory $\mathcal{E}$ of $\mathcal{S}$ whose objects have the form $\Sigma_t^n JA$, where $n \in \mathbf{Z}$ and A is a finite CW-complex. The isomorphism sends $\Sigma_t^n JA$ into $(A, -n)$ in the notation of **[3]** Chapt. II, 2.1 or **[19]** p. 81. It takes embeddings in $\mathcal{E}$ into embedings in $\mathcal{F}_s$ (which Boardman calls inclusions). Following Boardman we denote the subcategory of embeddings always by putting an $\mathcal{J}$ in front of the symbol for whole category.

BOARDMAN obtains his stable category $\mathcal{F}_{sw}$ by applying his w-construction to the pair $(\mathcal{F}_s, \mathcal{J}\mathcal{F}_s)$. Roughly this means that one adjoins to $\mathcal{F}_{sw}$ colimits of all directed diagrams in $\mathcal{J}\mathcal{F}_s$. We have

8.1. Proposition. *Let $\mathcal{C}$ be $\mathcal{S}p$ or $\mathcal{S}$. Any directed diagram in $\mathcal{J}\mathcal{C}$ has a colimit in $\mathcal{J}\mathcal{C}$ which is also a colimit in $\mathcal{C}$.*

The proof is trivial for $\mathcal{C} = \mathcal{S}p$. It is more subtle for $\mathcal{C} = \mathcal{S}$. For lack of space we cannot give it here.

By the basic properties of the w-construction 8.1 implies that there is one and only one functor G which makes

$$\begin{array}{ccc} (\mathcal{F}_s, \mathcal{I}\mathcal{F}_s) & \subset & (\mathcal{F}_{sw}, \mathcal{I}\mathcal{F}_{sw}) \\ \| & & \downarrow G \\ (\mathcal{E}, \mathcal{I}\mathcal{E}) & \subset & (\mathcal{S}, \mathcal{I}\mathcal{S}) \end{array}$$

commutative and preserves colimits of directed diagrams of embeddings. In **[17]** 2.5 TIERNEY has given conditions under which G is an equivalence of categories. It is easy to verify these conditions in the present situation.

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(Received 16 10. 1972)

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