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Maxwell's equations and electromagnetism II.

Example: Current through a wire. $E = 0$. From this data we can derive Ampere's law: $\int B \cdot ds = j_e/(\epsilon_0 c^2) = |B| \cdot 2\pi$. Hence $|B| = |j|/(2\pi\epsilon_0 c^2)$.

Example: Solenoid. We assume that $B_o = 0$. $\int B \cdot ds = j_e/(\epsilon_0 c^2) = N|j|/(\epsilon_0 c^2)$. Hence $|B| = n|j|/(\epsilon_0 c^2)$.

Potentials are a nice way to cook up solutions to Maxwell's equations. Since $\operatorname{div} B = 0$ we can write $B = \operatorname{curl} A$ for some non-unique vector potential A if there are no topological obstructions. Also $\operatorname{curl}(E+\partial_t A) = 0$, hence $E + \partial_t A = -\operatorname{div}\phi$ for some non-unique scalar potential ϕ.

Gauge transformations: $A \mapsto A + \operatorname{div}\psi$ and $\phi \mapsto \phi - \partial_t\psi$.

By choosing a particular A and ϕ we are "fixing our gauge". For example, $\operatorname{div} A = -c^{-2}\partial_t\phi$ is called the Lorentz gauge.

Relativistic formulation.

Motivation. Consider two wires with current. Lorentz' law: $F = q(E + v \times B)$ (force on a test particle with velocity v and charge q). (We cannot derive it from Maxwell's equations.) Hence the wires are attracted to each other if the have the same direction of current and repelled otherwise.

Now we move into a frame of reference in which electrons in wires are at rest. We observe that now we have electric field instead of magnetic.

We introduce some forms: $B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$, $E = E_x dx + E_y dy + E_z dz$, and $F = B + E \wedge dt$ (electromagnetic field). $J = \rho dt + j_x dx + j_y dy + j_z dz$. Maxwell's equations: $dF = 0$ and $*d*F = J$, where $*$ is the Hodge star.

In the vacuum, given a solution F, $*F$ is another solution. In particular we have self-dual and anti-self-dual solutions: $F = \pm * F$.

Self-dual solutions are left circularly polarized wave solutions and anti-self-dual are right circularly polarized (for given orientation on $\mathbf{R}^{3+1}$ determined by $*$.

Lagrangian formulation of Maxwell's equations: fields are closed 2-forms. Lagrangian (vacuum case): $\int F \wedge *F$.

We can also take $A \in \Omega^1(\mathbf{R}^{3+1})$ and $L = \int dA \wedge *dA + *J \wedge A$.

Aharonov-Bohm effect: $S_q = \int \exp(i\hbar^{-1} q \int_\gamma A)$.