Topology

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1 Preface

These notes offer an elementary introduction to topology. Why bother writing a new text when so many exist already? Two main features distinguish this text from all others known to the author:

- The selection of material is governed by its applications outside of topology proper. In particular, we cover topics such as K-theory and sheaf cohomology, which are typically omitted from the more traditional expositions.
- We do not hesitate to use modern machinery when it enhances clarity and simplifies the exposition. In particular, the following modern tools are used.
  - Simplicial sets are used because they provide the most rapidly accessible introduction to homology, cohomology, and fundamental groups. In particular, computations can be made once basic definitions are given, unlike for singular homology. Additionally, we can omit the rather intricate subtleties of general topology, such as the fact that typical categories of topological spaces (e.g., compactly generated weakly Hausdorff topological spaces) are neither locally presentable nor locally cartesian closed, which becomes troublesome when performing many common constructions, such as the small object argument, constructing the space of sections of a bundle, etc.
  - Homotopy limits and homotopy colimits are already omnipresent in classical treatments in their specific incarnations, such as constructions with mapping cylinders, mapping path spaces, mapping telescopes, etc. We give a systematic treatment, which simplifies the presentation and makes it easier to organize the acquired knowledge. Additionally, it allows for a simplified treatment of topics such as generalized homology theories.
  - Model categories make it easier to systematically treat the numerous derived constructions such as the homotopy (co)limits mentioned above, derived mapping spaces, homological algebra constructions, etc. In particular, they eliminate many repetitive technical arguments with resolutions. Additionally, they make it easy to set up higher algebra in spaces and spectra.

1.1. *Applications*

To illustrate the power of the subject, we state several theorems that will be proved using the machinery developed below:

- (Analysis.) The metric spaces $\mathbf{R}^m$ and $\mathbf{R}^n$ are homeomorphic if and only if $m = n$.
- (Differential equations.) A differential equation with continuous bounded coefficients in a bounded region always has a solution.
- (Partial differential equations.) If $\Omega \subset \mathbf{R}^n$ is open and bounded, $\partial\Omega$ is smooth, and $f\colon \mathbf{R} \to \mathbf{R}$ is a bounded continuous function, then the boundary value problem

$$\begin{cases} -\Delta u = f \circ u & \text{on } \Omega; \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

  has a weak solution $u \in \mathrm{W}_0^{1,2}(\Omega)$, i.e.,

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega (f \circ u) \cdot \varphi \, \mathrm{d}x$$

  for all $\varphi \in \mathrm{W}^{1,2}(\Omega)$.
- (Differential geometry.) There are $2^c + 8d - 1$ linearly independent vector fields on an n-dimensional sphere, and no more. Here $n + 1 = (2a + 1)2^b$, $b = c + 4d$, $0 \le c < 4$. In particular, a vector field on a sphere must vanish.
- (Complex analysis.) The vector space of meromorphic functions on a compact Riemann surface of genus $g \ge 2$ that have a single pole of order at most $p \ge 3$ has dimension $p - 1$.
- (Complex analysis.) A complex polynomial of positive degree has a complex root.
- (Algebra.) The algebras of real numbers, complex numbers, and quaternions are the only real algebras with division.
- (Algebra.) Subgroups of free groups are free. Moreover, index e subgroups of free groups of rank n are free of rank $1 + e(n - 1)$.

1.2. ***Hyperlinks***

As one might have noticed already, terms that are defined in the text, such as "model category" or "associative" are hyperlinked to their definition. Likewise for numbered items, such as Lemma 8.14. This allows one to easily recall definitions and statements of results.

Furthermore, every such numbered item contains a list of back references at the end (beginning with "Used in"). This allows one to see where a particular concept, such as simplicial set, or a result such as Lemma 8.14, is used.

Likewise, bibliographic references like [SHTgj] take one to the relevant bibliographic entry, which contains a list of back references at the end.

1.3. ***Table of notation***

(a,b)	ordered pair
$f\colon X \to Y$	f is a morphism (e.g., function) with domain X and codomain Y
$X \rightrightarrows Y$	a pair of morphisms $X \to Y$ (used for (co)equalizers)
id_X	the identity morphism on X
$g \circ f$	composition: first apply f, then g; for functions: $(g \circ f)(x) = g(f(x))$
Y^X	the set of all functions $X \to Y$
$\{a < b < c < \cdots\}$	an ordered set with elements a, b, c, ..., with the induced order
$\mathsf{U}(-)$	the underlying object (e.g., the underlying set of a group)
$\mathbf{m}$, $\mathbf{n}$	simplices
$\lvert\mathbf{m}\rvert$	geometric realization of a simplex
$\dim \mathbf{m}$	the dimension of a simplex

2 Supplementary sources

The most accessible texts on elementary simplicial homotopy theory are expository articles by Greg Friedman [EIISS] and Francis Sergeraert [ICHT].

A set of notes from 2008 by Joyal and Tierney [NSHT] is a good exposition of the topics that it covers, namely, the elementary theory of simplicial sets, operations on them, Kan complexes, fibrations and cofibrations, and simplicial weak equivalences. A 1999 book by Goerss and Jardine [SHTgj] is the only modern printed book on simplicial homotopy theory. Both of these sources require that one is familiar with elementary category theory, see Definition 11.1, Definition 12.1, Definition 14.9.

Four classical expositions of simplicial homotopy theory appeared between 1967 and 1971: Gabriel and Zisman [CFHT], May [SOAT], Curtis [SHTc], Lamotke [SAT]. These may be more difficult to read due to their extensive manual manipulation of face and degeneracy maps.

Once we construct the geometric realization and singular simplicial set functors and show they form an equivalence of homotopy theories, the traditional expositions that use topological spaces become accessible. The books by May [CCAT] (1999), Hatcher [ATh] (2001), and tom Dieck [ATd] (2008) are the current textbooks. Among the more classical textbooks, we single out Spanier [ATs] (1966), Dold [LAT] (1972), Switzer [ATHH] (1975). Munkres [EAT] (1984) offers a treatment using simplicial complexes, which is closer in spirit to our approach. Fulton [ATf] (1995) presents a more geometric approach. Davis and Kirk [LNAT] (2001) also cover more advanced topics.

3 Timeline of early homotopy theory

- 1847: Listing, “Vorstudien zur Topologie”: introduced the term “topology”.
- 1857: Riemann, “Theorie der Abel’schen Funktionen”: semirigorous definition of the rank of $\mathsf{H}^1(S, \mathbf{Z}/2)$ for a surface S and Poincaré duality for it.
- 1871: Betti, “Sopra gli glazi di un numero qualunque di dimensioni”: semirigorous definition of the rank of $\mathsf{H}^n(M, \mathbf{Z}/2)$ for a manifold M.
- 1895: Poincaré, “Analysis Situs”: definition of the rank of $\mathsf{H}^n(M, \mathbf{Z})$ using embedded submanifolds. Semirigorous proof of Poincaré duality.
- 1899: Poincaré, “Complément à l’Analysis situs”: simplicial homology of triangulated manifolds. First appearance of chain complexes and the Euler characteristic.
- 1900: Poincaré, “Second complément à l’analysis situs”: torsion in homology.
- 1913: Veblen and Alexander, “Manifolds of n dimensions”: proof of Poincaré duality for mod 2 Betti numbers.
- 1915: Alexander, “A proof of the invariance of certain constants of analysis situs”: topological invariance of Betti numbers and torsion coefficients.
- 1923: Künneth, “Über die Bettischen Zahlen einer Produktmannigfaltigkeit”: Künneth formula.
- 1925: Emmy Noether, “Ableitung der Elementarteilertheorie aus der Gruppentheorie”: homology groups.
- 1929: Mayer, “Über abstrakte Topologie”: definitions of chain complexes and their homology.
- 1935: Hurewicz: higher homotopy groups, Hurewicz homomorphism.
- 1938: Whitney, “Tensor products of abelian groups”: definition of tensor products.

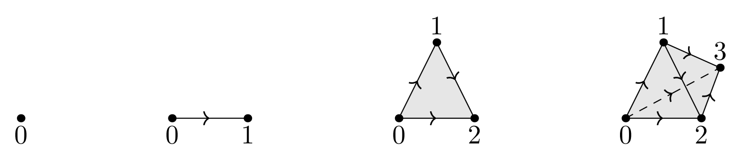
Elementary theory of simplicial sets

4 Simplices

Supplementary sources: [EIISS, §2, §3], [ICHT, §2, §3.1].

