L^p-SPACES ASSOCIATED WITH AN ARBITRARY VON NEUMANN ALGEBRA

by

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Abstract: To any von Neumann algebra M, we associate Banach spaces $L^p(M)$, $1 \leq p \leq \infty$, which generalize the classical Banach spaces $L^p(\Omega,\mu)$ of functions on a measure space (Ω,μ). We show that $L^\infty(M) \cong M$, $L^1(M) \cong M_*$, and that $L^2(M)$ is isomorphic to the Hilbert space of M in its standard form. When M is semifinete, the $L^p(M)$-spaces are isometric isomorphic to the spaces $L^p(M,\tau)$ introduced by Dixmier, Segal and Kunze in 1953-1958. The $L^p(M)$-spaces are constructed as certain spaces of unbounded operators affiliated with the crossed product $R(M,\sigma^\varphi)$ of M with the modular automorphism group associated with a fixed weight φ on M. The construction turns out to be independent (up to unitary equivalence) of the choice of φ.

RESUMÉ A toute algèbre de Von Neumann M nous associons des espaces de Banach $L^p(M)$, $1 \leq p \leq \infty$, qui généralisent les espaces de Banach classiques $L^p(\Omega,\mu)$ de fonctions sur un espace mesuré (Ω,μ). Nous montrons que $L^\infty(M) \cong M$, $L^1(M) \cong M_*$, et que $L^2(M)$ est isomorphe à l'espace de Hilbert de la représentation standard. Si M est semi-finie les espaces $L^p(M)$ sont isométriquement isomorphes aux espaces $L^p(M,\tau)$ introduits par Dixmier, Segal et Kunze en 1953-1958. Les espaces $L^p(M)$ sont construits comme espaces d'opérateurs non bornés affiliés aux produits croisés $R(M,\sigma^\varphi)$ de M avec l'automorphisme modulaire associé à un poids fixe φ sur M. La construction s'avère indépendante (à une équivalence unitaire près) du choix de φ.

Introduction

This note contains an outline of a forthcoming paper.

In [4], [11] and [8] J. Dixmier, I. Segal and R. Kunze have constructed the L^p-spaces $L^p(M,\tau)$ associated with a semifinite von Neumann algebra M, which generalize the classical Banach spaces $L^p(\Omega,\mu)$. The L^p-spaces we construct in this note will consist of operators affiliated not with M itself but with a bigger algebra, namely the crossed product $M_0 = R(M,\sigma^\varphi)$ of M with a modular automorphism group. M_0 has a trace τ satisfying $\tau\circ\theta_s^\varphi = e^{-s}\tau$ where θ_s^φ is the dual action. $L^p(M)$ is defined as the set of τ-measurable operators h affiliated with M_0 satisfying

$$\begin{cases} \theta_s^\varphi\, h = \exp(-\frac{s}{p})h & p < \infty \\ \theta_s^\varphi\, h = h & p = \infty \end{cases}$$

equipped with a suitable norm. Since the triple $(M_0, \tau, \theta^\varphi)$ is independent (up to unitary equivalence) of the choice of φ, the L^p-spaces are independent of φ.

We have $L^\infty(M) = M$ and $L^1(M) \simeq M_*$. $L^2(M)$ is a Hilbert space, and the representation of $L^\infty(M)$ on $L^2(M)$ defined by left multiplication is standard. If M is semifinite the L^p-spaces constructed in this way are isometric- and orderisomorphic to $L^p(M,\tau_0)$ for any n.f.s. trace τ_0 on M.

§ 1 Construction of the L^p-spaces

Let M be a von Neumann algebra. We will identify M with its natural injection in the crossed product $M_o = R(M, \sigma^{\varphi_o})$ where φ_o is a fixed weight on M. By construction M_o is generated by M and a one parameter group of unitaries $\lambda(t)$ such that for $x \in M$, $\sigma_t^{\varphi_o}(x) = \lambda(t) x \lambda(t)^*$.

Let T be the operator valued weight, $T: M_o^+ \to \hat{M}_+$, given by

$$T(x) = \int_{\hat{R}} \theta_s(x)\, ds\ , \quad x \in M_o^+ .$$

where $\theta_s = \theta_s^{\varphi_o}$ is the dual action on M_o (cf [6],[7]). The weight $\varphi_o \circ T$ on M_o is 2π times the dual weight to φ_o. Hence there exists a trace τ on M_o, such that $\varphi_o \circ T = \tau(h\cdot)$ where h is the positive selfadjoint operator affiliated with M_o determined by $h^{it} = \lambda(t)$ (cf [12], proof of lemma 8.2). The trace τ satisfies

$$\tau \circ \theta_s = e^{-s}\tau, \ s \in \hat{R}$$

1.1 Definition

For any normal semifinite weight φ on M we put $\tilde{\varphi} = \varphi \circ T$, and we let h_φ be the Radon-Nikodym derivative of $\tilde{\varphi}$ with respect to the trace τ on M_o, i.e. $\tilde{\varphi} = \tau(h_\varphi \cdot)$.

