Sugaku Expositions
Volume 4, Number 2, December 1991

INDEX THEORY FOR OPERATOR ALGEBRAS

HIDEKI KOSAKI

An operator algebra is a natural generalization of the matrix algebra $M_n(\mathbf{C})$. When one operator algebra is sitting in a larger one, by their index we mean a "ratio" between them. In other words, we would like to count how many copies of the smaller one are in the larger one. For example, $M_4(\mathbf{C}) \cong M_2(\mathbf{C}) \otimes M_2(\mathbf{C})$ contains the subalgebra $M_2(\mathbf{C}) \otimes 1$ isomorphic to $M_2(\mathbf{C})$.

$$M_4(\mathbf{C}) \supseteq M_2(\mathbf{C}) \otimes 1 = \left\{ \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & A \end{bmatrix} ;\ A \in M_2(\mathbf{C}) \right\}.$$

In this case, the index should be the ratio between the dimensions (as linear spaces) of the algebras

$$\dim M_4(\mathbf{C}) / \dim M_2(\mathbf{C}) = 4^2/2^2 = 4.$$

However, all the interesting operator algebras are of infinite dimension so that the above simple-minded method gives us ∞/∞ and makes no sense. We would like to define the index in a reasonable way and to develop meaningful index theory.

In the epoch-making article [32] V. Jones proved a striking result (see §2) and laid the foundation of index theory for a very important class of operator algebras (II_1-factors). Since then remarkable progress in the theory of operator algebras has been made. Furthermore, based on Jones' theory new invariants for knots and links were discovered [33], [19], and the relationship between operator algebras and various other fields (such as Virasoro algebras, solvable lattice models, and so on) has been clarified. (For this, see the exposition written by T. Kobayashi.) This unexpected connection to other fields is beyond the scope of the present exposition; hence, we will focus our attention just on operator algebras and explain the Jones index theory together with subsequent development. Although we tried to collect literature related to index theory as much as possible, we may have overlooked some literature. However, the material that has already been covered in standard textbooks (see the references [67]–[73]) has been omitted.

1. von Neumann algebras

The theory of operator algebras was initiated by Murray and von Neumann

This article originally appeared in Japanese in Sūgaku **41**(4)(1989), 289–304.
1991 *Mathematics Subject Classification.* Primary 46L10, 46L35.

in the 1930s. In the introduction of their first article, as motivation for opening the new field, quantum mechanics, representation theory, operator theory, and abstract ring theory are mentioned. An operator algebra is a $*$-algebra of (bounded) linear operators on a Hilbert space (over $\mathbf{C}$). Here, $*$ refers to the adjoint of an operator, which is a correct generalization of an adjoint matrix. Consequently, our theory deals with infinite-dimensional and noncommutative objects. The algebra structure being available, results are often formulated in algebraic language. However, due to infinite dimensionality, approximation arguments (i.e., analysis) are unavoidable, and we have to deal with several topologies. When an operator algebra contains the identity operator 1 and is closed relative to the strong operator topology (can be replaced by other topologies), it is called a von Neumann algebra. Here, a net $\{x_\iota\}$ of operators converges to x in the strong operator topology if and only if

$$\|x_\iota \xi - x\xi\| \to 0$$

for an arbitrary vector ξ. On the other hand, an operator algebra that is closed under the topology determined by the operator norm

$$\|x\| = \sup\{\|x\xi\|;\ \|\xi\| \leq 1\}$$

is called a C^*-algebra. (Index theory for C^*-algebras has been developed by Watatani [62].) We remark that a Hilbert space-free characterization of these algebras is also possible. In fact, in the late 1940s C^*-algebras were introduced by Gelfand and Naimark in such an abstract form.

A von Neumann algebra can be expressed as the direct integral of factors (central decomposition)—this is analogous to decomposing unitary representations into irreducible representations. Here, a von Neumann algebra is called a factor if its center (as an algebra) reduces to $\mathbf{C}1$. The full matrix algebra $M_n(\mathbf{C})$ and $B(H)$ (the set of all bounded operators on a Hilbert space H) are obviously factors. When a von Neumann algebra M $(\subseteq B(H))$ is given, its commutant M' is defined by

$$M' = \{x \in B(H);\ xy = yx,\ \forall y \in M\}.$$

(Note that M' makes sense only when the Hilbert space on which M acts is specified.) Remark that $M \subseteq M''$ is trivial. An important fact is that an operator algebra M is a von Neumann algebra if and only if $M = M''$. This characterization (the double commutation theorem) obtained by von Neumann in 1929 was a starting point of our theory. This result guarantees that the polar decomposition $(x = u|x|,\ |x| = (x^*x)^{1/2})$ and the spectral decomposition $(|x| = \int_0^{\|x\|} \lambda\, de_\lambda)$—fundamental tools for dealing with operators—can be performed inside the von Neumann algebra in question. In particular, a von Neumann algebra contains an abundance of projections. (This is false for a C^*-algebra.) By analyzing the structure of the projections (under a certain equivalence relation), Murray and von Neumann classified factors into the following types:

$$\begin{cases} \text{type } \mathrm{I}_n, & n = 1, 2, \ldots, \infty, \\ \text{type } \mathrm{II}_1, & \\ \text{type } \mathrm{II}_\infty, & \\ \text{type } \mathrm{III}. & \end{cases}$$

A factor of type I_n is isomorphic to $M_n(\mathbf{C})$ ($B(H)$ if $n = \infty$), and a factor of type II_∞ can be written as the tensor product of a II_1-factor and $B(H)$.

A II_1-factor has "continuous dimensions", which was a surprise to specialists. For example, consider $M_2(\mathbf{C})$ equipped with the normalized trace $\mathrm{tr}([a_{ij}]) = 2^{-1}(a_{11} + a_{22})$. The (normalized) dimensions (i.e., trace values) of the projections (i.e., subspaces) are obviously 0, $\frac{1}{2}$, and 1. From the tensor product $M_2(\mathbf{C}) \otimes M_2(\mathbf{C})$ equipped with $\mathrm{tr} \otimes \mathrm{tr}$ we get 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1. The more tensor components we add, the more trace values of projections we get. Completing the infinite tensor product $\bigotimes_{n=1}^{\infty} M_2(\mathbf{C})$ relative to $\bigotimes_{n=1}^{\infty} \mathrm{tr}$, we obtain a II_1-factor. It is not so hard to imagine that the trace values of the projections fill up the entire closed interval $[0, 1]$. A II_1-factor always admits a (unique) normalized ($\mathrm{tr}(1) = 1$) trace with this property, and this can be considered as an alternative definition of a II_1-factor. A linear functional $\mathrm{tr}\colon M \to \mathbf{C}$ (with suitable continuity requirement) on a factor M is called a trace if

(i) $\mathrm{tr}(x^*x) \geq 0$, $x \in M$ (positivity),
(ii) $\mathrm{tr}(xy) = \mathrm{tr}(yx)$, $x, y \in M$ (tracial property).

From a highly noncommutative countable group (ICC group such as S_∞, the finite permutations on $\mathbf{N}$, or the free group F_2 with two generators) one can construct a II_1-factor (the von Neumann algebra generated by the left regular representation, i.e., a certain completion of the group ring). Also (by making use of the group-measure-space construction) one obtains a II_1-factor from a (probability) measure-preserving ergodic transformation.

On the other hand, a factor of type III does not possess a trace (even if one allows $+\infty$ as a trace value like the usual trace on $B(H)$). Starting from $\bigotimes_{n=1}^{\infty} M_2(\mathbf{C})$ one obtains a factor of type III by performing a certain completion different from the previous one. Most examples of factors arising naturally from quantum statistical mechanics are known to be of type III. A factor arising from a nonsingular (but not admitting an invariant measure) ergodic transformation is also of type III. However, due to the fact that traces do not exist, type III factors had been very difficult to handle before the appearance of the Tomita-Takesaki theory (1967). It was their theory that made analysis on type III factors possible. Based on this theory Connes further classified type III factors into

$$\begin{cases} \text{type } \mathrm{III}_0, & \\ \text{type } \mathrm{III}_\lambda & (0 < \lambda < 1), \\ \text{type } \mathrm{III}_1. & \end{cases}$$

Also several structure theorems were obtained by Araki, Connes, and Takesaki. (The final step was marked by Takesaki [59].) It can be said that the analysis of type III factors reduces to the study of II_∞-algebras and that of automorphism groups on these algebras. Partly due to this fact, the study of automorphism groups has been (and is still being) actively carried out by several authors.

