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VECTOR FIELDS ON MANIFOLDS

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Approved: ______________________
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INTRODUCTION

A classical theorem of Poincaré-Hopf asserts, in part, that if a compact manifold M has Euler-Poincaré characteristic $\chi(M) \neq 0$ then every flow on M has a stagnation point. If the manifold has additional structure (e.g. Riemannian or complex) it is natural to consider the properties of the special group of transformations that preserve the additional structure. In particular, one may ask whether or not M admits a structure-preserving group action without fixed points.

Since flows are generated by vector fields, for Riemannian manifolds the question may be rephrased as follows: What topological or geometric properties of a Riemannian manifold M allow one to conclude that every Killing field -- an infinitesimal generator of isometries -- must vanish somewhere on M? In this connection, Marcel Berger [3] has shown that on a closed Riemannian manifold of even dimension every Killing field of a metric of positive sectional curvature must vanish somewhere. In Chapter II a number of results along these lines are presented. Among these are: (1) If an even dimensional closed manifold admits a metric of positive Ricci curvature and satisfies a certain topological condition then it admits no nonvanishing Killing fields for <u>any</u> metric, and (2) If the second Betti number of M is nonzero and M admits a metric with curvature satisfying a certain positive "pinching condition" then it admits no nonvanishing Killing fields for <u>any</u> metric. These results are proved by means of a fixed point theorem for circle actions.

For complex manifolds, Matsushima [21] proved that if M is projective algebraic and $H^1(M,R) = 0$ then every flow on M that preserves the complex structure has a stagnation point, i.e. every holomorphic vector field on M vanishes somewhere. Carrell and Lieberman [9] extended Matsushima's result to all closed Kähler manifolds. In Chapter III Matsushima's theorem is generalized to all closed complex manifolds and a number of related results are proved. We mention only: If M^n is a closed complex manifold and the Hodge numbers of M satisfy $\sum_{p \le n-1} h^{p,p+1} = 0$ then M admits no nonvanishing holomorphic vector field.

Chapter IV focuses on a question raised by Chern: If $\chi(M) = 0$ then M admits a nonvanishing vector field. One may ask whether or not M admits a vector field that is parallel with respect to some Riemannian metric. Chern proved that if such a metric exists then the first Betti number $b_1 \ge 1$ and the second Betti number $b_2 \ge b_1 - 1$, and he conjectured that these conditions were not sufficient [10]. One of the main results of Chapter IV is the existence of additional necessary conditions: $b_{k+1} \ge b_k - b_{k-1}$ for all $1 \le k \le n-1 = \dim M-1$. These conditions are refined when the vector field is parallel with respect to a Kähler metric. In addition, a number of generalizations of a classical theorem of Hurwitz [17] are presented. For example: If a closed

Riemannian n-manifold with Ricci curvature ≤ 0 fails to satisfy $b_{k+1} \geq b_k - b_{k-1}$ for some value of $1 \leq k \leq n-1$ then its isometry group is finite.

In Chapter V the following generalization of a theorem of Poincaré is proved: If a complex manifold of dimension 2n+1 admits a nonvanishing (2n+1,0) form then the arithmetic genus of M is zero. This theorem and Hurwitz' theorem are both proved by using the theory of elliptic operators.

I. PRELIMINARIES

The purpose of this chapter is to fix notation, collect some facts from real and complex differential geometry, and prove some preliminary results.

1. Riemannian manifolds.

1.1. M is a closed differentiable n-manifold if it is compact and $\partial M = \emptyset$. If M is an oriented Riemannian n-manifold with metric g and Riemannian volume form vol_g there exists an operator $*$, called the "Hodge-star operator", mapping p-forms to n-p forms: $*: \Lambda^p \to \Lambda^{n-p}$. Furthermore, $** \alpha = (-1)^{p(n-p)} \alpha$ if $\alpha \in \Lambda^p$, and $* 1 = vol_g$. If $\{\omega_j\}$ are orthonormal 1-forms $*1 = \omega_1 \wedge \ldots \wedge \omega_n$ if $\omega_1 \wedge \ldots \wedge \omega_n$ lies in the orientation class. A pointwise metric is defined on Λ^p by extending the following form bilinearly: $\langle \xi_{i_1} \wedge \ldots \wedge \xi_{i_p}, \eta_{i_1} \wedge \ldots \wedge \eta_{i_p} \rangle_p \overset{def}{=} \det g(\xi_{i_j}, \eta_{i_k})$ where $\xi_{i_j}, \eta_{i_k} \in \Lambda^1(T_p^*M)$. Then, for $\alpha \in \Lambda^p$, $\alpha \wedge * \alpha = (\alpha,\alpha)_p vol_g$. The pointwise inner product may be integrated to define a global L^2 inner product on p-forms: $(\alpha,\beta) = \int_M \langle\alpha,\beta\rangle_p \, vol_g$.

If d is the operator of exterior differentiation let $\delta \overset{def}{=}$ formal adjoint of d in the L^2 product above: $(d\alpha,\beta) = (\alpha,\delta\beta)$. It follows that $\delta: \Lambda^p \to \Lambda^{p-1}$ and, in fact, $\delta = (-1)^{\gamma} * d *$ where $\gamma = n(n+p)+1$ when δ is applied to p-forms. The Laplacian $\Delta \overset{def}{=} d\delta + \delta d: \Lambda^p \to \Lambda^p$ and with our choice of signs $\Delta = -\sum D_{x_i}^2$ for $M = R^n$ with the Euclidean metric. Cf. [4],[31].

1.2. If X is a vector field and ω is a p-form, $p \geq 1$, we define the interior product of X and ω by $i_X\omega \overset{\text{def}}{=} \omega(X,\ldots)$. Thus $i_X\colon \Lambda^p \to \Lambda^p$. On $\Lambda^0 \equiv C\ (M)$, i_X is defined as the zero operator. If η is the one form dual to a vector field E in the metric g then $i_E\eta = \eta(E) = g(E,E)$. For $\alpha \in \Lambda^r$, $\beta \in \Lambda^s$, and a vector field X we have

$$i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^{\deg \alpha}\, \alpha \wedge i_X\beta$$

i.e. i_X is an antiderivation of degree -1 on the exterior algebra $\Lambda^*(M)$. For a Riemannian manifold with $\eta \in \Lambda^1$ we sometimes write i_η in place of i_E where E = contravariant form of η. Let ℓ_η denote the operation of left multiplication by $\eta \in \Lambda^1$. It should be noted that $i_\eta\ell_\eta + \ell_\eta i_\eta\colon \Lambda^p \to \Lambda^p$ and, moreover, for $\alpha \in \Lambda^p$:

$$(i_\eta\ell_\eta + \ell_\eta i_\eta)(\alpha) = i_\eta(\eta \wedge \alpha) + \eta \wedge i_\eta\alpha = (i_\eta\eta)\alpha = g(\eta,\eta)\alpha$$

Here $i_\eta\ell_\eta + \ell_\eta i_\eta$ = multiplication by the function $g(\eta,\eta) \geq 0$, Also, it is easy to check that $\langle\eta \wedge \alpha,\beta\rangle_p = \langle\alpha, i_\eta\beta\rangle_{p-1}$ [28], [31].

1.3. Let $\mathcal{X}$ denote the linear space of vector fields on M. A connection or covariant differentiation on M is a map $\mathcal{X} \times \mathcal{X} \xrightarrow{\nabla} \mathcal{X}$ such that for $X,Y,Z \in \mathcal{X}$ and $f \in C^\infty$:

(a) $\nabla_{fX+Y}Z = f\nabla_X Z + \nabla_Y Z$

(b) $\nabla_X fY = (Xf)Y + f\nabla_X Y$

where $\nabla_X Y = \nabla(X,Y)$, etc.

The torsion of a connection ∇ is the tensor field T of type (1,2) given by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, where $[X,Y]$ is the commutator of X and Y. A connection is Riemannian (with respect to g) if $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$. It is well known that every Riemannian manifold (M,g) admits a unique torsion free ($T = 0$) connection called the <u>Levi-Civita connection</u>. Henceforth, ∇ always denotes the Levi-Civita connection. A <u>parallel vector field</u> is a vector field X satisfying $\nabla_Y X = 0$ for all $Y \in \mathcal{X}$ [14].

<u>1.4</u>. The commutator $[X,Y]$ is also called the <u>Lie derivative</u> of Y in the direction of X and denoted $L_X Y$. Both ∇_X and L_X can be extended to mappings of $\Lambda^p \to \Lambda^p$ and of tensor fields of type $(r,s) \to$ tensor fields of type (r,s) as follows:

(a) If $\omega \in \Lambda^p$ and $Y_1, \ldots, Y_p \in \mathcal{X}$

$$\nabla_X \omega(Y_1, \ldots, Y_p) = X[\omega(Y_1, \ldots Y_p)] - \sum_{j=1}^{p} \omega(Y_1, \ldots, \nabla_X Y_j, \ldots, Y_p)$$

(b) If $\omega \in \Lambda^p$ and $Y_1, \ldots, Y_p \in \mathcal{X}$

$$L_X \omega(Y_1, \ldots, Y_p) = X[\omega(Y_1, \ldots, Y_p)] - \sum_{j=1}^{p} \omega(Y_1, \ldots, L_X Y_j, \ldots Y_p)$$

and more generally:

(c) If T is a tensor field of type (r,s) and $Y_1, \ldots, Y_s \in \mathcal{X}$, $\omega_1, \ldots, \omega_r \in \Lambda^1$, then

$$(L_X T)(\omega_1, \ldots \omega_r; Y_1, \ldots, Y_s) = X[T(\omega;Y)] - \sum_{j=1}^{r} T(\omega_1, \ldots, L_X \omega_j, \ldots \omega_r; Y)$$
$$- \sum_{k=1}^{s} T(\omega; Y_1, \ldots, L_X Y_k, \ldots, Y_s).$$

If g is a Riemannian metric, we have thus defined $L_X g$, and if $L_X g = 0$ X is called a Killing vector field (with respect to g). It is known that a Killing field is divergence free (in the metric): $\operatorname{div}_g X = 0$.

If ϕ_t is the flow generated by a vector field X, it can be shown that $L_X T = \frac{d}{dt} \phi_t^*(T)\Big|_{t=0}$ for any tensor field T. $L_X T = 0$ means that T is invariant under ϕ_t, and so if X is a Killing vector its flow ϕ_t is a 1-parameter group of isometries of (M,g).

The Lie derivative and interior multiplication on p forms are connected by the following formula: $L_X = i_X \circ d + d \circ i_X$ cf. [13],[19].

1.5. The covariant differentiation induces a tensor of type (1,1): If $X \in \mathcal{X}$ let $A_X \overset{\text{def}}{=} L_X - \nabla_X = -\nabla_{(\cdot)} X$. The following property of Killing fields is well known.

Lemma 1 A_X is a skew adjoint transformation of $T\,M$ if X is a Killing field on (M,g).

Proof: Cf. [19], page 237.

We will need the following extension.

Lemma 2 If X is a Killing field on (M,g) then L_X is a skew adjoint transformation $\Lambda^k \to \Lambda^k$ (with the induced metric).

