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On Two Definitions of Measurable and Locally Measurable Operators.

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Sunto. – *In questo lavoro vengono confrontate le due diverse definizioni di operatore misurabile rispetto ad un'algebra di von Neumann semifinita date da Segal e da Nelson e le loro estensioni, che forniscono la nozione di operatore localmente misurabile. Vengono date condizioni necessarie e sufficienti per l'equivalenza di queste definizioni e, per quella di Nelson, viene studiata la dipendenza dalla particolare traccia definita sull'algebra di von Neumann.*

Section 1.

Two different definitions of a measurable operator with respect to a semifinite von Neumann algebra $\mathfrak{A}$ on which a normal, semifinite faithful trace tr is defined have been given by Segal [1] and Nelson [2]. The former is pureley algebraic and coincides with the usual one for the $L^\infty(\Gamma)$ case, while the latter depends on the particular trace tr; on the other hand it seems better suited for the generalization of the theory to the case of weights. The first part of this note is denoted mainly to a comparison of the two definitions. In the second part we consider the definition of a locally measurable operator given by Yeadon [3], which is an extension of the notion of measurable operator according to Segal, and compare it with the corresponding extension of measurability according to Nelson. We give necessary and sufficient conditions for the equivalence and find they are very often met. Moreover the second definition does not depend on the particular trace chosen. We only warn that alla traces considered in the following will be faithful normal, and semifinite. All von Neumann algebras will be supposed semifinite.

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For all basic definitions and results not explicitly recalled in the sequel, we refer to [1] and [3].

Section 2.

(2.1) DEFINITION. – A closed operator T affiliated with a von Neumann algebra $\mathfrak{A}$ will be called S-measurable with respect to $\mathfrak{A}$ if its domain is strongly dense with respect to $\mathfrak{A}$ (see [1], def. 2.1).

(2.2) DEFINITION. – A closed operator T affiliated with a von Neumann algebra $\mathfrak{A}$ on which a trace tr is defined is called N-measurable with respect to tr (or tr N measurable) if it lies in the closure of $\mathfrak{A}$ in the measure topology defined in [2], at the beginning of paragraph 2.

Any closed densely defined operator T has polar decomposition $T = U|T|$. If T is affiliated with a von Neumann algebra $\mathfrak{A}$, $U \in \mathfrak{A}$ and if $|T| = \int_0^\infty \lambda \, dE_\lambda$ the spectral projections $t_\lambda = \int_0^\lambda dE_\lambda$ of $|T|$ are in $\mathfrak{A}$.

(2.3) LEMMA. – *Let T be a closed densely defined operator affiliated with a von Neumann algebra $\mathfrak{A}$. Then T is* tr-*N-measurable iff* $\operatorname{tr}(t_\lambda^\perp) < \infty$ *for some λ, and T is S-measurable iff $t_\lambda^\perp$ is algebrically finite for some λ.*

PROOF. – The first part of the lemma is proved in [2], at the end of paragraph 2, and the proof of the second part is similar.

(2.4) PROPOSITION. – *Let $\mathfrak{A}$ be a von Neumann algebra and* tr *a trace on $\mathfrak{A}$. Then all* tr-*N-measurable operators with respect to $\mathfrak{A}$ are S-measurable with respect to $\mathfrak{A}$.*

PROOF. – By lemma (2.3), first statement, there is a λ such that $\operatorname{tr}(t_\lambda^\perp) < \infty$. Since tr is faithful, this implies $t_\lambda^\perp$ is algebrically finite for the same λ and hence by lemma (2.3), second statement, T is S-measurable.

(2.5) PROPOSITION. – *Let $\mathfrak{A}$ be a von Neumann algebra,* tr *a trace on $\mathfrak{A}$ and T a S-measurable operator. Then T is also* tr-*N measurable iff there is a projection P in $\mathfrak{A}$ such that* $\operatorname{tr} P < \infty$ *and $T\,P^\perp$ is bounded.*

PROOF. – Since the sum of two tr N-measurable operators with orthogonal support is tr N measurable, and TP is S-measurable and bounded and therefore in $\mathfrak{A}$ and so tr N measurable, we can assume $P = I$ in the if part of the proposition with no loss of generality. Then, if $\mathrm{tr}\, I < \infty$, we have $\mathrm{tr}\, t_{\lambda_0^\perp} = \mathrm{tr}\, I < \infty$ and so by lemma (2.3) T is tr N measurable.