When a factor is generated by an increasing sequence of finite-dimensional subalgebras, it is called AFD (approximately finite dimensional). Almost all factors appearing in applications are AFD. Very recently, the classification of AFD factors has been completed:

type I_n	$M_n(\mathbf{C})$, $B(H)$,	
type II_1	unique R	(Murray, von Neumann)
type II_∞	unique $R \otimes B(H)$	(Connes)
type III_0	equivalent to classification of ergodic flows (flows of weights)	(Connes, Krieger)
type III_λ	unique (for each λ); the Powers factor	(Connes)
type III_1	unique	(Haagerup)

The uniqueness of an AFD II_1-factor (this factor is denoted by R) is especially important. It was further proved by Connes that R is the smallest infinite-dimensional factor. Recall the II_1-factor constructed from the infinite tensor product of $M_2(\mathbf{C})$ and the one arising from S_∞. Obviously, both of them possess a structure of finite-dimensional nests so that they are isomorphic to R. On the other hand, the II_1-factor arising from F_2 is not AFD. From a given type III factor a certain nonsingular ergodic flow (the flow of weights [17]) can be constructed in a functorial way. In the AFD case the flow of weights serves as the complete invariant for the isomorphism class. A factor is of type III_1 (resp. III_λ) if and only if the associated flow of weights is trivial (resp. periodic with period $-\log\lambda$). All other ergodic flows correspond to type III_0-factors.

The study of non-AFD factors is equally important; however, definite results (such as the above classification) have not been obtained. Since AFD factors have been classified, we would like to attack the classification problem for subfactors (in a given AFD factor). Jones' index theory serves as an important tool for such a problem.

As indicated above, the theory of von Neumann algebras is closely related to ergodic theory on nonsingular transformations. From the viewpoint of operator algebras the notion of orbit-equivalence (which is weaker than the usual notion of equivalence in ergodic theory) plays an important role. Such ergodic theory has been developed by many authors (Dye, Krieger, Connes, and many others).

2. Jones' index theory

Here, Jones' theory on index for II_1-factors will be explained. Let $M \supseteq N$ be a pair of II_1-factors, and $\mathrm{Tr} = \mathrm{Tr}_M$ be the (unique) normalized trace on M. When M is acting on a Hilbert space H, one can regard H as an M-module. Hence, the quantity "$\dim_M H$" can be introduced by using Tr (theory on coupling constants). Jones defined the index $[M : N]$ based on coupling constants, but in this exposition we will explain the index without mentioning $\dim_M H$ so explicitly.

To motivate the reader, let us assume that $M_2(\mathbf{C})$ is acting on $\mathbf{C}^2$ in the usual way. The commutant is $\mathbf{C}1$, which is too small compared with $M_2(\mathbf{C})$. By letting $M_2(\mathbf{C})$ act on a larger Hilbert space, we will obtain a larger commutant having the "same size" as the original algebra $M_2(\mathbf{C})$. Let Tr be the usual (unnormalized) trace on $M_2(\mathbf{C})$. Then $M_2(\mathbf{C})$ is a 4-dimensional Hilbert space under the inner product

$$(x, y) \in M_2(\mathbf{C}) \times M_2(\mathbf{C}) \to \mathrm{Tr}(y^*x) \in \mathbf{C}.$$

Forgetting the original action of $M_2(\mathbf{C})$ on $\mathbf{C}^2$, we let $M_2(\mathbf{C})$ act on the above Hilbert space $M_2(\mathbf{C})$ by left multiplication. If we denote a vector

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \in M_2(\mathbf{C})$$

by ${}^t(x_{11}, x_{21}, x_{12}, x_{22}) \in \mathbf{C}^4$, the action (i.e., left multiplication) of

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

in the algebra $M_2(\mathbf{C})$ on the Hilbert space $M_2(\mathbf{C}) \cong \mathbf{C}^4$ is expressed as the 4×4 matrix

$$\begin{bmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & & \mathbf{0} \\ & & a_{11} & a_{12} \\ \mathbf{0} & & a_{21} & a_{22} \end{bmatrix} \quad \left(= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes 1 \right).$$

A matrix commuting with the above matrices must be of the form

$$\begin{bmatrix} b_{11} & 0 & b_{12} & 0 \\ 0 & b_{11} & 0 & b_{12} \\ b_{21} & 0 & b_{22} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{bmatrix} \quad \left(= 1 \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right)$$

(which is a matrix representing right multiplication). Note that $M = M_2(\mathbf{C})$ and M' have the same size now. Let us make this statement more precise. The tracial property $\mathrm{Tr}(x^*x) = \mathrm{Tr}(xx^*)$ implies that the map $x \to x^*$ is a unitary involution of the Hilbert space $M_2(\mathbf{C})$. The matrix representation of this unitary involution J is

$$J = \begin{bmatrix} J_0 & 0 & 0 & 0 \\ 0 & 0 & J_0 & 0 \\ 0 & J_0 & 0 & 0 \\ 0 & 0 & 0 & J_0 \end{bmatrix},$$

where J_0 denotes complex conjugation. It is easy to see that $JMJ = M'$, that is, $x \to Jx^*J$ gives us an anti-isomorphism from M onto M'.

Let us return to II_1-factors $M \supseteq N$. As above, M is a pre-Hilbert space relative to the inner product

$$(x, y) \in M \times M \to \mathrm{tr}(y^*x) \in \mathbf{C}.$$

Let us denote the Hilbert space completion by $L^2(M)$ $(= L^2(M; \mathrm{tr}))$, the standard Hilbert space of M. As before M acts on $L^2(M)$ by left multiplication, and from now on we understand that M (hence the subfactor N as well) is acting on $L^2(M)$. Again $x \in M \to x^* \in M$ (after the extension) gives us a unitary involution J on $L^2(M)$ satisfying $JMJ = M'$. We obviously get $N' \supseteq M'$. Generally, the commutant of a II_1-factor is a factor of type either II_∞ or II_1. The commutant N' $(\subseteq B(L^2(M)))$ is of type II_∞ when N is very small in M. Therefore, in this case the index $[M : N]$ is defined to be $+\infty$. Let us now assume that N' is a factor of type II_1 (with the unique normalized trace $\mathrm{tr}_{N'}$). Let e_N be the orthogonal projection from $L^2(M)$ $(= \overline{M}$ by the definition) onto the closed subspace $\overline{N}$. This subspace being invariant under the action of N, the projection e_N belongs to N' and $\mathrm{tr}_{N'}(e_N) \in (0, 1]$ makes sense.

Definition (*Jones index* $[M : N]$). When N' is of type II_1, we set

$$[M : N] = (\mathrm{tr}_{N'}(e_N))^{-1}.$$

Let us take $N = \mathbf{C}1$ in the previous example $M = M_2(\mathbf{C})$ (although this is not of type II_1). Then $N' = M_4(\mathbf{C})$, and the closed subspace $\overline{N}$ and the projection e_N are

$$\left\{ \begin{bmatrix} x \\ 0 \\ 0 \\ x \end{bmatrix} ; x \in \mathbf{C} \right\}, \qquad e_N = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

respectively. Thus, the (normalized) trace value of e_N is $\frac{1}{4}(\frac{1}{2} + 0 + 0 + \frac{1}{2}) = \frac{1}{4}$ and the index is 4 as expected.