Proof: Let L_X^t denote the adjoint of L_X. Suppose $\xi, \eta \in \Lambda^k$. We will show that $0 = ((L_X + L_X{}^t)\xi, \eta) =$ $= \int \langle (L_X + L_X{}^t)\xi, \eta \rangle_k (x) \operatorname{vol}_g$ where vol_g = Riemannian volume form $= \sqrt{\det g}\; dx_1 \wedge \dots \wedge dx_n$. We may assume that ξ and η

are decomposable at $x \in M$ so that:

$$\langle\xi,\eta\rangle_k(x) = \langle\xi_1 \wedge \dots \wedge \xi_k, \eta_1 \wedge \dots \wedge \eta_k\rangle_k(x)$$

$$= \det g(\xi_i,\eta_j)(x)$$

Using a well known fact about differentiating determinants we have

$$X\langle\xi,\eta\rangle = X \det \begin{vmatrix} g(\xi_1,\eta_1) & \dots & g(\xi_1,\eta_k) \\ \vdots & & \vdots \\ g(\xi_k,\eta_1) & \dots & g(\xi_k,\eta_k) \end{vmatrix}$$

$$= \sum_{j=1}^{k} \det \begin{vmatrix} g(\xi_1,\eta_1) & \dots & g(\xi_1,\eta_k) \\ \vdots & & \\ Xg(\xi_j,\eta_1) & \dots & Xg(\xi_j,\eta_k) \\ \vdots & & \\ g(\xi_k,\eta_1) & \dots & g(\xi_k,\eta_k) \end{vmatrix}$$

$$= \sum_{j=1}^{k} \langle\xi_i \wedge \dots \wedge L_X\xi_j \wedge \dots \wedge \xi_k, \eta_i \wedge \dots \wedge \eta_k\rangle_k$$

$$+ \sum_{j=1}^{k} \langle\xi_i \wedge \dots \wedge \xi_k, \eta_1 \wedge \dots \wedge L_X\eta_j \dots \wedge \eta_k\rangle_k$$

$$= \langle L_X\xi,\eta\rangle_k(x) + \langle\xi,L_X\eta\rangle_k(x)$$

since

$$L_X(\xi_1 \wedge \dots \wedge \xi_k) = \sum_{j=1}^{k} \xi_1 \wedge \dots \wedge L_X\xi_j \wedge \dots \wedge \xi$$

and $L_X g = 0$.

Thus, by Stokes' theorem,

$$0 = \int d \circ i_X[\langle\xi,\eta\rangle_k \, vol_g] = \int L_X[\langle\xi,\eta\rangle_k vol_g] =$$

$$= \int X\langle\xi,\eta\rangle_k \mathrm{vol} + \langle\xi,\eta\rangle_k L_X \mathrm{vol}_g = \int \langle (L_X + L_X{}^t)\xi,\eta\rangle_k \mathrm{vol}_g$$

since

$$L_X(\mathrm{vol}_g) = d \circ i_X(\mathrm{vol}_g) = d \circ i_X(\sqrt{G}\, dx^1 \wedge \dots \wedge dx^n) \ , \ G = \det(g_{ij})$$

$$= d\Big(\sum_{j=1}^{n} (-1)^{j-1} \sqrt{G}\, X^j\, dx^1 \wedge \dots \widehat{dx^j} \wedge \dots \wedge dx^n\Big)$$

$$= \sum_{j=1}^{n} \frac{\partial}{\partial x^j} (\sqrt{G}\, X^j)\ \ dx^1 \wedge \dots \wedge dx^n = (\mathrm{div}_g X)\mathrm{vol}_g = 0$$

□

1.6. Set $Z_d^k = \{\alpha \in \Lambda^k : d\alpha = 0\}$ and $B_d^k = \{\alpha \in \Lambda^k : \alpha = d\beta$, $\beta \in \Lambda^{k-1}\}$. Then $H_{dR}^k(M)$ = the k^{th} deRham cohomology group $\overset{\text{def}}{=}$ Z^k mod B^k. The k-th Betti number of M $\overset{\text{def}}{\equiv}$ $\dim H_{dR}^k(M)$ is denoted b_k. The Euler characteristic $\chi(M) \overset{\text{def}}{=} \sum_{k=0}^{n} (-1)^k b_k$. The Betti numbers b_k are topological invariants of M. According to the Hodge theorem $\dim \ker \Delta_k = b_k$ where Δ_k denotes the Laplacian $\Delta: \Lambda^k \to \Lambda^k$, cf. [31], [33].

1.7. The curvature transformation R of a Riemannian manifold (M,g) is the linear transform of T_pM defined by:

$$R(X\ Y)\,Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z$$

If P is a 2-plane in T_pM spanned by orthonormal vectors X_1, X_2 then the sectional curvature of P is $K(P) \overset{\text{def}}{=} \langle R(X_1,X_2)X_2,X_1\rangle$. A Riemannian manifold has nonpositive (resp. nonnegative) sectional curvature if $K(P) \leq 0$ (resp. $K(P) \geq 0$) for all

2-planes P at all points of M.

The Ricci tensor (X,Y) is given by $\mathrm{Ricc}(X,Y) \stackrel{\mathrm{def}}{=}$ trace of the operator $V \to R(V,X)Y$, i.e. for an orthonormal basis $\{E_i\}$ of T_pM we have $\mathrm{Ricc}(X,Y) = \sum_{i=1}^{n} g(R(E_i,X)Y,E_i)$ for $X,Y \in T_pM$. The Ricci tensor is positive definite ("Ricci > 0") if $\mathrm{Ricc}(X,X)_p > 0$ (if $X_p \neq 0$) and it is nonnegative if $\mathrm{Ricc}(X,X) \geq 0$.

Example: The Riemannian product of two positively curved manifolds has positive definite Ricci tensor but does not have positive curvature.

The scalar curvature τ of g is defined as the trace of the Ricci tensor: $\tau(p) \stackrel{\mathrm{def}}{=} \sum \mathrm{Ricc}(E_i,E_i)$ if $\{E_i\}$ is an orthonormal basis of T_pM cf. [19].

Note that sectional curvatures ≥ 0 (resp. ≤ 0) $\Rightarrow$ Ricci ≥ 0 (resp. ≤ 0) $\Rightarrow$ scalar curvature ≥ 0 (resp. ≤ 0).

One effect of curvature on the topology of M is seen in:

Myers' Theorem [26]. If M is Riemannian with Ricci $\geq c > 0$ then M is compact, the fundamental group of M is finite, and $b_1 = 0$.

Bochner's Theorems [5]. (a) If (M,g) is closed, $b_1 \neq 0$, and Ricci ≥ 0 then there is a parallel vector field on M. (b) If Ricci < 0 then (M,g) admits no Killing fields and if Ricci ≤ 0 then every Killing field is parallel.

1.8. A symplectic (or Hamiltonian) manifold is a 2n-manifold with a closed 2-form Ω of rank n ($\Omega^n \neq 0$) called a symplectic form. Such manifolds are important in the geometrical description of mechanics, cf [13].

2. Complex Manifolds

2.1. Suppose M is a closed complex manifold of dimension over $C = m$. M can be viewed as a differentiable 2m-manifold with a tensor field J of type (1,1) satisfying $J^2 = -1$ and also a certain "integrability" criterion, cf. [33]. Any such manifold admits a hermitian metric i.e. a Riemannian metric h satisfying $h(X,Y) = h(JX,JY)$ for all (real) vector fields X,Y. The tensor h is a Kähler metric if J is invariant under parallelism: $\nabla_X J = 0$ for all $X \in \mathcal{X}$. The complexified tangent bundle $TM^{2m} \otimes C$ splits as $TM^{1,0} \oplus TM^{0,1}$ where a section Z of $TM^{1,0}$ (resp. $TM^{0,1}$) has the form $Z = X - iJX$ (resp. $Z = X + iJX$) and is called a complex vector field of type (1,0) (resp. type (0,1)). In local complex coordinates $(z^1,\ldots,z^m)$ vector fields of type (1,0) have the form $Z = \sum a_k \frac{\partial}{\partial z_k}$ where $a_k \in C^\infty_C$. Like the tangent bundle, the spaces Λ^k split as $\oplus\, \Lambda^{p,q}$, $p + q = k$ and $d = \bar\partial + \partial$ with $d^2 = \bar\partial^2 = \partial^2 = 0$, cf. [19].

A holomorphic vector field Z is a vector field of type (1,0) that has a local expression $Z = \sum a_k \frac{\partial}{\partial z^k}$ with $\bar\partial a_k = 0$. $Z = X - iJX$ is a holomorphic vector field iff its real part X generates a local 1-parameter group of holomorphic transformations of M.

If h is a hermitian metric then it may be extended by linearity to complex vector fields Z,W: $h(Z,\bar W) = \sum h_{i\bar j} Z^i \bar W^j$ with $h_{i\bar j}$ a hermitian matrix. It is known that h is Kähler iff the (1,1) form $\omega_h = h_{i\bar j}\, dz^i \wedge d\bar z^j$ is closed. Thus every Kähler manifold is symplectic, cf. [19].

2.2. For hermitian manifolds the analogue of the Hodge-star operator is $\bar{*}$ where $\bar{*}(\alpha) = \overline{*\alpha}$ and $*$ is extended to complex valued differential forms by linearity. It follows that $\bar{*}: \Lambda^{p,q} \to \Lambda^{m-p,m-q}$ where $m = \dim_C M$. As for Riemannian manifolds, the local inner product of (p,q) forms α,β is $\bar{*}\langle\alpha,\beta\rangle = \alpha \wedge \bar{*}\beta$ and the global L^2 inner product is $(\alpha,\beta) = \int \alpha \wedge \bar{*}\beta = \int \langle\alpha,\beta\rangle \, vol_g$. The formal adjoint of $\bar{\partial}$ is $\theta = -\bar{*}\bar{\partial}\bar{*}$ and the complex Laplacian is $\bar{\square} \overset{def}{=} \bar{\partial}\theta + \theta\bar{\partial}$. Let $\bar{\square}_{p,q}$ denote $\bar{\square}: \Lambda^{p,q} \to \Lambda^{p,q}$. The Hodge numbers $h^{p,q} \overset{def}{=}$ dimension of the sheaf cohomology groups $H^q(M,\Omega^p)$ and vi the theorems of Dolbeault and Hodge-Kodaira we may take $h^{p,q} \overset{def}{=} \dim \text{kernel } \bar{\square}_{p,q} = \dim \{\alpha \in \Lambda^{p,q}: \bar{\partial}\alpha = 0\}$ mod $\{\alpha = \bar{\partial}\beta: \beta \in \Lambda^{p,q-1}\}$.

For Kähler metrics it is known that $\bar{\square}$ is a "real" operator and the real and complex Laplacians are related by $2\bar{\square} = \Delta$. As a consequence, it is not hard to show that for Kähler manifolds $b_k = \sum_{p+q=k} h^{p,q}$ and $h^{p,q} = h^{q,p}$. Thus $\frac{1}{2} b_1 = h^{0,1} = h^{1,0}$ = dimension of the space of holomorphic differentials.

The χ^p genus of a complex manifold is $\chi^p \overset{def}{=} \sum_{q=0} (-1)^q h^{p,q}$ and the arithmetic genus $\overset{def}{=} \chi^0 = \sum_{q=0}^{m} (-1)^q h^{0,q}$.
For $m = 1$, i.e. a Riemann surface, the arithmetic genus $= h^{0,0} - h^{0,1} = h^{0,0} - h^{1,0} = 1 - \text{genus} = \frac{1}{2}$ (Euler characteristic), cf. [15].

2.3. As in the case of real manifolds, define interior multiplication i_Z, where Z is a complex vector field, as follows: (a) $i_Z f = 0$ if $f \in C^\infty \equiv \Lambda^0$, (b) $i_Z \nu = \nu(Z,\dots)$ for $\nu \in \Lambda^k_C$, $k \geq 1$. Then $i_Z(\nu \wedge \mu) = i_Z\nu \wedge \mu + (-1)^k \nu \wedge i_Z\mu$ if $k = \deg \nu$. Note that if Z is of type $(1,0)$ then $i_Z\colon \Lambda^{p,q} \to \Lambda^{p-1,q}$. Let ℓ_ζ denote the operation of left multiplication by $\zeta \in \Lambda^{1,0}$. Then as in Section 1.2 we have $\ell_\zeta \circ i_Z + i_Z \circ \ell_\zeta\colon \Lambda^{p,q} \to \Lambda^{p,q}$ and in fact its just multiplication by $\zeta(Z) \in C^\infty_C$.

For a complex vector field Z we set $\bar{L}_Z = \bar{\partial} \circ i_Z + i_Z \circ \bar{\partial}$:

Lemma 1 If Z is a holomorphic vector field then $\bar{L}_Z = 0$ as an operator on $\Lambda^*_C(M)$.

