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On Selfdual Hilbert Modules

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Abstract. We establish some connections between operator space theory, and the theory of selfdual modules. Every selfdual C*−module over a W*−algebra is a dual operator space in a canonical way. We find appropriate W*−versions of some results from our previous paper "A new approach to Hilbert C*−modules". For instance we prove that the composition $Y\bar{\otimes}_\theta Z$ of correspondences Y and Z, is completely isometrically isomorphic, via a W*-homeomorphism, to the "extended Haagerup tensor product" of Y and Z. Also we can identify the (unique) predual of this composition as the module operator space projective tensor product of Z_*, the operator space predual of Z, and $\bar{Y}$, the conjugate module of Y. This predual is also completely isometrically isomorphic to the module operator space projective tensor product of N_*, $\bar{Z}$ and $\bar{Y}$, where N_* is the predual of the W*−algebra with respect to which Z is a C*-module. These are not true with the Banach module projective tensor product.

1 Introduction

Selfdual Hilbert C*−modules over a W*−algebra, which we shall abbreviate in this paper to *W*−modules*, were defined and developed by W. L. Paschke [1973, 1976], with later contributions by M. A. Rieffel [1974a, 1982]. More recently, they have played a significant role in noncommutative topology and geometry, under different guises (for instance in the study of correspondences, or the basic construction of Jones). A (right) W*−module Y over a W*−algebra M then, is a right C*−module over M which is *selfdual*. The last term means that every bounded right M-module map $f : Y \to M$ is of the form $f(y) = \langle z|y \rangle$, for some $z \in Y$. Selfdual modules behave almost exactly like Hilbert spaces (as opposed to general C*−modules); for instance there exist orthonormal bases, Gram-Schmidt, the Parseval identity, and a good orthogonality theory. There are several nice characterizations of W*−modules in the literature. In this paper we show how the ideas

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of Blecher [1995] translate into the W*−module setting. Since W*−modules are so well behaved, and in particular have orthonormal bases, our methods do not yield as many new results as in the C*−module case. Indeed, after submitting and circulating an earlier version of this paper we received a reprint of a recent paper of Denizeau and Havet [1994], which contained several of our corollaries. Since our methods are different and give another approach to some of their results, we have for the most part not removed these overlaps, but indicate where they occur.

We shall assume that the reader is familiar with the basic ideas explained in Blecher [1995]. In particular, Section 2 there summarizes most of the necessary background and notation on operator spaces and operator modules which we shall need. Every W*−module is a dual Banach space (Paschke [1973]), we will obtain versions of the results in Blecher [1995] which are compatible with the weak*-topology. The main difference is that the "Haagerup tensor product" is replaced with the "extended Haagerup tensor product" which was developed in Magajna [1994a] (Magajna originally called this the full Haagerup tensor product, but he has informed us that he has changed the name). This is the module version of the "weak*-Haagerup tensor product" of Blecher and Smith [1992]. The definition and relevant properties of the 'extended' Haagerup tensor product are summarised below in Section 2, along with other preliminaries. We also list there the connections between operator spaces and W*−modules (and their preduals).

In Section 3 below we describe in operator space language the 'composition' of correspondences. For instance we prove that the composition $Y\bar{\otimes}_\theta Z$ of correspondences Y and Z, is completely isometrically isomorphic, via a w*-homeomorphism, to the extended Haagerup tensor product of Y and Z. We give another proof of Denizeau and Havet's result that this composition is isometrically isomorphic, via the natural map, to the space $CB_M(\bar{Y}, Z)$ of completely bounded module maps from $\bar{Y}$ to Z. In fact this is a completely isometric isomorphism. From this we get a useful explicit form of the predual operator space of $Y\bar{\otimes}_\theta Z$. Namely, it is completely isometrically isomorphic to the module operator space projective tensor product of Z_*, the operator space predual of Z, and $\bar{Y}$, the conjugate module of Y. Thus it is also completely isometrically isomorphic to the module operator space projective tensor product of N_*, $\bar{Z}$ and $\bar{Y}$, where N_* is the predual of the W*−algebra w.r.t. which Z is a C*-module. These are not true with the Banach module projective tensor product. We also define a W*−module version of the exterior tensor product of C*−modules, and show that this may be identified completely isometrically with $CB(Y_*, Z)$, for W*−modules Y and Z. In Section 3 we also prove the basic HOM-tensor relations for W*−modules, and get better results than in Blecher [1995]. One example of these relations, (although this one is more or less in Blecher [1995]), is that the bounded module maps $\mathbb{B}_N(Y\bar{\otimes}_\theta Z, W)$ from a composition of correspondences Y and Z into a third W*−module W, is completely isometrically isomorphic to $\mathbb{B}_M(Y, \mathbb{B}_N(Z, W))$ (where $\mathbb{B}_M$ are the completely bounded module maps - see Section 2). In Section 4 we make some remarks on the Eilenberg-Watts and fundamental Morita theorem for W*−module categories. Many of the ideas contained herein may be found in Paschke [1973, 1976], Rieffel [1974a], Ghez, Lima and Roberts [1985], and Denizeau and Havet [1994], and some of what we do below can be construed as rephrasing part of what these authors did in operator space language. However, it is to be hoped that the operator space machinery will prove rewarding, and we do obtain many new results.

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2 Preliminaries

A dual operator space is an operator space which is completely isometric to the standard dual (see Blecher [1992a], Blecher and Paulsen [1991] or Effros and Ruan [1991a]) of some other operator space. In Blecher [1992a] the reader may find a simple proof of the fact that a dual operator space is w*-homeomorphic and completely isometric to a weak*-closed subspace of $B(H)$ (the predual of this embedding is explicitly listed after Proposition 3.1 there). Conversely, a weak*-closed subspace of $B(H)$ is a dual operator space (this is proved in Blecher [1992a] for von Neumann algebras, but by using this in conjunction with Corollary 2.4 there, we get it for any weak*-closed subspace). A w*-homeomorphism between dual spaces is a homeomorphism for the weak*-topologies. Similarly, a w*-isometry or w*-isomorphism shall mean an isometry or isomorphism which is a w*-homeomorphism. A normal *-homomorphism from one W*$-$algebra to another is assumed to be unital, and will also be referred to as a normal representation.