To prove the only if part of the proposition, we apply again lemma (2.3) and take $P = t_\lambda^\perp$ for an index λ for which $\mathrm{tr}\,(t_\lambda^\perp) < \infty$.

(2.6) PROPOSITION. – *Let $\mathfrak{A}$ be a von Neumann algebra,* tr_1 *and* tr_2 *traces on $\mathfrak{A}$, and T a tr-N-measurable operator. Then T is also* tr_2-N-*measurable iff there is a projection P in $\mathfrak{A}$ such that TP is bounded and for all projections QP, we have* $\mathrm{tr}_1 Q < \infty \Rightarrow \mathrm{tr}_2 Q < \infty$.

PROOF. – As in proposition (2.5) we can assume, with no loss of generality, $P = I$ in the if part of the proposition. Then $\mathrm{tr}_1\,(t_\lambda^\perp) < \infty$ for some λ by lemma (2.3) and therefore $\mathrm{tr}_2\,(t_\lambda^\perp) < \infty$ and T is tr_2 N measurable applying the same lemma. On the other hand, if the condition in the statement is not satisfied, this means that for all λ there is a projection Q_λ such that $Q_\lambda \leqslant t_\lambda^\perp$ and $\mathrm{tr}_2 Q_\lambda = \infty$. Then $\mathrm{tr}_2\, t_\lambda^\perp = \infty$ and, again by lemma (2.3) T is not tr_2 M measurable.

(2.7) LEMMA. – *Let P be a projection in a von Neumann algebra $\mathfrak{A}$, such that* $\mathrm{tr}\, P = \infty$, *with* tr *a trace on $\mathfrak{A}$. There is then an infinite sequence $\{P_n\}_{n=1}^\infty$ of orthogonal subprojections of P such that* $\mathrm{tr}\, P_n < \infty$ *for all n and* $\mathrm{tr} \sum_{n=1}^{\infty} P_n = \infty$.

PROOF. – We prove first by induction that for each n there is a partition $\mathcal{T}_n$ $(n = 0, 1, \ldots)$ of P in $n+1$ orthogonal projections $\left\{P_1, \ldots, P_n, P - \sum_{i=1}^{\infty} P_i\right\}$ such that $1 \leqslant \mathrm{tr}\, P_i < \infty$ for $i = 1, \ldots, n$. We can set $P_0 = \{P\}$. Let us assume now $\mathcal{T}_n$ exists; obviously $\mathrm{tr}\left(P - \sum_{i=1}^{\infty} P_i\right) = \infty$, but because of the semifiniteness of the trace the projection $P - \sum_{i=1}^{n} P_i$ is the supremum of projections with finite trace. There is therefore a projection $P_{n+1} < P - \sum_{i=1}^{n} P_i$ such that $1 \leqslant \mathrm{tr}\, P_{n+1} < \infty$.

Then the partition $\mathcal{T}_{n+1}$ of P defined by $\left\{P_1, \ldots, P_n, P_n, P_{n+1}, P - \sum_{n=1}^{n+1} P\eta\right\}$ satisfies all our conditions.

It is easy now to check that the sequence of projections $\{P_n\}$ obtained by taking the n-th projection from the partition $\mathcal{T}_n$ satisfies the conditions in the statement.