Proof: Note that $\bar{L}_Z$ is a derivation on $\Lambda^*_C(M)$: $\bar{L}_Z(\nu \wedge \mu) = \bar{L}_Z\nu \wedge \mu + \nu \wedge \bar{L}_Z\mu$. Also, it's local. Hence, it suffices to check that $\bar{L}_Z(C^\infty) = \bar{L}_Z(dz^k) = \bar{L}_Z(d\bar{z}^k) = 0$:

(a) For $f \in C^\infty$, $\bar{L}_Z f = i_Z(\bar{\partial} f) = \bar{\partial} f(Z) = 0$ since Z is of type $(1,0)$;

(b) $\bar{L}_Z(dz^k) = \bar{\partial} i_Z(dz^k) + i_Z \bar{\partial}\partial z^k = \bar{\partial}(a_k) - i_Z \partial\bar{\partial} z^k = \bar{\partial} a_k$ if $Z = \sum a_k \frac{\partial}{\partial z}$; since Z is holomorphic $\bar{\partial} a_k = 0$;

(c) $\bar{L}_Z(d\bar{z}^k) = \bar{\partial} i_Z(d\bar{z}^k) + i_Z \bar{\partial} d\bar{z}^k = 0 + i_Z \bar{\partial}^2 \bar{z}^k = 0$.

3. Elliptic Operators

3.1 Let E,F be vector bundles over a compact manifold M, and P a partial differential operator $P\colon \Gamma(E) \to \Gamma(F)$ where Γ denotes the space of smooth sections. The m-symbol of P, $\sigma_m(P)$, is the map $P^*M \setminus 0 \to \mathrm{Hom}(E,F)$ defined for $(x,\xi) \in T^*M$

by $\sigma_m(P)(x,\xi)s_x = P(\phi^m s)|_x$ where $\phi \in C^\infty$ such that $d\phi_x = \xi$, $\phi(x) = 0$, and $s \in \Gamma(E)$. P is of order k if $\sigma_{k+1}(P) \equiv 0$ and $\sigma_k(P) \not\equiv 0$. P <u>is elliptic of order</u> m if $\sigma_m(P)(x,\xi)$ is nonsingular for all $(x,\xi) \in T^*M \setminus 0$. It is easy to check that $D \overset{\text{def}}{=} d + \delta\colon \bigoplus_{p=0}^{[n/2]} \Lambda^{2p} \to \bigoplus_{p=0}^{[n/2]} \Lambda^{2p+1}$ and $\bar{D}_p \overset{\text{def}}{=} \bar{\partial} + \theta\colon \bigoplus_q \Lambda^{p,2q} \to \bigoplus_q \Lambda^{p,2q+1}$ are elliptic of order 1.

For P elliptic, $P\colon \Gamma(E) \to \Gamma(F)$, the <u>index of P</u> (denoted ind(P)) $\overset{\text{def}}{=}$ dim ker P - dim coker P is an integer as a consequence of the theory of elliptic partial differential equations.

The theories of Fredholm operators and elliptic equations together imply that if $P\colon \Gamma(E) \to \Gamma(F)$ is elliptic of order m and if A is a differential operator of order < m then ind (P + A) is again defined, and in fact, ind(P+A) =ind(P).

<u>Lemma 1</u> If Z is a vector field of type (1,0) then $\bar{L}_Z$ is a differential operator of order zero: $\Lambda^*_C(M) \to \Lambda^*_C(M)$.

Proof: Choose $\phi \in C^\infty$ such that $\phi(x) = 0$, $d\phi_x = \xi = \xi^{1,0} + \xi^{0,1} = \bar{\partial}\phi + \partial\phi$ and let $\omega \in \Lambda^*$. Then $\bar{\partial} i_Z(\phi\omega)|_x = \bar{\partial}\phi \wedge i_Z\omega|_x$ and $i_Z\bar{\partial}(\phi\omega) = i_Z(\bar{\partial}\phi \wedge \omega + \phi\,\bar{\partial}\omega)|_x$ $= \bar{\partial}\phi(Z) \wedge \omega - \bar{\partial}\phi \wedge i_Z\omega$. It follows that $\bar{L}_Z(\phi\omega)|_x = \bar{\partial}\phi_x(Z_x)\omega_x$ $= 0$. $\square$

<u>3.2</u>. It is a consequence of the Hodge (resp. Hodge-Kodaira) theorem for $\Delta = (d + \delta)^2$ (resp. for $\bar{\Box} = (\bar{\partial} + \theta)^2$) that ind(D) = Euler characteristic $\overset{\text{def}}{=} \sum (-1)^k b_k$ (resp. $\mathrm{ind}(\bar{D}_p) = \chi^p$ and $\mathrm{ind}\, D_0$ = arithmetic genus), cf. [15], [28].

II. GROUP ACTIONS AND KILLING FIELDS: ANALOGUES OF BERGER'S THEOREM

1. In this chapter we consider analytical and topological properties of compact differentiable manifolds that obstruct the free action of compact Lie groups. Let $\mathrm{Diff}(M^n)$ be the group of diffeomorphisms of M and let G be a compact connected subgroup of $\mathrm{Diff}(M^n)$. Since G is a compact connected Lie group it contains a circle group S^1 as a subgroup. Hence M admits a free action by some compact connected Lie group if and only if it admits a free circle action.

Suppose M^n is a compact differentiable manifold with Riemannian metric g. If S^1 acts on M then we may consider the metrics $g_\gamma = \gamma^*(g)$ where $\gamma \in S^1$ and, averaging, we can form $\bar{g} = \int_{S^1} \gamma^*(g)\, d\gamma$. It is well known, and easy to see, that S^1 acts isometrically with respect to the metric $\bar{g}$.

It is natural to ask when a flow ϕ_t (i.e. a one parameter group of diffeomorphisms) may act freely on M^n. Of course, if the generator of the flow ϕ_t vanishes at a point of M that point is fixed under ϕ_t. Thus if $\chi(M) \neq 0$ every 1-parameter group ϕ_t has a fixed point. In general, the question must be tackled via the special properties of the flow. In this direction, Berger [3] has shown that if M is an even dimensional, Riemannian manifold with strictly positive sectional curvature, then every isometric flow leaves some point fixed. We note that, in one sense, this theorem is very sharp for the Hopf-fibration $S^1 \to S^3 \to CP^1$ yields a free isometric flow on $M^6 = S^3 \times S^3$, a simply connected, oriented, even dimensional manifold with positive Ricci curvature. On the other hand, it is unknown whether sectional curvature > 0 implies that $\chi(M) > 0$, and in any event, $1 \geq$ sectional curvature $> \frac{1}{4}$ implies (for even dimensions) that M is topologically a sphere (the Rauch-Berger-Klingenberg "Sphere Theorem" [19]) $\Rightarrow \chi(M) = 2 \neq 0$. (In the case that the flow preserves a complex structure there are other obstructions. We take up this in the next chapter.)

2. We make the following

<u>Definition</u> A differentiable n-manifold is of <u>class T(k)</u> if it is orientable and there exists a product $\prod_j \omega_j^{p_j}$ of closed forms $\omega_j \in Z^{k_j}(M)$ such that (i) $\prod_j \omega_j^{p_j} \in \Lambda^n(M)$ (i.e. $\sum_j p_j k_j = n$), (ii) $\int_M \prod \omega_j^{p_j} \neq 0$, and (iii) $\max_j [\text{degree } \omega_j] = k$.

Thus, every orientable n-manifold is of class T(n).

<u>Theorem</u> Let M^{2n} be a closed, 2n dimensional manifold of class T(2). Then if a compact connected Lie group acts freely on M, $H^1(M) \neq 0$.

<u>Corollary 1</u> If M^{2n} is a closed symplectic manifold with $b_1 = 0$ (e.g. simply connected) then no compact connected Lie group acts freely on M.

The proof of the theorem will require the following lemma.

Lemma 1 If a connected compact Lie group G acts freely on M^{2n} then there exists a Riemannian metric on M^{2n} and a one form $\xi \in \Lambda^1(M)$ such that $i_\xi \, d\xi = 0$.

Proof of Lemma: Choose a circle $S^1 \subset G$ and any metric g_0 on M_m. Average g_0 over S^1 (as above) to get a metric g_1 with respect to which S^1 acts freely by isometries. Let X be the infinitesimal generator of S^1, viewed as a vector field on M, i.e. a Killing field in the metric g_1. Choose a new metric $g = \frac{1}{u} g_1$, where $u = g_1(X,X)$. Thus $g(X,X) = 1$. Let ξ be the 1-form dual to X in the metric g i.e. $\xi = g(X,\cdot)$. Then $i_\xi \xi = g(X,X) = 1$. For any vector fields A,B on M we have:

$$d\xi(A,B) = A\xi(B) - B\xi(A) - \xi([A,B]) = g(\nabla_A X,B) - g(\nabla_B X,A)$$

since the torsion of ∇ is zero. Since X is a Killing field $\nabla_{(\cdot)} X = -[\nabla_{(\cdot)} X]^t$ (see Chapter I, Section 1.5) and we have $d\xi(A,B) = 2g(\nabla_A X,B)$. Hence, $i_\xi \, d\xi = d\xi(X,\cdot) = 2g(\nabla_X X,\cdot)$.

To complete the proof of the lemma we note that for any vector field A

$$0 = Ag(X,X) = 2g(\nabla_A X,X) = -2g(A,\nabla_X X)$$

i.e. $\nabla_X X \equiv 0$. $\square$

Proof of the Theorem: Since M is of class $T(2)$ there exists a product of forms $\Pi\, \omega_j^{p_j}$ (such that $\int \Pi\, \omega_j^{p_j} \neq 0$) which may be rewritten $\Pi\, \omega_j^{p_j} = \Pi\, \alpha_j$, where $\alpha_j \in Z^2(M)$. By the

Hodge theorem -- for the metric g chosen in the lemma above -- each α_j has the form $\alpha_j = h_j + d\eta_j$ where h_j is g-harmonic. Hence $\prod_j \alpha_j = \prod_{j=1}^{n} h_j$ + an exact form. By Stokes' theorem we have

$$\int \Pi\, \alpha_j = \int \Pi\, h_j \neq 0$$

Let ℓ (resp. i) denote left exterior (resp. interior) multiplication by the one form ξ chosen in Lemma 1. Then, as an operator on forms, $T \overset{\mathrm{def}}{=} \ell\circ i + i\circ\ell\colon \Lambda^p \to \Lambda^p$ is just multiplication by $g(\xi,\xi) = 1$, i.e. the identity operator (see Chapter I, Section 1.2). Since $\Pi\, h_j \in \Lambda^n$ we have

$$\prod_{j=1}^{n} h_j = T\left(\prod_{j=1}^{n} h_j\right) = \ell\circ i\left(\prod_{j=1}^{n} h_j\right) = \xi \wedge \left[\sum_{k=1}^{n} i(h_k) \prod_{j\neq k} h_j\right]$$

Thus $i(h_k) \not\equiv 0$ for at least one k. Moreover,

$$di(h_k) = di(h_k) + i\circ dh_k = L_X h_k = 0$$

since harmonic forms are invariant under isometries. To conclude that $H^1(M) \neq 0$ it suffices to show that there do not exist $f_k \in C^\infty$ such that $i(h_k) = df_k$, for all k. So suppose $i(h_k) = df_k$. Let $\Pi\,(p_1,\ldots,p_k) \overset{\mathrm{def}}{=} h_1 \wedge \ldots \wedge \hat{h}_{p_1} \wedge \ldots \wedge \hat{h}_{p_k} \wedge \ldots \wedge h_n$ where the caret ^ indicates omission. Extend the definition of Π to make Π symmetric in the p_j's. Let

$$\textstyle\sum_k \overset{\mathrm{def}}{=} \displaystyle\sum_{1\leq p_1<p_2<\ldots<p_k\leq n} f_{p_1} \cdots f_{p_k}\, \Pi\,(p_1,\ldots,p_k)\ , \quad 1 \leq k \leq n,$$

and $\sum_0 \overset{\mathrm{def}}{=} \prod_{j=1}^{n} h_j$.