If Y is an operator space, and I, J are cardinal numbers, then we write $M_{I,J}(Y)$ for the set of $I \times J$ matrices whose finite submatrices have uniformly bounded norm (see Effros and Ruan [1988, 1990]). Such a matrix is normed by the supremum of the norms of its finite submatrices. We write $M_I(Y) = M_{I,I}(Y), C_I^w(Y) = M_{I,1}(Y)$, and $R_I^w(Y) = M_{1,I}(Y)$. We shall write $(y_i)_{i\in I}$ and $[y_i]_{i\in I}$ for typical elements of $C_I^w(Y)$ and $R_I^w(Y)$ respectively. Generally $[\cdot]$ denotes a row, and $(\cdot)$ a column. We write $C_I(Y)$ (resp. $R_I(Y)$) for the subspace of $C_I^w(Y)$ (resp. $R_I^w(Y)$) which is the norm closure of the set of elements with only a finite number of nonzero entries. If $Y = \mathbb{C}$ then $C_I^w(Y) = C_I(Y)$, and we write this column Hilbert space as C_I. Similarly, $R_I = R_I(\mathbb{C})$. In Wittstock [1980], Blecher, Muhly and Paulsen [1994], Blecher [1995], or Magajna [1994b], and elsewhere, it is explained how every C*$-$module has a canonical operator space structure, as may be seen, for example, by viewing it as the 2-1 corner of the linking C*$-$algebra. Similarly, every W*$-$module Y over a W*$-$algebra M is the 2-1 corner of the linking W*$-$algebra of 2×2 matrices

$$\begin{pmatrix} a & \bar{x} \\ y & b \end{pmatrix}$$

for $a \in M, y \in Y, \bar{x} \in \bar{Y}, b \in \mathbb{B}(Y)$. This linking W*$-$algebra may also be regarded as $\mathbb{B}(M \oplus Y)$ (which is defined later). Since a W*$-$module is thus a corner of a W*$-$algebra, it follows by duality principles that its predual is unique. If Y_* is the Banach space predual of a W*$-$module Y then it inherits an operator space structure from Y^*, and with this operator space structure, $(Y_*)^* = Y$ completely isometrically. Indeed, using canonical operator space identifications (see Blecher and Paulsen [1991] or Effros and Ruan [1991a]), we can write down the predual operator space explicitly as

$$Y_* = M_* \mathbin{\widehat{\otimes}_M} \bar{Y}, \tag{†}$$

where M_* is the operator space predual of M, $\bar{Y}$ is the conjugate W*$-$module of Y, and $\widehat{\otimes}_M$ is the module operator space projective tensor product. For completeness we give the easy argument. We refer the reader to Blecher and Paulsen [1991] or Effros and Ruan [1991a] for the definition of the operator space projective tensor product; the latter contains the 'internal' norm formula for this space - we shall not need this formula explicitly here, but it is nice to know that such a formula exists. We recall (see Blecher [1993, inpreparation]) that if V and W are respectively left and right operator modules over A then $V \widehat{\otimes}_A W$ may be defined to be the quotient of the operator space projective tensor product (see Blecher and Paulsen [1991] or Effros and Ruan [1991a]) $V \widehat{\otimes} W$ by the closed subspace generated by elements $v \otimes aw - va \otimes w$. As in Blecher, Muhly and Paulsen [1994] it can be shown that the module operator space projective tensor product is associative, and it linearizes jointly completely bounded balanced bilinear maps. Thus by definition $M_* \widehat{\otimes}_M \bar{Y}$ is a complete quotient of $M_* \widehat{\otimes} \bar{Y}$. Hence by duality, $(M_* \widehat{\otimes}_M \bar{Y})^* \subset (M_* \widehat{\otimes} \bar{Y})^*$ completely isometrically. However the latter space is $CB(\bar{Y}, M)$ by Blecher and Paulsen [1991] or Effros and Ruan [1991a], and it is straightforward that the range of the inclusion above consists of M-module maps. Conversely, any c.b. M-module map $\bar{Y} \to M$ obviously defines an element of $(M_* \widehat{\otimes}_M \bar{Y})^*$. This proves (†), since $CB_M(\bar{Y}, M) = \mathbb{B}_M(\bar{Y}, M) \cong Y$ completely isometrically (see next paragraph and Blecher [1995]). More generally, if Y is a C*−module over a W*−algebra M then the operator space predual of the selfdual completion of Y is $M_* \widehat{\otimes}_M \bar{Y}$. We remark that the selfdual completion of a C*−module over a W*−algebra M, as defined to be a W*−module over M containing Y as a weak*-dense submodule, is unique (see Rieffel [1974a], Theorem 6.10). If $T : Y \to Z$ is a surjective isometric module map between W*−modules over M, then T is unitary (see Lance [1995] or Blecher [1995]). Also, T is a w*-homeomorphism, and the unique preduals of Y and Z are completely isometrically isomorphic via the module map T_*.

If Y is a right W*−module over M, and if Z is a right operator module over M in the sense explained in Effros and Ruan [1988] or Blecher [1995], then we write $\mathbb{B}(Y, Z)$ or $\mathbb{B}_M(Y, Z)$ for the space $CB_M(Y, Z)$ of completely bounded module maps $Y \to Z$. If Z is also a C*−module over M then all bounded module maps $Y \to Z$ are adjointable, with norm = c.b. norm, so that in this case $\mathbb{B}(Y, Z)$ coincides with $B_M(Y, Z)$ isometrically, and thus it means the traditional thing. As explained in Blecher [1995] for example, $\mathbb{B}(Y, Z)$ is an operator space. Moreover, if Z is also a W*−module then $\mathbb{B}(Y, Z)$ is a dual operator space; the operator space predual (which again is unique, since $\mathbb{B}(Y, Z)$ is a corner of the W*-algebra $\mathbb{B}(Y \oplus Z)$) is easily shown by an argument similar to that for (†) above, to be

$$\mathbb{B}(Y, Z)_* \cong Y \widehat{\otimes}_M Z_* \cong Y \widehat{\otimes}_M M_* \widehat{\otimes}_M \bar{Z} \tag{††}$$

completely isometrically. The following standard result shall be used several times without comment: a bounded net T_λ in $\mathbb{B}(Y, Z)$ converges in the weak*−topology to $T \in \mathbb{B}(Y, Z)$ iff it converges to T in the point weak* topology. This may be seen using the first $\cong$ in (††). Indeed this $\cong$, and consequently also the net convergence result is true if Y is any right operator module and if Z is a *dual operator module,* by which we mean an operator module which is a dual operator space such that the module action is separately weak*-continuous in each variable. For such a module, Z_* is also a module with the obvious action, and is a submodule of Z^*.

The notation $\mathbb{B}(Y, Z)$ for the completely bounded module maps is convenient for this paper, but in the case that Z is not a C*−module it is perhaps not such a good notation. For instance, the reader should be warned that it does disagree with the notation used in the last identity of Section 8 of Blecher [1995].

Since pairs of W*−algebras answer the slice map problem affirmatively (see Kraus [1991]), so do pairs of W*−modules, being corners. From this we conclude by (Blecher [1992b], Theorem 2.5) (the part of this theorem which we are using here was also proved independently by Ruan) that for two W*−modules Y and Z we have

$$Y \bar{\otimes} Z \cong (Y_* \widehat{\otimes} Z_*)^* \qquad (\dagger\dagger\dagger)$$

completely isometrically and w*-homeomorphically, where $\widehat{\otimes}$ is the operator space projective tensor product mentioned above, and $Y \bar{\otimes} Z$ is the weak*-closure of $Y \otimes Z$ in the von Neumann algebra spatial tensor product of any containing von Neumann algebras. This formula shows that as a dual operator space, $Y \bar{\otimes} Z$ is independent of the particular containing von Neumann algebras, and we call it the weak*-spatial tensor product of Y and Z. We see in Section 3 that $Y \bar{\otimes} Z$ is also a W*−module. It is well known (see Effros and Ruan [1988]) that $C_I^w(Y) \cong C_I \bar{\otimes} Y$ and $R_I^w(Y) \cong R_I \bar{\otimes} Y$ as dual operator spaces, if Y is a dual operator space.