Simplicial sets are geometric shapes assembled of simplices that stick together like blocks in a construction toy. The goal of this section and the next two sections is to explain what simplices are. This knowledge will then be used to define simplicial sets.

In this section we formalize the following pictures:



Such objects are known as simplices. An n-dimensional simplex has $n+1$ vertices, typically numbered from 0 to n. We only record the combinatorial information about simplices, which in this case amounts to recording the set of vertices and their ordering. In the above picture, the ordering is indicated by drawing an arrow from a vertex with a smaller number to a vertex with a larger number. The numbers of vertices can be reconstructed from the arrows, starting with the lowest numbers: the vertex 0 only has outgoing arrows and no incoming arrows; the vertex 1 has a single incoming arrow from the vertex 0, all other arrows are outgoing; and so on up to the vertex n, which has no outgoing arrows. Accordingly, we do not record the numbers of vertices below, but only their ordering.

Recall that a (totally) *ordered set* is a pair $(S, \le)$, where S is a set and $\le$ is a binary relation on S (i.e., a subset of $S \times S$) such that $a \le b$ and $b \le a$ implies $a = b$, $a \le b$ and $b \le c$ implies $a \le c$, and $a \le b$ or $b \le a$ is true for any a and b. We define $a < b$ to mean $a \le b$ and $a \ne b$. A *morphism of ordered sets* $f\colon A = (S, \le_A) \to B = (T, \le_B)$ is an *order-preserving* (alias *nondecreasing*) map of sets $g\colon S \to T$, i.e., $a \le b$ implies $f(a) \le f(b)$. Any finite ordered set is isomorphic to an initial segment of natural numbers, i.e., $\{0 < 1 < \cdots < n-1\}$. Thus, its elements can be compared by comparing their numbers.

Definition 4.1. A *simplex* is a finite nonempty ordered set, whose elements are known as *vertices*. A *morphism of simplices* (alias *map of simplices*) is a morphism of ordered sets. Used in 1.3*, 4.0*, 4.1*, 4.2, 4.3, 4.3*, 4.4, 4.5, 4.6, 4.8, 4.9, 5.1*, 5.4, 6.0*, 6.1, 6.2, 6.4, 6.4*, 6.8, 6.9, 6.10, 6.13, 6.14*, 7.0*, 7.1, 7.2, 7.4, 7.5, 7.7, 7.10, 7.11, 7.12, 8.1, 8.3, 8.8, 8.11, 8.14*, 8.16, 8.16*, 8.17, 8.21, 8.22, 9.1, 10.1, 11.0*, 11.10, 12.8, 12.15, 13.7*, 15.6, 15.8*, 17.8*, 17.9, 17.11*, 17.15, 17.22*, 21.7*, 21.28, 21.29, 25.3, 28.1, 28.2, 28.8, 29.15, 34.1.

Up to an isomorphism of simplices (defined below), the only simplices are $\{0 < 1 < 2 < \cdots < n\}$ for all $n \geq 0$. We often stress this fact by using the bold letter $\mathbf{n}$ for such a simplex, and by abuse of notation also for any simplex whose underlying set has $n+1$ elements.

Remark 4.2. One may wonder why we defined simplices as finite nonempty ordered sets instead of simply saying that a simplex is a set $\{0, 1, \ldots, n\}$ and a map of simplices is a nondecreasing map of sets. The principal reason for the above definition is that we want to be able to remove a vertex or several vertices from a simplex and obtain a new simplex. For instance, removing vertices 2 and 4 from the simplex $\{0, 1, 2, 3, 4, 5\}$ yields the simplex $\{0, 1, 3, 5\}$. The naive definition would force us to renumber the vertices of this simplex as $\{0, 1, 2, 3\}$. Such renumberings would in general be quite difficult to keep track of. However, we only really need the relative ordering of vertices and not their numbers, which motivates the above definition.

Exercise 4.3. Prove the following properties of maps of simplices.

- The identity map $\mathrm{id}_{\mathbf{m}}\colon \mathbf{m} \to \mathbf{m}$ is a morphism of simplices.
- If $f\colon \mathbf{l} \to \mathbf{m}$ and $g\colon \mathbf{m} \to \mathbf{n}$ are morphisms of simplices, then their composition $g \circ f\colon \mathbf{l} \to \mathbf{n}$ is also a morphism of simplices.
- The associativity property is satisfied: $h \circ (g \circ f) = (h \circ g) \circ f$ for any morphisms of simplices $f\colon \mathbf{k} \to \mathbf{l}$, $g\colon \mathbf{l} \to \mathbf{m}$, $h\colon \mathbf{m} \to \mathbf{n}$.
- The unitality property is satisfied: $\mathrm{id}_{\mathbf{l}} \circ f = f \circ \mathrm{id}_{\mathbf{k}} = f$ for any map of simplices $f\colon \mathbf{k} \to \mathbf{l}$.

Used in 11.0*, 11.10.

These properties imply that the composition of finitely many morphisms of simplices does not depend on the order of composition and is again a morphism of simplices, so we can simply denote it by $f_n \circ \cdots \circ f_1$.

Definition 4.4. An *isomorphism of simplices* is a morphism of simplices $f\colon \mathbf{m} \to \mathbf{n}$ for which there is a morphism $g\colon \mathbf{n} \to \mathbf{m}$ such that $g \circ f = \mathrm{id}_{\mathbf{m}}$ and $f \circ g = \mathrm{id}_{\mathbf{n}}$. Used in 4.1*, 4.6, 15.6.

Definition 4.5. If $\mathbf{m} = (V, \leq)$ is a simplex, we set $\mathsf{U}(\mathbf{m}) = V$ and refer to it as the *underlying set of a simplex* $\mathbf{m}$. Likewise, if $f\colon \mathbf{m} \to \mathbf{n}$ is a morphism of simplices from $\mathbf{m} = (V_{\mathbf{m}}, \leq_{\mathbf{m}})$ to $\mathbf{n} = (V_{\mathbf{n}}, \leq_{\mathbf{n}})$, then we set $\mathsf{U}(f)\colon \mathsf{U}(\mathbf{m}) \to \mathsf{U}(\mathbf{n})$ to the underlying map of sets $V_{\mathbf{m}} \to V_{\mathbf{n}}$ and refer to it as the *underlying map of a morphism of simplices* f. Used in 12.5.

Exercise 4.6. Show that a map of simplices is an isomorphism of simplices if and only if its underlying map of sets is a bijection.

From the definition of U we immediately see that $\mathsf{U}(g \circ f) = \mathsf{U}(g) \circ \mathsf{U}(f)$ and $\mathsf{U}(\mathrm{id}_{\mathbf{m}}) = \mathrm{id}_{\mathsf{U}(\mathbf{m})}$.

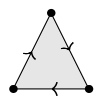
The figures of simplices above indicate that an n-dimensional simplex has $n+1$ vertices, e.g., a triangle is 2-dimensional and has 3 vertices.

Definition 4.7. The *dimension of a simplex* $\mathbf{m}$ is an integer number, denoted by $\dim \mathbf{m}$ and defined to be $\#\mathsf{U}(\mathbf{m}) - 1$, where $\#$ denotes cardinality. Used in 1.3*, 4.8, 6.1, 6.11, 7.13, 10.4*, 15.8*, 15.9, 15.10, 16.10, 17.17*, 17.22*, 19.1, 19.3, 19.4*.

Remark 4.8. It may be unclear why one would want an ordering on the set of vertices of a simplex. After all, the geometric pictures do not seem to indicate the existence of such an ordering. Indeed, one could drop the data of an ordering altogether, obtaining *symmetric simplices*, which give rise to *symmetric simplicial sets*. The homotopy theory of symmetric simplicial sets is equivalent (in the sense defined later) to the homotopy theory of simplicial sets, so from an abstract point of view there is no difference between the two notions. However, there is a substantial practical difference, which is manifested in the fact that for any simplex $\mathbf{m}$ there is exactly one isomorphism $\mathbf{m} \to \mathbf{m}$, namely, $\mathrm{id}_{\mathbf{m}}$, whereas if $\mathbf{m}$ was a symmetric simplex, any permutation of $\mathsf{U}(\mathbf{m})$ would give such an isomorphism. Taken together, such isomorphisms would form a symmetric group of order $\dim \mathbf{m} + 1$, a nontrivial group. Having a trivial group of automorphisms makes the exposition considerably simpler, which is why we do not use symmetric simplices. The idea of using ordered simplices was introduced by Eilenberg in 1943 [SHTe]. His paper discusses the historical context of this definition.