Lemma 2. $i(\sum_k) = d(\sum_{k+1}), \quad 1 \leq k \leq n-1.$

We postpone the proof of the lemma until the proof of the theorem is concluded. Now $\Pi\, h_j = T(\Pi\, h_j) = \ell \circ i(\sum_0)$ $= \xi \wedge d\sum_1 = -d(\xi \wedge \sum_1) + d\xi \wedge \sum_1$ and $T(d\xi \wedge \sum_1) = \xi \wedge d\xi \wedge i(\sum_1)$ by Lemma 1. In fact, we have in general

$$\Pi\, h_j = T^m(\Pi\, h_j) = \xi \wedge (d\xi)^{m-1}\, d\textstyle\sum_m + \text{an exact form}$$

$$= (d\xi)^m \wedge \textstyle\sum_m + \text{an exact form}$$

as follows easily by induction. $\int \Pi\, h_j = \int (d\xi)^m \wedge \sum_m$ for all $m \leq n$ and if we choose $m = n$ we have

$$\int f_1 \ldots f_n\, (d\xi)^n \neq 0$$

However, this is absurd since

$$f_1 \ldots f_n (d\xi)^n = T(f_1 \ldots f_n (d\xi)^n) = \xi \wedge (f_1 \ldots f_n)\, i_\xi (d\xi)^n = 0.$$

To complete the proof of the theorem we must prove Lemma 2.

Proof of Lemma 2:

$$i(\textstyle\sum_k) = \displaystyle\sum_{p_1 < \ldots < p_k} f_{p_1} \cdots f_{p_k}\, i\, \Pi\left((p_1, \ldots, p_k)\right)$$

$$= \sum_{p_1 < p_2 < \ldots < p_k} f_{p_1} \cdots f_{p_k} \left[\sum_{\substack{q=1 \\ q \neq p_j}}^{n} df_q\, \Pi\, (p_1, \ldots, p_k, q) \right]$$

$$= \sum_{q=1}^{n} \sum_{\substack{p_1 < \ldots < p_k \\ p_j \neq q}} f_{p_1} \cdots f_{p_k}\, df_q\, \Pi\, (p_1, \ldots, p_k, q)$$

$$= \sum_{\substack{p_j\text{'s and } q \text{ in} \\ \text{increasing order}}} d(f_{p_i} \cdots f_k f_q)\, \Pi\, (p_1,\ldots,p_k,q)$$

$$= \sum_{p_1<\cdots<p_{k+1}} d(f_{p_1} \cdots f_{p_{k+1}})\, \Pi\, (p_1,\ldots,p_{k+1})$$

$$= d(\Sigma_{k+1}) \qquad \text{since the } h_j \text{ are closed.} \qquad \square$$

Proof of Corollary: Let $\Omega \in Z^2$ be the symplectic form. Then Ω^n = a volume form and so $\int \Omega^n \neq 0$. Thus, every symplectic manifold is of class T(2).

3. Remarks

(1) It is well known that in order for a compact manifold M^{2n} to admit a symplectic structure it is necessary that (i) M^{2n} admits an "almost complex structure" i.e. a (1,1) tensor field J such that $J^2 = -I$, and (ii) the even degree cohomology groups $H^{2k}(M)$ have dimension ≥ 1. Corollary 1 says that there are additional necessary topological conditions. For example, if M is simply connected it cannot admit a free action by a connected compact Lie group.

(2) In connection with Remark 1, we note that Thurston (cf. [32]) has constructed an example of a 4 dimensional symplectic manifold with a free torus-action. In fact, it is a bundle of a torus over a torus. In his example, $H^1(M) = 3$ in agreement with Corollary 1 and also showing that his bundle cannot

admit a Kähler metric. Up to now, every known simply connected symplectic structure comes from a Kähler structure.

(3) The Pontryagin numbers of a 4m-dimensional compact manifold are defined as $\mathrm{Pont}(p,j) = \int_M \prod_k p_k^{j_k}$ where p_k, the k-th "Pontryagin class" of M, is a certain cohomology class $\in H^{4k}(M)$ and $\sum j_k k = m$ [24]. For a manifold of dimension $\equiv 0 \pmod 4$ Bott [7] has shown that the existence of a nonvanishing Killing field (i.e. a free action of a connected compact Lie group) implies that all the Pontryagin numbers vanish. The theorem above can be viewed as a somewhat analogous result for manifolds of dimension $\equiv 0 \pmod 2$.

(4) The Chern numbers of a compact complex manifold are defined as $\mathrm{Ch}(c,j) = \int \prod_k c_k^{j_k}$ where c_k, the k-th "Chern class" of M, is a certain cohomology class $\in H^{2k}(M)$ and $2 \sum j_k k = \dim_R M$ [24]. Bott [7] has shown that the existence of a free action by a connected group that preserves the complex structure implies that all the Chern numbers vanish. We will return to this question in the next chapter. We note here the following analogue of Bott's theorem for differentiable S^1 actions on a complex manifold:

Corollary 2 Let M^{2n} be a closed complex manifold with $b_1 = 0$ (e.g. simply connected). Then if M^{2n} admits a free smooth circle action the Chern number $\int c_1{}^n = 0$.

(5) In the spirit of Berger's result cited above we note that we have:

Corollary 3 If M^{2n} is a closed manifold of class $T(2)$ that admits some metric with Ricci > 0 then for any metric every Killing field must vanish somewhere.

Furthermore, it should be noted that as a consequence of the results of Tsagas [30] there exist positive constants $C(n)$ (depending only on n) such that if

(i) $b_2(M) \neq 0$, and

(ii) For some metric on M: $C(n) <$ sectional curvature≤ 1, then M^{2n} is of class $T(2)$. Since sec $> 0 \Rightarrow b_1 = 0$ we have:

Corollary 4 If M^{2n} is orientable (i) $b_2 \neq 0$, and (ii) $C(n) <$ sectional curvature ≤ 1 for some metric, then no metric admits nonvanishing Killing fields.

III. NONVANISHING HOLOMORPHIC VECTOR FIELDS AND MATSUSHIMA'S THEOREM

1. In this chapter we consider analytical properties of compact complex manifolds which are obstructions to the free holomorphic action of connected compact complex Lie groups. Let $H(M)$ be the biholomorphisms of M. If M is compact it is known that $H(M)$ is a Lie group but it is not in general compact [6]. In any event, $H(M)^0$ (the identity component of $H(M)$) is generated by vector fields which when "transferred" to M are holomorphic i.e. locally of the form $Z = \sum a_k \frac{\partial}{\partial z_k}$ where the a_k are holomorphic functions.

Given a one-parameter group ψ_t of biholomorphisms of M, we ask whether or not it may act freely, i.e. must its generator vanish somewhere? In this direction, Matsushima [21] proved that for a compact Hodge manifold (which according to a theorem of Kodaira [20] is projective algebraic) to admit a nonvanishing holomorphic vector field it is necessary that the first Betti number of M be nonzero. Matsushima's proof depended on properties of the group of biholomorphisms of CP^n. Carrell and Lieberman [9] extended Matsushima's result to all Kähler manifolds. We generalize these results to all compact complex manifolds.

2. We make the following

<u>Definition</u> A compact complex manifold of real dimension $2n$ is of <u>class</u> $T_{\bar\partial}(k)$ if there exists a product $\Pi\mu_j^{r_j}$ of $\bar\partial$-closed forms μ_j such that (i) $\Pi\,\mu_j^{r_j} \in \Lambda^{n,n}$, (ii) $\int \Pi\,\mu_j^{r_j} \neq 0$ and (iii) $\max_j$ [degree μ_j] $= k$. M is of <u>class</u> $T'_{\bar\partial}(2p)$ if there exists a $\bar\partial$-closed form $\mu \in \Lambda^{p,p} \subset \Lambda^{2p}$ such that $\mu^r \in \Lambda^{n,n}$ and $\int \mu^r \neq 0$. Thus $T'_{\bar\partial}(2p) \subset T_{\bar\partial}(2p)$ and every complex manifold of $\text{dimension}_{\mathbf{C}} = n$ is of class $T'_{\bar\partial}(2n) \subset T_{\bar\partial}(2n)$.

Denote the space of $\bar\partial$-closed (p,q) forms by $Z^{p,q}_{\bar\partial}$. Thus if $M \in T_{\bar\partial}(k)$ there exist differential forms $\mu_j \in Z^{p_j,q_j}_{\bar\partial}$ and integers r_j such that $\sum_j r_j(p_j + q_j) = 2n$ and $\max_j\,[p_j + q_j] = k$.

According to Dolbeault's theorem [11], $H^{p,q} = \{\alpha \in \Lambda^{p,q}\colon \bar\partial\alpha = 0\} \bmod \{\alpha \in \Lambda^{p,q}\colon \alpha = \bar\partial\beta\}$ has dimension $h^{p,q}$, and if $h^{p,q} = 0$ then every $\bar\partial$-closed (p,q) form is $\bar\partial$-exact.

<u>Theorem 1</u> Let M^n be a closed complex manifold of $\text{dimension}_{\mathbf{C}} = n$ and class $T_{\bar\partial}(2)$. If M admits a free holomorphic action by a connected complex Lie group then $h^{1,0} + h^{0,1} \neq 0$.

<u>Corollary 1</u> (Matsushima). If a closed Hodge manifold M admits a nonvanishing holomorphic vector field then $H^1(M) \neq 0$.

<u>Corollary 2</u> (Carrell and Lieberman). If a closed Kähler manifold admits a nonvanishing holomorphic vector field then $H^1(M) \neq 0$.

The proof of the theorem will require the following lemma.

Lemma 1 If a closed complex manifold admits a free holomorphic action by a connected Lie group G then there exists a hermitian metric on M and a complex differential form ζ of type (1,0) such that $i_\zeta(\bar\partial\zeta)$, a differential form of type (0,1), vanishes identically.

Proof of Lemma: Let Z be a nonvanishing holomorphic vector field on M that generates a one-parameter subgroup of G. Let h_0 be any hermitian metric on M. Let $u = h_0(Z,\bar Z) \neq 0$ and let $h = \frac{1}{u} h_0$. In the metric h, Z has length $\equiv 1$. Let ζ be the (1,0) form dual to Z in the metric h i.e. $\zeta = h(\ ,\bar Z)$. Clearly $\zeta(Z) = 1$ and it is easy to see that ζ is a complex differential form of type (1,0). Since Z is a holomorphic vector field $i_Z \circ \bar\partial = -\bar\partial \circ i_Z$ (see Chapter I, Section 2) and so $i_Z \bar\partial\zeta = -\bar\partial i_Z \zeta = -\bar\partial(\zeta(Z)) = 0$. □

Proof of the theorem: We use the notation introduced above the statement of the theorem. Thus $\int \Pi\, \mu_j^{r_j} \neq 0$ and we may decompose the product $\Pi\, \mu_j^{r_j}$ as

$$\Pi\, \mu_j^{r_j} = \prod_j F^{p_j,q_j} = F^{0,1} \wedge F^{1,0} \wedge F^{1,1} \wedge F^{0,2} \wedge F^{2,0}$$

where F^{p_j,q_j} = product of $\bar\partial$-closed forms in Z^{p_j,q_j}. We may assume $F^{1,0} \equiv 1$, for otherwise there exist $\mu \in Z^{1,0}$ i.e. $h^{1,0} \neq 0$. Since the total degree of $\Pi\, F^{p_j,q_j}$ is $\dim_R M = 2n$ it follows that $F^{0,1}$ is the product of an even number of (0,1) forms and taking two of them at a time we may consider $F^{0,1}$ as included in $F^{0,2}$. Thus we have

$$\Pi\, \mu_j^{r_j} = F^{1,1} \wedge F^{0,2} \wedge F^{2,0} .$$

Let i (resp. ℓ) denote interior (resp. left exterior) multiplication by the (1,0) form ζ whose existence is asserted in Lemma 1. Note that $i(F^{0,2}) = 0$ since i: $\Lambda^{p,q} \to \Lambda^{p-1,q}$. We may assume that $i(F^{2,0}) = 0$, for if not $\gamma = i(F^{2,0}) \in \Lambda^{1,0}$ is a holomorphic one-form (since $\bar\partial\gamma = -i\bar\partial F^{2,0} = 0$) and $h^{1,0} \neq 0$. Writing F for $F^{1,1}$ and R for the remaining factors $F^{0,2} \wedge F^{2,0}$ we have the product $P \overset{\text{def}}{=} \Pi\, \mu_j^{r_j} = F \wedge R$ where F is a product of $\bar\partial$-closed (1,1) forms and R is annihilated by i. Now factor F as $F = \prod_{j=1}^{m} \alpha_j$, $\alpha_j \in Z^{1,1}$.

