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Analyse fonctionnelle/*Functional Analysis*

Index theory and Galois theory for infinite index inclusions of factors

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Abstract — We study inclusions of von Neumann algebras in the case when the index is infinite. Some of the main notions of the index theory initiated by Jones such as the standard projections and Pimsner-Popa bases can be extended to this context. The corresponding structural results, such as Takesaki duality, hold.

We develop in the infinite index context results from the Galois theory for algebra inclusions of the second author and obtain an intrinsic characterization of twisted crossed products by a discrete group in terms of the position of the initial algebra in the crossed product algebra.

Théorie de Galois et de l'indice de Jones pour les inclusions de facteurs

Résumé — On étudie des inclusions d'algèbres de von Neumann dans le cas où l'indice est infini. Dans ce contexte on peut généraliser les constructions les plus importantes de la théorie de l'indice de Jones, telles que les projections standard et les bases de Pimsner-Popa. Des résultats structuraux correspondants, comme la dualité de Takesaki, restent vrais dans ce contexte.

On étend au cas d'indice infini des résultats de la théorie de Galois pour les inclusions d'algèbres du second auteur, et on obtient une caractérisation intrinsèque du produit croisé par un groupe discret en termes de la position de l'algèbre initiale dans l'algèbre produit croisé.

Version française abrégée — On commence par l'étude des inclusions des algèbres de von Neumann d'indice infini. On travaille avec des facteurs semi-finis $\mathcal{M} \supseteqq \mathcal{N}$ à predual séparable ayant des traces tr et τ telles qu'il existe un poids opératoriel T avec $\mathrm{tr} = \tau \circ \mathrm{T}$. Dans ces conditions il existe une base de $\mathcal{M}$ comme module sur $\mathcal{N}$. Si $\mathcal{M}$ est un facteur fini et si l'indice est fini on retrouve la base de Pimsner et Popa.

THÉORÈME. — *Il existe une collection au plus dénombrable d'éléments* $\{\lambda_i\}$ *de* $\mathcal{M}$ *telle que chaque* $x \in \mathcal{M}$ *vérifie* $x = \sum \mathrm{T}(x\lambda_i^*)\lambda_i$.

On utilise un procédé de type Gram Schmidt par rapport à la forme bilinéaire $(x, y) \to \mathrm{T}(xy^*)$. Pour v dans $\mathcal{M}$ bien choisi, tout élément r de $[\mathcal{N} v]$ s'écrit comme $r = \mathrm{T}(r\lambda^*)\lambda$, avec $\lambda = \mathrm{T}(vv^*)^{-1/2} v$.

INCLUSIONS DISCRÈTES ET COMPACTES. — Une inclusion de facteurs semi-finis $\mathcal{M}_0 \subseteqq \mathcal{M}_1$ est appelée discrète si la trace de $\mathcal{M}_1$ se restreint (à un scalaire près) en la trace de $\mathcal{M}_0$. L'inclusion $\mathcal{M}'_0 \subseteqq \mathcal{M}_1$ est appelée compacte si, pour l'extension canonique $\mathcal{M}_0 \subseteqq \mathcal{M}_1 \subseteqq \mathcal{M}_2$, l'inclusion $\mathcal{M}_1 \subseteqq \mathcal{M}_2$ est discrète. Une inclusion à la fois discrète et compacte est d'indice fini.

THÉORÈME. — *Soit* $\mathcal{M}_1$ *le produit croisé de* $\mathcal{M}_0$ *par un groupe localement compact abélien* G. *L'inclusion* $\mathcal{M}_0 \subseteqq \mathcal{M}_1$ *est discrète* (*compacte*) *si et seulement si* G *est discret* (*compact*).

On caractérise de manière directe les inclusions compactes comme suit.