Remark 4.9. If we allow the empty ordered set as a simplex, we get *augmented simplices*. These give rise to *augmented simplicial sets*, which are an important ingredient in many constructions, but their homotopy theory is not equivalent to that of simplicial sets.

Warning 4.10. One must remember that not every picture that looks like a simplex specifies a simplex. The picture below does not correspond to any simplex because the arrows do not specify an antisymmetric relation.



5 Geometric realization of simplices

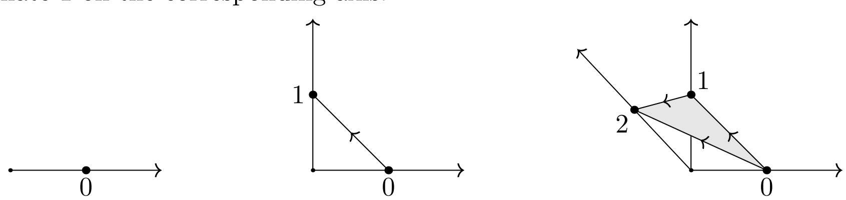
Supplementary sources: [EIISS, §2, §3], [ICHT, §2].

Recall that the vector space $\mathbf{R}^5$ can be thought of as the set of functions $\{0,1,2,3,4\} \to \mathbf{R}$. Below, the set $\{0,1,2,3,4\}$ is replaced by the finite set $\mathsf{U}(\mathbf{m})$.

Definition 5.1. The *geometric realization of a simplex* $\mathbf{m}$ is the set $|\mathbf{m}| = \{x\colon \mathsf{U}(\mathbf{m}) \to \mathbf{R}_{\geq 0} \mid \sum_{s\in\mathsf{U}(\mathbf{m})} x_s = 1\}$. The *geometric realization of a map of simplices* $f\colon \mathbf{m} \to \mathbf{n}$ is the map of sets $|f|\colon |\mathbf{m}| \to |\mathbf{n}|$ that sends $x \in |\mathbf{m}|$ to $y \in |\mathbf{n}|$ such that $y_t = \sum_{s\in\mathsf{U}(\mathbf{m}):f(s)=t} x_s$. Used in 1.3*, 6.0*, 6.6, 7.0*, 12.6, 17.9, 17.11*.

Observe that $\sum_{t\in\mathsf{U}(\mathbf{n})} y_t = \sum_{t\in\mathsf{U}(\mathbf{n})} \sum_{s\in\mathsf{U}(\mathbf{m}):f(s)=t} x_s = \sum_{s\in\mathsf{U}(\mathbf{m})} x_s = 1$, so the above formula indeed defines a map $|\mathbf{m}| \to |\mathbf{n}|$.

We examine the low-dimensional cases of simplices $\mathbf{m} = \{0 < 1 < \cdots < m\}$ for $m \leq 2$. The set $|\mathbf{0}| = \{1\} \subset \mathbf{R}^1$ is a point. In particular, maps $\mathbf{0} \to \mathbf{m}$ pick some vertex of $\mathbf{m}$ and their geometric realization is a map $|\mathbf{0}| \to |\mathbf{m}|$, i.e., a point in $|\mathbf{m}|$, which we refer to as a *geometric vertex* of $|\mathbf{m}|$. We have $|\mathbf{1}| = \{(x, 1-x) \mid x \in [0,1]\}$ and the two vertices of $|\mathbf{1}|$ are $\mathrm{e}_0 = (1,0)$ and $\mathrm{e}_1 = (0,1)$. Finally, $|\mathbf{2}| = \{(x,y,1-x-y) \mid x,y,x+y \in [0,1]\}$ and the vertices are $\mathrm{e}_0 = (1,0,0)$, $\mathrm{e}_1 = (0,1,0)$, $\mathrm{e}_2 = (0,0,1)$. We record these observations in the following pictures, where labels denote the vertices of $\mathbf{m}$ and each geometric vertex has coordinate 1 on the corresponding axis:



Thus, the geometric idea behind the definition of geometric realization is clear by now: an m-dimensional simplex with vertices $\{0 < 1 < 2 < \cdots < m\}$ is realized as a subset of $\mathbf{R}^{m+1}$. Any vertex $i \in \mathsf{U}(\mathbf{m})$ is realized by the ith unit vector e_i (the ith coordinate is 1 and the others are 0). Furthermore, any point in $|\mathbf{m}|$ is a unique convex combination of vertices. (A convex combination is a linear combination with nonnegative coefficients that sum to 1.) Given a morphism of simplices $f\colon \mathbf{m} \to \mathbf{n}$, it gives rise to a unique linear map $\mathbf{R}^f\colon \mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$ that sends the unit vector $\mathrm{e}_i \in \mathbf{R}^{m+1}$ corresponding to a vertex $i \in \mathsf{U}(\mathbf{m})$ to the unit vector $\mathrm{e}_{f(i)} \in \mathbf{R}^{n+1}$. We have $\mathbf{R}^f(|\mathbf{m}|) \subset |\mathbf{n}|$ and the (co)restriction of $\mathbf{R}^f$ to $|\mathbf{m}|$ and $|\mathbf{n}|$ is precisely $|f|$.

Remark 5.2. We have $|\mathrm{id}_{\mathbf{m}}| = \mathrm{id}_{|\mathbf{m}|}$ and $|g \circ f| = |g| \circ |f|$ for any $f\colon \mathbf{m} \to \mathbf{n}$ and $g\colon \mathbf{n} \to \mathbf{p}$. For the latter relation, observe that evaluating both sides on some $x \in |\mathbf{m}|$ and taking the uth component ($u \in \mathsf{U}(\mathbf{p})$) yields

$$\sum_{s\in\mathsf{U}(\mathbf{m}):g(f(s))=u} x_s = \sum_{t\in\mathsf{U}(\mathbf{n}):g(t)=u} \sum_{s\in\mathsf{U}(\mathbf{m}):f(s)=t} x_s,$$

which holds because s runs over identical sets in both cases. Used in 12.6.

Remark 5.3. Depending on the situation at hand, one may want to equip the set $|\mathbf{m}|$ with a structure of a topological space, smooth manifold, etc. In algebraic geometry $\mathbf{R}_{\geq 0}$ does not make sense, so one uses instead $\mathbf{A}^1$, the affine line. Used in 12.6.

Exercise 5.4. Suppose $f\colon \mathbf{m} \to \mathbf{n}$ is a map of simplices. Show that $\mathsf{U}(f)$ is injective if and only if $|f|$ is. Show that $\mathsf{U}(f)$ is surjective if and only if $|f|$ is. Used in 6.6.

6 Maps of simplices

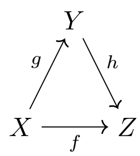
Supplementary sources: [EIISS, §2, §3], [ICHT, §2, §3.1].

We now examine in more detail the notion of a map of simplices and its geometric realization.

Definition 6.1. We say that a map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ is injective respectively surjective if $\mathsf{U}(f)$ is. The *relative dimension* of f is defined as $\dim f = \dim \mathbf{m} - \dim \mathbf{n}$ and the *relative codimension* of f is defined as $\operatorname{codim} f = -\dim f = \dim \mathbf{n} - \dim \mathbf{m}$. A *face inclusion* (alias *coface map*) is an injective map of simplices of relative codimension 1. A *degenerate map* (alias *edge collapse*, *codegeneracy map*) is a surjective map of simplices of relative dimension 1. Used in 6.1, 6.2, 6.4*, 6.5, 6.6, 6.7, 6.8, 6.9, 6.9*, 6.10, 7.0*, 7.4, 7.5, 7.13*, 7.14, 8.11, 8.22, 10.2, 10.3, 39.9*.

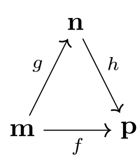
Example 6.2. The map of simplices $\{0 < 1\} \to \{0 < 1 < 2\}$ that sends $0 \mapsto 0$ and $0 \mapsto 2$ is an injective map of simplices of relative codimension 1, hence a face inclusion. Geometrically, it is a map from an interval to a triangle that covers the side opposite of the vertex 1. The map of simplices $\{0 < 1 < 2 < 3\} \to \{0 < 1\}$ that sends $0, 1 \mapsto 0$ and $2, 3 \mapsto 1$ is a surjective map of simplices of relative dimension 2, hence not a degenerate map.