The Jones index $[M : N]$ satisfies all the expected properties such as (1) $M \supseteq N \supseteq L$ implies $[M : L] = [M : N][N : L]$, (2) when a finite group G acts on N as (outer) automorphisms, the crossed product (semidirect product) $M = N \rtimes G$ satisfies $[M : N] = \#G$.

An obvious (and probably the most important) question is what the possible values of $[M : N]$ are. When $M = M_n(\mathbf{C})$, then $[M : N]$ has to be the square of an integer (since M and N have the same unit). On the other hand, the characteristic of a II_1-factor being of continuous dimensions, an arbitrary value seems to be possible. Jones proved the following surprising result:

Theorem [32]. *The Jones index* $[M : N]$ *belongs to*

$$\left\{ 4\cos^2 \frac{\pi}{n} : n = 3, 4, \ldots \right\} \cup [4, +\infty].$$

Furthermore, any value in the above set can be realized as the index value of a subfactor in the AFD II_1*-factor* R.

The last half of the theorem being quite technical, in what follows we will briefly explain the first half. It is not so hard to prove that N' is generated (as a von Neumann algebra) by M' and e_N. Since $\overline{N}$ is invariant under J (N is a $*$-subalgebra), we see $Je_NJ = e_N$. Consequently, $JN'J$ is generated by $JM'J = M$ and $Je_NJ = e_N$. We set $M_1 = JN'J$, the basic extension of $M \supseteq N$. Hence, starting from $M \supseteq N$, we have constructed (basic construction) the new inclusion $M_1 \supseteq M$ of II_1-factors (if $[M : N] < \infty$), and this procedure preserves the index value ($[M_1 : M] = [M : N]$). The extension M_1 was obtained as a factor acting on $L^2(M)$. However, by letting M_1 act on its own standard Hilbert space $L^2(M_1)$, one can construct the next basic extension M_2 of $M_1 \supseteq M$:

$$N \subseteq M \subseteq M_1 = \langle M, e_N \rangle \subseteq M_2 = \langle M, e_N, e_M \rangle .$$

Iterating this procedure, we obtain the following canonical tower of II_1-factors:

$$N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots .$$

The index value between any neighboring factors is always $[M : N]$. Note that each basic construction produces the projection and that we get the sequence $\{e_i\}_{i=0,1,\dots}$ of projections ($e_N = e_0$, $e_M = e_1$, and M_k is generated by M_{k-1} and e_{k-1} with $M_0 = M$). These projections satisfy the following beautiful relations:

$$\begin{cases} e_ie_j = e_je_i, & |i-j| \geq 2, \\ e_ie_{i\pm1}e_i = [M : N]^{-1}e_i, & \\ \operatorname{tr}(e_iw) = [M : N]^{-1}\operatorname{tr}(w), & \text{where } w \text{ is a word} \\ & \text{on } e_0, e_1, \dots, e_{i-1}. \end{cases}$$

Similar relations appear in various fields of mathematics (Hecke algebras, braid groups, etc.). This is the reason why the Jones theory became a bridge between operator algebras and various other fields.

The index $[M : N]$ being finite means that N is quite large in M. Thus, the relative commutant $M \cap N'$ turns out to be finite-dimensional. Furthermore, N is irreducible in M, i.e., $M \cap N' = \mathbf{C}1$ as long as $[M : N] < 4$. A very interesting question is: What are the possible index values of irreducible subfactors (when $[M : N] > 4$)? These values form a closed set, but at this stage we do not know how large this set is.[1] Based on representations of the Hecke algebra of type A_n, H. Wenzl [64] constructed a sequence of index values tending to n^2 ($n = 3, 4, \dots$). As far as values close to 4 are concerned, Jones obtained $3 + 3^{1/2}$ ($\cong 4.73205\dots$). Very recently, Haagerup and Schou [25] obtained infinitely many values in the open interval $(4, 5)$ (such as $2^{-1}(5+\sqrt{17}) \cong 4.56155\dots$ and $2^{-1}(5+\sqrt{13}) \cong 4.30278\dots$). Also Ocneanu obtained a value approximately equal to $4.02642\dots$ (by using the graph E_{10}). In these examples a larger factor is always R. A factor-subfactor pair arises as a certain limit of increasing pairs of finite-dimensional algebras, and the notion

[1] For the footnotes [1] to [5] see the Appendix which has been added in the translation.

of commuting squares is important. This notion was originally introduced by S. Popa to obtain his deep results on II_1-factors, and is playing a fundamental role in index theory as well [22], [48], [54], [64]. In the construction of these examples, considering various graphs is crucial and the index appears as the square of the maximal eigenvalue (the Perron-Frobenius theorem) of the incidence matrix of a given graph. The importance of graphs had already been pointed out in Jones' first article [32] and especially in [35].

The Pimsner-Popa article [48] has been playing an important role in the subsequent development of index theory. Computation of the relative commutant is important, but unfortunately quite difficult. Therefore, as a substitute (see [54] for explanation) Pimsner and Popa investigated the relative entropy $H(M/N)$. (This notion was introduced by Connes and Størmer in the process of obtaining their noncommutative Kolmogoroff-Sinai type theorem.) Generally, the inequality $H(M/N) \le \log[M : N]$ holds. They obtained several necessary and sufficient conditions (which have been proved to be extremely useful) for the equality, clarified the relation between $H(M/N)$ and $M \cap N'$, and computed $H(M/N)$ for several examples. Here, the so-called Pimsner-Popa basis is fundamental so that we will briefly explain this notion. The projection $e_N : L^2(M) = \overline{M} \to \overline{N}$ sends M into N so that $E_N = e_N|_M : M \to N$ can be considered. This is called a normal conditional expectation and satisfies the following properties (together with suitable continuity):

$$\begin{cases} E_N(x) = x, & x \in N \text{ (projection property)}, \\ E_N(y_1 x y_2) = y_1 E_N(x) y_2; & y_i \in N,\ x \in M \text{ (bimodule property)}. \end{cases}$$

Such a map was originally introduced in [60] as a noncommutative version of a conditional expectation in probability theory. We easily observe that $e_0 = e_N$ satisfies $e_0 x e_0 = E_N(x) e_0$, $x \in M$. When $[M : N] < \infty$, it was proved in [48] that there exists a family $\{\lambda_i\}_{i=1,2,\dots,n}$ in M $(n \le [M : N] + 1)$ satisfying

$$\sum_{i=1}^{n} \lambda_i^* e_0 \lambda_i = 1.$$

This family is called a Pimsner-Popa basis. Since $x \in M$ satisfies

$$e_0 x = e_0 x 1 = \sum_{i=1}^{n} e_0 x \lambda_i^* e_0 \lambda_i = \sum_{i=1}^{n} E_N(x\lambda_i^*) e_0 \lambda_i,$$

by dividing both sides by e_0 (as can be justified) we get

$$x = \sum_{i=1}^{n} x_i \lambda_i, \qquad x_i = E_N(x\lambda_i^*) \in N.$$

In other words, $\{\lambda_i\}_{i=1,2,\dots,n}$ indeed forms a basis (when M is regarded as an N-module). Based on this expression, Pimsner and Popa obtained the following estimate (the Pimsner-Popa inequality):

$$E_N(x^* x) \ge [M : N]^{-1} x^* x, \qquad x \in M.$$

This means that the difference is positive as an operator. The estimate is very powerful. In fact, it turns out to provide a characterization of the value $[M : N]$. The above-mentioned inequality on the relative entropy also follows from this.

When $[M : N] < \infty$, the infinite-dimensional spaces M and N have a finite-dimensional difference. Hence M and N are expected to share many properties in common. In fact, several results have been obtained in this direction, and almost always the technique in [48] turns out to be a key.

