Peter Teichner.

Hamiltonian formalism.

Data for Lagrangian field theory: Smooth bundle $E \to M$, M is the space-time, $\Phi = \Gamma(E)$ are the fields on M, local Lagrangian $\lambda \in \Omega^{n,0}(JE)$, $S\colon \Phi \to \mathbf{R}$, $S := \int_M \lambda$ is the classical action.

$M \times \Phi \to JE$ is the obvious map. We can canonically lift vector fields from M to JE. So we have $j^*\colon \Omega^{r,s}(JE) \to \Omega^r M \otimes \Omega^s \Phi$. J^1E is an affine bundle with associated vector bundle $\mathrm{Hom}(\pi^* TM, T_v E)$.

We have $d_H \to d_M$ and $d_V = \delta \to d_\Phi$.

$\lambda \in \Omega^{n,0} \to \delta(\lambda) \in \Omega^{n,1} = d_H(\alpha) + E(\lambda)$, where $E(\lambda) \in \Omega^{n+1}_H(JE \to E) \cap \Omega^{n,1} \subset \Omega^{n+1}(JE)$ is the Euler-Lagrange form.

Definition. The field ϕ satisfies the classical equation of motion (Euler-Lagrange equation) if $E(\lambda)_{[x,\phi]} = 0$ for all $x \in M$.

Theorem. (Takens.) There is a spectral sequence such that $E_0^{r,s} = \Omega^{r,s}(JE)$, $E_1^{r,s} = H^*(\Omega^{r,s}(JE))$, $E_2^{r,s} = H^r M \otimes H^s(F)$. The horizontal rows of this spectral sequence are exact except for the edges: $\Omega^{r,0}$, $\Omega^{r,s}$. Hence H^*E can be computed from the chain complex $\cdots \to \Omega^{k,0} \to \cdots \to \Omega^{n,0} \to E^1 \to E^2 \to \cdots \to E^k$. Here $\Omega^{n,k} = \mathrm{im}(d_H) \oplus E^k$.

$\delta E(\lambda) = 0$ is the Helmholtz equation. This is the first obstruction for $E(\lambda)$ being an Euler-Lagrange equation. The second obstruction is $[E(\lambda)] \in H^{n+1}(E)$. Uniqueness: LFT correspond to $H^n(E)$.

If $Y^r \subset M$ is a compact oriented submanifold, then we have the following diagrams $Y \times \Phi \to M \times \Phi \to JE$. The composition $\Omega^{r,s} JE \to \Omega^r Y \otimes \Omega^s \Phi \to \Omega^s \Phi$ is the integral $\int_Y$.

Key diagram: $d_H\colon \Omega^{r-1,s} \to \Omega^{r,s}$, $\int_{\partial Y}\colon \Omega^{r-1,s} \to \Omega^s\Phi$, $\int_Y\colon \Omega^{r,s} \to \Omega^s\Phi$, $\delta = d_V\colon \Omega^{r,s} \to \Omega^{r,s+1}$, $d_\Phi\colon \Omega^s\Phi \to \Omega^{s+1}\Phi$, $\int_Y\colon \Omega^{r,s+1} \to \Omega^{s+1}\Phi$.

Applications: (a) $Y = M$ is closed: $\Omega^0\Phi \ni S = \int_M \lambda$, $\lambda \in \Omega^{n,0}$. $d_\Phi S = \int_M (\delta\lambda) = \int_M (d_H(\alpha) + E(\lambda)) = \int_M E(\lambda)$. Hence the classical solutions are the extrema of S.

(b) $Y^{n-1} \subset M$, M is closed. ("Space.") $\Omega^1\Phi \ni a(Y) := \int_Y \alpha$ (independent of α!). $\Omega^2\Phi \ni \omega(Y) = da(Y) = \int_Y \delta\alpha$ gives in good cases a symplectic form on $\Phi(Y)$.

(c) If $\partial\Sigma^n = Y^{n-1}$ then we get $\Omega^1(\Phi(\Sigma) \ni d(S_\Sigma) = \int_\Sigma \delta\lambda_\Sigma$.