THÉORÈME. — *L'inclusion de facteurs semi-finis* $\mathcal{M}_0 \subseteqq \mathcal{M}_1$ *est compacte si et seulement si pour une* (*et alors pour toute*) *base* $\{\lambda_i\}$ *de* $\mathcal{M}_1$ *sur* $\mathcal{M}_0$ *la somme* $\sum \lambda_{i*}\lambda_i$ *est finie.*

L'argument principal est que $\mathcal{M}_2$, définie comme $\mathrm{J}_{\mathcal{M}_1} \mathcal{M}'_0 \mathrm{J}_{\mathcal{M}_1}$ sur l'espace de Hilbert standard de $\mathcal{M}_1$, est égale au produit croisé de $\mathcal{M}_1$ avec elle-même au-dessus de $\mathcal{M}_0$. Avec ce résultat on montre que dans la tour $(\mathcal{M}_n)$ la propriété d'être discrète (compacte) est de période 2 par rapport à n.

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APPLICATION AUX PRODUITS CROISÉS. — On obtient la caractérisation intrinsèque suivante du produit tensoriel par un groupe discret.

THÉORÈME. — *Soient* $\mathcal{M}_0 \subseteqq \mathcal{M}_1$ *une inclusion de facteurs et* $\mathcal{M}_0 \subseteqq \mathcal{M}_1 \subseteqq \mathcal{M}_2 \subseteqq \mathcal{M}_3$ *son extension canonique. L'algèbre* $\mathcal{M}_1$ *est un produit croisé de* $\mathcal{M}_0$ *par une action extérieure perturbée par un cocycle d'un groupe discret si et seulement si* $\mathcal{M}'_0 \cap \mathcal{M}_1$ *est scalaire,* $\mathcal{M}'_0 \cap \mathcal{M}_2$ *est abélienne et* $\mathcal{M}'_0 \cap \mathcal{M}_3$ *est un facteur.*

In the first part of this Note we study inclusions of von Neumann algebras for which the index is infinite. We extend to this framework some of the main results of the theory initiated by Jones [4] and developed in [10] and [2]. Previous generalizations of the theory by Kosaki [5] and Phan Loi [6] study the case of general algebras with finite index. The principal role in our study is not played by the algebras, which are kept semifinite. Instead we study for the first time what happens when the index becomes infinite. The main ingredients of the finite index theory, such as the Jones projections, no longer make sense in this context. We show however, using appropriate substitutes, that the main relations and technical properties of the finite index constructions can be generalized to the infinite index setting. Consider as a model the case of a sequence of inclusions with finite index in which the index tends to infinity. The trace of the Jones projection tends to zero and the corresponding idempotent does not exist in the infinite index case. A scalar multiple of the Jones projection, normalized as in statistical mechanics [15] so that its trace is one, would yield at the limit an operator-valued weight [3] satisfying an analogue of the Jones relations. We then show that the basis of Pimsner and Popa [10] can be obtained by a Gram-Schmidt orthogonalization process with respect to an operator valued inner product and is naturally associated to modules rather than algebra inclusions. The spanning property of the Jones projections is replaced by the density of a tensor product construction as in Connes [1] and Sauvageot [12]. Using these tools, we prove that properties such as discreteness and compactness of inclusions, which we introduce in this framework, display a period two occurrence in the tower of extensions which generalizes Takesaki duality [14].

The second part of this Note develops in the infinite index context results from the Galois theory for algebra inclusions of the second author ([7], [8]). As in that context, we obtain an intrinsic characterization of the twisted crossed product by a discrete group in terms of the position of the initial algebra in the crossed product algebra. We use, however, a different approach, in which information coming from the tower of relative commutants is translated into the property that the normalizer of the initial algebra in the larger algebra is spanning. This yields then the crossed product structure following Sutherland [12].

GENERAL THEORY. — We take as our setting a pair of semifinite factors $\mathcal{M} \supseteqq \mathcal{N}$ with traces tr and τ related by an operator valued weight T such that $\mathrm{tr} = \tau \circ \mathrm{T}$. Under the condition that the predual be separable we prove the existence of a basis for $\mathcal{M}$ over $\mathcal{N}$.