1.2 Theorem

1) The set $\{h_\varphi \mid \varphi$ normal, semifinite$\}$ is equal to the set of positive selfadjoint operators h affiliated with M_o, which satisfies
$$\theta_s h = e^{-s} h \qquad s \in \hat{R}$$

2) If $\int_0^\infty \lambda\, de_\lambda^\varphi$ is the spectral decomposition of h_φ then
$$\tau((e_\lambda^\varphi)^\perp) = \tfrac{1}{\lambda}\varphi(1)\ , \quad \lambda > 0.$$

   In particular h_φ is τ-measurable iff φ is bounded (cf [9] p. 111).

1.3 Theorem

The map $\varphi \to h_\varphi$, $\varphi \in M_*^+$ has a unique extension to a linear map of M_* onto the set of τ-measurable operators h affiliated with M_o, satisfying

$$\theta_s h = e^{-s} h .$$

(Note that the set of τ-measurable operators on M_o is an algebra with respect to strong sum and strong product, cf [11]).

1.4 Definition

i) We let $L^1(M)$ denote the set of τ-measurable operators h affiliated with M_o, for which $\theta_s h = e^{-s}h$, $s \in \hat{R}$.

ii) We define a linear functional tr on $L^1(M)$ by $\mathrm{tr}(h_\varphi) = \varphi(1)$.

1.5 Proposition

The map $\varphi \to h_\varphi$ of M_* onto $L^1(M)$ is an isometry with respect to the norm $\| h\|_1 = \mathrm{tr}(|h|)$ on $L^1(M)$.

1.6 Remarks

a) The trace τ is infinite on any non vanishing operator in $L^1(M)_+$.

b) If M is a factor of type III_1, M_o is a II_∞- factor. In this case any normal trace on M_o is proportional to τ. Hence tr is not in general the restriction of a trace on M_o.

1.7 Definition

We put $L^p(M) = \{h, \tau\text{-measurable aff. with } M_o \mid \theta_s h = \exp(-\frac{s}{p})h\}$
and $L^\infty(M) = \{h, \tau\text{-measurable aff. with } M_o \mid \theta_s h = h\}$

1.8 Remarks

a) If $p \neq q$ then $L^p(M) \cap L^q(M) = \{0\}$.

b) If $p < \infty$ then any non vanishing L^p- operator is unbounded.

c) $L^\infty(M)$ consists only of bounded operators. Hence

$$L^\infty(M) = \{h \in M_o \mid \theta_s h = h, \ s \in \hat{R}\} = M .$$

1.9 Proposition

Let $p \in [1,\infty[$ and let a be a closed, densely defined operator affiliated with M_o, and let $a = u|a|$ be its polar decomposition. The following conditions are equivalent:

(1) $a \in L^p(M)$

(2) $u \in L^\infty(M)$ and $|a|^p \in L^1(M)$.

1.10 Definition

On $L^p(M)$ we define $\| \ \|_p$ by

$$\|a\|_p = \mathrm{tr}(|a|^p)^{\frac{1}{p}} \qquad p < \infty$$

$$\|a\|_\infty = \|a\|$$

For $p = 1,\infty$ $\| \ \|_p$ is a norm (cf. prop. 1.5). It will be proved later, that $\| \ \|_p$ is also a norm for $1 < p < \infty$.

1.11 Lemma

Let $h,k \in L^1(M)_+$. The function $\alpha \to h^\alpha k^{1-\alpha} \in L^1(M)$ is analytic in the open strip $\{\alpha \in \mathbb{C} \mid \operatorname{Re}\alpha \in]0,1[\}$.

1.12 Proposition

Let $p,q \in [1,\infty]$, $\frac{1}{p} + \frac{1}{q} = 1$, and let $a \in L^p(M)$ and $b \in L^q(M)$, then $ab,\ ba \in L^1(M)$ and $\operatorname{tr}(ab) = \operatorname{tr}(ba)$.