Note that $T \overset{\text{def}}{=} \ell \circ i + i \circ \ell : \Lambda^{p,q} \to \Lambda^{p,q}$ = multiplication by $|\zeta|_h^2$ on $\Lambda^{p,q}$ is the identity operator since $|\zeta|^2 = 1$.

Since P is an (n,n) form

$$P = TP = \ell \circ i\, P = \zeta \wedge i(F) \wedge R = \zeta \wedge \left[\sum_{k=1}^{m} i(\alpha_k) \prod_{j \neq k} \alpha_j \right] \wedge R.$$

Hence for some k $i(\alpha_k) \not\equiv 0$. Now $\bar\partial i(\alpha_k) = -i\bar\partial(\alpha_k) = 0$, so $i(\alpha_k) \in Z^{0,1}$. To show $h^{0,1} \neq 0$ it suffices to show that there do not exist $\{f_k\}_{k=1}^{m}$, $f_k \in C_C^{\infty}$, such that $\bar\partial f_k = i(\alpha_k)$ for all k. Suppose such f_k did exist. Let $\Pi\,(p_1,\ldots,p_k) \overset{\text{def}}{\equiv} \alpha_1 \ldots \wedge \hat\alpha_{p_1} \wedge \ldots \wedge \hat\alpha_{p_2} \wedge \ldots \wedge \hat\alpha_{p_k} \wedge \ldots \wedge \alpha_m$ where the caret ^ indicates omission. Extend the definition of Π so that $\Pi\,(p_1,\ldots,p_k)$ is symmetric in $(p_1,\ldots,p_k)$. Let

$$\textstyle\sum_k^{1,1} = \sum_{1 \le p_1 < \ldots < p_k \le m} f_{p_1} \ldots f_{p_k}\, \Pi\,(p_1,\ldots,p_k) \;,\quad 1 \le k \le m,$$

and let $\sum_0 = F = \prod_{j=1}^{m} \alpha_j$.

Lemma 2 $\quad i(\sum_k^{1,1}) = \bar\partial(\sum_{k+1}^{1,1})$.

We postpone the proof of the lemma until the proof of the theorem is completed. Now

$$F \wedge R = T(F \wedge R) = \zeta \wedge i(F) \wedge R = \zeta \wedge i(\textstyle\sum_0^{1,1}) \wedge R$$
$$= \zeta \wedge \bar\partial(\textstyle\sum_1^{1,1}) \wedge R = -\bar\partial(\zeta \wedge \sum_1^{1,1} \wedge R) + \bar\partial\zeta \wedge \sum_1^{1,1} \wedge R$$
$$= \bar\partial\zeta \wedge \textstyle\sum_1^{1,1} \wedge R + \bar\partial\text{-exact form}$$

Applying T again we have $F \wedge R = T^2(F \wedge R) = \zeta \wedge i(\bar\partial\zeta \wedge \sum_1^{1,1}) \wedge R$ $+ \bar\partial$-exact form $= \zeta \wedge \bar\partial\zeta \wedge i(\sum_1) \wedge R$ $+ \bar\partial$-exact form $= (\bar\partial\zeta)^2 \wedge \sum_2^{1,1} \wedge R + \bar\partial$-exact form. It is easy to see that continuing in this fashion we have for $1 \le k \le m$:

$$F \wedge R = \bar\partial\omega_k + (\bar\partial\zeta)^k \wedge \textstyle\sum_k \wedge R\,, \quad \text{where} \quad \omega_k \in \Lambda^{n,n-1} .$$

Choosing $k = m$ in this last expression we have:

$$P = F \wedge R = \bar\partial\omega_m + (\bar\partial\zeta)^m \prod_{j=1}^{m} f_j \wedge R$$
$$= d\omega_m + \prod_{j=1}^{m} f_j (\bar\partial\zeta)^m \wedge R$$

since $d\omega = \bar\partial\omega$ if $\omega \in \Lambda^{n,n-1}$. Applying Stokes' theorem we find that

$$\int P = \int \prod_{j=1}^{m} f_j (\bar\partial\zeta)^m \wedge R \neq 0 .$$

However, this is absurd since

$$\prod_{j=1}^{m} f_j(\bar\partial\zeta)^m \wedge R = T\left[\prod_{j=1}^{m} f_j(\bar\partial\zeta)^m \wedge R\right]$$

$$= \prod_{j=1}^{m} f_j\zeta \wedge [i(\bar\partial\zeta)^m \wedge R + (\bar\partial\zeta)^m \wedge iR] = 0$$

To complete the proof it only remains to prove Lemma 2.

Proof of Lemma 2:

$$i(\sum\nolimits_k^{1,1}) = \sum_{1\le p_1<\dots<p_k\le m} f_{p_1} \cdots f_{p_k}\, i\big(\Pi(p_1,\dots,p_k)\big)$$

$$= \sum_{\substack{p_1<p_2<\dots<p_k \\ q\ne p_j}} f_{p_1} \cdots f_{p_k}\left[\sum_{q=1}^{m} \bar\partial f_q \quad \Pi(p_1,\dots,p_k,q)\right]$$

$$= \sum_{q=1}^{n} \sum_{\substack{1\le p_j<\dots<p_k\le m \\ q\ne p_\ell}}^{n} f_{p_1} \cdots f_{p_k}\, \bar\partial f_q\, \Pi\, (p_1,\dots,p_k,q)$$

$$= \sum_{\substack{p_j\text{'s and } q \text{ in} \\ \text{ascending order}}} \bar\partial(f_{p_1} \cdots f_{p_k}\, f_q)\, \Pi\, (p_1,\dots,p_k,q)$$

$$= \sum_{1\le p_1<\dots<p_{k+1}\le m} \bar\partial(f_{p_1} \cdots f_{p_{k+1}})\, \Pi\, (p_1,\dots,p_{k+1})$$

$$= \bar\partial \sum f_{p_1} \cdots f_{p_{k+1}}\, \Pi\, (p_1,\dots,p_{k+1}) \quad \text{since the } \alpha_j \text{ are } \bar\partial\text{-closed}$$

$$= \bar\partial \quad \sum\nolimits_{k+1}^{1,1}$$

Proof of corollaries: Corollary 1 follows from Corollary 2. To prove Corollary 2 we note that if M is a Kähler manifold with Kähler form $\omega \in \Lambda^{1,1}$ then $d\omega = 0$ $\Rightarrow \bar{\partial}\omega = 0$. Since $\int \omega^n = n!\ \mathrm{vol}\ (M)$ it follows that M is of class $T_{\bar{\partial}}(2)$. Since, for Kähler manifolds, $h^{0,1} + h^{1,0} = b_1(M)$ we are done.

Remarks.

(1) Whereas in the previous chapter we dealt with compact group actions in this section the one parameter group need not be compact.

(2) As noted in i.e. Chapter II, Section 3 Bott [7] has shown that the existence of a nonvanishing holomorphic vector field implies that all the Chern numbers of M must vanish. Theorem 1 shows that if $h^{0,1} = h^{1,0} = 0$ then the existence of a nonvanishing holomorphic vector field implies that $\int \omega^n = 0$ for all $\bar{\partial}$-closed (1,1) forms ω, not just $\omega = c_1$.

Theorem 2. Suppose M^n is a closed, complex manifold and of class $T'_{\bar{\partial}}(2)$. If M^n admits a nonvanishing holomorphic vector field then $h^{0,1} \neq 0$.

The proof is exactly the same as the proof of the preceding theorem; however the possibility that $h^{0,1} = 0$ is eliminated since no factors of bidegree (1,0) or (2,0) can occur. Theorem 2 also implies the results of Matsushima, Carrell and Lieberman.

Note that for a Riemann surface S the Poincaré-Hopf theorem says, in part, that if Euler characteristic $= \int_S c_1 > 0$ ($\Rightarrow h^{0,1} = 0$) then every complex vector field of type (1,0) must vanish. This conclusion is false if dim = n and $\int c_1^n > 0$, say. The result of Bott [7] can be considered a generalization that is valid for holomorphic vector fields. Theorem 2 above allows us to give another generalization using a hypothesis that is closer to the Gauss Bonnet condition $\int_S c_1 = \int_S$ curvature > 0.

<u>Theorem 3</u>. Let M^n be a closed complex manifold with $h^{0,1} = 0$ and $\int c_1 \wedge \Omega^{n-1} \neq 0$ for some closed (1,1) form Ω. Then every holomorphic vector field on M must vanish somewhere.

The proof is immediate. We note that if n = 1 we have $\int c_1$ = Euler characteristic $\neq 0$. To see the relationship to "curvature" recall that if Ω_h denotes the (1,1) form associated to a hermitian metric h then for any $n = \dim_C M$ $\int_M c_1 \wedge \Omega_h^{n-1} = \int$ scalar curvature of the "hermitian connection" (cf. Berger, Gauduchon, Mazet [4]).

For all closed complex manifolds we can prove:

<u>Theorem 4</u> If a closed, complex manifold M ($\dim_C = n$) admits a nonvanishing holomorphic vector field then the Hodge numbers of M satisfy:

$$\sum_{0 \leq p \leq n-1} h^{p,p+1} \neq 0 .$$

Proof: Let ζ be the (1,0) form that appears in Lemma 1. If ℓ_ζ and i_Z denote left exterior and interior multiplication, then as in the proof of Theorem 1, $T \overset{\text{def}}{=} \ell_\zeta \circ i_Z + i_Z \circ \ell_\zeta$ is the identity operator. Let ω be an (n,n)-form such that $\int \omega \neq 0$; e.g. a volume form on M. We will assume that $\sum h^{p,p+1} = 0$.

<u>Lemma 3</u> If $\sum\limits_{p \leq n-1} h^{p,p+1} = 0$ then there exists a sequence of differential forms $\alpha_q \in \Lambda^{q,q}$ such that $\bar{\partial}\alpha_q = i_Z \alpha_{q+1}$ for $0 \leq q \leq n-1$.

Proof: Set $\alpha_n = \omega$. Since $\bar{\partial} i\omega = -i\bar{\partial}\omega = 0$ and $h^{n-1,n} = 0$ there exists some $\alpha_{n-1} \in \Lambda^{n-1,n-1}$ such that $\bar{\partial}\alpha_{n-1} = i\omega$. Since $\bar{\partial} i\alpha_{n-1} = -i\bar{\partial}\alpha_{n-1} = -i^2\omega = 0$ and $h^{n-2,n-1} = 0$ there exists $\alpha_{n-2} \in \Lambda^{n-2}$ such that $\bar{\partial}\alpha_{n-2} = i\alpha_{n-1}$. In general, if there exist $\alpha_n, \ldots, \alpha_{q+1}$ such that $\bar{\partial}\alpha_{q+1} = i\alpha_{q+2}$ then $\bar{\partial} i\alpha_{q+1} = -i\bar{\partial}\alpha_{q+1} = -i^2\alpha_{q+2} = 0$. Since $h^{q,q+1} = 0$, there exists $\alpha_q \in \Lambda^{q,q}$ such that $\bar{\partial}\alpha_q = i\alpha_{q+1}$ and this completes the proof of the lemma. □

Using the forms $\alpha_q \in \Lambda^{q,q}$ of the lemma note that

$$\begin{aligned}
\omega = T\omega = \zeta i\omega = \zeta \wedge \bar{\partial}\alpha_{n-1} &= \bar{\partial}\zeta \wedge \alpha_{n-1} + \text{an exact form} \\
&= T^2\omega = \zeta \wedge \bar{\partial}\zeta \wedge i\alpha_{n-1} + \text{an exact form} \\
&= \zeta \wedge \bar{\partial}\zeta \wedge \bar{\partial}\alpha_{n-2} + \text{an exact form} \\
&= (\bar{\partial}\zeta)^2 \wedge \alpha_{n-2} + \text{an exact form.}
\end{aligned}$$

In general,

$$\omega = (\bar{\partial}\zeta)^k \wedge \alpha_{n-k} + \text{an exact form}, \qquad 1 \leq k$$

as follows easily by induction. Choosing $k = n$ we have

$$\omega = \alpha_0 (\bar{\partial}\zeta)^n + \text{an exact form, and } \alpha_0 \in C^\infty_C .$$

Applying T one more time we have:

$$\omega = \alpha_0 \zeta \wedge i(\bar{\partial}\zeta)^n + \text{an exact form} = \text{an exact form.}$$

It follows then from Stokes' theorem that $\int \omega = 0$, in contradiction to the choice of ω. □

Remark It follows from the result of Bott [7] that the Chern numbers $\chi^0 = \chi^1 = 0$ if M admits a nonvanishing holomorphic vector field. Thus, if dim M = 3

$$1 - h^{0,1} + h^{0,2} - h^{0,3} = 0$$

and

$$h^{1,0} - h^{1,1} + h^{1,2} - h^{1,3} = 0$$

So $h^{0,1} + h^{1,2} = 1 + h^{0,2} + h^{1,1} + h^{1,3} - (h^{0,3} + h^{1,0})$. According to the theorem above we may conclude in addition that $h^{0,1} + h^{1,2} \neq 0$.