(2.8) THEOREM. – *Let $\mathfrak{A}$ be a von Neumann algebra and* tr, tr_1, tr_2 *traces on $\mathfrak{A}$. Then* tr *N-measurability and S measurability with respect to $\mathfrak{A}$ are equivalent iff all algebrically finite projections have finite trace;* tr_1-*N measurability and* tr_2 *N measurability are equivalent iff* $\mathrm{tr}_1 P < \infty \Leftrightarrow \mathrm{tr}_2 P < \infty$ *for all projections in $\mathfrak{A}$.*

PROOF. – By proposition (3.4) tr N measurability implies S measurability. On the other hand, if all algebrically finite projections have finite trace, for any closed densely defined operator T affiliated with $\mathfrak{A}$ the conditions stated in proposition (2.5) are met, with $P = 0$ and therefore S measurability implies tr N measurability. Conversely, let P be a finite projection in $\mathfrak{A}$ with $\mathrm{tr}\, P = \infty$, and $\{P_n\}_{n=1}$ be a sequence as in lemma (2.7). Since P is finite, so is $\sum_{n=1}^{\infty} P_n < P$ and each of the P_n: Let $T = \sum_{n=1}^{\infty} nP_n$ T is obviously closed and affiliated with $\mathfrak{A}$. Moreover T is S-measurable by lemma (2.3) since $t_\lambda^\perp \leqslant \sum_{n=1}^{\infty} P_n$ for every λ and is therefore finite.

It is not tr-N measurable, again by lemma (2.3) since $t_\lambda^\perp \sum_{n=N} P_n$ for $\lambda > N-1$, and $\mathrm{tr} \sum_{n=N}^{\infty} P_n = \infty$ for all $N \in \boldsymbol{N}$.

In order to prove the second statement it is enough to prove that tr_1-N measurability implies tr_2 N measurability iff $\mathrm{tr}_1 P < \infty \Rightarrow \Rightarrow \mathrm{tr}_2 P < \infty$ for all projection P in $\mathfrak{A}$. If $\mathrm{tr}_1 P < \infty \Rightarrow \mathrm{tr}_2 P < \infty$, then for all tr_1-N-measurable operators T proposition (2.6) applies, with $P = I$ and so tr_1 N measurability implies tr_2 N measurability.

On the other hand, let P be such that $\mathrm{tr}_1 P < \infty$ but $\mathrm{tr}_2 P < \infty$ and $\{P_n\}_{n=1}^{\infty}$ a sequence as in lemma (2.7). Then the operator $T = \sum_{n=1}^{\infty} P_n$ is tr_1-N measurable but not tr_2 N measurable by lemma (2.3) since it is affiliated with $\mathfrak{A}$, densely define and closed, $\mathrm{tr}_1 t_\lambda^\perp < < \mathrm{tr}_1 \sum_{n=1}^{\infty} P_n < \mathrm{tr}_1 P < \infty$ for all λ, but $\mathrm{tr}_2 t_\lambda^\perp \geqslant \mathrm{tr} \sum_{n=N}^{\infty} P_n$ for $\lambda > N-1$.

(2.8) EXAMPLES. – *a*) If $\mathfrak{A}$ is a semifinite factor, then tr N measurability and S measurability are equivalent, since if a projection P in a factor is finite then $\mathrm{tr}\, P < \infty$. Also N measurability does not depend on the particular trace considered, since two traces differ only by a multiplicative constant.

b) Let $\mathfrak{A} = L^\infty(\boldsymbol{R})$, and tr be the usual Lebesgue measure on $\boldsymbol{R}$. Then $f \in L^\infty(\boldsymbol{R})$ is S-measurable iff f is measurable in the ordinary sense as a function (see [1], theorem 2), but is is tr N measurable if it is measurable as a function *and* bounded in a neighbourhood of infinity.

Section 3.

(3.1) Definition. – A closed densely defined operator T affiliated with a von Neumann algebra $\mathfrak{A}$ is S-locally measurable if there exist projections Q_n in the center $\mathfrak{Z}$ of $\mathfrak{A}$ such that $Q_n \uparrow I$ and TQ_n is S-measurable for each n.

(3.2) Definition. – A closed densely defined operator T affiliated with a von Neumann algebra $\mathfrak{A}$ on which a trace tr is defined is tr N locally measurable if there exist projections Q_n in $\mathfrak{Z}$ such that $Q_n \uparrow I$ and TQ_n is tr N measurable for all n.

(3.3) Remark. – Definition (3.1) is due to Yeadon [2, def. 2.2], who gives also four more equivalent conditions [2, th. 2.1]. Definition (3.2) is the corresponding definition from the point of view of N-measurability. It is straightforward to verify that the conditions corresponding to the four conditions given by Yeadon are again equivalent to def. (3.2). It is also easy to see that while tr N measurability implies local tr N measurability, the converse is not true, with examples similar to the example at the end of [4]. It is immediate that $|T|$ is locally N measurable iff T is locally N measurable.