Definition 6.3. A *factorization* of a map (of objects of any type) $f\colon X \to Z$ is a triple (Y, g, h), where Y is an object of the same type as X and Z and $g\colon X \to Y$ and $h\colon Y \to Z$ are maps such that $f = h \circ g$.



Used in 6.4, 6.4*.

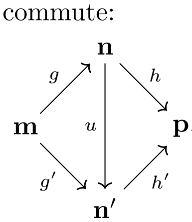
Lemma 6.4. Any map of simplices $f\colon \mathbf{m} \to \mathbf{p}$ admits a unique factorization $(\mathbf{n}, g, h)$, where $\mathbf{n}$ is a simplex and $g\colon \mathbf{m} \to \mathbf{n}$ and $h\colon \mathbf{n} \to \mathbf{p}$ are maps of simplices such that $f = h \circ g$, g is surjective and h is injective.



Used in 6.14*, 10.4*.

Proof. For existence, construct a simplex $\mathbf{n}$ by setting its underlying set to the image of $\mathsf{U}(f)$ and equipping it with the ordering induced from $\mathbf{p}$. The map $g\colon \mathbf{m} \to \mathbf{n}$ is obtained by restricting the codomain of $f\colon \mathbf{m} \to \mathbf{p}$ to $\mathbf{n}$. The map $h\colon \mathbf{n} \to \mathbf{p}$ is the inclusion map. By construction, $f = h \circ g$, the map g is surjective, and h is injective.

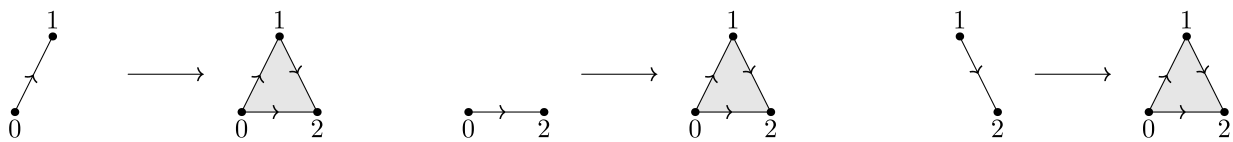
For uniqueness, suppose that $(\mathbf{n}', g', h')$ is another such factorization. We claim that there is a unique $u\colon \mathbf{n} \to \mathbf{n}'$ that makes the following triangles commute:



The requirement that u is unique is the precise sense in which the factorization is unique. A posteriori, the map u will turn out to be an isomorphism. We claim that h and h' have the same image in $\mathbf{p}$, which allows us to construct an isomorphism u between their sources. Indeed, the image of h coincides with the image of $h \circ g = f$ because g is surjective. Likewise for h'. If we corestrict the codomains of h and h' to their images in $\mathbf{p}$, the resulting maps H and H' are isomorphisms because h and h' are injective. Thus, we can take $u = (H')^{-1} \circ H$, which makes the right triangle commute automatically. It remains to verify that the left triangle commutes. Indeed, since h' is an injection, we have $u \circ g = g'$ if and only if $h' \circ u \circ g = h' \circ g'$. We have $h' \circ u \circ g = h \circ g = f$ and $h' \circ g' = f$, as desired. ▮

There is a counterpart of the above lemma that we will not need: any map of simplices $f\colon \mathbf{m} \to \mathbf{p}$ admits a factorization $(\mathbf{n}, g, h)$, where $\mathbf{n}$ is a simplex and $g\colon \mathbf{m} \to \mathbf{n}$ and $h\colon \mathbf{n} \to \mathbf{p}$ are maps of simplices such that $f = h \circ g$, g is injective and h is surjective. (Surjective and injective are exchanged, and uniqueness is dropped compared to the previous lemma.)

For a fixed $\mathbf{n}$, injective maps $f\colon \mathbf{m} \to \mathbf{n}$ can be identified with nonempty subsets of $\mathsf{U}(\mathbf{n})$. Indeed, the image of $\mathsf{U}(f)$ is a nonempty subset of $\mathsf{U}(\mathbf{n})$. Different injective maps yield different subsets, and any nonempty subset of $\mathsf{U}(\mathbf{n})$ can be equipped with the induced order thereby giving rise to an injective map of simplices. Thus, an n-dimensional simplex admits exactly $2^{n+1} - 1$ injective maps into it, which correspond to the $2^{n+1} - 1$ nonempty subsets of the set $\mathsf{U}(\mathbf{n})$, where $\#\mathsf{U}(\mathbf{n}) = n + 1$. Of these maps, exactly $\binom{n+1}{k+1}$ maps have domain of dimension k because the image of such a map must be a subset of $\mathsf{U}(\mathbf{n})$ of cardinality $k + 1$, and there are $\binom{n+1}{k+1}$ such subsets. For instance, a 0-simplex $\mathbf{0}$ has a single injective map with image $\{0\}$ (itself), a 1-simplex $\mathbf{1} = \{0 < 1\}$ has maps with images $\{0\}$, $\{1\}$, and $\{0 < 1\}$, a 2-simplex $\mathbf{2} = \{0 < 1 < 2\}$ has maps with images $\{0\}$, $\{1\}$, $\{2\}$, $\{0 < 1\}$, $\{0 < 2\}$, $\{1 < 2\}$, and $\{0 < 1 < 2\}$. Here are the three face inclusions for $\mathbf{2}$:



The images of 0-simplices can be identified with vertices, depicted by dots in our pictures. The images of 1-simplices are given by pairs of vertices $v_0 < v_1$ and are depicted by arrows. The images of 2-simplices are specified by a triple vertices $v_0 < v_1 < v_2$ and are depicted by shaded triangles. We have no good way to depict simplices of dimension 3 and higher, so this information must be inferred from the context.

Exercise 6.5. Prove that an injective map of simplices of codimension $d > 0$ can be presented as a composition of d face inclusions. Is such a presentation unique? Used in 6.14*.

Example 6.6. By Exercise 5.4, the geometric realization of a surjective map of simplices is also surjective. The easiest examples are given by maps $\mathbf{m} \to \mathbf{0}$ that send all vertices of $\mathbf{m}$ to the only vertex of $\mathbf{0}$. The next easiest example are given by two maps $f, g\colon \mathbf{2} \to \mathbf{1}$ that send $0 \mapsto 0$, $2 \mapsto 1$, and $1 \mapsto 0$ respectively $1 \mapsto 1$. These maps can be depicted by the following horizontal projection maps:

Used in 7.11.

Exercise 6.7. Prove that any surjective map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ of relative dimension $d > 0$ can be presented as a composition of degenerate maps. Is such a presentation unique? Used in 6.14*.

Exercise 6.8. Suppose $f\colon \mathbf{m} \to \mathbf{n}$ is a map of simplices. Prove that f is surjective if and only if there is a map of simplices $g\colon \mathbf{n} \to \mathbf{m}$ such that $f \circ g = \mathrm{id}_{\mathbf{n}}$. Prove that f is injective if and only if there is a map of simplices $g\colon \mathbf{n} \to \mathbf{m}$ such that $g \circ f = \mathrm{id}_{\mathbf{m}}$. Used in 10.4*.

Remark 6.9. One may question the desirability of having degenerate maps in the first place. Indeed, one can allow only injective maps of simplices as morphisms, which give rise to *semisimplicial sets*. The homotopy theory of semisimplicial sets is equivalent (in the sense defined later) to the homotopy theory of simplicial sets, so from an abstract point of view there is no difference between the two notions. However, there is a substantial practical difference, which is manifested in the fact that the model structure (to be defined later) on semisimplicial sets is not right proper, and a semisimplicial set that is not weakly contractible in this model structure must have infinitely many simplices, which makes computations difficult. Used in 7.5.

We finish this section by introducing notation for face inclusions and degenerate maps and establishing some identities between them.