If $\{Y_\alpha\}$ is a collection of W*−modules, then we shall write $\oplus_\alpha^w Y_\alpha$ for their W*−module direct sum (called the ultraweak direct sum in Paschke [1973]). If Y is a W*−module then what we called $C_I^w(Y)$ earlier coincides with the ultraweak direct sum of I copies of Y, for any cardinal I.

We say that a submodule Z of a C*−module Y is 1-complemented if it is the range of a contractive idempotent module map P. It follows (see Blecher [95] for instance) that P is an adjointable projection in $\mathbb{B}(Y)$, and that there exists another closed submodule W of Y which is orthogonal to Z, with $Y = Z + W$, so that Y is the internal C*−module direct sum $Z \oplus W$. Thus '1-complemented' is the same as the usual C*−module notion of a complemented submodule. We express it this way so as to remove the inner product from view. If Y is a W*−module then a 1-complemented submodule is also a W*−module, by an obvious argument. The following well known characterization of W*−modules corresponds to Theorem 3.1 in Blecher [1995]:

Theorem 2.1 *Suppose Y is a Banach space and a right module over a W*−algebra M. Then the following are equivalent*

(i) *Y is a W*−module,*

(ii) *There exists an index set I and contractive M-module maps $\phi : Y \to C_I^w(M)$ and $\psi : C_I^w(M) \to Y$ such that $\psi\phi = Id_Y$,*

(iii) *Y is isometrically module isomorphic to a 1-complemented submodule of $C_I^w(M)$ for some set I.*

In this case, the inner product on Y is $< y|z > = < \phi(y)|\phi(z) >$, if ϕ is as in (ii).

Proof That (i) implies (ii) and (iii) is proved in Theorem 3.12 of Paschke [1973]. That (iii) implies (i) is a consequence of the fact mentioned above that a complemented submodule of a W*−module is a W*−module. That (ii) implies (iii) is clear. □

Paschke proves Theorem 3.12 (Paschke [1973]) by first showing that a W*−module Y has an o.n.b., that is, a subset $\{x_\alpha\}_{\alpha \in J}$ of Y which is maximal w.r.t. the properties 1) $< x_\alpha|x_\alpha >$ is a projection in M, and 2) $< x_\alpha|x_\beta > = 0$ if $\alpha \neq \beta$. It then follows that $y = \sum_\alpha x_\alpha < x_\alpha|y >$ in the weak*-topology for all $y \in Y$, once one establishes the Bessels and Parseval identities. We shall refer to this o.n.b. $\{x_\alpha\}$ in the sequel. The element $[x_\alpha]_\alpha$ of $R^w_J(M)$ clearly has norm 1. Then we may take $\phi(y) = (< x_\alpha, y >)$ and $\psi((m_\alpha)) = \sum_\alpha x_\alpha m_\alpha$ in (ii) above. The reader is referred to Denizeau and Havet [1994] for the related, but slightly weaker, notion of a quasi-basis: if ϕ, ψ are as in (ii) above, and if $\{e_\alpha\}$ is the usual basis of $C^w_I(M)$, then $\{\psi(e_\alpha)\}$ is a quasi-basis for Y.

Most of the following is in Paschke [1973], before Theorem 3.12 there (see also Ghez et al. [1985]):

Theorem 2.2 *Suppose that $\{Y_n\}_{n\in I}$ is a collection of W*−modules over M, that Y is a fixed W*−module over M, and that there exist contractive M module maps $i_n : Y_n \to Y$, $\pi_n : Y \to Y_n$ with $\pi_n \circ i_m = \delta_{nm} Id_{Y_m}$ for all n, m. Here δ_{nm} is the Kronecker delta. Then $i_n^* = \pi_n$, and Y is unitarily equivalent to the W*−module direct sum $Z \oplus (\oplus^w_n Y_n)$, where Z is a W*−module over M. If $\sum_n i_n \pi_n = Id_Y$ in the weak*-topology of $\mathbb{B}(Y)$, then $Z = (0)$.*

Proof The ranges $i_n(Y_n)$ are mutually orthogonal 1-complemented submodules of Y. The sum $\sum_n i_n \pi_n$ is a increasing net of contractive projections in $\mathbb{B}(Y)$, indexed by the finite subsets of I directed upwards by inclusion. Hence it converges in the weak*-topology in $\mathbb{B}(Y)$ to a contractive projection P in $\mathbb{B}(Y)$. Let $Z = Ran(I - P)$. The obvious map $Z \oplus (\oplus^w_n Y_n) \to Y$ is well defined, isometric and surjective (see Paschke [1973]), hence unitary. □

We shall need the weak*-Haagerup tensor product which we introduced in Blecher and Smith [1992], and its module version which was developed in Magajna [1994a]. We defined the weak*-Haagerup tensor product for dual operator spaces, but it has been pointed out subsequently by several authors (see particularly Effros and Ruan [1992], Effros, Kraus and Ruan [1993]) that it makes sense for general operator spaces. Indeed the module version of this tensor product was defined by Magajna on the class of "full operator modules". Unfortunately, in the present context, Magajna's nomenclature was bound to be confused with the usual notion of a full C*−module, so that we believe that he has changed the name to "strong operator module". We now describe this notion.

A subspace Y of $B(H)$ which is right invariant under multiplication from a von Neumann algebra $M \subset B(H)$ is called a 'strong' module over M if $\sum_n y_n m_n$ converges strongly (or equivalently, in the weak*-topology) in $B(H)$ to an element of Y, whenever $[y_n]_n \in R^w_I(Y)$, and $(m_n)_n \in C^w_I(M)$. More generally, if Y is an operator module over a W*−algebra M, then we say that Y is a *strong operator module* (or simply a *strong* module) over M if

(i) there exists a linear completely isometry $\Phi : Y \to B(H)$, and a normal representation $\phi : M \to B(H)$ such that $\Phi(y \cdot m) = \Phi(y)\phi(m)$ for all $y \in Y, m \in M$, and

(ii) $\Phi(Y)$ is a strong module in the above sense over $\phi(M)$.

If (i) holds then we say that (Φ, ϕ) is a representation of (Y, M), and Magajna proves that if (ii) holds for some representation then it holds for all representations, and that ϕ may be taken to be faithful. The class of strong modules over the scalars $\mathbb{C}$ is exactly the class of (norm closed) operator spaces. For most of the applications in this paper, Y will be a dual operator space. In fact Y will mostly be a dual operator module, from which it follows (from the proof of Theorem 3.4 in Effros and Ruan [1988]) that there is a representation (Φ, ϕ) as above for which Φ and ϕ are w*-homeomorphisms. Thus a dual operator module is a strong operator module. Nonetheless, it will be convenient to summarize some results which were established by Magajna for general strong operator modules.