(3.4) Proposition. – *Local N measurability implies local S measurability.*

Proof. – If T is tr N locally measurable there is a sequence of central projections Q_n as in def. (3.2). But if TQ_n is tr N measurable it is also S measurable by prop. (2.4) and so the same sequence satisfies the requirements.

(3.5) Theorem. – *Local S measurability and local* tr *N measurability are equivalent for a von Neuman algebra $\mathfrak{A}$ on which a trace* tr *is defined iff $P_{\mathrm{II}}\mathfrak{A}$ has a countable decomposable center, where P_{II} is the maximal central projection in $\mathfrak{A}$ such that $P_{\mathrm{II}}\mathfrak{A}$ is type* II.

Proof. – Local tr N measurability always implies local S measurability by proposition (3.4).

To show the if part of the thesis let us first show that if $\mathfrak{A}$ is either type I_n or it has a countably decomposable center the thesis is satisfied.

Let $\mathfrak{A}$ be type I_n and T an S measurable operator. Then there is a finite projection P in $\mathfrak{A}$ such that $T(I-P)$ is bounded; it is therefore possibile to assume I finite with no loss of generality, and therefore $n \in \boldsymbol{N}$.

There is then a family of orthogonal abelian projections $\{F_i\}_{i=1}^n$ in $\mathfrak{A}$, with $\sum_{i=1}^n F_i = I$ and $C_{F_i} = I$ for all i, where C_{F_i} is the central carrier of F_i.

Let us set $TF_i = T_i$ and call $C_{i\lambda\perp}$ the central carrier of the spectral projection $(t_i)_\lambda^\perp$. Set $C_\lambda = (I - C_{1\lambda\perp})(I - C_{2\lambda\perp})(I - C_{n\lambda\perp})$.

We claim that $C_\lambda \uparrow I$ and that TC_λ is bounded for all λ-Obviously $C_{\lambda\perp}$ is central for all λ. Indeed, for a fixed i we have $C_{i\lambda\perp}\downarrow 0$ because $C_{i\lambda\perp}\downarrow 0 = \sum_{j=1}^n F_j C_{i\lambda\perp} = \sum_{j=1}^n V_{ji} F_i C_{i\lambda\perp} = \sum_{i=1}^n V_{j,i}(t_i)_\lambda^\perp$ and $(t_i)_\lambda^\perp \downarrow 0$ for $\lambda \to \infty$ where V_j is the partial isometry with initial projection F_i and final projection F_j.

This implies $I - C_{i\lambda}\text{-}\uparrow I$ for all I, and so $C_\lambda \uparrow I$. We also have

$$\|TC_\lambda\| = \|T(I - C_{1\lambda\perp})(I - C_{2\lambda\perp}) \dots (I - C_{n\lambda\perp})\| =$$

$$= \left\| \sum_{i=1}^n [TF_i(I - C_{1\lambda\perp})(I - C_{2\lambda\pm}) \dots (I - C_{n\lambda\perp})] \right\| \leqslant$$

$$\leqslant \sum_{i=1}^n \|TF_i(I - C_{1\lambda\perp})(I - C_{2\lambda\perp}) \dots (I - C_{n\lambda\perp})\| \leqslant$$

$$\leqslant \sum_{i=1}^n \|TF_i(I - C\eta_{\lambda\perp})\| = \sum_{i=1}^n \|T_i F_i (I - C_{i\lambda\perp})\| =$$

$$= \sum_{i=1}^n \|T_i(I - (t_i)_\lambda^\perp)\| = \|T_i(t_i)_\lambda\| \leqslant \sum_{i=1}^n \lambda =$$

$$= n\lambda < \infty .$$

Obviously TC_λ is then tr N measurable and so the projections C_λ meet the requirements of def. (3.2) and T is locally tr N measurable.