Definition 6.10. Suppose $\mathbf{m}$ is a simplex and $i \in \mathsf{U}(\mathbf{m})$ is a vertex of $\mathbf{m}$. Denote by $\mathrm{d}^{\mathbf{m},i}\colon \mathbf{m} \setminus i \to \mathbf{m}$ the face inclusion that includes the simplex $\mathbf{m} \setminus i$ obtained by removing the vertex i from $\mathbf{m}$ (so that we have $\cdots < i-2 < i-1 < i+1 < i+2 < \cdots$) into the simplex $\mathbf{m}$, retaining the relative ordering of vertices. Denote by $\mathrm{s}^{\mathbf{m},i}\colon \mathbf{m} \sqcup i \to \mathbf{m}$ the degenerate map that sends the simplex $\mathbf{m} \sqcup i$ obtained by repeating the vertex i in $\mathbf{m}$ (so that we have $\cdots < i-1 < i' < i'' < i+1 < \cdots$) into the simplex $\mathbf{m}$, sending both i' and i'' into i. Used in 7.4.

Notation 6.11. The traditional notation for $\mathrm{d}^{\mathbf{m},i}$ and $\mathrm{s}^{\mathbf{m},j}$ is d^i and s^j. In the traditional notation, $\mathbf{m}$ must be inferred from the context. Furthermore, in the traditional notation i and j are no longer elements of $\mathsf{U}(\mathbf{m})$, but rather integer numbers in $[0, \dim \mathbf{m}]$. Thus, $\mathrm{d}^i = \mathrm{d}^{\mathbf{m},v_i}$, where v_i denotes the ith element of $\mathbf{m}$, with the smallest element being the 0th element. Used in 7.4.

Example and warning 6.12. Suppose $m = 3$ (hence $\mathbf{m} = \{0 < 1 < 2 < 3\}$) and $i = 2$. Then $\mathrm{d}^{\mathbf{m},i}\colon \{0 < 1 < 3\} \to \{0 < 1 < 2 < 3\}$ is the inclusion map. Notice how the source no longer has the standard numbering. We can renumber it and obtain a map $\{0 < 1 < 2\} \to \{0 < 1 < 2 < 3\}$ that sends $0 \mapsto 0$, $1 \mapsto 1$, $2 \mapsto 3$. In particular, $\mathrm{d}^2 = \mathrm{d}^{\{0<1<3\},3}$ because the vertex 3 has number 2 in $\{0 < 1 < 3\}$. Likewise, $\mathrm{s}^{\mathbf{m},i}\colon \{0 < 1 < 2' < 2'' < 3\} \to \{0 < 1 < 2 < 3\}$ is the obvious map, which sends $2'$ and $2''$ to 2. Again, the numbering of the source is nonstandard, and if we renumber it, we get the map $\{0 < 1 < 2 < 3 < 4\} \to \{0 < 1 < 2 < 3\}$ that sends $0 \mapsto 0$, $1 \mapsto 1$, $2 \mapsto 2$, $3 \mapsto 2$, $4 \mapsto 3$.

Example 6.13. Suppose $\mathbf{m}$ is a simplex and $i, j \in \mathsf{U}(\mathbf{m})$ are two different vertices. We have the following commutative diagram of simplices and maps of simplices, where all maps are inclusions:

$$\begin{array}{ccc} \mathbf{m} \setminus \{i,j\} & \xrightarrow{\mathrm{d}^{\mathbf{m}\setminus\{i\},j}} & \mathbf{m} \setminus \{i\} \\ {\scriptstyle \mathrm{d}^{\mathbf{m}\setminus\{j\},i}}\downarrow & & \downarrow{\scriptstyle \mathrm{d}^{\mathbf{m},i}} \\ \mathbf{m} \setminus \{j\} & \xrightarrow{\mathrm{d}^{\mathbf{m},j}} & \mathbf{m}. \end{array}$$

Thus, $\mathrm{d}^{\mathbf{m},j} \circ \mathrm{d}^{\mathbf{m}\setminus\{j\},i} = \mathrm{d}^{\mathbf{m},i} \circ \mathrm{d}^{\mathbf{m}\setminus\{i\},j}$. In the traditional notation, we must omit the simplex. Furthermore, vertices must be replaced by their numbers (the numbering starts from 0). In the simplex $\mathbf{m} = \{0, \ldots, m\}$ vertices coincide with their numbers, so $\mathrm{d}^{\mathbf{m},i} = \mathrm{d}^i$ and $\mathrm{d}^{\mathbf{m},j} = \mathrm{d}^j$. Suppose without the loss of generality that $i < j$ (we could always exchange them if $i > j$). Then in the simplex $\mathbf{m} \setminus \{j\} = \{0, \ldots, j-1, j+1, \ldots, m\}$ the vertex i has number i because $0 \le i \le j-1$, so $\mathrm{d}^{\mathbf{m}\setminus\{j\},i} = \mathrm{d}^i$. However, in the simplex $\mathbf{m} \setminus \{i\} = \{0, \ldots, i-1, i+1, \ldots, m\}$ the vertex j has number $j-1$ because $i+1 \le j \le m$ and the vertex i has been removed, so the numbers of vertices following it are shifted by 1. Thus, $\mathrm{d}^{\mathbf{m}\setminus\{i\},j} = \mathrm{d}^{j-1}$. Accordingly, in the traditional notation the commutativity of the above diagram is expressed as $\mathrm{d}^j\mathrm{d}^i = \mathrm{d}^i\mathrm{d}^{j-1}$, which may obscure the fact that both sides work with the same pair of vertices in $\mathbf{m}$.

Exercise 6.14. Verify the following *cosimplicial identities* by expanding the definitions of d^i and s^j:

$$\mathrm{d}^j\mathrm{d}^i = \mathrm{d}^i\mathrm{d}^{j-1} \qquad (i < j)$$

$$\mathrm{s}^j\mathrm{s}^i = \mathrm{s}^i\mathrm{s}^{j+1} \qquad (i \le j)$$

$$\mathrm{s}^j\mathrm{d}^i = \begin{cases} \mathrm{d}^i\mathrm{s}^{j-1}, & i < j \\ \mathrm{id}, & i = j \text{ or } i = j+1 \\ \mathrm{d}^{i-1}\mathrm{s}^j, & i > j+1. \end{cases}$$

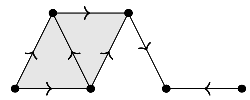
Here i and j are assumed to refer to vertices using the standard numbering, i.e., for a simplex of dimension n we take all integers between 0 and n inclusive. In particular, one must take into account the above warning about the renumbering of sources of s and d. Used in 6.14*, 15.8*.

The significance of these identities lies in the fact that any map of simplices can be presented as a composition of maps of the form d^i and s^j, as shown in Lemma 6.4, Exercise 6.5, and Exercise 6.7. One can show that the cosimplicial identities generate all possible equalities between formal compositions of maps d^i and s^j, i.e., if the compositions of two different chains of such maps are equal, then we can transform one chain into another by applying some sequence of simplicial identities. This fact can be used to give a very different definition of a simplicial set (and simplicial maps) than the one given below: a simplicial set is given by a sequence of sets X_n for all integer $n \geq 0$ together with maps $\mathrm{d}_i\colon X_n \to X_{n-1}$ for all $0 \leq i \leq n$ $(n > 0)$ and $\mathrm{s}_j\colon X_n \to X_{n+1}$ for all $0 \leq j \leq n$ $(n \geq 0)$ such that the *simplicial identities* are satisfied, which are obtained from the cosimplicial identities by reversing the order of composition and replacing superscripts by subscripts. This definition was in fact commonly used during the early period of development of simplicial methods, which obscured their geometric nature to newcomers.

7 Simplicial sets

Supplementary sources: [EIISS, §2, §3], [ICHT, §2, §3, §4].

Our goal in this section is to formalize pictures like this:



Namely, we have a bunch of simplices that may overlap only if their intersection is again a simplex, and the ordering of vertices is respected, i.e., we cannot glue edges with opposite orientations. We only record the combinatorial discrete data that shows how simplices stick together to each other, not their spatial position or orientation. Such objects are known as *simplicial sets*.

Likewise, *maps* of simplicial sets (alias *simplicial maps*) can be thought of as piecewise-linear continuous maps that map simplices to simplices just like in the definition of a geometric realization. However, once again we only record the combinatorial discrete data (i.e., what simplex maps to what simplex), and not the actual continuous map. This is entirely analogous to how we talk about maps of simplices instead of their geometric realizations.