Suppose that Y is a strong right module over a W*−algebra M, and that Z is a strong left module over M. Let I be a cardinal number bigger than that of some Hilbert space on which Y and Z can be represented in the sense of (i) above. One may define the "extended Haagerup tensor product" (see Remark 3.7 in Magajna [1994a]) $Y\bar{\otimes}_{hM}Z$ to be equivalence classes of pairs $(y, z) \in R_I^w(Y) \times C_I^w(Z)$, where two pairs (y, z) and (x, w) are equivalent iff there exists a projection $p \in M_{2I}(M)$ such that $[y, x]p = 0$ and $p^{\perp}(z, -w) = 0$. Denote the equivalence class of (y, z) as $y \odot_M z$. The sum $y \odot_M z + x \odot_M w$ is defined to be $[y, x] \odot_M (z, w)$ (by the earlier stated convention, $[y, x]$ means, for example, the element in $R_{2I}^w(Y)$ given by y followed by x). Finally, we norm the equivalence class of (y, z) by $\inf\{\|x\| \|w\|\}$, the infimum taken over all equivalent pairs (x, w). A similar formula defines $M_n(Y\bar{\otimes}_{hM}Z)$, making $Y\bar{\otimes}_{hM}Z$ into an operator space.

A pair (y, z) should be regarded as a formal sum $\sum_k y_k \otimes z_k$, if $y = [y_k]_k, z = (z_k)_k$. We shall see that in the cases which we are interested in, when Y is a W*−module and Z a dual operator module, then $Y\bar{\otimes}_{hM}Z$ is a dual operator space, and has as predual operator space $Z_* \widehat{\otimes}_M \bar{Y}$, and the formal sums above defining the extended Haagerup tensor product converge in the weak*-topology.

Another equivalent way of defining the extended Haagerup tensor product $Y\bar{\otimes}_{hM}Z$, which is perhaps more practical, is as follows. We suppose that Y and Z are represented in the sense of (i) above in the same $B(H)$, which is always possible. Let $[y_n]_n \in R_I^w(Y)$ and $(z_n)_n \in C_I^w(Z)$. Identify the formal sum $\sum_n y_n \otimes z_n$ with the operator in $CB(M', B(H))$ given by $r' \mapsto \sum_n y_n r' z_n$. We identify two such formal sums if they give the same operator in $CB(M', B(H))$, and we norm such a formal sum by the c.b. norm of the associated operator, which turns out to be the same as $\inf\{\| \sum_n y_n y_n^* \|^{\frac{1}{2}} \| \sum_n z_n^* z_n \|^{\frac{1}{2}}\}$, the infimum taken over all formal sums which represent the same operator in $CB(M', B(H))$. (The reader may be confused at the mention of '*' and multiplication in an operator space, however it turns out that these expressions give the same number when performed in any containing W*−algebras, in fact for instance $\| \sum_n z_n^* z_n \|^{\frac{1}{2}}$ is simply the norm of $(z_n)_{n \in I}$ in $C_I^w(Z)$.) W.l.o.g. we can take I in these formal sums to be the cardinality of the Hilbert space H. The space of such formal sums, up to identification as operators in $CB(M', B(H))$, is denoted by $Y\bar{\otimes}_{hM}Z$. Finally, it turns out that $Y\bar{\otimes}_{hM}Z$ is independent, up to completely isometric isomorphism, of the way that Y and Z are represented in the same $B(H)$.

Magajna remarks that $Y\bar{\otimes}_{hM}Z$ contains the module Haagerup tensor product $Y \otimes_{hM} Z$ completely isometrically. If $M = \mathbb{C}$, and if Y and Z are dual operator spaces, then $Y\bar{\otimes}_{hM}Z$ is exactly the weak*-Haagerup tensor product of Blecher and Smith [1992]. We remark that these definitions may seem cumbersome at first, but in practice it allows us to get our hands on elements in a very convenient way. These expressions may be manipulated using slice maps, or using projection techniques in the W*−algebra or its amplifications.

Magajna shows that if, further, Z is a strong right module over a W*−algebra N, then $Y\bar{\otimes}_{hM}Z$ is again a strong right operator module. Moreover the tensor product is ASSOCIATIVE:

$$(Y\bar{\otimes}_{hM}Z)\bar{\otimes}_{hN}W \cong Y\bar{\otimes}_{hM}(Z\bar{\otimes}_{hN}W)$$

completely isometrically. The following result is mentioned in Magajna [1994a], and may be easily proved by the methods of that paper.

Theorem 2.3 *(The strong Haagerup tensor product is FUNCTORIAL). If $T_1 : X_1 \to Y_1$ is a completely bounded right M-module map between strong right M-modules, if $T_2 : X_2 \to Y_2$ is a completely bounded left M-module map between strong left M-modules, then $T_1 \otimes T_2$ is completely bounded from $X_1\bar{\otimes}_{hM}X_2 \to Y_1\bar{\otimes}_{hM}Y_2$, and moreover $\|T_1 \otimes T_2\|_{cb} \leq \|T_1\|_{cb}\|T_2\|_{cb}$.*

Here $T_1 \otimes T_2$ is the map $\sum_k x_k \otimes z_k \mapsto \sum_k T_1(x_k) \otimes T_2(z_k)$, where $\sum_k x_k \otimes z_k$ is one of the typical sums in $X_1\bar{\otimes}_{hM}X_2$ (that this is well defined emerges from the proof).

The following result, which is essentially Theorem 4.2 and the remark afterwards in Effros and Ruan [1988], is used to prove the above, and also used later in our paper:

Theorem 2.4 *If $T : Y \to Z$ is a completely bounded module map between strong operator modules, then $\sum_n T(y_n)m_n = T(\sum_n y_n m_n)$, for all I and $[y_n] \in R_I^w(Y)$, and $(m_n) \in C_I^w(M)$.*

If Y is an operator space then $C_I^w(Y) \cong C_I\bar{\otimes}_h Y$ and $R_I^w(Y) \cong Y\bar{\otimes}_h R_I$, and $C_J\bar{\otimes}_h Y\bar{\otimes}_h R_J \cong M_{JI}(Y)$ completely isometrically, via the canonical identifications. Thus, for instance, if Z is a strong left M-module then $C_I^w(M)\bar{\otimes}_{hM}Z \cong C_I^w(Z)$ completely isometrically. This is because $C_I^w(M)\bar{\otimes}_{hM}Z \cong C_I\bar{\otimes}_h M\bar{\otimes}_{hM}Z \cong C_I\bar{\otimes}_h Z \cong C_I^w(Z)$.

This completes our summary of pertinent facts about the strong Haagerup tensor product. For further details see Magajna [1994a]. We now return our attention to W*−modules. Every W*−module is a strong operator module in the above sense, as may be seen by viewing it as a subspace of the linking W*−algebra. Similarly every $M - N$ correspondence (see Section 3) is strong as a left and as a right operator module, and in fact is a dual left and a dual right operator module. Thus the important modules in the theory of selfdual Hilbert modules are strong in the above sense.