If T is not S measurable, but locally S measurable, let $Q^{(h)}$ be a sequence of central projections in $\mathfrak{A}$ as in def. (3.1). Then $TQ^{(h)} = T^h$ is S-measurable and, using the preceding paragraph, also locally tr N measurable.

There is then for each k a sequence of central projections $Q_l^{(k)}$ in $\mathfrak{A}$ such that $\{Q_l^{(k)}\}_{l=1}^\infty \uparrow Q^{(h)}$ and $T_k Q^{(h)}$ is tr N measurable for each k and l.

It is easy to check that the sequence of projections $\{Q_k^{(k)}\}_{k=1}^\infty$ satisfies then the conditions in def. (3.2) and therefore T is locally tr N measurable.

Let now $\mathfrak{A}$ be a finite von Neuman algebra with a countably decomposable center. We first note that, because of the semifiniteness of the trace, it is possible to find a countable family $\{Q_k\}_{k=1}^\infty$ of orthogonal projections in the center of $\mathfrak{A}$ such that $\operatorname{tr} Q_k < \infty$ for each k and $\sum_{k=1}^{\infty} P_k = I$. If T is a locally S measurable operator in $\mathfrak{A}$, there is again a sequence $\{C_l\}_{l=1}^{\infty}$ of central projections in $\mathfrak{A}$ such that TC_l is S measurable and $C_l \uparrow I$.

Let us set $P_n = \left(\sum_{k=1}^{n} Q_k\right) C_n$. We clain that the projections P_n satisfy the conditions in def. (3.2) and therefore T is locally tr N-measurable.

Obviously the P_n are central projections in $\mathfrak{A}$ and

$$(P_{n+1} - P_n) = \left(\sum_{1}^{n+1} Q_k\right) C_{n+1} - \left(\sum_{k=1}^{n} Q_k\right) C_n =$$

$$= Q_{n+1} C_{n+1} + \sum_{k=1}^{n} Q_k (C_{n+1} - C_n) \geqslant 0 ,$$

and so the $P_n \cdot s$ form an increasing sequence.

It is now easy to check that $P_n \uparrow I$ and since $\operatorname{tr} P_n = \operatorname{tr}\left(\sum_{k=1}^{n} Q_k\right) C \leqslant \operatorname{tr}\left(\sum_{k=1}^{n} Q_k\right) = \sum_{k=1}^{n} \operatorname{tr}(Q_k) < \infty$, by theorem 2.7. S-measurability and tr N measurability are equivalent on $\mathfrak{A}P_n$ and so TP_n is tr N measurable for all P.

We can end now the proof of the if part of the theorem by noticing that in a general von Neumann algebra $\mathfrak{A}$ with the properties stated in the statement, there are central projections Q_i ($i \in N$), Q_0, and Q_{-1}, such that $\mathfrak{A}Q_i$ is type I_i, $\mathfrak{A}Q_0$ has a countably decomposable center, $\mathfrak{A}Q_{-1}$ is type I_∞ and $\sum_{i-1}^{\infty} Q_i = I$.

If T is locally S measurable, then so is TQ_k ($k = -1, 0, 1, 2, \ldots$) for all k and then, by one of the preceding paragraphs, also locally tr N measurable.

Let for k fixed $\{Q_h^l\}$ be a sequence of central projections in $\mathfrak{A}$ such that $\{Q_h^l\} \uparrow Q_k$ and $Q_h^l T = Q_h^l T_k$ is tr N measurable. It is easy again to check that the sequence $P_l = \sum_{k=1}^{l} Q_h^l$ satisfies the properties in def. (3.2) and T is locally tr N measurable.

Let now $\mathfrak{A}$ be a type II von Neumann algebra with uncountably decomposable center. Let $\{Q_\infty\}_{\alpha \in A}$ be a family of uncountably many central projections in $\mathfrak{A}$ and P_α a finite subprojection of Q_α for each. Then $P = \sum P_\alpha$ is a finite projection in $\mathfrak{A}$ and $P\mathfrak{A}P$ is a type II_1 von Neumann algebra with uncountably decomposable center. Therefore we can assume, with no loss of generality, that $\mathfrak{A}$ is type II_1 in building our counterexample. Let us now define by induction a sequence of projections $\{P_n\}$ in $\mathfrak{A}$ by setting $P_0 = 0$ and P_{n+1} a projection such that $P_{n+1} < I - \sum_{k=1}^{n} P_k$ and $P_{n+1} \sim I - \sum_{k=1}^{n+1} P_k$.