IV. PARALLEL VECTOR FIELDS

1. According to the classical theorem of Hopf, on a closed differentiable n-manifold there exists a nowhere vanishing vector field if and only if the Euler-Poincaré characteristic is zero, $\chi(M) = 0$. If M is Riemannian one may ask whether or not it admits a vector field that is parallel. Chern has shown that necessary conditions are that the first Betti number $b_1 \geq 1$ and that the second Betti number $b_2 \geq b_1 - 1$ and he has conjectured that these conditions are not sufficient [10]. Since parallel fields are Killing fields it follows from the result of Bott [7] cited at the end of Chapter II that all the Pontryagin numbers of such a manifold are zero. Of course, this is a restriction only if dimension $M \equiv 0 \bmod 4$. Below, we derive additional necessary topological conditions that extend those of Chern. Thus, we are able to exhibit a family of manifolds with $\chi = 0$ that satisfy the conditions of Chern and Bott but still cannot admit parallel fields whatever the metric. If the manifold admits a complex structure we refine our results to deduce additional conditions that are necessary for M to admit a vector field that is parallel with respect to a Kähler metric. We apply these results to the topology of compact homogeneous spaces (supplementing some similar results of Hurewicz and de Rham). Finally, we give some n-dimensional generalizations of some classical results of Hurwitz on Riemann surfaces of genus ≥ 2 and also consider the effect of curvature on topology.

2. We begin by recalling that if V is a parallel vector field on M then V is also a Killing field. In fact, if $\nabla_Y V = 0$ for all Y then $(L_V g)(Y,Z) = Vg(Y,Z) - g(L_V Y,Z) - g(Y,L_V Z) = g(\nabla_Y V,Z) + g(Y,\nabla_Z V) = 0$. As a consequence, if V is parallel it is divergence free, $\mathrm{div}_g\, V = 0$ (see Chapter I). Also, if $\nu = g(V,\cdot)$ = the covariant form of V, $d\nu = 0$, since

$$\begin{aligned} d\nu(Y,Z) &= Yg(V,Z) - Zg(V,Y) - g(V,L_Y Z) \\ &= g(\nabla_Y V,Z) - g(\nabla_Z V,Y) = 0\ , \quad \text{for all } Y,Z. \end{aligned}$$

Thus ν is a harmonic 1-form and $b_1 \geq 1$.

<u>Theorem 1</u> If a closed differentiable n-manifold M admits a vector field that is parallel with respect to some Riemannian metric then the Betti numbers of M satisfy:

$$b_1 \geq 1 \quad \text{and} \quad b_{k+1} \geq b_k - b_{k-1} \quad \text{for} \quad 1 \leq k \leq n-1\ .$$

Notice that when $k = 1$ we have Chern's condition $b_2 \geq b_1 - 1$.

Proof: Suppose that V is parallel with respect to some metric g. Let ν be the covariant form of V, $\nu = g(V,\cdot)$. Define an operator $T\colon \Lambda^*(M) \to \Lambda^*(M)$ by $T = \ell_\nu + i_V$. Let $D = d + \delta$ and let $\mathcal{H}^k(M)$ denote the finite dimensional vector space of g-harmonic k-forms. We need the following anti-commutation property:

Lemma 1 $DT + TD \equiv 0$ as an operator on Λ^k.

Proof: $DT + TD = (d + \delta)(\ell + i) + (\ell + i)(d + \delta)$
$= d\circ i + i\circ d + \delta\circ\ell + \ell\circ\delta + d\ell_\nu + \ell_\nu d + i\circ\delta + \delta\circ i$
$= L_V + L_V^t + A + A^t$ where for $\omega \in \Lambda^k$, $A\,\omega = d(\nu \wedge \omega) + \nu \wedge d\omega = d\nu \wedge \omega$ and A^t is the transpose of A. Since V is a Killing field $L_V + L_V^t = 0$ (see Section I-1.5) and since ν is closed $A = A^t = 0$.

Returning to the proof of the theorem, note that

$$T\colon \Lambda^k \to \Lambda^{k-1} \oplus \Lambda^{k+1}$$

We now claim that T is a linear injection of the finite dimensional space $H^k \subseteq \Lambda^k$ into $H^{k-1} \oplus H^{k+1} \subseteq \Lambda^{k-1} \oplus \Lambda^{k+1}$. Thus by the Hodge Theorem, $b_k = \dim H^k \leq \dim H^{k-1} + \dim H^{k+1} = b_{k-1} + b_{k+1}$ and this will complete the proof of the theorem. To see that T does, indeed, have the property claimed observe first that if $\alpha_k \in H^k$ then $D\alpha_k = d\alpha_k + \delta\alpha_k = 0$. So, by the lemma, $D[T(\alpha_k)] = dT\alpha_k + \delta T\alpha_k = 0$ and (since the ranges of d and δ are orthogonal) we see that $T\alpha_k$ is harmonic. Thus $T(H^k) \subseteq H^{k-1} \oplus H^{k+1}$. To check that T is an injection note that if $T\alpha = 0$ then $0 = T^2\alpha = (\ell\circ i + i\circ\ell)\alpha = |V|^2\alpha$ and since $V \neq 0$ we have $\alpha = 0$. The proof of the theorem is complete. □

Suppose now that M^{2m} has a complex structure, and a Kähler metric h. If a vector field V is parallel with respect to such a Kähler metric it is called Kähler-parallel.

Recall that for a Kähler manifold the Hodge numbers $h^{p,q}$ satisfy $b_k = \sum_{p+q=k} h^{p,q}$. We have the following refinement of Theorem 1

Theorem 2 If a closed 2m-manifold with Kähler metric admits a Kähler-parallel vector field then the Hodge numbers of M satisfy:

$$h^{p+1,q} \geq h^{p,q} - h^{p-1,q} \quad \text{for} \quad 0 \leq p \leq n,\ 0 \leq q \leq n.$$

Proof: Suppose that V is parallel with respect to some Kähler metric h. Consider the vector field of type (1,0) $Z = V - iJV$ and its dual (1,0) form $\zeta = h(\cdot,\bar{Z})$ so that $\zeta(Z) = |Z|_h^2 \neq 0$. Let T be the operator $T: \Lambda_C^* \to \Lambda_C^*$ defined by $T\omega = \ell_\zeta \omega + i_Z \omega$ for $\omega \in \Lambda^{p,q}$ and let $\bar{D} = \bar{\partial} + \theta$.

Lemma 2 $\bar{D}T + T\bar{D} \equiv 0$ as an operator on Λ_C^*.

Proof: $\bar{D}T + T\bar{D} = \bar{\partial}\circ i_Z + i_Z\circ\bar{\partial} + \theta\circ\ell_\zeta + \ell_\zeta\circ\theta + \bar{\partial}\ell + \ell\bar{\partial} + i\circ\theta + \theta\circ i = \bar{L}_Z + \bar{L}_Z^t + \bar{A} + \bar{A}^t$, where $\bar{A}$ denotes left multiplication by $\bar{\partial}\zeta$. Thus it is sufficient to check that $\bar{L}_Z = \bar{A} = 0$. Since ζ is a (1,0) form $\theta\zeta = 0$ and so $\bar{\partial}\zeta = 0 \Rightarrow \bar{\Box}\zeta = 0$. Since h is Kähler $\bar{\Box} = \frac{1}{2}\Delta$ and $\bar{\Box}\zeta \Rightarrow$ the real and imaginary parts of ζ are harmonic with respect to the Riemannian metric h. Now $\zeta = h(\cdot,\bar{Z}) = h(\cdot,V) + ih(\cdot,JV) = \mathrm{Re}\,\zeta + i\,\mathrm{Im}\,\zeta$. As noted at the beginning of this section, to verify that $\Delta\,\mathrm{Re}\,\zeta = \Delta\,\mathrm{Im}\,\zeta = 0$ it is sufficient to check that $\nabla_X \mathrm{Re}\,\zeta = \nabla_X \mathrm{Im}\,\zeta = 0$ for every vector field X. This is, in fact, the case since $\nabla_X h = \nabla_X J = \nabla_X V = 0$. To complete the proof of the lemma we need only check that Z is holomorphic so $\bar{L}_Z = 0$ by Lemma 1 of Section I-2. To see that Z is holomorphic let $\nabla_{\bar{\alpha}} = \nabla_{\partial/\partial\bar{z}}$ where $\{z^\alpha\}$ are local complex coordinates on M. Then $Z = \sum Z^j \frac{\partial}{\partial z^j}$, since $Z_p \in T_pM^{1,0}$, and Z is holomorphic iff $\partial Z_j/\partial\bar{z}^\alpha = 0$ for all α, j for some choice of local coordinates $\{z^\alpha\}$. Since the metric is Kähler we may choose local coordinates that are "normal", (i.e. the Christoffel symbols vanish) and so $\nabla_{\bar{\alpha}}\Big|_p = \frac{\partial}{\partial\bar{z}^\alpha}\Big|_p$. In these coordinates Z is holomorphic iff $\nabla_{\bar{\alpha}} Z^j = 0$. Since Z^j is a linear combination of components of the tensors J and V which are parallel the proof of the lemma is complete.

Returning to the proof of the theorem, note that

$$T: \Lambda^{p,q} \to \Lambda^{p-1,q} \oplus \Lambda^{p+1,q}$$

As in the proof of the previous theorem, we show that T injects the finite dimensional space $\mathcal{H}^{p,q} \overset{\text{def}}{=} \ker \bar{\Box} \cap \Lambda^{p,q} \subset \Lambda^{p,q}$ into $\mathcal{H}^{p-1,q} \oplus \mathcal{H}^{p+1,q}$. By the lemma, $D[T(\mathcal{H}^{p,q})] = 0$ and so for $\alpha_{p,q} \in \mathcal{H}^{p,q}$: $\bar{\partial}T\alpha + \theta T\alpha = 0 \Rightarrow T\alpha$ is $\bar{\Box}$-harmonic. To check that T is an injection we just note that $T^2 = \ell_\zeta \circ i_Z + i_Z \circ \ell_\zeta$ = multiplication by $h(Z,\bar{Z})$ and $Z = V - iJV$ is nonvanishing. Since by the Hodge-Kodaira theorem $\dim \mathcal{H}^{p,q} = h^{p,q}$ the proof is complete. $\Box$

As an application of Theorem 1 we can construct examples of manifolds that satisfy the conditions of Bott and Chern but still do not admit parallel fields for any metric. One family of such manifolds is:

$$S(p,q,r) \equiv S^1 \times S^p \# S^q \times S^r$$

with $3 \leq q < r-1$ and $p+1 = q+r \equiv 1 \mod 2$. Here S^k denotes the sphere of dimension k and $\#$ is the operation of "connected sum" (cf. [23]). It is easy to check that $b_1 = 1$ and $\chi = 0$ while $b_{r+1} < b_r - b_{r-1}$.