If Y is a W*−module over M, and if Z is a strong right module over M, then there is a natural inclusion of $Z \otimes_M \bar{Y}$ into $\mathbb{B}(Y, Z)$. The following corresponds to Theorem 3.10 in Blecher [1995]:

Theorem 2.5 *If Y is a right W*−module over M, and if Z is a strong left (resp. right) module over M, then $Y \overline{\otimes}_{hM} Z \cong CB_M(\bar{Y}, Z)$ completely isometrically (resp. $Z \overline{\otimes}_{hM}\bar{Y} \cong CB_M(Y, Z)$ completely isometrically), via the canonical map. If in addition Z is a W*−module with respect to the module action above, then these identifications are also w*-homeomorphisms. Putting $Z = Y$, we have that $Z\bar{\otimes}_{hM}\bar{Z}$ with the natural multiplication*

$$(z \otimes \bar{y})(z' \otimes \bar{y'}) = z < y|z' > \otimes \bar{y'}$$

is a W−algebra ($\cong \mathbb{B}(Z)$).*

Proof Define a map $\Theta : Z \bar{\otimes}_{hM} \bar{Y} \to CB_M(Y,Z)$, by $\Theta(u)(y) = \sum_n z_n < y_n, y >$, if $u = \sum_n z_n \otimes y_n$. Note that $\Theta(u)(y) = m \circ (Id_Z \otimes g_y)(u)$, where m is the module action $Z \bar{\otimes}_{hM} M \to Z$, and $g_y : \bar{Y} \to M : \bar{x} \mapsto < x|y >$. This shows that Θ is well defined. It is easy to see that Θ is contractive, and indeed completely contractive. Now if $T \in CB_M(Y,Z)$, define $u = \sum_\alpha T(e_\alpha) \otimes \bar{e_\alpha}$, where $\{e_\alpha\}$ is an o.n.b. for Y. Since T is completely bounded it follows from Theorem 2.3 (the functoriality of the tensor product), that u is a well-defined element of $Z \bar{\otimes}_{hM} \bar{Y}$. Clearly, $\|u\| \leq \|T\|_{cb} \| \sum_\alpha e_\alpha \otimes \bar{e_\alpha}\|_{Y \bar{\otimes}_{hM} \bar{Y}} \leq \|T\|_{cb}$. For this u we have that $\Theta(u)(y) = \sum_\alpha T(e_\alpha) < e_\alpha | y >= T(\sum_\alpha e_\alpha < e_\alpha | y >) = T(y)$, using Theorem 2.4. This shows that Θ is onto, and an isometry. Showing that Θ is a complete isometry is similar, or it may be deduced from the isometry by a standard argument. If now Y, Z are right W$^*-$modules, then to prove the w*-homeomorphism $Z \bar{\otimes}_{hM} \bar{Y} \cong CB_M(Y,Z)$ we may assume by taking the direct sum if necessary, that $Y = Z$. The space $Z \bar{\otimes}_{hM} \bar{Z}$ has (unique) predual by the above. It is easily checked that with the natural multiplication described in the statement, the identification $Z \bar{\otimes}_{hM} \bar{Z} \cong \mathbb{B}(Z)$ described above, is a homomorphism. Thus, $Z \bar{\otimes}_{hM} \bar{Z}$ is a W$^*-$algebra with this multiplication. By the uniqueness of predual this identification is necessarily a w*-homeomorphism. □

As consequences of the previous theorem we obtain the following useful facts:

(1) if Z is also a strong right (resp. left) module over N, then $CB_M(\bar{Y}, Z) = \mathbb{B}_M(\bar{Y}, Z)$ (resp. $CB_M(Y,Z) = \mathbb{B}_M(Y,Z)$) is a strong right (resp. left) operator module over N;

(2) the infima in the expression for the norm in the strong Haagerup tensor product in the cases discussed in the theorem are actually achieved;

(3) if Z is also a dual operator module, then by (††) and the discussion below it, an operator space predual of $Y \bar{\otimes}_{hM} Z$ is $Z_* \widehat{\otimes}_M \bar{Y}$. Moreover, from this observation it is easy to see that the formal sums defining $\bar{\otimes}_{hM}$ converge in the weak*-topology of $Y \bar{\otimes}_{hM} Z$ induced by this predual.

(4) If Y, Z are right W$^*-$modules over M, then $T \in \mathbb{B}(Y,Z)$ with $\|T\| \leq 1$ if, and only if, there is an indexing set I such that we can write $T = \sum_{k \in I} |z_k >< y_k|$, where $\| \sum_{k \in I} |z_k >< z_k| \| \leq 1$, and $\| \sum_{k \in I} |y_k >< y_k| \| \leq 1$. Here we use the Dirac notation $|y >< z|$ for the operator $y < z| \cdot >$. All the sums here converge in the weak*-topology (and the last two converging as such implies the first does too). This result, however, is well known, and follows from standard methods.

Results similar to 3) and 4) hold for left modules.

3 Tensor products of W$^*-$modules

Let M and N be W$^*-$algebras. A (right) $M - N$ correspondence is a right W$^*-$module Z over N, which is a left M-module via a normal representation $\theta : M \to \mathbb{B}(Z)$. A left $M - N$ correspondence is a left W$^*-$module over M which admits a normal right action of N. The notion of a correspondence is due to Connes. His definition Connes [94] looks dissimilar to ours (which is what Rieffel calls a selfdual normal N-rigged M-module in Rieffel [74a]); we are not sure whom to attribute for the fact that they are equivalent, probably Rieffel [1974a] and Baillet et al. [1988] (where it is made explicit). It follows as remarked in Section 2 from looking at the linking W*−algebra for such a Z, that Z is a strong left module over M, and a strong right module over N. In fact it is easy to see that Z is a 2-sided dual operator module. If in addition Y is a right W*−module over M then the *composition* tensor product $Y\bar{\otimes}_\theta Z$ is defined to be the selfdual completion of the C*−module interior tensor product $Y \otimes_\theta Z$ of Y and Z.

We now show how to use the preceding theory to obtain some results in Denizeau and Havet [94], and some other results.