THEOREM 1. — *There exists a collection of elements* $\{ \lambda_i \}$ *in* $\mathcal{M}$*, at most countable, such that every* x *in* $\mathcal{M}$ *can be written as* $x = \sum \mathrm{T}(x \lambda_i^*) \lambda_i$. *Given any such basis* $\{ \lambda_i \}$*, the matrix* $(\mathrm{T}(\lambda_i \lambda_j^*))_{ij}$ *is a projection in the matrices over* $\mathcal{N}$*, uniquely determined up to projection equivalence. Moreover if the index is an integer or infinite, the basis can be chosen to satisfy* $\mathrm{T}(\lambda_i \lambda_j^*) = \delta_{ij} \mathrm{I}$.

In the case where $\mathcal{M}$ is a finite factor there exists a projection $E_{\mathcal{N}}$ from $\mathcal{M}$ onto $\mathcal{N}$ and corresponding Hilbert space projection $e_{\mathcal{N}}$. In that case the basis is the one constructed by Pimsner and Popa.

THEOREM 1′. — *Each element x in* M *can be written as* $x=\sum E_{\mathcal{N}}(x\lambda_i^*)\lambda_i$ *with respect to a basis* $\{\lambda_i\}$ *as in the Theorem* 1. *Moreover the basis satisfies* $\sum \lambda_i^* e_{\mathcal{N}} \lambda_i = I$.

The proof of Theorem 1′ relies on the following result.

PROPOSITION 2. — *If n is the left ideal associated to an operator valued weight between factors* $\mathcal{M}$ *and* $\mathcal{N}$ *on a separable Hilbert space, then the identity can be written as* $I=\sum p_i$, *where the* p_i *are in n.*

This proposition is applied to the pair of factors $\mathcal{M}$ and $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ where the latter is the fundamental construction of Jones [4]. In the case of infinite index $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ is a semifinite non-finite factor.

The proof of Theorem 1 is based on a Gram-Schmidt procedure for choosing an orthogonal basis in $\mathcal{M}$ as a module over $\mathcal{N}$ with respect to the bilinear form $(x, y) \to T(xy^*)$. The argument for a cyclic submodule uses the following idea, which can be made into a formal argument. For v in $\mathcal{M}$ carefully chosen, any element r in the left $\mathcal{N}$-module $[\mathcal{N} v]$ can be written as $r=T(r\lambda^*)\lambda$, where $\lambda=T(vv^*)^{-1/2}v$. The proof follows from the equality

$$T(xvv^*T(vv^*)^{-1/2})T(vv^*)^{-1/2}v=xT(vv^*)T(vv^*)^{-1/2}T(vv^*)^{-1/2}T(vv^*)^{-1/2}v=xv.$$

With additional arguments one gets $T(\lambda_i\lambda_j^*)=\delta_{ij}I$.

To show the relation between two bases $\{\lambda_i\}$ and $\{\underline{\lambda}_i\}$, we express one in terms of the other as $\lambda_i=\sum T(\lambda_i\underline{\lambda}_j^*)\underline{\lambda}_j$. Then the matrix $(T(\lambda_i\underline{\lambda}_j^*))_{ij}$ is a partial isometry W and $WW^*=(T(\lambda_i\lambda_j^*))_{ij}$ and $W^*W=(T(\underline{\lambda}_i\underline{\lambda}_j^*))_{ij}$ are the support projections for the respective bases.

DISCRETE AND COMPACT INCLUSIONS. — We study two different types of inclusions, modelled on the idea of crossed products by outer actions of compact and discrete groups. The test used is whether the trace on the algebra extends to a semifinite trace on the crossed product.