3. A classical theorem of Hurwitz may be restated as: "A closed two-dimensional manifold of genus ≥ 2 has a finite isometry group for any Riemannian structure, and it admits no one-parameter group of holomorphic transformations for any complex structure" cf [5]. If a two dimensional manifold has scalar curvature $\tau(x) \leq 0$ (and $\not\equiv 0$) then by the Gauss-Bonnet theorem $\frac{1}{2}$ Euler characteristic = arithmetic genus = $h^{0,0} - h^{0,1} = 1 -$ genus < 0 i.e. $b_2 < b_1 - 1$ or $0 = h^{0,2} < h^{0,1} - h^{0,0}$. Bochner [5] has extended Hurwitz' theorem under the hypothesis of negative definite Ricci tensor (Ricci < 0) in place of $\tau(x) \leq 0$ or genus ≥ 2. By explicitly introducing topological conditions (in place of using the Gauss-Bonnet theorem) we can get the same result for Ricci ≤ 0:

<u>Theorem 3</u> If a closed Riemannian n-manifold (M^n, g) has Ricci ≤ 0 and satisfies:

$$b_{k+1} < b_k - b_{k-1} \quad \text{for some} \quad 1 \leq k \leq n-1$$

then the isometry group of (M,g) is finite.

Proof: According to a well known result of Myers and Steenrod [27] the isometry group of M is a compact Lie group

I(M). The finiteness of I(M) will then follow from the fact that its Lie algebra, which is isomorphic to the set of Killing fields on M, is empty, i.e. dim I(M) = 0. To see that (M,g) has no Killing fields observe that it follows from Bochner's theorem (cited in Section I-1.7) that any Killing field must be parallel and it follows from Theorem L that M admits no parallel fields.

The analogous result for holomorphic vector fields is

<u>Theorem 4</u> If a closed Kähler manifold M^n has Ricci ≤ 0 and satisifes:

$$h^{p,q+1} < h^{p,q} - h^{p,q-1} \quad \text{for some} \quad 0 \leq p \leq n, \quad 1 \leq q \leq n$$

then it admits no connected group of holomorphic transformations.

Proof: If M admits a connected group of holomorphic transformations then it admits a nontrivial holomorphic vector field Z. Let h be the Kähler metric and $\zeta = h(\,,\bar{Z})$ the (1,0) form dual to Z. Set $T = \ell_\zeta + i_V$ and $\bar{D} = \bar{\partial} + \theta$. Then, as in the proof of Lemma 2, $\bar{D}T + T\bar{D} = \bar{A} + \bar{A}^t$ where $\bar{A}$ denotes left multiplication by $\bar{\partial}\zeta$, and A^t is the hermitian transpose of $\bar{A}$.

<u>Lemma 3</u> Under the hypotheses of the theorem and with notation as above $\bar{\partial}\zeta = 0$.

We postpone the proof of the lemma until the proof of the theorem is completed. Returning to the theorem we have

$\bar{D}T + T\bar{D} = 0$ and so, as in the proof of Theorem 2 $T: H^{p,q} \to H^{p-1,q} \oplus H^{p+1,q}$. To check that we still have an injection note that T^2 = multiplication by $h(Z,\bar{Z})$ and, since Z is holomorphic, $h(Z,\bar{Z}) \neq 0$ almost everywhere. To complete the proof of the theorem we only need the

Proof of Lemma 3: We use an elementary integral formula for which we refer to [22], p. 17: For a (0,1)-form ξ on a closed Kähler manifold $(\Box\xi,\xi)_{L^2} = \int_M \mathrm{Ricc}(\xi,\xi)\ \mathrm{vol} + \int (\bar{\nabla}\xi,\ \bar{\nabla}\xi)(x)\ \mathrm{vol}$ where $\bar{\nabla}\xi$ is the complex tensor field

$$(\bar{\nabla}\xi)_{AB} = \begin{cases} \nabla_{\bar{\lambda}}\xi_{\bar{\mu}} & \text{if } A = \bar{\lambda},\ B = \bar{\mu} \\ 0 & \text{otherwise} \end{cases}$$

and $\xi = \sum \xi_{\bar{\mu}}\, d\bar{z}^{\mu}$. Let $\xi = \bar{\zeta} = \overline{h(\ ,\bar{Z})} = h(\cdot,Z) \in \Lambda^{0,1}$. Since Z is holomorphic $\bar{\nabla}\xi = 0$ cf. [22]. Thus $(\bar{\Box}\bar{\zeta},\bar{\zeta}) = \int \mathrm{Ricc}(\bar{\zeta},\bar{\zeta}) \leq 0$. It follows that $\bar{\Box}\bar{\zeta} = 0$. Since the metric is Kahler $\bar{\Box} = \Box = \partial\circ\partial^t + \partial^t\circ\partial$ and so $\bar{\Box}\bar{\zeta} = \Box\,\bar{\zeta} = \partial^t\partial\bar{\zeta}$ since $\bar{\zeta} \in \Lambda^{0,1}$. Thus $\partial\bar{\zeta} = 0 \Rightarrow \bar{\partial}\zeta = 0$ as claimed. This completes the proof of the theorem.

Remark. For a manifold with real dimension = 2, these conditions on Betti and Hodge numbers just mean genus ≥ 2, Hurwitz' condition.

If we strengthen our hypotheses we can prove:

<u>Theorem 5</u> If a closed Riemannian n-manifold (M^n,g) has nonpositive curvature and Betti numbers satisfying

$$b_{k+1} < b_k - b_{k-1} \quad \text{for some} \quad 1 \leq k \leq n-1$$

then Diff $(M)^0$ (the identity component of the group of diffeomorphisms of M) contains no nontrivial isometries.

Proof: Let $\phi\colon M \to M$ be a nontrivial isometry in Diff$(M)^0$ and ϕ_t a family of diffeomorphisms of M connecting ϕ_0 = identity and $\phi_1 = \phi$. Let $\tilde{M} \xrightarrow{\pi} M$ denote the universal Riemannian covering of M. π is a local isometry and $\tilde{M}$ has nonpositive curvature. Let $\tilde{\phi}_t$ denote the unique lift of ϕ_t to $\tilde{M}$ so that for $\tilde{p} \in \tilde{M}$: $\pi\circ\phi_t(\tilde{p}) = \phi_t(\pi(\tilde{p}))$. Set $\tilde{\phi} = \tilde{\phi}_1$. It is easy to see that $\tilde{\phi}$ is an isometry since $\pi\circ\tilde{\phi} = \phi\circ\pi$. $\tilde{\phi}$ commutes with deck transformations since $\pi\circ\tilde{\phi}_t = \phi_t\circ\pi$ for all t and the set of deck transformations is discrete. It follows from the compactness of M that $\text{dist}(\tilde{p},\tilde{\phi}(p))$ is a bounded function on M. We may now apply the results of J. Wolf [34] to conclude that the vector field V determined by $V_{\tilde{p}}$ = tangent vector to the minimal geodesic connecting $\tilde{p}$ and $\tilde{\phi}(\tilde{p})$ is a parallel vector field on $\tilde{M}$. Since $\tilde{M}$ has nonpositive curvature and is simply connected it follows that for any deck transformation T we have $T_*V_{\tilde{p}} = V_{T(\tilde{p})}$. Hence the vector field V projects to a parallel vector field on M. Now apply Theorem 1. □

For $n = 2$ the theorem is due to Hurwitz [18]. We have used an idea of Frankel [12] who proves a similar result.

4. As remarked in [16], it is known that the Betti numbers of compact homogeneous spaces satisfy: (a) $b_j \le \binom{\dim}{j}$ (deRham) and (b) if $b_1 \neq 0$ then $b_j \ge \binom{b_1}{j}$ (Hurewicz). We may apply the theorem of Section 1 to prove:

Corollary 1 If G/H is a homogeneous space of compact Lie groups with $b_1(G/H) \neq 0$ then the Betti numbers of G/H satisfy

$$b_{k+1} \ge b_k - b_{k-1} \quad \text{for all} \quad 1 \le k \le n-1$$

Proof: According to Samelson [29], every compact homogeneous space admits a metric with nonnegative sectional curvature $\Rightarrow$ Ricci ≥ 0. If $b_1 \neq 0$ then Bochner's theorem (see Chapter I, Section 1.7) guarantees the existence of a parallel vector field and the proof is concluded by applying Theorem 1.

Remark. It follows that no compact Lie group can act transitively on the manifolds $S(p,q,r)$ of Section 2.

5. As a final application of Theorem 1 we have

Theorem 6 If a closed n-manifold has nonzero first Betti number and satisfies

$$b_{k+1} < b_k - b_{k-1} \quad \text{for some} \quad 1 \le k \le n-1$$

then it admits no Riemannian structure with Ricci ≥ 0.

Proof: Again by Bochner's theorem, (see Chapter I, Section 1.6), Ricci ≥ 0 and nonzero first Betti number imply that M admits a parallel vector field. The conclusion then follows from Theorem 1.

<u>Remark</u>. It is a well known conjecture of Calabi [8] that a compact Kähler manifold with vanishing first Chern class admits a Ricci-flat (i.e. Ricci = 0) metric. Assuming the validity of the Calabi conjecture, Theorem 6 implies topological conditions for Kähler manifolds with vanishing canonical class. In particular, suppose a Kähler manifold M admits a nonvanishing holomorphic vector field or a free S^1 action. Then either $c_1 \neq 0$ or $b_{k+1} \geq b_k - b_{k-1}$ for all $1 \leq k \leq n-1$.

V. ELLIPTIC OPERATORS AND VECTOR FIELDS

The purpose of this chapter is to prove some results connecting the existence of some special differential forms and vector fields with topological and geometric properties of manifolds by means of the theory of elliptic operators.

1. A classical theorem of Poincaré asserts that if a 2-dimensional manifold admits a nonvanishing vector field then $\chi(M) = 0$. A 2-manifold M can be viewed as a Riemann surface and every vector field of type $(1,0)$ is of the form $Z = X - iJX$ for a real vector field X. Furthermore, for a Riemann surface the Euler characteristic is twice the arithmetic genus (cf. Chapter I, Section 2). Thus, the Poincaré result can be restated as: If a Riemann surface admits a nonvanishing $(1,0)$ vector field (or $(1,0)$-form) then its arithmetic genus $= 0$. We generalize this as follows:

<u>Theorem</u> If a complex manifold M of complex dimension $2n+1$ admits a nonvanishing $(2n+1,0)$ form then its arithmetic genus $= 0$.

<u>Corollary</u> If M admits $2n+1$ independent $(1,0)$ vector fields then arithmetic genus $= 0$.

For $n > 1$ the arithmetic genus and Euler characteristic are not connected in any simple manner.

Proof: Let $\mathcal{E}^{p,\text{even}}$ (resp. $\mathcal{E}^{p,\text{odd}}$) $\overset{\text{def}}{=} \oplus \Lambda^{p,2q}$ (resp. $\oplus \Lambda^{p,2q+1}$). Choose a hermitian metric and let $\bar\partial+\theta = D_p$: $\mathcal{E}^{p,\text{even}} \to \mathcal{E}^{p,\text{odd}}$ (see Chapter I. Section 3). By the Serre Duality Theorem (cf. [33]) $h^{p,q} = h^{n-p,n-q}$ and $\chi^p = (-1)^m \chi^{m-p}$ if $\dim M = m$. Since $m = 2n+1$: $\chi^0 = -\chi^m$: Let $\Gamma(E) = \mathcal{E}^{0,\text{even}} \oplus \mathcal{E}^{m,\text{odd}}$ and $\Gamma(F) = \mathcal{E}^{0,\text{odd}} \oplus \mathcal{E}^{m,\text{even}}$ and $\bar{\mathcal{D}} = D_0 \oplus D_m^*$. Then $\bar{\mathcal{D}}$: $E \to F$ is an elliptic operator and $\operatorname{ind}(\bar{\mathcal{D}}) = \operatorname{ind} D_0 + \operatorname{ind} D_m^* = \operatorname{ind} D_0 - \operatorname{ind} D_m = \chi^0 - \chi^m = 2\chi^0$ $= 2$ arithmetic genus. We will show that if there exists $\omega \in \Lambda^{m,0}$ and $\omega \neq 0$ then $\operatorname{ind}(\bar{\mathcal{D}}) = 0$. Let ℓ_ω denote left exterior multiplication by ω, ℓ_ω^* the adjoint operator and $\Omega = \ell_\omega \oplus \ell_\omega^*$. Then Ω: $E \to F$ and we have the diagram:

$$\begin{tikzcd}
\mathcal{E}^{0,\text{even}} \arrow[r, "D_0"] \arrow[ddr, "\ell"] & \mathcal{E}^{0,\text{odd}} \\
\oplus & \oplus \\
\mathcal{E}^{m,\text{odd}} \arrow[r, "D_m"'] \arrow[uur, "\ell^*"'] & \mathcal{E}^{m,\text{even}}
\end{tikzcd}$$

<u>Lemma</u> $\mathcal{D}^*\Omega + \Omega^*\mathcal{D} =$ differential operator of order zero.