Theorem 3.1 *Let Y be a right W*−module over a W*−algebra M, let Z be a right W*−module over a W*−algebra N, and let $\theta : M \to \mathbb{B}(Z)$ be a normal representation. Then the composition tensor product $Y\bar{\otimes}_\theta Z$ is completely isometrically w*-isomorphic to $Y\bar{\otimes}_{hM}Z$. The conjugate W*−module $(Y\bar{\otimes}_\theta Z)^{\bar{}} \cong \bar{Z}\bar{\otimes}_{hM}\bar{Y}$ completely w*-isometrically. The predual of $Y\bar{\otimes}_\theta Z$ is completely isometrically isomorphic to $Z_* \widehat{\otimes}_M \bar{Y} \cong N_* \widehat{\otimes}_N \bar{Z} \widehat{\otimes}_M \bar{Y}$.*

Proof Let $Y \otimes_\theta Z$ be the C*−module interior tensor product. By Corollary 8.2 in Blecher [1995] we have

$$CB_N(Y \otimes_\theta Z, N) \cong CB_M(Y, CB_N(Z, N)) \cong CB_M(Y, \bar{Z}) \cong \bar{Z}\bar{\otimes}_{hM}\bar{Y}$$

completely isometrically, using Theorem 2.5. Of course $B_N(Y\otimes_\theta Z, N) = CB_N(Y\otimes_\theta Z, N)$. Thus $\bar{Z}\bar{\otimes}_{hM}\bar{Y}$ is the selfdual completion of the conjugate C*−module of $Y \otimes_\theta Z$. Similarly, $Y\bar{\otimes}_{hM}Z$ is the selfdual completion of $Y \otimes_\theta Z$. Unravelling the identifications above we see that the embedding of $Y \otimes_\theta Z$ in its selfdual completion corresponds to the natural inclusion of $Y \otimes_\theta Z = Y \otimes_{hM} Z$ in $Y\bar{\otimes}_{hM}Z$. The last statement of the theorem now follows from Remark 3 at the end of Section 2, and (†). □

Since by Remark 3) after Theorem 2.5 the sums defining $Y\bar{\otimes}_{hM}Z$ converge in the (unique) weak*-topology of $Y\bar{\otimes}_{hM}Z$, and since the inner product on a W*−module is separately weak*-continuous, we have that the inner product on $Y\bar{\otimes}_\theta Z \cong Y\bar{\otimes}_{hM}Z$ may be expressed as

$$\begin{aligned} <\sum_{i\in I} y_i \otimes z_i | \sum_{j\in J} x_j \otimes w_j >&= \sum_i \sum_j < z_i | \theta(< y_i | x_j >) w_j > \\ &= \sum_j \sum_i < z_i | \theta(< y_i | x_j >) w_j >= \underline{z}^* [\theta(< y_i | x_j >)] \underline{w}, \end{aligned}$$

where $\underline{z} = (z_i), \underline{w} = (w_j)$, and the last matrix product is computed in the linking W*−algebra. The sums here converge in the w*-topology of M. Here I is an arbitrary set, so that the formula above explicitly gives the i.p. for the most general elements of $Y\bar{\otimes}_\theta Z$. Part of the following result may also be found in Denizeau and Havet [1994], Corollaire 1.2.6 and Lemme 2.1.5.

Corollary 3.2 *An element u in the composition tensor product $Y\bar{\otimes}_\theta Z$ has norm ≤ 1 if, and only if, u may be written as a sum $u = \sum_{k\in I} y_k \otimes z_k$, where $y_k \in Y, z_k \in Z$ satisfy $\| \sum_{k\in I} < z_k | z_k > \| \leq 1$, and $\| \sum_{k\in I} |y_k >< y_k| \| = \| [< y_i | y_j >]_{i,j} \| \leq 1$. All sums here converge in weak*-topology (and the last two w*-converging imply that the first does too).*

The following result is an immediate corollary of Theorem 3.1 and Theorem 2.5. Its main assertion, that the composition of correspondences may be identified with a space of completely bounded maps, is contained in Denizeau and Havet

[1994], Proposition 2.1.1 and Theorem 2.2.2. The first two lines of Theorem 3.1 contains our alternative proof of this result. B. Magajna has recently shown the author another route to proving 3.1, and consequently also to this result.

Corollary 3.3 *Let Y, Z be right W^*−modules over M and N respectively, and suppose that $\theta : M \to \mathbb{B}(Z)$ is a normal *-homomorphism. The composition tensor product $Y\bar{\otimes}_\theta Z \cong \mathbb{B}_M(\bar{Y}, Z)$ completely w^*-isometrically (via the canonical map on $Y \otimes Z$). For $u \in Y\bar{\otimes}_\theta Z$ write $\tilde{u}$ for the induced map $\bar{Y} \to Z$ (this is the obvious map if u is an elementary tensor). Then we have*

$$\|u\| = \|\tilde{u}\|_{cb} = min\{\|f\|\|g\|_{cb}\}$$

where the minimum is over all indexing sets I and (completely) bounded left M-module maps $f : \bar{Y} \to R_I^w(M)$, and $g : R_I^w(M) \to Z$ with $\tilde{u} = gf$. If, further, Z is also a left W^*−module over M, then we have $Y\bar{\otimes}_\theta Z \cong \mathbb{B}_M(\bar{Z}, Y)$ completely w^*-isometrically.

It is not sufficient in the first result in the corollary above to use the bounded norm on $\mathbb{B}_M(\bar{Z}, Y)$ as we noted in Blecher [1995].

As a consequence we note that if X is a left W*−module over M, and Z is a right $M - N$ correspondence, then $CB_M(X, Z) = \mathbb{B}(X, Z)$ is a right W*−module over N via the action $(T \cdot n)(x) = T(x)n$. The inner product here is $< S|T >= \sum_\alpha < S(e_\alpha)|T(e_\alpha) >$ where $\{e_\alpha\}$ is ANY o.n.b. for X. In particular, $CB_M(X, Z)$ is the set of module maps $T : X \to Z$ with $\sum_\alpha < T(e_\alpha)|T(e_\alpha) >$ weak*-convergent in N, and for such T we have $\|T\|_{cb} = \| \sum_\alpha < T(e_\alpha)|T(e_\alpha) > \|^{\frac{1}{2}}$. This apparent generalization of the Hilbert-Schmidt notion was investigated in Denizeau and Havet [1994], and would be an interesting object for further study.

We also obtain a description of the 'bounded operators' on a composition tensor product.

Corollary 3.4 *We have $\mathbb{B}(Y\bar{\otimes}_\theta Z) \cong Y\bar{\otimes}_{hM}\mathbb{B}(Z)\bar{\otimes}_{hM}\bar{Y}$ completely isometrically w^*-isomorphically. The operator space predual of $\mathbb{B}(Y\bar{\otimes}_\theta Z)$ is*

$$Y \widehat{\otimes}_M Z \widehat{\otimes}_N N_* \widehat{\otimes}_N \bar{Z} \widehat{\otimes}_M \bar{Y} \cong Y \widehat{\otimes}_M \mathbb{B}(Z)_* \widehat{\otimes}_M \bar{Y}$$

completely isometrically.