1.13 Corollary

(1) For any $h \in L^1(M)$ and any unitary $u \in L^\infty(M)$:

$$\operatorname{tr}(uhu^*) = \operatorname{tr}(h)$$

(2) For any $x \in L^2(M)$: $\operatorname{tr}(x^*x) = \operatorname{tr}(xx^*)$

1.14 Theorem

Let $p,q \in [1,\infty]$, $\frac{1}{p} + \frac{1}{q} = 1$, and let $a \in L^p(M)$ and $b \in L^q(M)$, then

$$\|ab\|_1 \le \|a\|_p \|b\|_q \quad \text{(Hölders inequality)}$$

1.15 Remarks

The proof of Theorem 1.14 is based on lemma 1.11 and the three line theorem for analytic functions (compare with [9]p. 113). Dixmiers proof of Hölders inequality in [4] can not be applied, because in our case the spectral projections of an L^p-operator is not in L^p.

1.16 Proposition

(1) Let $p,q \in [1,\infty]$, $\frac{1}{p} + \frac{1}{q} = 1$. For any $a \in L^p(M)$

$$\|a\|_p = \sup\{|\operatorname{tr}(ab)| \mid b \in L^q(M),\ \|b\|_q \le 1\}$$

(2) For $a,b \in L^p(M)$ $\|a+b\|_p \le \|a\|_p + \|b\|_p$. (Minkowskis inequality)

Hence $\|\ \|_p$ is a norm.

1.17 Proposition

(1) For any $p \in [1,\infty]$ the norm topology on $L^p(M)$ is equal to the topology of convergence in measure (cf [9] p. 106).

(2) For any $p \in [1,\infty]$ $L^p(M)$ is comlete in the p-norm.

(3) $L^2(M)$ is a Hilbert space with inner product $(a|b) = \operatorname{tr}(b^*a)$.

1.18 Lemma (cf. [4] lemma 5 p.30)

Let $p \in [2, \infty[$. For $a, b \in L^p(M)$

$$\|a + b\|_p^p + \|a - b\|_p^p \leq 2^{p-1}(\|a\|_p^p + \|b\|_p^p)$$

The proof of lemma 1.18 is based on lemma 1.11 and the three line theorem.

1.19 Theorem

Let $p \in [1, \infty[$ and put $q = \frac{p}{p-1}$. An operator $a \in L^q(M)$ defines a functional φ_a on $L^p(M)$ by $\varphi_a(x) = tr(ax)$. The map $a \rightarrow \varphi_a$ is an isometric isomorphism of $L^q(M)$ onto the dual Banach space of $L^p(M)$.

Proof: Same as [4] proof of theorem 7.

1.20 Proposition

Let $p, q \in [1, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1$ and let $a \in L^q(M)$. Then

$$a \geq 0 \iff tr(ab) \geq 0 \quad \forall\, b \in L^p(M)_+$$

i.e. the partial ordering of $L^q(M)_{sa}$ is the dual of the ordering of $L^p(M)_{sa}$. (sa = selfadjoint).

For $a \in M = L^\infty(M)$ and $x \in L^2(M)$ we put

$$\lambda(a)x = ax$$
$$\rho(a)x = xa .$$

1.21 Theorem

(1) λ (resp. ρ) is a normal, faithful representation (resp. anti-representation) of M on the Hilbert space $L^2(M)$.

(2) The von Neumann algebras $\lambda(M)$ and $\rho(M)$ are commutants of each other, and

$$\rho(M) = J\lambda(M)J$$

where J is the conjugate linear isometry $x \rightarrow x^*$ in $L^2(M)$.

(3) The quadruple $(\lambda(M), L^2(M), J, L^2(M)_+)$ is a standard form in the sense of [5].

§2 The semifinite case

Let M be a semifinite von Neumann algebra on a Hilbert space H, and let τ_0 be a n.f.s. trace on M. Identifying $L^2(\mathbb{R},H)$ with $H\otimes L^2(\mathbb{R})$ we have:

$$R(M, \sigma^{\tau_0}) = M \otimes U(\mathbb{R})$$

where $U(\mathbb{R})$ is the von Neumann algebra associated with the left regular representation of the group $\mathbb{R}$. Let F denote the Fourier-Plancherél operator $L^2(\mathbb{R}) \to L^2(\hat{\mathbb{R}})$

$$(Ff)(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} f(t)\, dt$$

then
$$U(\mathbb{R}) = F^* L^\infty(\hat{\mathbb{R}})F$$

where $L^\infty(\hat{\mathbb{R}})$ acts as multiplication operators on $L^2(\hat{\mathbb{R}})$. Hence

$$R(M,\sigma^{\tau_0}) = M \otimes F^*L^\infty(\hat{\mathbb{R}})F.$$

For any borel function $f(s)$ on $\hat{\mathbb{R}}$ we let $m(f)$ denote the closed, densely defined multiplication operator $g \to fg$ on $L^2(\hat{\mathbb{R}})$.

2.1 Theorem

Let $p\in[1,\infty[$. If $a\in L^p(M,\tau_0)$ then $a \otimes F^*m(\exp(\frac{s}{p}))F \in L^p(M)$ and the map $a \to a\otimes F^*m(\exp(\frac{s}{p}))F$ is an isometry of $L^p(M,\tau_0)$ onto $L^p(M)$.