3. Classification of subfactors in R (Galois theory for R)

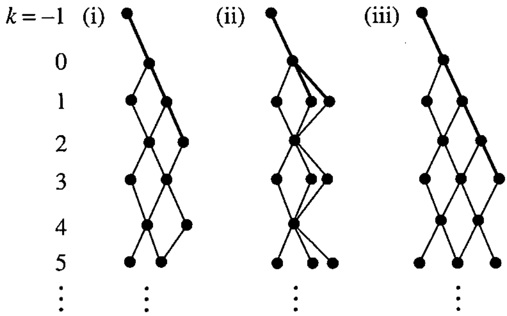
Since AFD factors have been classified, we would like to classify (finite-index) subfactors in R, for example. This problem has been actively investigated by A. Ocneanu and S. Popa. Here we will explain Ocneanu's approach (see [47] for details). A finite-index subfactor N of R is (as a factor) isomorphic to R (by Connes' theorem) so that classification of subfactors up to conjugacy is a main issue (i.e., Galois theory for R). This means that for an automorphism α of R the two pairs $M \supseteq N$ and $M \supseteq \alpha(N)$ are identified. It is routine to check $[M : N] = [M : \alpha(N)]$: hence, the Jones index is indeed an invariant for this classification. For example, when $[R : N] = 2$, Goldman's theorem (Jones' work was strongly motivated by this theorem) asserts that R is the crossed product of N $(\cong R)$ by a $\mathbf{Z}_2$-(outer) action. Hence, Connes' classification of $\mathbf{Z}_2$-action on R shows that an index 2 subfactor of R is unique up to conjugacy. Generally, the classification of subfactors and that of automorphisms are closely related.[2]

Based on Jones' index theory and deep analysis due to Pimsner and Popa, Ocneanu introduced an invariant (for subfactor classification), called a quantized group (paragroup or coupling system). His invariant can be considered as a "quantization" of a finite group. There are several equivalent ways of describing his invariant. Here, for simplicity we deal with the case $[M : N] < 4$ and explain Ocneanu's paragroup. A paragroup $(\mathfrak{G}_{M:N}, \{\gamma_k\}_{k=0,1,\dots})$ consists of a graph $\mathfrak{G}_{M:N}$ (called a principal graph) and a family $\{\gamma_k\}_k$ of anti-automorphisms on certain finite-dimensional algebras. Let us begin by explaining $\mathfrak{G}_{M:N}$. As remarked in §2, $M \supseteq N$ gives us the canonical tower of II_1-factors:

$$N = M_{-1} \subseteq M = M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq M_4 \subseteq \cdots.$$

Setting $A_k = M_k \cap N'$, we obtain the increasing sequence $\{A_k\}_{k=-1,0,1,\dots}$ of finite-dimensional von Neumann algebras (i.e., direct sums of matrix algebras) that contains an enormous amount of combinatorial data. Since $M \cap N' = \mathbf{C}1$ (when $[M : N] < 4$), we have $A_{-1} = A_0 = \mathbf{C}1$. However, A_k's $(k \geq 1)$ are getting larger and larger. In fact, A_k contains at least $e_0, e_1, \dots, e_{k-1}$. An increasing sequence of finite-dimensional algebras is described by the associated

Bratteli diagram. For example, let us consider the three cases (i) $[M : N] = 4\cos^2\frac{\pi}{5}$, (ii) $M = N \rtimes \mathbf{Z}_3$ (crossed product), and (iii) $M = R_{S_2} \supseteq N = R_{S_3}$ (R_{S_3} = the fixed point subalgebra of R under an S_3-outer action). The corresponding Bratteli diagrams are



Here, each vertex $\cdot$ represents a matrix algebra $M_n(\mathbf{C})$ (the size n to be determined shortly), and the number (could be more than one) of edges between vertices indicates how many copies of the smaller matrix algebra are sitting inside the larger one. The sizes of matrix algebras are determined as in Pascal's triangle (with multiplicities of edges counted). Thus the sizes of the involved matrix algebras in the three cases are

```
k=-1  (i) 1          (ii) 1          (iii) 1
           \               \                \
  0         1               1                1
           / \             / \\             / \
  1       1   1           1   1 1          1   1
           \ / \           \ //             \ / \
  2         2   1           3                2   1
           / \ /           / \\             / \ / \
  3       2   3           3  3 3           2   3   1
           \ / \           \ //             \ / \ /
  4         5   3           9                5   4
           / \ /           / \\             / \ / \
  5       5   8           9  9 9           5   9   4
  ⋮         ⋮               ⋮                ⋮
```

Hence, for example the A_2's in the three cases are

(i) $M_2(\mathbf{C}) \oplus \mathbf{C}$,
(ii) $M_3(\mathbf{C})$,
(iii) $M_2(\mathbf{C}) \oplus \mathbf{C}$,

while the A_3's are

(i) $M_2(\mathbf{C}) \oplus M_3(\mathbf{C})$,
(ii) $M_3(\mathbf{C}) \oplus M_3(\mathbf{C}) \oplus M_3(\mathbf{C})$,
(iii) $M_2(\mathbf{C}) \oplus M_3(\mathbf{C}) \oplus \mathbf{C}$.

Furthermore, in each of the three cases A_2 is sitting in A_3 as the following subalgebra:

$$\text{(i)}\quad \left\{ \begin{bmatrix} [A] & \\ & \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ 0 & 0 & \alpha \end{bmatrix} \end{bmatrix} ; A \in M_2(\mathbf{C}),\ \alpha \in \mathbf{C} \right\},$$

$$\text{(ii)}\quad \left\{ \begin{bmatrix} [A] & & \\ & [A] & \\ & & [A] \end{bmatrix} ; A \in M_3(\mathbf{C}) \right\},$$

$$\text{(iii)}\quad \left\{ \begin{bmatrix} [A] & & \\ & \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ 0 & 0 & \alpha \end{bmatrix} & \\ & & [\alpha] \end{bmatrix} ; A \in M_2(\mathbf{C}), \alpha \in \mathbf{C} \right\}.$$

Notice that if a certain edge appears between A_{k-1} and A_k, then its "mirror image" also appears at the next stage (i.e., between A_k and A_{k+1}). In the first few steps new edges might also be added besides the "mirror images" so that each diagram gets wider and wider. However, after a few steps each diagram stops getting wider (finite depth). In the above three cases depth is defined to be 3, 2, and 4 respectively. Depth might be generally infinite; however, depth is always finite as long as $[M : N] < 4$. Once the part of the Bratteli diagram denoted by thick edges is given, then the rest is completely recovered (by taking successive mirror images). This part of the diagram, denoted by $\mathfrak{G}_{M:N}$, is called a principal graph. The principal graphs in (i), (ii), (iii) are the Dynkin diagrams A_4, D_4, and A_5 respectively. (Index = the "order" of a paragroup = the square of the maximal eigenvalue of the incidence matrix of a principal graph.) When $M = N \rtimes G \supseteq N$ (G being a finite group of order n), the resulting principal graph is easily shown to be

$n-1$ vertices

Hence, A_1 is an n-dimensional abelian algebra $(\mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C} \oplus \cdots \oplus \mathbf{C})$. This happens precisely because $l^\infty(G)$ (with pointwise multiplication) shows up as $A_1 = M_1 \cap N'$, and the n vertices at the level $k = 1$ correspond bijectively to the group elements. This fact explains that a principal graph is a "quantization" of the underlying set (i.e., $\#G$-point set) of a finite group. Without the multiplication law this set alone is of limited interest. The second ingredient $\{\gamma_k\}_k$ corresponds to the multiplication law. Recall that M_1 was defined as $J_0 N' J_0$ (acting on L^2 $(M = M_0)$), where J_0 is the unitary involution of $L^2(M_0)$. Therefore, the map $\gamma_0 : x \to J_0 x^* J_0$ is an anti-automorphism of $A_1 = M_1 \cap N'$. In the above "group case" γ_0 sends g to g^{-1}, where