Proof The first relation is because

$$\mathbb{B}(Y\bar{\otimes}_\theta Z) \cong (Y\bar{\otimes}_\theta Z)\bar{\otimes}_{hN}\overline{(Y\bar{\otimes}_\theta Z)}$$

$$\cong (Y\bar{\otimes}_{hM} Z)\bar{\otimes}_{hN}(\bar{Z}\bar{\otimes}_{hM}\bar{Y}) \cong Y\bar{\otimes}_{hM}\mathbb{B}(Z)\bar{\otimes}_{hM}\bar{Y}.$$

The weak*-isomorphism follows as in Theorem 2.5. To prove the assertion about the predual, we note that using the first relation in Theorem 3.6 we have:

$$\begin{aligned}\mathbb{B}(Y\bar{\otimes}_\theta Z, Y\bar{\otimes}_\theta Z) &\cong CB_M(Y, CB_N(Z, , Y\bar{\otimes}_\theta Z)) \\ &\cong CB_M(Y \widehat{\otimes}_M Z, Y\bar{\otimes}_\theta Z) \cong ((Y \widehat{\otimes}_M Z) \widehat{\otimes}_N (Y\bar{\otimes}_\theta Z)_*)^*.\end{aligned}$$

The last part uses (†). Thus from the last part of Theorem 3.1 the predual is

$$(Y \widehat{\otimes}_M Z) \widehat{\otimes}_N N_* \widehat{\otimes}_N \bar{Z} \widehat{\otimes}_M \bar{Y}.$$

Grouping the three middle terms and using (††) gives the last relation. □

If Z is also a W*–equivalence bimodule (see Rieffel [74a,74b]) for M , or indeed if θ is faithful with range $\mathbb{B}(Z)$, then it follows immediately from 3.4 that $\mathbb{B}(Y \bar{\otimes}_\theta Z) \cong \mathbb{B}(Y)$ since $\mathbb{B}(Z) \cong M$. This is well known (Denizeau and Havet [1994], Proposition 1.2.3 (i)). We have been told that the following result is well known too, however we include it together with its one line proof since we are unable to give a specific reference.

Corollary 3.5 *A bounded net T_λ in $\mathbb{B}(Y \bar{\otimes}_\theta Z)$ converges weak* to $T \in \mathbb{B}(Y \bar{\otimes}_\theta Z)$ if, and only if, for all $y, y' \in Y, z, z' \in Z$ we have*

$$< T_\lambda(y \otimes z)|y' \otimes z' > \rightarrow < T(y \otimes z)|y' \otimes z' >$$

weak in N.*

The necessity here is obvious, the sufficiency follows from Corollary 3.4 since the hypothesis extends easily to finite rank tensors, which are norm dense in the last space mentioned in the statement of 3.4.

We observe here too, because we have not seen this in print, both left and right distributivity of the composition tensor product over arbitrary W*-module direct sums. This is easy using the universal property 2.2 and the previous corollary.

Next we transfer the HOM - tensor relations in Chapter 20 of Anderson and Fuller [1992] into our context, with HOM $= \mathbb{B}(-)$, the completely bounded module maps.

Theorem 3.6 *Let M and N be W*–algebras. We have the following completely isometric identifications:*

(1) $\mathbb{B}_N(Y \bar{\otimes}_\theta Z, W) \cong \mathbb{B}_M(Y, \mathbb{B}_N(Z, W))$ *if Z is an $M - N$ correspondence, Y is a right W*–module over M, and W is a strong right operator module over N.*

(2) $\mathbb{B}_M(Y, (Z \bar{\otimes}_{hN} W)) \cong Z \bar{\otimes}_{hN} \mathbb{B}_M(Y, W)$ *if Y is a right W*–module over M, if W is a strong $N - M$ operator bimodule and if Z is a strong right N-operator module.*

(3) $\mathbb{B}_N(\mathbb{B}_M(Y, W), X) \cong Y \bar{\otimes}_{hM} \mathbb{B}_N(W, X)$ *if Y is a right W*–module over M, X is a strong left N-operator module, and W is a left $N - M$ correspondence.*

(4) $\mathbb{B}_M(X, \mathbb{B}_N(Z, W)) \cong \mathbb{B}_N(Z, \mathbb{B}_M(X, W))$ *if X, Z are left and right W*–modules over M and N respectively, and if W is a strong $M - N$ operator bimodule.*

As in Blecher [1995] we remark that these are (right) versions, of the four HOM - tensor relations in Chapter 20 of Anderson and Fuller. In fact (4) is a symmetric condition, but (1)-(3) have 'left' versions whose statements we leave to the reader.

The proofs all follow easily from Theorem 2.5, therefore we shall prove only one of the relations above, namely (3):

$$\begin{aligned} \mathbb{B}_N(\mathbb{B}_M(Y, W), X) &\cong \mathbb{B}_N(W \bar{\otimes}_{hM} \bar{Y}, X) \cong \overline{(W \bar{\otimes}_{hM} \bar{Y})} \bar{\otimes}_{hN} X \\ &\cong Y \bar{\otimes}_{hM} \bar{W} \bar{\otimes}_{hN} X \cong Y \bar{\otimes}_{hM} \mathbb{B}_N(W, X). \end{aligned}$$

One would expect these identifications to be w*-homeomorphic whenever that makes sense. We checked all of these except (2), using the 'canonical preduals' $\mathbb{B}(Y, Z)_* \cong Y \widehat{\otimes}_M Z_*$ for an operator module Y and a dual operator module Z (see remarks after (††), and Remark 3 after Theorem 2.5). As illustrated by example in Blecher [1995], these HOM-tensor relations are not generally true without completely bounded maps. As a corollary of (1) above we see that $\mathbb{B}_N(Y\bar{\otimes}_\theta Z, W) = \mathbb{B}_N(Y \widehat{\otimes}_M Z, W)$ completely isometrically, where Y, Z, W are as in (1). We have already used this relation in Corollary 3.4.

We have not seen the following definition in the literature, although it is the selfdual completion of the construction in Rieffel [74a] Proposition 8.5. Suppose that Y is a right W*$-$module over a W*$-$algebra M, and that Z is a right W*$-$module over a W*$-$algebra N. We define the W*-exterior tensor product $Y\bar{\otimes}Z$, which is to be a W*$-$module over $M\bar{\otimes}N$ as follows. Let $L(Y)$ (resp. $L(Z)$) be the linking W*$-$algebra for Y (resp. Z). Identify $Y \otimes Z$ with the obvious subspace of the W*$-$algebra tensor product $L(Y)\bar{\otimes}L(Z)$. Write $Y\bar{\otimes}Z$ for its completion in the weak*-topology of $L(Y)\bar{\otimes}L(Z)$. As we mentioned in († † †) in Section 2, we have $Y\bar{\otimes}Z \cong (Y_* \widehat{\otimes} Z_*)^*$ completely isometrically. Thus we have from canonical operator space identifications that $Y\bar{\otimes}Z \cong CB(Y_*, Z) \cong CB(Z_*, Y)$ completely isometrically. By viewing $Y\bar{\otimes}Z$ as in the tensor product of the linking W*$-$algebras, we see that it is obviously a $M\bar{\otimes}N$ module, and has a natural $M\bar{\otimes}N$ valued inner product. Its selfduality may be seen in several ways, one of which is given in Theorem 3.8 below.