In particular, for any simplex $\mathbf{m}$ we expect to have an associated simplicial set, denoted by $\Delta^{\mathbf{m}}$, which "looks" just like $\mathbf{m}$. Likewise, any map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ should give rise to a map of simplicial sets, denoted by $\Delta^f\colon \Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$. Furthermore, we expect that *any* map of simplicial sets of the form $\Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$ is equal to Δ^f for some $f\colon \mathbf{m} \to \mathbf{n}$. Thus, while simplices and simplicial sets are objects of a different type, we could consider simplices to be a special case of simplicial sets: the set of maps of simplicial sets $\Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$ can be identified with the set of maps of simplices $\mathbf{m} \to \mathbf{n}$.

Suppose now that somebody else has managed to construct simplicial sets and simplicial maps as described above. This means, specifically, that we are given a collection of things, called simplicial sets, and for any pair of simplicial sets X, Y we are given a set $\mathsf{hom}(X, Y)$ of maps $X \to Y$. We have no way of examining the internal structure of simplicial sets or maps between them. (For instance, we know the cardinality of $\mathsf{hom}(X, Y)$, but if we pick a particular element of $\mathsf{hom}(X, Y)$, we have no way to say anything specific about this element.) We also assume that we are given all compositions of simplicial maps, namely, for any simplicial sets X, Y, Z, we have a map $\mathsf{hom}(Y, Z) \times \mathsf{hom}(X, Y) \to \mathsf{hom}(X, Z)$ that performs the role of composition. (Again, this map is a black box: we put in an element of $\mathsf{hom}(Y, Z)$ and $\mathsf{hom}(X, Y)$ and it spits out an element of $\mathsf{hom}(X, Z)$ for us.)

Additionally, we assume that we are given simplicial sets $\Delta^{\mathbf{m}}$ that behave as described above, in particular, we have bijective maps $\{\mathbf{m} \to \mathbf{n}\} \to \mathsf{hom}(\Delta^{\mathbf{m}}, \Delta^{\mathbf{n}})$, where $\{\mathbf{m} \to \mathbf{n}\}$ denotes the set of all maps of simplices $\mathbf{m} \to \mathbf{n}$.

Even though we do not yet know what simplicial sets are, we could look at the set of simplicial maps of the form $\Delta^{\mathbf{m}} \to X$, which we denote by $X_{\mathbf{m}}$. For instance, for the above picture, $X_{\mathbf{0}}$ has 6 elements, corresponding to the 6 vertices in the picture. The set $X_{\mathbf{1}}$ has $6+7$ elements, where the 6 elements correspond to the 6 maps $\Delta^{\mathbf{1}} \to X$ that are given by the compositions $\Delta^{\mathbf{1}} \to \Delta^{\mathbf{0}} \to X$ for each of the 6 possible maps $\Delta^{\mathbf{0}} \to X$, whereas the other 7 elements correspond to the 7 edges in the picture. The set $X_{\mathbf{2}}$ has $6+7+7+2$ elements, where the 6 elements correspond to the 6 maps given by the compositions $\Delta^{\mathbf{2}} \to \Delta^{\mathbf{0}} \to X$ for each of the 6 possible maps $\Delta^{\mathbf{0}} \to X$, the $7+7$ elements correspond to the $7 \cdot 2$ maps given by the compositions $\Delta^{\mathbf{2}} \to \Delta^{\mathbf{1}} \to X$ for each of the 7 possible maps $\Delta^{\mathbf{1}} \to X$ and 2 possible surjective maps $\Delta^{\mathbf{2}} \to \Delta^{\mathbf{1}}$, and the remaining 2 elements correspond to the 2 solid triangles in the picture.

Notice how we managed to extract quite a bit of information about the above picture just by looking at the cardinalities of sets $X_{\mathbf{m}}$ for various $\mathbf{m}$. For instance, we already know that our picture must contain 6 vertices, 7 edges, and 2 triangles. What we do not know yet is how these vertices, edges, and triangles stick together. This is where the ability to compose simplicial maps comes in. Suppose $f\colon \mathbf{m} \to \mathbf{n}$ is a map of simplices. An element of $X_{\mathbf{n}}$, i.e., a simplicial map $\Delta^{\mathbf{n}} \to X$, can be composed with the simplicial map $\Delta^f\colon \Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$, yielding a simplicial map of the form $\Delta^{\mathbf{m}} \to X$. This gives us a map of sets $X_f\colon X_{\mathbf{n}} \to X_{\mathbf{m}}$, which is known as a simplicial structure map. For instance, take f to be the only map of simplices $\mathbf{1} \to \mathbf{0}$. The associated simplicial structure map $X_f\colon X_{\mathbf{0}} \to X_{\mathbf{1}}$ sends the 6 elements of $X_{\mathbf{0}}$ to their 6 counterparts in $X_{\mathbf{1}}$, as we already described above. Likewise, the $6+7+7$ out of the $6+7+7+2$ elements of $X_{\mathbf{2}}$ can be obtained via the map $\Delta^{\mathbf{2}} \to \Delta^{\mathbf{0}}$ and the two maps $\Delta^{\mathbf{2}} \to \Delta^{\mathbf{1}}$.

The information about the endpoints of edges and faces of triangles can be extracted in a similar manner, using injective maps of simplices $f\colon \mathbf{m} \to \mathbf{n}$. For instance, the two maps $\mathrm{d}^1, \mathrm{d}^0\colon \mathbf{0} \to \mathbf{1}$ yields two maps $\mathrm{d}_1, \mathrm{d}_0\colon X_{\mathbf{1}} \to X_{\mathbf{0}}$. The visual meaning is as follows: an edge $e \in X_{\mathbf{1}}$ is depicted by an arrow that goes from the vertex $\mathrm{d}_1(e)$ to the vertex $\mathrm{d}_0(e)$. (Here the index of d denotes the vertex that is *removed* from the edge, i.e., d_1 removes the 1st vertex and leaves the 0th vertex.) Similarly, the three maps $\mathrm{d}_0, \mathrm{d}_1, \mathrm{d}_2\colon X_{\mathbf{2}} \to X_{\mathbf{1}}$ send an element of $X_{\mathbf{2}}$ (depicted by a triangle) to its edge that is opposite to the initial, middle, or terminal vertex respectively.

Thus, we see that the collection of sets $X_{\mathbf{m}}$ for all simplices $\mathbf{m}$ and maps of sets X_f for any map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ captures pretty much everything we want to know about a simplicial set: we know what the simplices in the picture are and how they stick together. Since we are not concerned with a particular embedding or orientation of a simplex in any kind of ambient space, this is really all we want to know. Thus, we could say that a simplicial set could be *reconstructed* from $X_{\mathbf{m}}$ and X_f. "Reconstructed" does not mean that we can recover the original simplicial set, but rather something isomorphic to it.

Not every collection of sets $X_{\mathbf{m}}$ and maps of sets X_f could possibly come from a simplicial set X. For instance, given maps of simplices $f\colon \mathbf{m} \to \mathbf{n}$ and $g\colon \mathbf{n} \to \mathbf{p}$, we could compose the resulting maps $X_f\colon X_{\mathbf{n}} \to X_{\mathbf{m}}$ and $X_g\colon X_{\mathbf{p}} \to X_{\mathbf{n}}$, obtaining a map $X_f \circ X_g\colon X_{\mathbf{p}} \to X_{\mathbf{m}}$. When applied to an element of $X_{\mathbf{p}}$, i.e., a simplicial map $\alpha\colon \Delta^{\mathbf{p}} \to X$, we get

$$(X_f \circ X_g)(\alpha) = X_f(X_g(\alpha)) = X_f(\alpha \circ g) = (\alpha \circ g) \circ f = \alpha \circ (g \circ f) = X_{g \circ f}(\alpha).$$

Thus, $X_f \circ X_g = X_{g \circ f}$. Analogously, for any $\beta \in X_{\mathbf{m}}$ we have $X_{\mathrm{id}_{\mathbf{m}}}(\beta) = \beta \circ \mathrm{id}_{\mathbf{m}} = \beta = (\mathrm{id}_{X_{\mathbf{m}}})(\beta)$, so $X_{\mathrm{id}_{\mathbf{m}}} = \mathrm{id}_{X_{\mathbf{m}}}$. These two properties taken together are referred to as the functoriality property.

Thus, in order for a collection of sets $X_{\mathbf{m}}$ and maps of sets X_f to come from an actual simplicial set, they must satisfy the functoriality property given above. The definition of a simplicial set below relies on two insights:

- The data of $X_{\mathbf{m}}$ and X_f is sufficient to reconstruct X;
- The properties of $X_{\mathbf{m}}$ and X_f can be formulated without any reference to the nature of elements of $X_{\mathbf{m}}$, e.g., we do not have to assume that elements of $X_{\mathbf{m}}$ are maps of any sort.