Given an inclusion of semifinite factors $\mathcal{M}_0 \subseteqq \mathcal{M}_1$ we call the inclusion *discrete* if the trace of $\mathcal{M}_1$ restricts (up to a scalar multiple) to that of $\mathcal{M}_0$, or equivalently, if there exists a conditional expectation of $\mathcal{M}_1$ onto $\mathcal{M}_0$ preserving the trace. The inclusion $\mathcal{M}_0 \subseteqq \mathcal{M}_1$ will be called *compact* if upon forming the standard extension $\mathcal{M}_0 \subseteqq \mathcal{M}_1 \subseteqq \mathcal{M}_2$, the inclusion $\mathcal{M}_1 \subseteqq \mathcal{M}_2$ is discrete. An inclusion is both discrete and compact if and only if it has finite index, which is shown using the lower bound for conditional expectations ([11], Theorem 2.2).

These definitions are justified by the following properties of crossed products, stated below for abelian groups.

THEOREM 3. — *Suppose that* $\mathcal{M}_1$ *is the crossed product of* $\mathcal{M}_0$ *by a locally compact abelian group* G. *The inclusion* $\mathcal{M}_0 \subseteqq \mathcal{M}_1$ *is discrete if and only if the group* G *is discrete. The inclusion* $\mathcal{M}_0 \subseteqq \mathcal{M}_1$ *is compact if and only if the group* G *is compact.*

To prove that the group must be compact if the inclusion is, we use the existence of the conditional expectation from $\mathcal{M}_2$ onto $\mathcal{M}_1$ to obtain a finite Haar measure. It is interesting to verify that if u_t are the unitaries in the crossed product implementing the given action, then the corresponding spectral projections serve as a basis, albeit not the obvious one when the group is finite.

We now provide a global condition on the basis for $\mathcal{M}_1$ over $\mathcal{M}_0$ such that the inclusion is compact.

THEOREM 4. — *Given an inclusion of semifinite factors $\mathcal{M}_0 \subseteq \mathcal{M}_1$, the inclusion is compact if for a given (and hence for any) basis $\{\lambda_i\}$ for $\mathcal{M}_1$ over $\mathcal{M}_0$ the sum $\sum \lambda_i^* \lambda_i$ is finite.*

The first argument in the proof is the fact that the extension $\mathcal{M}_2$, defined as $J_{\mathcal{M}_1} \mathcal{M}'_0 J_{\mathcal{M}_1}$ on the standard Hilbert space of $\mathcal{M}_1$, is equal to a tensor product of $\mathcal{M}_1$ with itself over $\mathcal{M}_0$.

Suppose that $\mathcal{M}_0 \subseteq \mathcal{M}_1$ is a pair of semifinite factors with traces τ_0 and τ_1. Let n_T be the dense ideal in $\mathcal{M}_1$ for the operator valued weight T (in the sense of [3]), relating the traces of $\mathcal{M}_1$ and $\mathcal{M}_0$. On the Hilbert space of $\mathcal{M}_1$ we consider the operators $y \otimes x$ where y, x belong to n_T, n_T^* respectively. Their operation is defined by $y \otimes x(z) = y\,T(xz)$ for z in $\mathcal{M}_1$. These operators extend to bounded operators on the whole Hilbert space. Denote the algebra generated by these operators by $\mathcal{M}_1 \otimes_{\mathcal{M}_0} \mathcal{M}_1$, extending the corresponding notion in [2]. Then we have

THEOREM 5. — *$\mathcal{M}_2 = J\mathcal{M}'_0 J = \mathcal{M}_1 \otimes_{\mathcal{M}_0} \mathcal{M}_1$, where $J = J_{\mathcal{M}_1}$ is the conjugate linear isometry such that $J\mathcal{M}_1 J = \mathcal{M}'_1$.*

To get a trace on the semifinite factor $\mathcal{M}_1 \otimes_{\mathcal{M}_0} \mathcal{M}_1$ we use the basis for $\mathcal{M}_1$ over $\mathcal{M}_0$. In general the element $\sum \lambda_i^* \lambda_i$ is a scalar, which may be infinite.