Let $\mathcal{D}_t = \mathcal{D} + t\Omega$. Then $t \to \mathcal{D}_t$ is a continuous path of Fredholm operators (in suitable spaces) connected to $\mathcal{D}$. Hence $\operatorname{ind}(\mathcal{D}_t) = \operatorname{ind}(\mathcal{D})$. We will show that there exists a t_0 with $\operatorname{ind}(\mathcal{D}_{t_0}) = 0$. In fact, for t sufficiently large it will be seen that $\ker \mathcal{D}_{t_0} = \ker \mathcal{D}_{t_0}^* = \emptyset$.

First, observe that $\ker \mathcal{D}_t = \ker \mathcal{D}_t^* \mathcal{D}_t$ $= \ker [\mathcal{D}^*\mathcal{D} + t^2\Omega^*\Omega + t(\mathcal{D}^*\Omega + \Omega^*\mathcal{D})]$. Note that $\mathcal{D}^* = (\bar\partial + \theta)^* = (\theta + \bar\partial) = \mathcal{D}$ and $\mathcal{D}^*\mathcal{D} = \bar\Box$ while $\Omega^*\Omega = \ell^*\ell + \ell\ell^*$. If $u = u^{0,\text{even}} \oplus u^{m,\text{odd}} \in E$ then

$$\Omega^*\Omega u = \ell^*\ell\, u^{0,\text{even}} \oplus \ell\, \ell^*\, u^{m,\text{odd}} .$$

At $p \in M$, $\ell_\omega u = (\zeta_1 \wedge \dots \wedge \zeta_n)u$ for some $\zeta_j \in \Lambda^{1,0}$ with $\zeta_j \perp \zeta_k$, $j \neq k$. Then

$$\ell_\omega^* = (\ell_{\zeta_1} \circ \dots \circ \ell_{\zeta_n})^* = i_{\zeta_n} \circ \cdots \circ i_\zeta \text{ in } T_pM$$

(cf. [31]), and we have $\ell_\omega^*\ell_\omega$ = multiplication by $\Pi\, |\zeta_j|^2$ in a neighborhood of p. Similarly for $\ell_\omega^*\ell_\omega$. If we put $\mathcal{D}_t^*\mathcal{D}_t + t^2\Omega^*\Omega = A(t)$ it follows that for the usual inner product on E and F there exists $c > 0$ such that

$$\|A(t)u\|\|u\| \geq (A(t)u,u) = (\bar\Box u,u) + t^2(\Omega^*\Omega u,u)$$

$$\geq (\bar\Box u,u) + t^2 c(u,u) \geq t^2 c\|u\|^2$$

Thus $\ker A(t) = 0$. Since $A(t)$ has closed range and the same estimate shows that $\ker A^*(t) = 0$ it follows that $A(t)^{-1}$ is a bounded operator.

It follows from the estimate $\|A(t)u\| \geq ct^2\|u\|$ that $\|A(t)^{-1}\| \leq \alpha t^{-2}$, $\alpha > 0$, and by Lemma 2 we may choose $t = t_0$ such that $t_0(\mathcal{D}^*\Omega + \Omega^*\mathcal{D}) \circ A^{-1}(t_0)$ is a bounded operator C with norm < 1. Suppose $u \in \ker \mathcal{D}^*_{t_0}\mathcal{D}_{t_0}$. Then $u \in C^\infty$ and $u = A(t_0)^{-1} \in v$ and

$$0 = \mathcal{D}^*_{t_0}\mathcal{D}_{t_0}u = [A(t_0) + t_0(\mathcal{D}^*\Omega + \Omega\mathcal{D})]u$$

$$= (I + C)\ v$$

Since $\|C\| < 1$ it follows that $v = 0 \Rightarrow u = 0$.
Thus, we have shown $\ker \mathcal{D}_{t_0} = 0$. The same method works for the adjoint to show that $\ker \mathcal{D}^*_{t_0} = 0$, and we conclude that $\operatorname{ind}(\bar{\mathcal{D}}) = 2$ arithmetic genus $= 0$.

To complete the proof we prove the lemma: $\mathcal{D}^*\Omega + \Omega^*\mathcal{D}\colon E^{0,\text{even}} \oplus E^{m,\text{odd}} \to E^{0,\text{odd}} \oplus E^{m,\text{even}}$ and if $u = u^0 \oplus u^m$ then

$$(\mathcal{D}^*\Omega + \Omega^*\mathcal{D})(u^0 \oplus u^m) = [\ell(\bar\partial+\theta) + (\bar\partial+\theta)\ell]u^0 \oplus [\ell^*(\bar\partial+\theta)+\ell^*(\bar\partial+\theta)]u^m$$

$$= [R + S^*)u^0 \oplus [S + R^*]u^m$$

where $R = \ell\bar\partial + \ell\bar\partial$ = left multiplication by $\bar\partial\Omega \in \Lambda^{2n+1,1}$ and $S = (\ell^*\bar\partial + \bar\partial\ell^*)$.

R is certainly a zero order operator and we claim that the same is true of S. In fact, let $\{U\}$ be a collection of small coordinate patches on M over which E is trivial. Then, clearly, S is of order zero if S restricted to sections over U is of order zero for all U. In U we may assume that $\omega = \zeta_1 \wedge \ldots \wedge \zeta_{2n+1}$, $\zeta_j \in \Lambda^{1,0}$ and the ζ_j are orthogonal. Then $\ell^* = i_{\zeta_{2n+1}} \circ \ldots \circ i_{\zeta_1}$ (cf. Section I-1.2) and

$$S_{\mathcal{U}} = \ell^{*}\bar{\partial} + \bar{\partial}\ell^{*}\Big|_{\mathcal{U}} = (i_{\zeta_{2n+1}} \circ \ldots \circ i_{\zeta_1})\bar{\partial} + \bar{\partial}(i_{\zeta_{2n+1}} \circ \ldots \circ i_{\zeta_1})$$

By Lemma 1 of Chapter I, Section 3, $i_{\zeta_1} \circ \bar{\partial} = -\bar{\partial} i_{\zeta_1} + Z_1$ where Z_1 is a zero order operator and so

$$S = -i_{2n+1} \circ \cdots \circ i_2 \cdot \bar{\partial} \cdot i_1 + \bar{\partial} \circ i_{2n+1} \circ \cdots \circ i_1 + \text{zero order operator}$$

Continuing the transpositions $i_{\zeta_j} \leftrightarrow \bar{\partial}$ in the first term on the right-hand side we find

$$S = -\bar{\partial} \circ i_{\zeta_{2n+1}} \circ \cdots \circ i_{\zeta_1} + \bar{\partial}\zeta_{2n+1} \circ \cdots \circ i_{2n+1} + \text{zero order operator}$$

$$= \text{zero order operator.}$$

<u>Remarks</u> (1) It is possible to prove the theorem by applying some characteristic class theory and the full force of the Hirzebruch-Atiyah-Singer Riemann-Roch theorem for arbitrary compact complex manifolds (whose only proof is via the Atiyah-Singer Index theorem). The proof above is elementary. In any event, the result does not appear to have been remarked in the literature.

(2) It should be observed that the proof showed that (i) arithmetic genus = $\mathrm{ind}(\mathcal{D})$ and (ii) $\mathrm{ind}(\mathcal{D}) \le 0$ and $\mathrm{ind}(\mathcal{D}) \ge 0$. It is conceivable that similar considerations will show in some cases either that $\mathrm{ind}(\mathcal{D}) \le 0$ or $\mathrm{ind}(\mathcal{D}) \ge 0$ (e.g. when Ω is bounded below but Ω^* is not). For this reason we have avoided a different method of attack, employed by Atiyah in [1],

that can only show that an elliptic operator has index zero.

(3) Essentially the same method of proof shows that if a differentiable manifold M admits a nonvanishing vector field X then $\chi = \mathrm{ind}(D) = \mathrm{ind}(D + \ell + i) \leq 0$ and ≥ 0 i.e. $= 0$. Here D is the operator $D = d + \delta\colon \oplus \Lambda^{2p} \to \oplus \Lambda^{2p-1}$.

2. In this section we give a proof of Hurwitz' theorem cited in Chapter I.

<u>Theorem</u> (Hurwitz-Bochner). If M is a closed 2-manifold with $\chi(M) < 0$, then the isometry group of M is finite for any Riemannian structure.

Proof: Let g be any metric on M. It will be shown that $\chi < 0$ implies that (M,g) admits no Killing vectors. It then follows from the compactness of the isometry group that the group is finite.

Suppose that (M,g) did admit a Killing vector K. Let $\omega_K \overset{\mathrm{def}}{=} g(K,\ )$ be the covariant form of K, and consider the differential operator $D_k = d + \delta + \ell_\omega + i_K\colon \Lambda^0 \oplus \Lambda^2 \to \Lambda^1$. Since $\ell_\omega + i_K$ is a differential operator of order zero D_K is a Fredholm operator of order 1 and $\mathrm{ind}(D_K) = \mathrm{ind}(d+\delta) = \chi < 0$ (cf. Chapter I, Section 3). Now $\mathrm{ind}(D_K) < 0$ implies that dim cokernel $D_K = \dim\ker D_K^* > 0$ so that there exists $u \in \Lambda^1$ such that $D_K^* u = 0$. $D_K^* = (d + \delta)^* + (\ell_\omega + i_K)^* = \delta + d + i_K + \ell_\omega = D$ since i_K is the adjoint of left multiplication by ℓ_ω (cf.I-1.2). Thus $u \in \ker D_K^*$ iff $u \in \ker D_K D_K^*$ iff $D_K D_K^* u = [(d+\delta)^2 + (\ell+i)^2 + di + id + \delta\ell + \ell\delta + d\ell + \ell d + i\delta + \delta i]u =$

$= (\Delta + L_X + L_X^t + A + A^t + |K|^2)u$ where $Au = d\xi \wedge u$.
By Lemma 2 of Chapter I and the fact that $d\xi \wedge u = 0$ we have $(\Delta + |K|^2)u = 0$. Taking the inner product of both sides with u:

$$0 = (\Delta u,u) + \int (|K|^2) \wedge *u = |du|^2 + |\delta u|^2 + \int |K|^2(u \wedge *u)$$

It follows that $\int |K|^2 u \wedge *u = 0$ and since the set zero(K) on which K vanishes has no interior (since K satisfies an elliptic equation with Δ = highest order term [5]) it follows that $u \wedge *u = 0$ a.e. Since $u \in C^\infty$ we have $u \equiv 0$. The proof is complete.

<u>Remark</u> Recall that the Hsiang <u>degree of symmetry</u> of a manifold M is the maximal dimension of compact subgroup of Diff(M). Since any such compact subgroup can be made to act by isometries it follows that if $\chi(M^2) < 0$ the Hsiang degree of symmetry of M^2 is zero. This point of view is of interest in view of the fact that Atiyah and Hirzebruch [2] have shown that the degree of symmetry of a 4k-dimensional spin-manifold is zero if its $\hat{A}$-genus is nonzero.

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