Global theory of elliptic operators

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§ 1. Fredholm operators and K-theory.

In recent years the problem of computing the index of an elliptic differential operator on a closed manifold (or on a manifold with boundary) has been successfully solved [3] by the use of a new branch of algebraic topology (K-theory). Further investigation has brought to light many fundamental connections between elliptic operators and K-theory and in this lecture I want to present a new view-point on these matters.

The best place to start is with abstract functional analysis which provides the natural meeting ground of algebraic topology and partial differential equations. So let H be complex Hilbert space (infinite-dimensional and separable) and let us recall that a Fredholm operator T on H is a bounded linear operator on H such that

(1.1) T and T^* have finite-dimensional null-spaces (or kernels)

and

(1.2) T has closed range (so that $\mathrm{Im}(T) = (\mathrm{Ker}\, T^*)^{\perp}$).

The index of T is then defined by

$$\text{index}\, T = \dim \mathrm{Ker}\, T - \dim \mathrm{Ker}\, T^* .$$

The basic properties of the index of Fredholm operators are

(1.3) *If T is Fredholm and A is compact, then $T+A$ is Fredholm and*

$$\text{index}(T+A) = \text{index}\, T ;$$

(1.4) *if T is Fredholm, and if S is a bounded operator sufficiently near in norm to T, then S is also Fredholm and*

$$\text{index}\, S = \text{index}\, T .$$

If we topologise $\mathcal{F}$ (the space of Fredholm operators on H) by the norm topology, then (1.4) asserts that

$$\text{index} : \mathcal{F} \to \boldsymbol{Z}$$

is a continuous map to the integers $\boldsymbol{Z}$. It is trivial that this map is surjective and not hard to show that two operators with the same index are in the same

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component of $\mathscr{F}$. Thus the index induces a bijection

$$\pi_0(\mathscr{F}) \to \boldsymbol{Z} \tag{1.5}$$

where π_0 denotes the set of path-components.

Suppose now that X is a compact space and that $T: X \to \mathscr{F}$ is a continuous map, so that T_x is a family of Fredholm operators depending continuously on the parameter $x \in X$. If $\dim \operatorname{Ker} T_x$ is independent of x the family of vector spaces $\operatorname{Ker} T_x$ forms a vector bundle $\operatorname{Ker} T$ over X and similarly for $\operatorname{Ker} T^*$. We can then define the index of the family by

$$\operatorname{index} T = [\operatorname{Ker} T] - [\operatorname{Ker} T^*] \in K(X)$$

where $K(X)$ is the abelian group generated by the semi-group of vector bundles over X (up to isomorphism). If $\operatorname{Ker} T_x$ has a variable dimension we can compose with a projection (see [1] or [2]) to reduce to the previous case. It turns out (see [1]) that $T \mapsto \operatorname{index} T$ defines a bijection of the set $[X, \mathscr{F}]$ (of homotopy classes of maps $X \to \mathscr{F}$) onto $K(X)$. The composition of operators in $\mathscr{F}$ corresponds to the addition in $K(X)$ and adjoints correspond to negatives. For X a point we then have

$$K(\text{point}) \cong \pi_0(\mathscr{F}) \cong \boldsymbol{Z}.$$

The group $K(X)$ is a contravariant functor of X, depending only on the homotopy of X. It has many of the formal properties of cohomology, and one can prove [1] that

$$K(X) \otimes_{\boldsymbol{Z}} \boldsymbol{R} \cong H^{\mathrm{ev}}(X; \boldsymbol{R})$$

where $\boldsymbol{R}$ is the real field and H^{ev} denotes the direct sum of all even-dimensional cohomology groups. In fact if we replace the algebra of operators on H by a von Neumann factor of type II_∞ and define Fredholm operators as in [4], the corresponding K-groups turn out to be[1)] actually isomorphic to H^{ev}.

Now, as is well-known, a topological space not only has cohomology groups $H^*(X)$, it also has homology groups $H_*(X)$. These are covariant functors of X and are, in a certain sense, dual of cohomology. If we take real coefficients and make some reasonable assumptions on X (for example that it is a compact polyhedron)[2)], then $H_q(X; \boldsymbol{R})$ is in fact the dual of the finite-dimensional vector space $H^q(X; \boldsymbol{R})$. For integer coefficients the relation is more complicated but one can define $H_q(X; \boldsymbol{Z})$ by putting

$$H_q(X; \boldsymbol{Z}) = H^{N-q}(D_N X; \boldsymbol{Z})$$

where $D_N X$ is a Spanier-Whitehead N-dual of X: we embed X in some Eucli-

1) See a forthcoming article by I.M. Singer published by Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen, Düsseldorf, 1969.

