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Back to the variational complex.

The case $E = M \times N \to M$ (trivial bundle). "Sigma-models" $\Phi = \Gamma(M, E) = C^\infty(M, N)$.

The easiest Lagrangian has kinetic term only: M and N are Riemannian or Lorentz manifolds.

$L\colon J^1E = \mathrm{Hom}(TM, TN) \to \mathbf{R} = (f \mapsto (x \mapsto -\|f(x)\|^2/2))$.

V_g is the volume form on M. We have $\lambda = LV_g \in \Omega^{d,0}JE$. Hence $d_V\lambda = d_H(\alpha) + E(\lambda)$ for some $\alpha \in \Omega^{d-1,1}JE$.

Case 1: $M = N = \mathbf{R}$. We have $x\colon M \to \mathbf{R}$, $y\colon M \times N \to \mathbf{R}$, $\dot{y}\colon J^1E \to \mathbf{R}$: $\dot{y}[x, \phi] = \partial_x\phi(x) = \dot{\phi}(x)$.

Hence $J^1E = \mathbf{R}^3$ has coordinate functions x, y, $\dot{y}$. $L = -\dot{y}^2/2$, $\lambda = -\dot{y}^2dx/2$. $\theta = dy - \dot{y}dx \in \Omega^{0,1}$ and $\dot{\theta} = d\dot{y} - \ddot{y}dx$. $\delta\lambda = -\dot{y}\dot{\theta}dx = -\dot{y}d\dot{y}dx$. Set $\alpha = -\dot{y}\theta$. Hence $d_H\alpha = -(\ddot{y}dy + \dot{y}d\dot{y})dx$. $\delta\lambda = d_H\alpha + \ddot{y}dy\,dx$ so that $E(\lambda) = \ddot{y}dy\,dx$. $E(\lambda)$ contains no terms $d\dot{y}$, $d\ddot{y}$, etc.

From $E(\lambda)$ we get $E(\Sigma^1) \in \Omega^1(\Phi)$ by integration $E(\Sigma)_\phi(\psi) = \int E_\phi(\psi)$. $E(\Sigma)$ vanishes at those fields whose second derivative is 0. Hence the classical solutions are linear maps $\mathbf{R} \to \mathbf{R}$.

Case 2: M is arbitrary, $N = \mathbf{R}$. $E(\Sigma)_\phi(\psi) = \int_\Sigma(\Delta_g\phi(x))\psi(x)V_g$.

Case 3: $M = \mathbf{R}$, N is arbitrary. $E(\Sigma)_\phi(\psi) = \int_\Sigma(\langle\nabla_{\dot{\phi}(x)}\dot{\phi}(x), \psi(x)\rangle V_g$.

Case 4: $E(\Sigma)_\phi(\psi) = \int_\Sigma\langle\mathrm{tr}_g\,\nabla(T\phi), \psi\rangle V_g$. Here $T\phi \in \Gamma(M, T^*M \otimes \phi^*TN)$, $\nabla T\phi \in \Gamma(M, T^*M \otimes T^*M \otimes \phi^*TN)$.

Now we want to find α. $\alpha = \dot{y}\theta$ gives $\alpha(0) \in \Omega(\Phi)$ by evaluating near 0.

Lemma: $\theta_{[x,\phi]}(\psi) = \psi(x)$ is independent of ϕ. $\ddot{\theta}_{(x,\phi)}(\psi) = \ddot{\psi}(x)$.

Hence $\alpha(0) \in \Omega^1(\Phi)$ comes from the bilinear pairing $\Phi \otimes \Phi \to \mathbf{R}$. It factors through $\mathbf{R} \times \mathbf{R} \subset \Phi$. $(\phi, \psi) \mapsto \dot{\phi}(0)\psi(0)$. $d\alpha \in \Omega^2(\Phi)$ is constant. $d\phi_\phi(\dot{\psi}^0, \dot{\psi}^1) = \langle\psi^0, \psi^1\rangle - \langle\psi^1, \psi^0\rangle$.

Generalization: For $M = \mathbf{R}$ we have the same formula. For $N = \mathbf{R}$ we have $\alpha(\Sigma)_\phi(\psi) = \int_\Sigma(*d\phi)_\Sigma \wedge \psi$.