$\{g\}_{g\in G}$ is the natural basis in $l^\infty(G)$. Hence γ_0 determines the "inverse operation" in a given paragroup. Originally M_{2k+1} was defined as $JM'_{2k-1}J$ (on $L^2(M_{2k})$). But it was proved in [49] that M_{2k+1} can be identified with the basic extension of the $(k+1)$-step inclusion $M_k \supseteq N$. In other words, when M_k is represented on $L^2(M_k)$ with the unitary involution J_k, $J_k N' J_k$ can be identified with M_{2k+1}. Consequently, as above we get the anti-automorphism $\gamma_k : x \to J_k x^* J_k$ of $A_{2k+1} = M_{2k+1} \cap N'$. In the "group case" γ_1 gives us information on the multiplication (and $\gamma_2, \gamma_3, \dots$ give no further information). Let us be more precise. The composition $\gamma_1 \circ \gamma_0$ is shown to send $A_1 \cong l^\infty(G)$ into $A_3 \cap A'_1$, and $A_3 \cap A'_1$ is isomorphic to $A_1 \otimes A_1 \cong l^\infty(G) \otimes l^\infty(G)$ (since depth $= 2$). The transpose of the linear map

$$\begin{array}{ccc} l^1(G) \otimes l^1(G) & \to & l^1(G) \\ \cup\!\!\!\!\!\!\in & & \cup\!\!\!\!\!\!\in \\ g \otimes h & \to & gh \end{array}$$

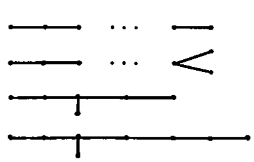
($\{g\}_{g\in G}$ also denotes the dual basis in $l^1(G)$) is precisely $\gamma_1 \circ \gamma_0$. In other words, $\gamma_1 \circ \gamma_0$ corresponds to comultiplication

$$\gamma_1 \circ \gamma_0(k) = \sum_{gh=k} g \otimes h = \sum_g g \otimes g^{-1}k.$$

When depth is two (but A_1 is not necessarily abelian), besides the natural algebra structure (coming from $M_1 \cap N'$) $A_1 = M_1 \cap N'$ is equipped with comultiplication $\gamma_1 \circ \gamma_0 : A_1 \to A_3 \cap A'_1 \cong A_1 \otimes A_1$. Consequently, A_1 is a finite-dimensional Hopf algebra. When depth ≥ 3, we have higher-order products γ_k as well so that a paragroup is more general than a Hopf algebra.

A paragroup satisfies certain properties, and it is possible to axiomatize the notion of a paragroup by them. Conversely, starting from an abstract paragroup, one can construct a pair of (AFD) II_1-factors (string algebras). Ocneanu showed that a paragroup serves as the complete invariant in the classification of subfactors (with indices < 4) of R and classified paragroups (whose orders are less than 4).[3] The notion of downward basic construction (basic extensions into a given subfactor) introduced by Jones [32] plays an important role. The classification result is as follows:

Theorem [47]. *For each of the Dynkin diagrams A_n $(n \geq 2)$, D_{2n} $(n \geq 2)$, there exists a unique subfactor of R. For each of the Dynkin diagrams E_6 and E_8, there are exactly two subfactors of R.*



A_n $(n \geq 2)$	$m = n+1$
D_{2n} $(n \geq 2)$	$m = 2n+2$
E_6	$m = 12$
E_8	$m = 30$

Here, the index is $4\cos^2 \frac{\pi}{m}$ with the Coxeter number m.

For each of the principal graphs A_n and D_{2n} there is a unique way of introducing γ_k's whereas for each of E_6, E_8 there are two ways of doing so. This is the reason why two subfactors are available from each of E_6, E_8.

4. Index theory for type III factors

The index for II_1-factors is defined by using coupling constants based on traces. Although general factors do not admit traces, Connes' spatial theory [11] gives us a substitute for coupling constants. Based on this theory the index between arbitrary factors (the type III case is of primary interest) can be defined. In this section we will explain such a notion of index and some related results.

Assume that a von Neumann algebra M is acting on a Hilbert space H. We do not require this action to be standard so that M and M' may not be of the same size. Let ψ be a faithful normal positive linear functional on M ($\psi(x^*x) \geq 0$ and the equality holds only when $x = 0$). Let ψ' be a similar functional on M'. The spatial theory gives us a certain (nonsingular) positive selfadjoint operator (denoted by $d\psi/d\psi'$ and called the spatial derivative) on H. This is a right substitute for a coupling constant in the general setup (see [11] for the definition and detail). Let us assume that a factor-subfactor pair $M \supseteq N$ and a normal conditional expectation $E : M \to N$ (see §2, but E may not arise from a trace as in §2) are given. For this E the index $\operatorname{Index} E$ will be defined. Fixing a faithful normal positive linear functional φ on N, we set $\psi = \varphi \circ E$ (functional on M). Starting from the inner product on M induced by ψ, one obtains the standard Hilbert space $L^2(M)$ $(= L^2(M\,;\psi))$ by completion as in §2. Also let M (hence N as well) act on $L^2(M)$ as left multiplication. (This representation of M is known as the GNS representation.) Note, however, that $x \in M \to x^* \in M$ does not give us a unitary involution as in §2 for lack of tracial property. (This map is an antilinear closable operator so that one can talk about the polar decomposition of the closure—this is a starting point of the Tomita-Takesaki theory. Let $J\Delta_\psi^{1/2}$ be the polar decomposition, where $J = J_\psi$ is the phase part, a unitary involution, and Δ_ψ is the square of the absolute value part, a nonsingular positive selfadjoint operator. The fundamental theorem of the Tomita-Takesaki theory states $JMJ = M'$ and $\Delta_\psi^{it} M \Delta_\psi^{-it} = M$ ($t \in \mathbf{R}$). Hence, $\operatorname{Ad} \Delta_\psi^{it}$ gives us the one-parameter automorphism group $\{\sigma_t^\psi\}_{t\in\mathbf{R}}$ of M, the modular automorphism group.) So far the subfactor N and the expectation E have not played any role. Recall that ψ' is a functional on M'. Although N is not acting standardly on $L^2(M)$, thanks to the spatial theory (and [23]) there exists a (unique) operator-valued weight $F : N' \to M'$ satisfying

$$d(\varphi \circ E)/d\psi' = d\varphi/d(\psi' \circ F).$$

An operator-valued weight is a map similar to a normal conditional expectation, but the projection property is not required and "$+\infty$" is allowed as its value. The general theory on operator-valued weights can be found in Haagerup's original article [23], but the reader might as well think that in the present context an operator-valued weight is just a normal conditional expectation multiplied by a positive scalar (could be $+\infty$). There is no canonical choice of φ, ψ' (like unique traces on II_1-factors), but it can been shown that the above F (and $L^2(M)$, J) is uniquely determined by E. We will denote this F by E^{-1}. (This notation is justified after some sample computations.) Our $\operatorname{Index} E$ is defined by using E^{-1}. If $M \supseteq N$ happen to be II_1-factors, the Jones index $[M : N]$ is exactly $\operatorname{Index} E_N$ with the normal conditional expectation E_N constructed from the unique trace (see §2).

Besides the obvious property $E^{-1}(1) \in (M')_+$ (or $+\infty$), $E^{-1}(1)$ satisfies

$$uE^{-1}(1)u^* = E^{-1}(u1u^*) = E^{-1}(1)$$

for an arbitrary unitary u in M' (the bimodule property). Consequently, $E^{-1}(1)$ belongs to the center of M', that is, $E^{-1}(1)$ is a scalar. We set

$$\operatorname{Index} E = E^{-1}(1) \quad (\in [1, \infty]).$$

Many important properties of the Jones index can also be proved for $\operatorname{Index} E$. Details can be found in [38]. When $M \cap N' = \mathbf{C}1$, a normal conditional expectation (if it exists) is unique. However, there are generally many normal conditional expectations, and the value $\operatorname{Index} E$ does depend on the choice of E. When $\operatorname{Index} E < +\infty$ (this property does not depend upon the choice of E), there is a unique normal conditional expectation $E_0 : M \to N$ such that $\operatorname{Index} E_0 = \operatorname{Min}\{\operatorname{Index} E ; E\}$. Furthermore this unique E_0 is characterized by the property

$$E_0^{-1} = (\operatorname{Index} E_0)E_0 \quad \text{on } M \cap N'$$

(see [31], [45]). One of the important properties of E_0 is that E_0 behaves like a trace on $M \cap N'$. (However, in the II_1-factor case, this E_0 is not necessarily E_N arising from the unique trace. This phenomenon is related to the behavior of the relative entropy $H(M/N)$.)