2) These assumptions will be made from now on.

dean space $\boldsymbol{R}^{N+1}$ and take $D_N X$ to be a deformation retract of the complement $\boldsymbol{R}^{N+1}-X$. By the same device we can define[3] a covariant K-functor, namely we put

$$K_0(X)=K(D_{2N}X).$$

In the next two sections I shall explain how this covariant—or homology type —K-functor is related to elliptic operators.

NOTE. Partly for reasons of symmetry and partly to help to distinguish the two different K-functors we shall now write K^0 instead of K. Thus K^0 is contravariant, K_0 is covariant.

§ 2. Elliptic operators.

I shall now recall a few basic facts concerning linear elliptic differential operators on closed manifolds. If P is such an operator (acting on scalar functions, vector-valued functions or more generally on sections of a vector bundle) it has an order m and if we introduce the Sobolev spaces H^s (functions f all of whose partial derivatives up to order s are in L^2) we can regard P as a bounded operator $P: H^s \rightarrow H^{s-m}$. The ellipticity of P implies that it has a parametrix (inverse modulo lower order) and hence that it is a Fredholm operator. It therefore has an index which (because of the regularity of solutions of elliptic equations) is independent of s. The index theorem of [3] shows how the index of P can be calculated in terms of the geometrical data provided by P and the underlying manifold X.

Of course for a differential operator (other than the trivial case of a multiplication) the order m is a positive integer. However using the larger class of pseudo-differential operators [3] one can reduce to the case of order 0, obtaining what used to be called singular integral operators.

An elliptic operator (of order zero) therefore gives rise to a Fredholm operator on Hilbert space. There is however more structure in the elliptic operator which has been ignored on passing to the Hilbert space. To reinstate this further structure we must make use of the fact that our Hilbert spaces are not just abstract vector spaces but are in fact *function spaces.* Thus they not only admit multiplication by complex scalars but also by complex scalar-valued functions on X. A differential operator does not of course commute with multiplication by functions, but it does commute modulo a lower order operator.

Thus if P is a differential operator of order m with C^∞ coefficients and if f is a C^∞ function, the commutator $Pf-fP$ is a differential operator of

3) There are also other (equivalent) definitions, see [7].

order $m-1$ with C^∞ coefficients. For pseudo-differential operators P of order zero one even has the following result: for any *continuous* function f, the commutator $Pf-fP$ is a *compact* operator.

We have now arrived at a property of pseudo-differential operators which can be abstracted out and applied to general topological spaces. Thus let X be any compact Hausdorff space, $C(X)$ the ring of continuous complex-valued functions on X, and let H_1, H_2 be two Hilbert spaces which are continuous $C(X)$-modules (that is, we have uniformly continuous *-algebra homomorphisms of $C(X)$ into the algebras of bounded operators on H_1 and H_2). A bounded linear operator

$$P: H_1 \to H_2$$

will be called an *operator on X* if

(2.1) for any $f \in C(X)$ the commutator $Pf-fP$ is a compact operator.

P will be called an *elliptic* operator on X if, in addition, it is a Fredholm operator. There is then another operator Q on X such that $QP-I$ and $PQ-I$ are both compact (where I denotes the identity operator).

The set of all elliptic operators on X will be denoted by $\mathrm{Ell}(X)$. This definition abstracts out the interaction between elliptic operators and multiplication by functions. It does not attempt to give any abstract analogue of the local representation (or symbol) of a differential operator. We shall return to this question later.

§ 3. From elliptic operators to K_0.

In this section I shall show how to define a map

$$\mathrm{Ell}(X) \to K_0(X),$$

that is we shall associate to each elliptic operator on X (in the sense of § 2) an element of $K_0(X)$. Moreover every element of $K_0(X)$ can be shown to arise in this way so that we can think of elliptic operators on X as representative "cycles" for the "homology" group $K_0(X)$. This will be explained in § 4.