Lemma 3.7 *Suppose for $k = 1, 2$ that $T_k : Y_k \rightarrow Z_k$ are completely bounded weak*-continuous maps. Then $T_1 \otimes T_2$ defines a unique completely bounded weak*-continuous map $Y_1\bar{\otimes}Y_2 \rightarrow Z_1\bar{\otimes}Z_2$. Moreover, $\|T_1 \otimes T_2\|_{cb} \leq \|T_1\|_{cb}\|T_2\|_{cb}$.*

Proof This is straightforward and well known, $T_1 \otimes T_2$ may be taken to be $((T_1)_* \otimes (T_2)_*)^*$, viewing $(T_1)_* \otimes (T_2)_* : (Z_1)_* \widehat{\otimes} (Z_2)_* \rightarrow (Y_1)_* \widehat{\otimes} (Y_2)_*$. The latter is completely bounded since $\widehat{\otimes}$ is functorial (see Blecher and Paulsen [1991]), with c.b. norm $\leq \|(T_1)_*\|_{cb}\|(T_2)_*\|_{cb} = \|T_1\|_{cb}\|T_2\|_{cb}$. This proves the result. □

Theorem 3.8 *The W*-exterior tensor product of W*$-$modules Y and Z is a W*$-$module. Its operator space predual is $Y_* \widehat{\otimes} Z_*$ completely isometrically. If $\{x_\alpha\}$ is an o.n.b. for Y and if $\{w_\beta\}$ is an o.n.b. for Z, then $\{x_\alpha \otimes w_\beta\}_{\alpha,\beta}$ is an o.n.b. for $Y\bar{\otimes}Z$.*

Proof Let $\phi : Y \rightarrow C_I^w(M)$ (resp. $\phi' : Z \rightarrow C_J^w(N)$) be as in Theorem 2.1 for Y (resp. for Z), coming from the o.n.b.'s. Note that

$$C_I^w(M)\bar{\otimes}C_J^w(N) \cong C_I\bar{\otimes}M\bar{\otimes}C_J\bar{\otimes}N \cong C_{IJ}^w(M\bar{\otimes}N)$$

completely isometrically. Via this identification of $C_I^w(M)\bar{\otimes}C_J^w(N)$ with $C_{IJ}^w(M\bar{\otimes}N)$, the inner product on $C_I^w(M)\bar{\otimes}C_J^w(N)$ is easily seen to be given by the formula

$$< y \otimes z|y' \otimes z' >=< y|y' > \otimes < z|z' >$$

for $y, y' \in C_I^w(M), z, z' \in C_J^w(N)$. Thus $C_I^w(M)\bar{\otimes}C_J^w(N)$ is a W*$-$module over $M\bar{\otimes}N$, from which we see that $Y\bar{\otimes}Z$ is a W*$-$module over $M\bar{\otimes}N$ using Theorem 2.1 and the functoriality of $\bar{\otimes}$, since $\phi \otimes \phi' : Y\bar{\otimes}Z \rightarrow C_I^w(M)\bar{\otimes}C_J^w(N)$ embeds $Y\bar{\otimes}Z$ in $C_I^w(M)\bar{\otimes}C_J^w(N)$ as a 1-complemented summand. The statement about the o.n.b. follows immediately from the explicit embedding just constructed. □

Since both operator space tensor products are functorial (see 3.7 and 2.3) we get a quick proof of the following result, most of the statements of which are known:

Corollary 3.9 *Suppose that Y_1, Z_1 are right W^*-modules over M, Y_2, Z_2 are right W^*-modules over N, and that $T_i : Y_i \to Z_i$ are bounded right module maps.*
(i) *Then $T_1 \otimes T_2$ is bounded as a map $Y_1 \bar{\otimes} Y_2 \to Z_1 \bar{\otimes} Z_2$.*

(ii) *If, further, θ and π are normal *-homomorphisms from M to $\mathbb{B}(Y_2)$ and $\mathbb{B}(Z_2)$ respectively, and if T_2 is a left M-module map, then $T_1 \otimes T_2$ is bounded as a map $Y_1 \bar{\otimes}_\theta Y_2 \to Z_1 \bar{\otimes}_\pi Z_2$.*

In both cases, $\|T_1 \otimes T_2\| \leq \|T_1\| \|T_2\|$. For (ii), we do not need T_2 to be a right module map if it is completely bounded.

4 The Eilenberg-Watts theorem for W*−modules

In Rieffel [1974a], it was shown that isomorphisms between categories of Hilbert spaces of normal representations of two W*−algebras are implemented by the composition tensor product with a fixed W*−equivalence bimodule. Moreover, he showed that the correspondence between such functorial isomorphisms and modules was bijective. That is, he proved a version of Morita's fundamental theorem in the setting of W*−algebras. En route, the Eilenberg-Watts theorem for such categories was established. An Eilenberg-Watts theorem should state that an additive covariant functor between categories of right modules (over two different rings) is naturally equivalent to tensoring with a fixed bimodule. In his paper it is made clear that no such theorem can exist for C*−algebras, without considering extra structure (and at present it is unknown what that extra structure could be). In Blecher [1995] we found the appropriate Eilenberg-Watts and fundamental Morita theorems for the category of C*−modules. As a consequence, we were able to give a functorial significance to the notion of strong Morita equivalence of C*−algebras, namely that it is indeed a Morita equivalence of some category. Our methods transfer easily to W*−modules, however here these results seem to be known although not explicitly written out (see Ghez et al. [1985] - we thank W. L. Paschke for bringing this reference to our attention). Because of this, and because of the lack of difficulty in this case, we omit the proof. We shall merely state the Eilenberg-Watts theorem, since it may be viewed as a functorial characterization of correspondences, complementing Rieffel's characterization in Rieffel [1974a].

If M is a W*−algebra, let W*MOD$_M$ be the category of right W*−modules over M with morphisms the bounded M-module maps. This is called Hmod(M) in Ghez et al. [1985], which we refer to for additional terminology. If Z is an $M - N$ correspondence as in Section 3, then the composition tensor product $-\bar{\otimes}_\theta Z$ is a normal *-functor from W*MOD$_M$ to W*MOD$_N$ (use Corollary 3.5 and the well known fact that a map is weak*-continuous iff its restriction to the unit ball is weak*-continuous, to get the normality). We want the converse of this.

Theorem 4.1 *(Eilenberg-Watts theorem for W*−modules.) Let M and N be W*−algebras, and let F be a normal *-functor W*MOD$_M$ → W*MOD$_N$. Then there is a right W*−module Z over N, and a normal *-homomorphism $\theta : M \to \mathbb{B}(Z)$, such that $F(-)$ is naturally unitarily isomorphic to the composition tensor product $-\bar{\otimes}_\theta Z$, and also to $\mathbb{B}_M(\bar{(-)}, Z)$. That is, there is a natural isomorphism between these functors, which implements a unitary isomorphism $F(Y) \cong Y \bar{\otimes}_\theta Z \cong \mathbb{B}_M(\bar{Y}, Z)$ for all $Y \in$ W*MOD$_M$.*

Just as in pure algebra, it is an easy exercise to show that the unitary isomorphism classes of normal *-functors W*MOD$_M$ →W*MOD$_N$ is in a 1-1 correspondence with the unitary equivalence classes of $M-N$-correspondences. Composition of such functors corresponds to the composition tensor product of the bimodules.

Finally we remark that it can be easily shown that two W*-algebras M and N are Morita equivalent if, and only if, there is a Hilbert space H such that $M\bar{\otimes}B(H) \cong N\bar{\otimes}B(H)$ *-isomorphically. This is not written down anywhere as far as the author can tell, although it is known.

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