In the rest of the paper we will assume that M and N are factors of type III. An obvious question is how well types (III_0, III_λ, III_1) of M and N are conserved under the assumption $\operatorname{Index} E < +\infty$. In [42] it was shown

$$\begin{cases} M \text{ type } \mathrm{III}_1 \Leftrightarrow N \text{ type } \mathrm{III}_1, \\ M \text{ type } \mathrm{III}_0 \Leftrightarrow N \text{ type } \mathrm{III}_0, \\ M, N \text{ type } \mathrm{III}_\lambda, \mathrm{III}_\mu, \ 0 < \lambda, \mu < 1, \text{ respectively}, \\ \qquad \Rightarrow \log\lambda / \log\mu \in \mathbf{Q}. \end{cases}$$

As mentioned in §1, types can be seen from flows of weights. Let (X_M, F_t^M), (X_N, F_t^N) be the flows of weights of M, N, respectively. When $\operatorname{Index} E < +\infty$, each of the flows restricts the other in the following sense [27]. One can construct a flow (X, F_t) that is a common finite-to-one extension of (X_M, F_t^M)

and (X_N, F_t^N) in the sense that

(i) there exists an m-to-1 ($m \le \operatorname{Index} E$) projection map $\pi_M : X \to X_M$ such that $F_t^M \circ \pi_M = \pi_M \circ F_t$,

(ii) there exists an n-to-1 ($n \le \operatorname{Index} E$) projection map $\pi_N : X \to X_N$ such that $F_t^N \circ \pi_N = \pi_N \circ F_t$.

Under a certain condition (for example, the flows are periodic) we further get $mn \le \operatorname{Index} E$. The common extension (X, F_t) need not be ergodic, but we have only finitely many ergodic components. Since a (partial) converse can be proved [28], the above restriction (on the flows under the assumption $\operatorname{Index} E < +\infty$) is optimal.

We now explain that an inclusion $M \supseteq N$ of type III factors splits into an "essentially type II" inclusion and a "purely type III" inclusion. It is natural to imagine that the (nonergodic) flow (X, F_t) appearing above is the flow of weights of a certain von Neumann algebra (which is not necessarily a factor). Indeed we have the following result [40]:

Let $M \supseteq N$ be factors of type III, but not of type III_1. Assume $\operatorname{Index} E < \infty$ and E gives us the minimum index value. Then there exist two von Neumann algebras $\mathfrak{A}$, $\mathfrak{B}$ such that

(i) $M \supseteq \mathfrak{A} \supseteq \mathfrak{B} \supseteq N$ and $\mathfrak{A}$, $\mathfrak{B}$ have the same center (they are factors if, for example, $M \cap N' = \mathbf{C}1$),

(ii) $\mathfrak{A}$, $\mathfrak{B}$ have the same flow of weights, which is exactly (X, F_t),

(iii) $E : M \to N$ splits into the three normal conditional expectations

$$M \xrightarrow{F} \mathfrak{A} \xrightarrow{G} \mathfrak{B} \xrightarrow{H} N,$$

(iv) $G : \mathfrak{A} \to \mathfrak{B}$ is an "essentially type II" inclusion.

The last statement (iv) requires some clarification. According to the structure theorem for type III factors, the crossed product $\widetilde{M} = M \rtimes_\sigma \mathbf{R}$ relative to the modular automorphism group $\{\sigma_t = \sigma_t^\psi\}$ is a von Neumann algebra of type II_∞ admitting the special action $\{\theta_s\}_{s \in \mathbf{R}}$, called the dual action. The original factor M can be recovered as the crossed product $\widetilde{M} \rtimes_\theta \mathbf{R}$ [59]. (It is also possible to express M as the crossed product by a Z-action if M is not of type III_1.) The above (iv) means that $\mathfrak{A} \supseteq \mathfrak{B}$ admit the simultaneous crossed product ($\mathbf{R}$ or $\mathbf{Z}$) representation: $\mathfrak{A} = \widetilde{\mathfrak{A}} \rtimes_\theta \mathbf{R} \supseteq \mathfrak{B} = \widetilde{\mathfrak{B}} \rtimes_\theta \mathbf{R}$. Then (ii) means that $\widetilde{\mathfrak{A}}$ and $\widetilde{\mathfrak{B}}$ have the same center ($\cong L^\infty(X)$) so that we have the simultaneous central decomposition

$$\widetilde{\mathfrak{A}} = \int_X^\oplus \widetilde{\mathfrak{A}}(\omega)\, d\omega \supseteq \widetilde{\mathfrak{B}} = \int_X^\oplus \widetilde{\mathfrak{B}}(\omega)\, d\omega.$$

Consequently, $\{\widetilde{\mathfrak{A}}(\omega) \supseteq \widetilde{\mathfrak{B}}(\omega)\}_{\omega \in X}$ is a field of inclusions of II_∞-factors. Since we can pull out a type I_∞-factor from a type II_∞-factor, we actually obtain a field of inclusions of II_1-factors. Thus, for each $\omega \in X$ we can compute the Jones index between the (fiber) II_1-factors. This index value is constant on each ergodic component of (X, F_t).

Although we have to look at the behavior of the dual action θ_s carefully, the above consideration shows that analysis of $\mathfrak{A} \supseteq \mathfrak{B}$ reduces to that of inclusions of II_1-factors (at least in principle.)[4] On the other hand, $M \supseteq \mathfrak{A}$ and $\mathfrak{B} \supseteq N$ are "purely type III" inclusions in the sense that the effect of inclusions appears at the level of the involved flows of weights. These two inclusions are "dual" to each other so that in what follows we will just consider $\mathfrak{B} \supseteq N$. For simplicity let us assume $M = \mathfrak{A} = \mathfrak{B} \supseteq N$. (Cutting by a suitable projection, one can reduce $\mathfrak{B} \supseteq N$ into this situation.) In this case the projection map $\pi_M : X \to X_M$ is one-to-one so that we simply obtain the factor map

$$(X_M, F_t^M) \overset{\pi_M}{\to} (X_N, F_t^N)$$

between the two flows. In other words, X_M is (as a measure space) isomorphic to $X_N \times \{1, 2, \dots, n\}$, and F_t^M can be expressed as the skew-product flow

$$F_t^M(\omega, i) = (F_t^N(\omega), \varphi_{\omega,t}(i)).$$

By direct calculation we observe $M \cap N' = \mathbf{C}1$ and $\mathrm{Index}\, E = n$. For AFD type III_λ $(0 \le \lambda < 1)$ factors, the classification of M $(= \mathfrak{A} = \mathfrak{B}) \supseteq N$ up to conjugacy is equivalent to that of the above factor map between the two flows [40]. Consequently, classification subfactors (with finite indices) of an AFD type III_λ $(0 \le \lambda < 1)$ factor reduce (at least in principle) to:

classification of subfactors of R (together with the effect of the dual action)
$+$
classification of factor maps between ergodic flows.

Let us return to the general case (AFD not required, but $M = \mathfrak{A} = \mathfrak{B} \supseteq N$). The following three conditions are equivalent:

(i) depth of $M \supseteq N$ (in the sense explained in §3) is two,
(ii) N is the fixed point subalgebra M_G of an outer action on M by a finite group G of order n,
(iii) the number (generally $\le n$) of ergodic components of the flow $(\omega, i, j) \to (F_t^N(\omega), \varphi_{\omega,t}(i), \varphi_{\omega,t}(j))$ (on $X_N \times \{1, 2, \dots, n\}^2$) is exactly n.