When X is a point, an elliptic operator on X is just an abstract Fredholm operator and hence defines an element of $K^0(\text{point})$. But rather formally one has

$$K_0(\text{point}) = K^0(\text{point})$$

and so we get our map $\mathrm{Ell}(\text{point}) \to K_0(\text{point})$. In fact, since $K^0(\text{point}) \cong \boldsymbol{Z}$, given by the index, our map just assigns to each operator its index.

Consider next the dependence of $\mathrm{Ell}(X)$ on X. If $f: X \to Y$ is a continuous map of compact spaces, we get a homomorphism of rings $f^*: C(Y) \to C(X)$. If H_1, H_2 are Hilbert space modules for $C(X)$ they can then be viewed, using f^*, as $C(Y)$-modules. In this way an elliptic operator on $P: H_1 \to H_2$ on X can be viewed as an elliptic operator on Y, thus f induces

$$f_*: \mathrm{Ell}(X) \to \mathrm{Ell}(Y),$$

so that elliptic operators depend covariantly on the underlying space. In particular, if Y is a point,

$$\mathrm{Ell}(X) \to \mathrm{Ell}(\text{point}) \to K_0(\text{point}) \cong \boldsymbol{Z}$$

is given by $P \mapsto \operatorname{index} P$.

The main construction we need is one which defines a "cap-product" between a "cycle" and a "cocycle". More precisely, given $P \in \mathrm{Ell}(X)$ and a vector bundle V on X we shall define a new element $P \cap V \in \mathrm{Ell}(X)$. For fixed P and variable V the map $V \mapsto \operatorname{index}(P \cap V)$ will then extend by linearity to a homomorphism

$$K^0(X) \to \boldsymbol{Z}.$$

In this way (varying P) we will obtain a map

$$\mathrm{Ell}(X) \to \mathrm{Hom}_{\boldsymbol{Z}}(K^0(X), \boldsymbol{Z}).$$

This is nearly what we want, because one has a homomorphism

$$K_0(X) \to \mathrm{Hom}_{\boldsymbol{Z}}(K^0(X), \boldsymbol{Z})$$

which becomes an isomorphism after tensoring with the rationals. Thus our construction will certainly give a map

$$\mathrm{Ell}(X) \to K_0(X) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}.$$

To refine this, and remove Q, a further argument is needed and we shall return to this later. For the moment we concentrate on defining our "cap-product".

Given a vector bundle V on X one can always find (see [**1**]) a complementary bundle, that is a vector bundle W on X so that $V \oplus W$ is isomorphic to a trivial bundle $X \times \boldsymbol{C}^N$. This is the same as saying that there is a continuous map of X into the projections in $\boldsymbol{C}^N$

$$T: X \to \mathrm{Proj}(\boldsymbol{C}^N)$$

so that V is isomorphic to the bundle of kernels of T. Given such a T and an elliptic operator $P: H_1 \to H_2$ on X we shall define a new elliptic operator Q on X. First we define projection operators T_1, T_2 on H_1^N, H_2^N respectively[4)] as follows. Let $e_1, \cdots, e_N$ be the standard basis of $\boldsymbol{C}^N$, then for each $x \in X$,

4) $H_i^N = H_i \otimes \boldsymbol{C}^N$ for $i = 1, 2$.

$T(x)$ is given by

$$T(x)e_i = \sum_j T^{ij}(x)e_j$$

where the $T^{ij}(x)$ are continuous functions of x. Since H_1 is a $C(X)$-module we can multiply any $u \in H_1$ by the function T^{ij} to get $T^{ij}u \in H_1$. We can therefore define a bounded operator T_1 on H_1^N by

$$T_1(u \otimes e_i) = \sum_j T^{ij}u \otimes e_j .$$

T_2 is defined similarly. Note that T_1 and T_2 are $C(X)$-module homomorphisms and so the decompositions

$$H_1^N = T_1 H_1^N \oplus (1 - T_1) H_1^N$$

$$H_2^N = T_2 H_2^N \oplus (1 - T_2) H_2^N$$

commute with the action of $C(X)$. Moreover, since $[P, f]$ is compact for any $f \in C(X)$, we deduce that $T_2\tilde{P} - \tilde{P}T_1$ is compact[5]. We now consider the operator $Q = T_2\tilde{P}T_1$ as an operator $T_1H_1^N \to T_2H_2^N$. For any $f \in C(X)$