Definition 7.1. A *simplicial set* X is specified as follows. For any simplex $\mathbf{m}$ we specify a set of *m-simplices* of X, denoted by $X_{\mathbf{m}}$. For any map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ we specify a *simplicial structure map* $X_f\colon X_{\mathbf{n}} \to X_{\mathbf{m}}$. We require that the following *functoriality property for simplicial sets* is satisfied: (1) $X_{\mathrm{id}_{\mathbf{m}}} = \mathrm{id}_{X_{\mathbf{m}}}$ for any simplex $\mathbf{m}$ and (2) $X_{g \circ f} = X_f \circ X_g$ for any maps of simplices $f\colon \mathbf{m} \to \mathbf{n}$ and $g\colon \mathbf{n} \to \mathbf{p}$. Used in 1.0*, 1.2*, 2.0*, 4.0*, 4.9, 6.9, 6.14*, 7.0*, 7.2, 7.3, 7.4, 7.5, 7.6, 7.6*, 7.7*, 7.8, 7.9, 7.10, 7.11, 7.13*, 7.14, 7.14*, 7.15, 8.2, 8.3, 8.7, 8.11, 8.13, 8.14, 8.14*, 8.15, 8.16, 8.19, 8.20, 8.21, 8.22, 9.0*, 9.2, 9.3, 9.3*, 9.7, 9.10, 9.11, 10.0*, 10.1, 10.2, 10.3*, 10.4, 11.11, 11.12, 11.13, 12.8, 12.15, 13.0*, 13.8, 13.10, 13.11, 13.12*, 13.19*, 13.20, 13.21*, 13.22*, 13.23, 13.24, 13.25*, 13.26, 13.28, 13.29, 14.4*, 14.13, S.0*, 15.0*, 15.2, 15.4*, 15.5, 15.7, 15.8, 15.12, 15.16, 16.0*, 16.13, 17.1*, 17.2*, 17.3, 17.3*, 17.5, 17.6*, 17.11*, 17.14*, 17.15, 17.17*, 17.21, 17.22*, 18.1, 18.4, 18.7, 19.2, 19.4, 19.5, 20.7, 20.8, 20.11, 20.12, 21.0*, 21.10, 21.13*, 21.20, 21.22, 21.24*, 21.26, 21.27, 21.29, 22.0*, 22.8, 22.9, 22.23, 23.1, 23.2, 23.5, 23.7, 23.11, 24.2, 24.5, 24.6, 24.7, 24.8, 24.9, 24.12, 25.3, 28.1, 28.3, 28.4, 28.4*, 28.5*, 28.7*, 28.8, 28.9, 29.16, 29.17, 29.24, 29.25, 30.5, 32.1, 33.1, 33.10, 33.11, 34.0*, 34.1, 34.3, 34.4, 35.2, 36.1, 36.2, 36.3, 36.4, 36.5, 36.7, 36.8, 36.10, 36.12, 37.1, 37.2, 38.1, 38.2, 38.3, 38.4, 39.1, 39.2, 39.5, 39.6*, 39.7, 39.8, 39.9, 39.9*, 39.10*, 39.14, 39.15, 39.19, 39.22, 40.2, 40.3, 40.11, 40.12, 40.13*, 41.2, 42.3, 42.4, 42.7, 42.8, 42.9, 43.3, 45.3*, 45.5, 45.6, 45.8*, 52.1.

Remark 7.2. The simplicial structure map $X_f\colon X_{\mathbf{n}} \to X_{\mathbf{m}}$ *reverses* the order of $\mathbf{m}$ and $\mathbf{n}$ in comparison to the map of simplices $f\colon \mathbf{m} \to \mathbf{n}$. We express this by saying that X_f is *contravariant* with respect to f. If such a reversal did not happen, we would say that X_f is *covariant* with respect to f. Later these observations will be formalized in the concept of covariant and contravariant functors.

Remark 7.3. Notice how this definition completely eliminated all references to the original intuition behind $X_{\mathbf{m}}$ and X_f: for us $X_{\mathbf{m}}$ is an "abstract" set and X_f is a map of "abstract" sets. In particular, we do *not* assume that elements of $X_{\mathbf{m}}$ are maps of the form $\Delta^{\mathbf{m}} \to X$. This is *necessary*: after all, we have not defined what a simplicial map is yet, nor have we defined the simplicial set $\Delta^{\mathbf{m}}$. This will be done below. We certainly could not define simplicial maps without defining simplicial sets first, so in order to break the vicious circle, we must not assume that elements of $X_{\mathbf{m}}$ are simplicial maps. However, the above does *not* mean that we throw away the original intuition. Rather, it will be brought back to us by the Yoneda lemma (Lemma 8.14), which uses the definition of simplicial sets $\Delta^{\mathbf{m}}$ in Definition 7.10 and simplicial maps given below to *prove* that $X_{\mathbf{m}}$ is *isomorphic* (not equal!) to the set of simplicial maps $\Delta^{\mathbf{m}} \to X$. Used in 7.12, 8.14*.

Remark 7.4. If $f\colon \mathbf{m} \to \mathbf{n}$ is a degenerate map of simplices, we refer to X_f as a *degeneracy map*. Likewise for face inclusions, which yield *face maps*. In Definition 6.10 we introduced a specific notation for face inclusions and degenerate maps, namely, $\mathrm{d}^{\mathbf{m},i}$ and $\mathrm{s}^{\mathbf{m},j}$ respectively, or simply d^i and s^j, where $\mathbf{m}$ must be deduced from the context. The associated simplicial structure maps for a simplicial set X are denoted by $\mathrm{d}_{X,\mathbf{m},i}$ and $\mathrm{s}_{X,\mathbf{m},j}$, or simply d_i and s_j, where X and $\mathbf{m}$ must be deduced from the context and i and j are now numbers whose meaning is explained in Notation 6.11. Used in 7.5, 7.11, 13.24.

Remark 7.5. Simplicial sets were defined in 1949 by Eilenberg and Zilber in [SSCSH, §8], where they are called *complete semi-simplicial complexes*. This terminology is no longer in use, but the word "complex" survives in many derivative names of constructions involving simplicial sets, such as "Kan complex" and "function complex". The adjective "complete" refers to the presence of degeneracy maps; a *semi-simplicial complex* (modern name: semisimplicial set, but here "semi" refers to the absence of degeneracy maps, *not* to the condition on vertices below) is defined just like a simplicial set, but requiring all morphisms of simplices to be injective, without any degeneracy maps. "Semi" refers to the fact that two different n-simplices can have the same $(n+1)$-tuple of vertices. An overview of the relationship between simplicial sets and simplicial complexes can be found in §17.1. Used in 7.6.

Remark 7.6. The advantages of simplicial sets over topological spaces in homotopy theory became clear soon after their introduction. In his review of Kan's 1957 paper [KanCSS] John C. Moore (of Borel–Moore homology, Eilenberg–Moore spectral sequence, and the Milnor–Moore theorem) writes "In recent years it has become evident that for most purposes in homotopy theory it is more convenient to use semi-simplicial complexes instead of topological spaces."

We proceed to define two interesting classes of examples of simplicial sets.

Definition 7.7. Any set S gives rise to a *discrete simplicial set* $\operatorname{dis} S$ such that $(\operatorname{dis} S)_{\mathbf{m}} = S$ for any simplex $\mathbf{m}$ and $(\operatorname{dis} S)_f = \mathrm{id}_S$ for any map of simplices f. Used in 7.7, 7.7*, 7.8, 8.6, 8.19, 10.5, 12.9, 13.8, 13.10, 13.26, 13.26*, 13.29, 21.8, 30.3.

The simplicial set $\operatorname{dis} S$ can be visualized as a bunch of isolated points indexed by the elements of S.

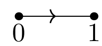
Definition 7.8. The *empty simplicial set* (alias *initial simplicial set*) is defined as $\operatorname{dis}\emptyset$. We abuse notation and denote this simplicial set again by $\emptyset$. The *point simplicial set* (alias *terminal simplicial set*) is defined as $\operatorname{dis} 1$, where 1 denotes any singleton set. Again we abuse notion and denote this simplicial set by 1. Used in 8.7, 26.8.