THEOREM 6. — *The map $y \to \sum \tau_0(T(\lambda_i y(\lambda_i^*)))$ defines a faithful, semifinite normal trace on $\mathcal{M}_1 \otimes_{\mathcal{M}_0} \mathcal{M}_1$. This trace restricts to a multiple of the trace on $\mathcal{M}_1$ if and only if $\sum \lambda_i^* \lambda_i$ is finite. Moreover the map $y \to (\Theta_{ij}(y))_{ij}$, where $\Theta_{ij}(y) = T(\lambda_i y(\lambda_j^*))$, provides an isomorphism between $\mathcal{M}_1 \otimes_{\mathcal{M}_0} \mathcal{M}_1$ and* $\mathrm{Mat}(\mathcal{M}_0)$.

Remark. — With $\{\lambda_i\}$ a basis for $\mathcal{M}_1$ over $\mathcal{M}_0$ it is possible to write down a basis for $\mathcal{M}_1 \otimes_{\mathcal{M}_0} \mathcal{M}_1$ over $\mathcal{M}_1$. In the finite index case this is ([7], [11]), up to a scalar, $\{e\lambda_i\}$, where e is the projection corresponding to the conditional expectation of $\mathcal{M}_1$ onto $\mathcal{M}_0$. In the infinite index case we have no such e, but in the tensor product picture $\{p_k \otimes \lambda_i\}$ is a basis, with projections p_k chosen as in Proposition 2, and the operator valued weight T from $\mathcal{M}_1 \otimes_{\mathcal{M}_0} \mathcal{M}_1$ to $\mathcal{M}_1$ is given by $T(y \otimes x) = yx$.

With these comments we do the same construction for the pair $\mathcal{M}_1 \subseteq \mathcal{M}_2$.

THEOREM 7. — *Following the same construction of the map as in Theorem 6, we obtain an isomorphism of $\mathcal{M}_3$ with* $\mathrm{Mat}(\mathcal{M}_1)$. *Moreover the restriction of this map to $\mathcal{M}_2$ coincides with the map of Theorem* 6.

This shows that in the tower of extensions discreteness and compactness have period two. The inclusion $\mathcal{M}_0 \subseteq \mathcal{M}_1$ is discrete if and only if $\mathcal{M}_1 \subseteq \mathcal{M}_2$ is compact, and the inclusion $\mathcal{M}_0 \subseteq \mathcal{M}_1$ compact if and only if $\mathcal{M}_1 \subseteq \mathcal{M}_2$ is discrete.

APPLICATIONS TO CROSSED PRODUCTS. — We give a characterization of twisted crossed products by discrete groups using the tools developed above. This extends the characterization obtained by one of us [7] for the case of finite index.

THEOREM 8. — *Let $\mathcal{M}_0 \subseteq \mathcal{M}_1$ be a discrete inclusion. Then $\mathcal{M}_1$ is a cocycle twisted crossed product of $\mathcal{M}_0$ by an outer action of a discrete group if and only if $\mathcal{M}'_0 \cap \mathcal{M}_1$ is scalar, $\mathcal{M}'_0 \cap \mathcal{M}_2$ is abelian and $\mathcal{M}'_0 \cap \mathcal{M}_3$ is a factor.*

The group is made to appear as unitaries normalizing $\mathcal{M}_0$ by using an extension of a proposition of Pimsner and Popa [10] relating such unitaries to projections in $\mathcal{M}'_0 \cap \mathcal{M}_2$. Of course the necessity of these conditions is well known [14]. More generally we conjecture the following characterisation if we drop the condition that the relative commutant of $\mathcal{M}_0$ in $\mathcal{M}_2$ be abelian.

CONJECTURE. — *Let $\mathcal{M}_0 \subseteqq \mathcal{M}_1$ be a discrete inclusion. Then $\mathcal{M}_1$ is a cocycle twisted crossed product of $\mathcal{M}_0$ by an outer action of a discrete Kac algebra if and only if $\mathcal{M}'_0 \cap \mathcal{M}_1$ is scalar and $\mathcal{M}'_0 \cap \mathcal{M}_3$ is a factor.*

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