$$[Q, f] = T_2[\tilde{P}, f]T_1$$

and so is compact. If P' is a parametrix for P (an inverse modulo compact operators) then $Q' = T_1\tilde{P}'T_2$ is a parametrix for Q:

$$Q'Q = T_1\tilde{P}'T_2T_2\tilde{P}T_1 = T_1\tilde{P}'\tilde{P}T_1 + T_1\tilde{P}'(T_2\tilde{P} - \tilde{P}T_1)T_1$$

$$\equiv T_1 \text{ modulo compact operators}.$$

$$QQ' \equiv T_2 \text{ similarly}.$$

Thus $Q \in \mathrm{Ell}(X)$ as required.

The operator Q is the cap-product we set out to define. Since it depends on the choice of the family T of projections and not just on the bundle V (of kernels of T) it is better to write $Q = P \cap T$, rather than $P \cap V$. However the various choices of T (and the associated integer N) turn out to be equivalent in an appropriate sense. In particular index$(P \cap T)$ depends on V and not on T and so $T \mapsto \mathrm{index}(P \cap T)$ still defines (for fixed P) a map

(3.1) $$K^0(X) \to \boldsymbol{Z}.$$

REMARK. The Hilbert spaces $T_1H_1^N$ and $T_2H_2^N$ depend only on V and not on T. In fact, the space of continuous sections of V is a $C(X)$-module $M(V)$, and

$$T_iH_1^N = H_i \otimes_{C(X)} M(V), \qquad i = 1, 2.$$

If P actually commuted with $C(X)$, then Q would simply be $P \otimes I$, where I is

5) We denote by $\tilde{P}: H_1^N \to H_2^N$ the natural extension $P \otimes I$ of P.

the identity on $M(V)$. Since P does not actually commute with $C(X)$, Q cannot be defined so simply, hence the need to introduce the complementary bundle W and the projections T.

We next generalise this cap-product construction as follows. In addition to our space X take another compact space Y (which will play the role of a parameter space). Given $P \in \mathrm{Ell}(X)$ and a vector bundle V on $X \times Y$ (with a defining map $T: X \times Y \to \mathrm{Proj}\, \boldsymbol{C}^N$) we perform the above construction over each point $y \in Y$. We get a continuous family of elliptic operators $Q(y)$ and hence (ignoring the $C(X)$-structures) a continuous family of abstract Fredholm operators,[6] in other words we get an element of $K^0(Y)$. For fixed P and varying V we get in this way a homomorphism

$$K^0(X \times Y) \to K^0(Y) \tag{3.2}$$

which generalises (3.1).

Although the homomorphism (3.1) does not determine an element of $K_0(X)$ (only an element of $K_0(X) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$) the homomorphisms (3.2)—for general Y—do determine an element of $K_0(X)$. First we remark that general cohomological theory shows that one always has a pairing

$$K_0(X) \otimes K^0(X \times Y) \to K^0(Y) \tag{3.3}$$

called the "slant product". Moreover if $Y = DX$ is a dual of X, as in § 1, there is a fundamental element[7] $\mu \in K^0(X \times DX)$, and multiplication by μ in (3.3) induces the duality isomorphism

$$K_0(X) \to K^0(DX)\,.$$

A general reference for these remarks is [7]. Hence applying the homomorphism (3.2) to the element $\mu \in K^0(X \times DX)$ we obtain finally the required element associated to P in $K^0(DX) = K_0(X)$. Thus we have defined a natural map

$$\mathrm{Ell}\,(X) \to K_0(X) \tag{3.4}$$

as required. Since (3.4) is obtained by assigning to each $P \in \mathrm{Ell}\,(X)$ the index of an associated family of Fredholm operators, we shall call the map (3.4) the K-index. Thus for $P \in \mathrm{Ell}\,(X)$, we have

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6) Actually the Hilbert spaces on which these operators act also vary with y but this is not a serious point: using Kuiper's Theorem [6] we can make the Hilbert spaces constant.

