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Classical field theory.

Goal: Describe a mathematical framework in which all real physical theories can be expressed.

A

- classical field theory consists of the following three data:
- space-time M (a Lorentz manifold of dimension $(d-1)+1$ or a Riemannian manifold of dimension d or an arbitrary semi-Riemannian manifold);
- field content $\Phi(M)$ (space of sections of a certain bundle over M);
- classical action $S\colon \Phi(M) \to \mathbf{R}$.

Roughly speaking, classically one only sees

- extrema of S. In quantum field theory a state is a superposition of all fields with weight $\exp(iS(\Phi)/\hbar)$.

For statistical field theory the probability density is $\exp(-S(\Phi)/T)$. So the equations become the same if $T = i\hbar$.

Examples:

(a) Mechanics: A particle moving in a configuration space N. Here (1) M is one-dimensional (just time); (2) $\Phi(M) = C^\infty(M, N)$; (3) $S(\phi) = \int_M L(\phi_r(m))dm$, where $\phi_r(m)$ is the r-jet of ϕ at point $m \in M$ and $L\colon J^r(N) \to \mathbf{R}$ is a Lagrangian, for example: $r = 1$, $J^1 = TN$, $L(v) = g(v, v) - f(\pi(v))$. Note that $J^r(N)$ is finite-dimensional manifold.

(b) Free boson: (1) M is any oriented semi-Riemannian manifold; (2) $\Phi(M) = C^\infty(M, \mathbf{R})$ (free scalar σ-model); (3) $S(\phi) = \int_M L(\phi_r(m))\omega$, where ω is a volume form, $L(\phi) = \Delta(\phi)\phi + m^2\phi^2$ and $\Delta = d^*d = -(*d*)d \geq 0$. Hence $L(\phi)\omega = (d\phi) \wedge (*d\phi) + m^2\phi \wedge *\phi$.

Key property:

- Locality of the classical action S. We have a map $\mathrm{ev}_r\colon M \times \Phi(M) \to J^r(\Phi)$. We require that $S = \int_M L$,

where L is a form of degree $d = \dim M$ on $M \times \Phi(M)$, which comes from $J^r(\Phi)$: L is the pullback of some form of degree d on the manifold $J^r(\Phi)$ along the map ev_r.