One can similarly find a necessary and sufficient condition (in terms of the factor map) for M to be the crossed product $N \rtimes \Gamma$ relative to an outer action on N by a finite group Γ of order n. When this happens, Γ is automatically abelian. Furthermore, the groups G and Γ can be determined by looking at the ergodic decomposition of the flow in (iii).

Let us explain the difference between index theory for type III_0 factors and that for type III_λ $(0 < \lambda < 1)$ factors. Assume $M = \mathfrak{A} = \mathfrak{B} \supseteq N$ as before. Depth may or may not be two in the III_0 case. When depth is two, any finite group G (of order n) can occur. On the other hand, in the III_λ $(0 < \lambda < 1)$ case depth is always two and the group G in (iii) is automatically the cyclic group $\mathbf{Z}_n$. Consequently, we always get $M = N \rtimes \mathbf{Z}_n$, or equivalently, $N = M_{\mathbf{Z}_n}$. This fact was directly showed in Loi's thesis [43]. He further proved the following uniqueness result: The AFD type III_λ $(0 < \lambda < 1)$ factor M admits a unique (up to conjugacy) subfactor N such that N is of type $\mathrm{III}_{\lambda^{1/n}}$ and $\operatorname{Index} E = n$. In our language this corresponds to the fact that a periodic flow (period $= -(\log\lambda)/n$) admits a unique (up to isomorphism as a factor map) n-to-1 extension.

Not only the classification of isomorphism classes of factors (see §1) but also the classification of subfactors are related to certain classification problems in ergodic theory. Several attempts of developing index theory in the context of ergodic theory (more precisely in terms of ergodic equivalence relations) have already been made [18], [29], [58]. When an inclusion of AFD type III_λ $(0 \leq \lambda < 1)$ factors does not contain an "essentially type II" inclusion, it can be described by certain ergodic equivalence relations. Construction of various inclusions based on subtle examples in ergodic theory and computation of various invariants in terms of ergodic theoretical data seem to deserve investigation.

For a pair $M \supseteq N$ of von Neumann algebras without finite traces, R. Longo constructed the canonical endomorphism $\gamma : M \to N$ (unique up to an inner perturbation) and obtained some remarkable results (see [44], for example). He has also developed index theory for type III factors based on this technique [45]. Although the index in [38] and that in [45] are equivalent, each approach seems to have its own advantage. Our approach explained so far is based on analysis of flows of weights so that results on type III_1 factors are quite difficult to obtain. On the other hand, quantum field theory provides us with natural inclusions of type III_1 factors, for which Longo's approach seems to be fitting. [5]

APPENDIX

The original exposition in Japanese was written more than a year ago. Index theory for operator algebras initiated by V. Jones is a very active area, and since then a lot of substantial contributions have been added to the subject matter. In particular, some remarkable results were announced in the Satellite Conference of ICM-90 on "Current Topics in Operator Algebras" (which was held at Nara, Japan, Aug. 16–19). The reader can find some of the latest results in the Proceedings of that conference to be published in the near future. To update this exposition, we here add several endnotes.

1. Quite decisive results were obtained by S. Popa for this problem. If factors are not required to be AFD, an arbitrary value (> 4) is possible [93]. For the AFD II_1-factor, this problem is related to a certain combinatorial problem. In particular, there is a gap for index values between 4 and $4.026\ldots$, the value obtained by Ocneanu.

2. This has been completely justified in [91]. A subfactor approach seems to be very powerful for analysis of actions (even for analysis of minimal actions of compact groups as was shown by Popa and Wasserman).

3. To be more precise, the result was announced, for example, in [47] without detailed proofs. A complete proof (with a slightly different approach—commuting squares and the Pimsner-Popa inequality) was published by Popa

[89]. Popa's approach also works for certain (= strongly amenable) infinite-depth cases. In particular, the complete classification of index 4 subfactors of R was announced in [91].

4. Some classification results were obtained in [87], [85] when $\mathfrak{A}$ and $\mathfrak{B}$ are AFD and of type III_λ, $0 < \lambda < 1$.

5. Jones and Wasserman have constructed some inclusions of type III_1 factors based on representations of loop groups.

REFERENCES

[1] M. Baillet, Y. Denizeau, and J. Havet, *Indices d'une esperance conditionnelle*, Compositio Math. **66** (1988), 199–236.

[2] J. Bion-Nadal, *von Neumann subalgebras of a type* II_1*-factor: Correspondences and property* T, preprint.

[3] M. Choda, *The conjugacy classes of subfactors and the outer conjugacy classes of the automorphism group*, Math. Japon. **32** (1987), 379–388.

[4] ——, *Full* II_1*-factors with non-integer index*, Math. Japon. **33** (1988), 663–669.

[5] ——, *Index for subfactors generated by Jones two sided projections*, Pacific J. Math. (to appear).

[6] ——, *Entropy for* **-endomorphisms and relative entropy for subfactors*, preprint.

[7] A. Delaroche, *On Connes' property* T *for von Neumann algebras*, Math. Japon. **32** (1987), 337–355.

[8] E. Christensen, *Subalgebras of a finite algebra*, Math. Ann. **243** (1979), 17–29.

[9] A. Connes, *Une classification des facteurs de type* III, Ann. Sci. École Norm. Sup. **6** (1973), 133–258.

[10] ——, *Classification of injective factors, Cases* II_1, II_∞, III_λ, $\lambda \neq 1$, Ann. of Math. (2) **104** (1976), 73–115.

[11] ——, *Spatial theory of von Neumann algebras*, J. Funct. Anal. **35** (1980), 153–164.

[12] ——, *Indices des sous facteurs, algèbres de Hecke et théorie des noeuds*, Séminaire Bourbaki, 37e année, 1984–1985, no. 647.

[13] ——, *Entropie de Kolmogoroff-Sinai et mecanique statistique quantique*, C. R. Acad. Sci. Paris **301** (1985), 1–6.

[14] A. Connes and D. Evans, *Embedding of* $U(1)$ *current algebras in non-commutative algebras of classical statistical mechanics*, preprint.

[15] A. Connes, H. Narnhofer, and W. Thirring, *Dynamical entropy of* C^**-algebras and von Neumann algebras*, Comm. Math. Phys. **112** (1987), 691–719.

[16] A. Connes and E. Størmer, *Entropy for automorphisms of* II_1 *von Neumann algebras*, Acta Math. **134** (1975), 288–306.

[17] A. Connes and M. Takesaki, *The flow of weights of factors of type* III, Tôhoku Math. J. **29** (1977), 473–575.

[18] J. Feldman, C. Sutherland, and R. Zimmer, *Subrelations of ergodic equivalence relations*, Ergodic Theory Dynamical Systems (to appear).

[19] P. Freyd, D. Yetter, J. Hoste, J. Hickorish, K. Millet, and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), 239–246.

[20] M. Goldman, *On subfactors of factors of type* II_1, Michigan J. Math. **7** (1960), 167–172.

[21] F. Goodman, P. de La Harpe and V. Jones, *Classification des matrices entières non-negatives de petites normes*, C. R. Acad. Sci. Paris **300** (1985), 463–465.

[22] ——, *Coxeter-Dynkin diagrams and towers of algebras*, Springer-Verlag, 1989 (to appear).

[23] U. Haagerup, *Operator valued weights in von Neumann algebras*. I, J. Funct. Anal. **32** (1979), 175–206; II, ibid., **33** (1979), 339–361.

[24] ——, *Connes' bicentralizer problem and uniqueness of the injective factor of type* III_1, Acta Math. **158** (1987), 95–147.

[25] U. Haagerup and J. Schou, *Some new subfactors of the hyperfinite* II_1*-factor*, preprint.

[26] T. Hamachi, *Jones index and Markov odometers*, preprint.

[27] T. Hamachi and H. Kosaki, *Index and flow of weights of factors of type* III, Proc. Japan Acad. **64A** (1988), 11–13.

[28] ——, *Inclusion of type* III *factors constructed from ergodic flows*, Proc. Japan Acad. **64A** (1988), 195–197.