§3 Applications to von Neumann algebras with a periodic weight.

Let M be a von Neumann algebra with a periodic NSF-weigt φ_0, and let T_0 be a period for φ_0, i.e. $\sigma^{\varphi_0}_{T_0}= 1$. Put $G = \mathbb{R}/T_0\mathbb{Z}$ and let $t \to \dot{t}$ be the quotient map $\mathbb{R} \to G$. Let α be the action $\alpha: G \to \operatorname{aut}(M)$ defined by $\alpha(\dot{t}) = \sigma^{\varphi_0}_t$. We will identify M with its injection in the crossed product $M_1 = R(M,\alpha)$. M_1 is generated by M and a group of unitaries $\lambda(g)$, $g\in G$, such that

$$\sigma^{\varphi_0}_t(x) = \lambda(\dot{t})x\lambda(\dot{t}) \qquad x\in M.$$

Let S denote the operator valued weight $M_1^+ \to \hat{M}_+$ given by

$$S(x) = \sum_{n=-\infty}^{\infty} \theta^n(x) \qquad x\in M_1^+$$

where θ^n, $n\in\mathbb{Z}$, is the dual action of $\hat{G}\simeq\mathbb{Z}$ on M_1. (cf. [7]).

Repeating the arguments from the start of §1 we get that M_1 has a (unique) n.f.s. trace τ such that $\varphi_0 \circ S = \tau(k\cdot)$ where k is the positive selfadjoint operator affiliated with M_1 determined by $k^{it} = \lambda(\dot{t})$, $t \in \mathbb{R}$. The trace τ satisfies:

$$\tau \circ \theta = \lambda \tau \quad \text{where} \quad \lambda = \exp(-\tfrac{2\pi}{T_0}).$$

3.1 Definition

For any normal semifinite weight φ on M, we put $\tilde{\varphi} = \varphi \circ S$, and we let k_φ be the Radon-Nikodym derivative of $\tilde{\varphi}$ with respect to τ, i.e. $\tilde{\varphi} = \tau(k_\varphi \cdot)$.

3.2 Proposition

(1) The set $\{k_\varphi \mid \varphi$ normal, semifinite$\}$ is equal to the set of positive selfadjoint operators k affiliated with M_1 for which $\theta k = \lambda k$ $\quad (\lambda = \exp(-\frac{2\pi}{T_0}))$.

(2) Let $k_\varphi = \int_0^\infty \mu \, de_\mu^\varphi$ be the spectral decomposition of k_φ, then for any $a > 0$:

$$\varphi(1) = \tau\left(\int_{\lambda a}^{a} \mu \, de_\mu^\varphi\right)$$

and

$$\frac{\lambda \varphi(1)}{(1-\lambda)a} \le \tau\left((e_a^\varphi)^\perp\right) \le \frac{\varphi(1)}{(1-\lambda)a}$$

in particular k_φ is τ-measurable iff φ is bounded.

We could now continue as in §1 and construct new L^p-spaces consisting of the τ-measurable operators affiliated with M_1 which satisfies:

$$\begin{cases} \theta k = \lambda^{\frac{1}{p}} k & p < \infty \\ \theta k = k & p = \infty \end{cases}$$

However it is not hard to prove that these spaces are isomorphic to the spaces $L^p(M)$ obtained from the general construction.
We will instead use proposition 3.2 to prove the following slight strengthening of a result due to Connes and Takesaki (cf. [3 , Chap.II, corollary 4.10]).

3.3 Theorem

Let M be a σ-finite factor of type III_{λ}, $\lambda \in]0,1[$.

(1) For any two normal, faithful states φ, ψ on M , there exists a unitary $u \in M$, such that $\lambda \psi \leq u \varphi u^* \leq \lambda^{-1} \psi$.

(2) For any two unbounded n.f.s. weights φ, ψ on M , there exists a unitary u M , such that $\lambda \psi \leq u \varphi u^* \leq \lambda^{-1} \psi$.

(The method of [3] gives $\lambda^2 \psi \leq u \varphi u^* \leq \lambda^{-2} \psi$ in the above inequalities).

3.4 Remark

It is easy to prove, that Theorem 3.3 is not valid for $\lambda = 1$.
A. Connes and E. Størmer [2] have recently proved, that any two normal states φ, ψ on a σ-finite type III_1-factor are almost equivalent in the sense, that there exists a sequence of unitaries $(u_n)_{n \in \mathbb{N}}$ in M , so that $\| \psi - u_n \varphi u_n^* \| \to 0$ for $n \to \infty$.

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