7) The element μ may be defined as follows. Recalling that $D_{2N}X$ is a deformation retract of $R^{2N+1} - X$, we define $f: X \times DX \to S^{2N}$ by taking $f(x, y)$ to be point where the directed line xy meets a very large sphere $S^{2N} \subset R^{2N+1}$. Then $\mu = f^*(a)$ where $a \in K^0(S^{2N})$ is the basic Bott element.

$$K\text{-index } P \in K_0(X).$$

In the next section we shall discuss how much is known about the K-index, and in particular, what its relation is to the index theorems for operators on differential manifolds.

§ 4. Operators on manifolds.

For a manifold, ordinary homology and cohomology are related by Poincaré duality. Corresponding relations exist in K-theory. For our present purposes these are best put in the following form.

Let TX, NX denote respectively the tangent bundle and normal bundle[8] of a compact differentiable manifold X. Let $K^0(TX)$, $K^0(NX)$ denote the K-groups with compact support (obtained by forming one-point compactifications as explained in [3]). Then there is a natural isomorphism

$$\varphi : K^0(TX) \to K^0(NX). \tag{4.1}$$

On the other hand, as a simple consequence of the definition given in § 1 of K_0, we have a natural isomorphism

$$\psi : K_0(X) \to K^0(NX). \tag{4.2}$$

Now let Diff-Ell(X) denote the class of pseudo-differential operators of order zero acting on C^∞ vector bundles as in [3]. Our motivation for introducing Ell(X) in § 2 was precisely because there is an inclusion

$$\text{Diff-Ell}(X) \subset \text{Ell}(X).$$

For $P \in$ Diff-Ell(X) one has a symbol $\sigma(P)$, given from the local expression for P, and this symbol defines an element $[\sigma(P)] \in K^0(TX)$ called the symbol class of P (see [3, § 6]).

We then have the following main theorem.

THEOREM (4.3). *If P is an elliptic pseudo-differential operator of order zero on a compact C^∞ manifold, we have*

$$\psi(K\text{-index}(P)) = \varphi[\sigma(P)].$$

In other words if we use the isomorphisms (4.1) and (4.2) to identify $K_0(X)$ and $K^0(TX)$, the K-index is "equal" to the symbol class. The significance of this is of course that the symbol class is defined by local data while the K-index is defined globally.

Theorem (4.3) gives an equality in the group $K^0(NX)$, or equivalently in $K_0(X)$. If we apply the homomorphism $K_0(X) \to K_0(\text{point}) \cong \boldsymbol{Z}$, (4.3) gives an

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8) Relative to some C^∞ embedding of X in $\boldsymbol{R}^n$, with $n \equiv \dim X \bmod 2$.

equality between integers which is precisely the formula for index P given in [3]. Theorem (4.3) is thus a generalisation of the main theorem of [3] and it is proved by essentially the same methods. In fact, it is really a consequence of the index theorem for families of elliptic operators.[9]

As a consequence of (4.3) it follows that, for a manifold X,

$$\mathrm{Ell}\,(X) \to K_0(X)$$

is surjective (since even Diff-Ell $(X) \to K_0(X)$ is surjective). On the other hand, a result of Conner-Floyd [5] implies that, for any space X and any element $\xi \in K_0(X)$, there exists a C^∞-manifold M, an element $\eta \in K_0(M)$ and a continuous map $f: M \to X$ so that $\xi = f_*(\eta)$. Since we have commutative diagram

$$\begin{array}{ccc} \mathrm{Ell}\,(M) & \longrightarrow & \mathrm{Ell}\,(X) \\ \downarrow & f_* & \downarrow \\ K_0(M) & \longrightarrow & K_0(X), \end{array}$$

it then follows that

(4.4) $$\mathrm{Ell}\,(X) \to K_0(X) \text{ is surjective.}$$

This result justifies us in thinking of elements of Ell (X) as representative cycles for the "homology" group $K_0(X)$.

A natural question which arises is whether it is possible to describe explicitly the equivalence relations which must be imposed on our "cycles" to produce the "homology". There are some obviously necessary relations including homotopy, but whether these are also sufficient seems to be a difficult question.

What I have outlined here is really just the beginning of an attempt at developing a significant theory of operators on spaces other than C^∞-manifolds. I hope that more can be done in particular on spaces such as piece-wise linear manifolds or complex analytic spaces where there is some further structure. The main problem would then be to find some substitute for the symbol in terms of local data.

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9) See a forthcoming paper by M.F. Atiyah and I.M. Singer, "The index of elliptic operators IV."

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