[29] ——, *Orbital factor map (Index theory for ergodic transformation groups)*, preprint.

[30] D. Handelman and H. Wenzl, *Closedness of index values for subfactors*, Proc. Amer. Math. Soc. **101** (1987), 277–282.

[31] F. Hiai, *Minimizing indices of conditional expectations onto a subfactor*, Publ. Res. Inst. Math. Sci. **24** (1988), 673–678.

[32] V. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25.

[33] ——, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (N.S.) (1985), 103–111.

[34] ——, *Index for subrings of rings*, Contemp. Math., no. 43, Amer. Math. Soc., Providence, R.I., 1985, pp. 181–190.

[35] ——, *Braid groups, Hecke algebras and type* II_1*-factors*, Geometric Methods in Operator Algebras, Res. Notes in Math., vol. 123, Pitman, Boston, 1986, pp. 242–273.

[36] ——, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) **126** (1987), 335–388.

[37] S. Kawakami and H. Yoshida, *Actions of finite groups on finite von Neumann algebras and the relative entropy*, J. Math. Soc. Japan **39** (1987), 609–629.

[38] H. Kosaki, *Extension of Jones' theory on index to arbitrary factors*, J. Funct. Anal. **66** (1986), 123–140.

[39] ——, *Characterization of crossed products (properly infinite case)*, Pacific J. Math. **137** (1989), 159–167.

[40] ——, *Index theory for type* III *factors*, Mappings of Operator Algebras (Proc. US-Japan Seminar, 1988) (to appear).

[41] W. Krieger, *On ergodic flows and the isomorphism of factors*, Math. Ann. **223** (1976), 19–70.

[42] P. H. Loi, *Sur la théorie de l'indices et les facteurs de type* III, C. R. Acad. Sci. Paris **305** (1987), 423–426.

[43] ——, *On the theory of index and type* III *factors*, Thesis, Penn. State Univ., 1988.

[44] R. Longo, *Simple injective factors*, Adv. in Math. **63** (1987), 152–171.

[45] ——, *Index of subfactors and statistics of quantum fields*, preprint.

[46] B. Mashhood and K. Taylor, *On continuity of the index of subfactors of finite type*, J. Funct. Anal. **76** (1988), 56–66.

[47] A. Ocneanu, *Quantized groups, string algebras and Galois theory*, Operator Algebras and Applications, Vol. II, London Math. Soc. Lecture Note Ser., no. 136, Cambridge Univ. Press, 1988.

[48] M. Pimsner and S. Popa, *Entropy and index for subfactors*, Ann. Sci. École Norm. Sup. **19** (1986), 57–106.

[49] ——, *Iterating the basic construction*, Trans. Amer. Math. Soc. **310** (1988), 127–133.

[50] ——, *Finite dimensional approximation of pairs of algebras and obstructions for the index*, preprint.

[51] S. Popa, *Notes on Cartan subalgebras in type* II_1 *factors*, Math. Scand. **57** (1985), 171–188.

[52] ——, *Some rigidity results in operator algebras*, preprint.

[53] ——, *Correspondences*, preprint.

[54] ——, *Relative dimension, tower of projections and commuting squares of subfactors*, Pacific J. Math. **137** (1989), 181–207.

[55] C. Skau, *Finite subalgebras of a von Neumann algebra*, J. Funct. Anal. **25** (1977), 211–235.

[56] V. S. Sunder, *A model for AF algebras and representation of the Jones projections*, J. Operator Theory **18** (1987), 289–302.

[57] ——, *Monomials in the Jones projections*, Proc. Amer. Math. Soc. **103** (1988), 761–764.

[58] C. Sutherland, *Nested pairs of equivalence relations and von Neumann algebras*, preprint.

[59] M. Takesaki, *Duality for crossed products and structure of von Neumann algebras of type* III, Acta Math. **131** (1973), 79–122.

[60] H. Umegaki, *Conditional expectation in an operator algebra*. I, Tôhoku Math. J. **6** (1954), 177–181.

[61] Y. Watatani, *L'indice d'une C^*-sous-algèbre d'une C^*-algèbre simple*, C. R. Acad. Sci. Paris **305** (1987), 23–26.

[62] ——, *Index for C^*-subalgebras*, preprint.

[63] A. Wasserman, *Coactions and Yang-Baxter equations for ergodic actions and subfactors*, Operator Algebras and Applications, Vol. II, London Math. Soc. Lecture Note Ser., no. 136, Cambridge Univ. Press, 1988.

[64] H. Wenzl, *Representation of Hecke algebras and subfactors*, Thesis, Univ. of Penn., 1985.

[65] ——, *Hecke algebras of type A_n and subfactors*, Invent. Math. **92** (1988), 349–383.

[66] ——, *On sequences of projections*, C. R. Math. Rep. Acad. Sci. Canada (IX) **1** (1987), 5–9.

[67] J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1969.

[68] R. Kadison and J. Ringrose, *Fundamentals of the theory of operator algebras*, Academic Press, 1986.

[69] G. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, 1979.

[70] S. Sakai, *C^*-algebras and W^*-algebras*, Springer-Verlag, 1971.

[71] S. Stratila, *Modular theory in operator algebras*, Abacuss Press, 1981.

[72] M. Takesaki, *Theory of operator algebras*. I, Springer-Verlag, 1979.

[73] ——, *Structure of operator algebras*, Iwanami, 1983. (Japanese)

Supplementary References

[74] J. Birmann and H. Wenzl, *Braids, link polynomials and a new algebra*, Trans. Amer. Math. Soc. **313** (1989), 249–273.

[75] D. H. Bisch, *On the existence of central sequences in subfactors*, preprint.

[76] M. Choda, *Entropy for canonical shifts*, preprint.

[77] M. Choda and F. Hiai, *Entropy for canonical shifts*. II, preprint.

[78] J. F. Havet, *Espérance conditionnelle minimale*, preprint.

[79] R. Herman and A. Ocneanu, *Index theory and Galois theory for infinite index inclusions*, C. R. Acad. Sci. Paris Sér. I **309** (1989), 923–927.

[80] F. Hiai, *Minimum index for subfactors and entropy*, preprint.

[81] ——, *Minimum index for subfactors and entropy*. II, preprint.

[82] P. Jolissante, *Index for pairs of finite von Neumann algebras*, preprint.

[83] ——, *Indice d'espérances conditionnelles et algèbres de von Neumann finies*, preprint.

[84] Y. Kawahigashi, *Automorphisms commuting with a conditional expectation onto a subfactor*, preprint.

[85] ——, *Remarks on Loi's classification of automorphisms of subfactors*, preprint.

[86] S. Kawakami, *Some remarks on index and entropy for von Neumann subalgebras*, Proc. Japan Acad. **65A** (1989), 323–325.

[87] P. H. Loi, *On automorphisms of subfactors*, preprint.

[88] R. Longo, *Index of subfactors and statistics of quantum fields*. II: *Correspondences, braid group statistics, and Jones polynomial*, Comm. Math. Phys. **130** (1990), 285–309.

[89] S. Popa, *Classification of subfactors: the reduction to commuting squares*, Invent. Math. **101** (1990), 19–43.

[90] ——, *Subfacteurs, actions des groupes et cohomologie*, C. R. Acad. Sci. Paris Sér. I **309** (1989), 771–776.

[91] ——, *Sur la classification des subfacteurs d'indice fini du facteur hyperfini*, preprint.

[92] ——, *Some rigidity results in* II_1*-factors*, preprint.

[93] ——, *Markov traces on universal Jones algebras and subfactors of finite index*, preprint.

[94] T. Sano and Y. Watatani, *Angles between subfactors*, preprint.

[95] S. K. Tsui, *Index of faithful normal conditional expectations*, preprint.

Translated by H. KOSAKI

DEPARTMENT OF MATHEMATICS, COLLEGE OF GENERAL EDUCATION, KYUSHU UNIVERSITY, FUKUOKA, 810, JAPAN