The operator $T = \sum_{n=0}^{\infty} nP_n$ is clearly S measurable, since it is clearly closed, densely defined, affiliated with $\mathfrak{A}$ and $t_n^{\perp} = \sum_{k=n}^{\infty} P_k < I$ which is finite.

On the other hand let $\{C_n\}_{n=1}^{\infty}$ be any sequence of central orthogonal projections in $\mathfrak{A}$ such that $\sum_{k=1}^{\infty} C_k = I$. For at least one k there is an uncountable family $\{C_h^{\alpha}\}_{\alpha\in A}$ of orthogonal central subprojections of C_n, since $\mathfrak{A}$ has an uncountably decomposable center. Then TC_k is not tr N measurable by lemma (2.3), since $Q_n^{k\alpha} = \left(\sum_{j=n}^{\infty} P_j\right) C_h^{\alpha} \neq 0$ and therefore $\operatorname{tr} Q_n^{k\alpha} \neq 0$ for all n and α, which implies $\operatorname{tr} t_n^{(k)} = \operatorname{tr} \sum_{\alpha\in A} Q_n^{k\alpha} = \sum_{\alpha\in A} \operatorname{tr}(Q_n^{k\alpha}) = \infty$ since the summands are uncountably many and all different from zero. But this implies T is not locally tr N measurable, since, if it were so, we could find a sequence $Q_n = C_{n+1} - C_n$ of orthogonal central projections in $\mathfrak{A}$ such that $\sum Q_n = I$, if the sequence C_n has the properties in def. (3.2).

(3.7) THEOREM. – *In a von Neumann algebra $\mathfrak{A}$ the notion of local tr N measurability is not dependent on the trace.*

PROOF. – Let tr_1 and tr_2 be traces on $\mathfrak{A}$. Let us suppose first T tr_1-N measurable. Then by lemma (2.3) for a certain $\bar{\lambda}$ we have $\operatorname{tr}_1(t_{\bar{\lambda}}^{\perp}) < \infty$. But this implies $\mathfrak{A}C_{t_{\bar{\lambda}}^{\perp}}$ to have a countably decomposable center, if $C_{t_{\bar{\lambda}}^{\perp}}$ is the central carrier of the projection $t_{\bar{\lambda}}^{\perp}$. Then, T is locally tr_2 N measurable, since $TC_{t_{\bar{\lambda}}^{\perp}} \in \mathfrak{A}C$-I, which is a von Neumann algebra with countably decomposable center and therefore on $\mathfrak{A}C_{t_{\bar{\lambda}}^{\perp}}$ local tr_1 N measurability and local tr_2 N measurability bounded.

If T is locally tr_1 N measurable, there is a sequence Q_n of central projections in $\mathfrak{A}$ such that $Q_n \uparrow I$ and TQ_n is tr_1 N measurable. Then TQ_n is locally tr_2 N measurable. Let Q_n^k be a sequence of central projections in $\mathfrak{A}$ such that $Q_n^k \uparrow Q_n$ and TQ_n^k is tr_2 N measurable.

It is immediate to check that the sequence Q_k^k converges monotonically to the identity and TQ_k^k is tr_2 N measurable. So T is locally tr_2 N measurable.

BIBLIOGRAPHY

[1] I. SEGAL, *A non commutative extension of abstract integration*, Ann. of Math., **57** (1973), pp. 401-457.
[2] E. NELSON, *Notes on noncommutative integration*, J. Functional Analysis, **15** (1974), pp. 103-116.
[3] J. DIXMIER, *Algebres d'operateurs dans l'espace Hilbertien*, Gautier-Villars.
[4] F. J. YEADON, *Convergence of measurable operators*, Proc. Cambridge philos. Soc., **74** (1973), pp. 257-268.
[5] S. SANKARAN, *The $*$-algebra of unbounded operators*, J. London Math. Soc., **34** (1959), pp. 337-433.

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