Remark 7.9. Below, we will define maps of simplicial sets, and will see that the simplicial set $\emptyset$ has a universal property: there is exactly one simplicial map $\emptyset \to X$ for any simplicial set X. Likewise, there is exactly one simplicial map $X \to 1$ for any simplicial set X. Later, we will define initial objects and terminal objects in any category, and the initial and terminal simplicial sets will turn out to be initial and terminal objects in the category of simplicial sets.

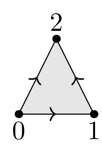
Definition 7.10. (The Yoneda embedding.) Given a simplex $\mathbf{p}$, we define a simplicial set $\Delta^{\mathbf{p}}$ as follows: $(\Delta^{\mathbf{p}})_{\mathbf{m}}$ is the set of morphisms of simplices $\mathbf{m} \to \mathbf{p}$ and $(\Delta^{\mathbf{p}})_f\colon (\Delta^{\mathbf{p}})_{\mathbf{n}} \to (\Delta^{\mathbf{p}})_{\mathbf{m}}$ for a morphism of simplices $f\colon \mathbf{m} \to \mathbf{n}$ is a map that sends an element $g \in (\Delta^{\mathbf{p}})_{\mathbf{n}}$ (i.e., a morphism $g\colon \mathbf{n} \to \mathbf{p}$) to the element $g \circ f \in (\Delta^{\mathbf{p}})_{\mathbf{m}}$. Used in 7.0*, 7.3, 7.10, 7.11, 7.12, 7.13, 7.13*, 8.8, 8.9, 8.10, 8.11, 8.13, 8.14, 8.14*, 8.15, 8.16, 8.16*, 8.17, 8.17*, 8.19, 8.20, 8.21, 8.22, 9.3*, 9.4, 9.4*, 10.0*, 10.1, 10.2, 10.3, 10.4, 10.4*, 10.5, 11.11, 12.7, 13.8, 13.21, 13.22*, 13.29, 14.4*, 15.0*, 15.5, 15.9, 16.10, 21.10, 21.11, 21.24*, 21.28, 22.6, 22.7, 22.8, 22.9, 22.10*, 22.15, 22.22, 23.3, 25.5, 28.2, 28.3, 28.4*, 28.5, 28.5*, 28.8, 28.9, 29.17*, 30.5, 31.1, 31.6*, 32.2, 32.3, 32.4, 32.5*, 32.6, 32.7*, 32.8*, 34.0*, 34.1, 34.2, 34.4, 34.5, 34.5*, 35.1, 35.2, 35.2*, 35.6*, 35.7*, 35.8, 35.8*, 35.10*, 36.2, 36.5, 36.12, 38.6, 39.1, 39.2, 39.3, 39.4, 39.6*, 39.7*, 39.9, 39.9*, 39.10, 39.10*, 39.22, 40.11, 42.4, 42.7, 43.3, 45.1, 45.5*, 46.1, 46.2*, 46.4*, 46.8.

Example 7.11. The simplicial set $\Delta^{\mathbf{p}}$ can be visualized in the same way that we previously visualized the simplex $\mathbf{p}$. We now show how the internal structure of a simplicial set can be visualized. First, we introduce a notation for individual simplices of $\Delta^{\mathbf{p}}$: such an $\mathbf{m}$-simplex is given by a map of simplices $f\colon \mathbf{m} \to \mathbf{p}$, and assuming $\mathbf{m} = \{0 < 1 < \cdots < m\}$ and $\mathbf{p} = \{0 < \cdots < p\}$ are standard simplices, we denote f by its values on all vertices of $\mathbf{m}$, taken in their given order. For instance, the map $f\colon [2] \to [3]$ that sends $0 \mapsto 1$, $1 \mapsto 1$, $2 \mapsto 3$ is denoted by 113. We now use this notation to illustrate the low-dimensional examples of $\mathbf{m}$-simplices and simplicial structure maps.

- The simplicial set $\Delta^{\mathbf{0}}$ is visualized as a point 0. The only $\mathbf{m}$-simplex of $\Delta^{\mathbf{0}}$ is denoted by a string of $(m + 1)$ zeros and can be visualized as a map that collapses the entire simplex $\mathbf{m}$ to the point 0. The simplicial structure maps are trivial in this case: a simplex collapsed to 0 is again mapped to a simplex collapsed to 0, both of which are represented by strings of zeros.
- The simplicial set $\Delta^{\mathbf{1}}$ is visualized as a line segment:



The 0-simplices are $\{0, 1\}$, visualized as the dotted points in the above picture. The only simplicial structure maps for 0-simplices are degeneracy maps: applying the degeneracy map s_0 several times to a 0-simplex 0 or 1 yields a string of several 0's or 1's, which represents an $\mathbf{m}$-simplex crushed to a point and mapped to the same vertex. The 1-simplices are $\{00, 01, 11\}$, where 00 and 11 should be visualized as 1-simplices crushed to points 0 and 1 respectively, whereas 01 is the interval in between. The face map d_1 takes the initial character: $\mathrm{d}_1(00) = \mathrm{d}_1(01) = 0$, $\mathrm{d}_1(11) = 1$. Likewise, d_0 takes the last character: $\mathrm{d}_0(00) = 0$, $\mathrm{d}_0(01) = \mathrm{d}_0(11) = 1$. The degeneracy maps duplicate the corresponding characters. For example, $\mathrm{s}_0(01) = 001$, $\mathrm{s}_1(01) = 011$. The 2-simplices are $\{000, 001, 011, 111\}$, where 000 and 111 are visualized as 2-simplices crushed to points 0 and 1 respectively, whereas 001 and 011 are visualized by the same pictures as in Example 6.6. The face maps throw away one character (corresponding to the vertex given by the subscript) and the degeneracy maps duplicate a character: $\mathrm{d}_0(011) = 11$, $\mathrm{d}_1(011) = \mathrm{d}_2(011) = 01$, $\mathrm{s}_0(011) = 0011$, $\mathrm{s}_1(011) = \mathrm{s}_2(011) = 0111$.
- The simplicial set $\Delta^{\mathbf{2}}$ is visualized as a triangle:



The 0-simplices are $\{0, 1, 2\}$, corresponding to three vertices in the picture. The 1-simplices are

$$\{00, 01, 02, 11, 12, 22\},$$

where 00, 11, 22 correspond to 1-simplices crushed to a point and mapped to the corresponding vertex, whereas 01, 02, and 12 correspond to the three edges of the triangle. The 2-simplices are

$$\{000, 001, 002, 011, 012, 022, 111, 112, 122, 222\},$$

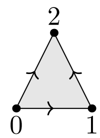
where 000, 111, 222 are visualized by 2-simplices crushed to a point and mapped to the corresponding vertex, 001, 002, 011, 022, 112, 122 are visualized by 2-simplices crushed to a 1-simplex as in Example 6.6, and then mapped to the corresponding edge of the triangle, and 012 represents the interior of the triangle.

Used in 13.24.

Remark 7.12. In this definition one can see the ideology of Remark 7.3 applied quite literally: $(\Delta^{\mathbf{p}})_{\mathbf{m}}$ was *defined* as the set of maps $\mathbf{m} \to \mathbf{p}$. The functoriality property follows immediately from the associativity and unitality properties for maps of simplices.

Exercise 7.13. Compute the cardinality of $(\Delta^{\mathbf{p}})_{\mathbf{m}}$ in terms of $\dim \mathbf{p}$ and $\dim \mathbf{m}$.

We now define a nontrivial simplicial set, the simplicial sphere S^n. An n-dimensional sphere is $\Delta^{\mathbf{n}}$ with its boundary (to be defined precisely below) "collapsed" to a point. We illustrate this with the simplicial set $\Delta^{\mathbf{2}}$:



In this case, the boundary consists of the three vertices and edges between them. Imagine that the entire boundary is gradually folded and then glued together. The result looks like a xiǎolóngbāo dumpling without a filling, or a sphere with a distinguished point, namely, the vertex to which the boundary was collapsed.

Soon we will define what it means to collapse the boundary in a completely general fashion, but for now we resort to an ad hoc definition. Thus, we should have simplicial maps (defined in the next section) $\Delta^{\mathbf{n}} \to \mathrm{S}^n$ and $\Delta^{\mathbf{0}} \to \mathrm{S}^n$, where the latter map picks the single vertex to which the boundary was collapsed. Any map $\Delta^{\mathbf{k}} \to \mathrm{S}^n$ should factor through the collapsing map $\Delta^{\mathbf{k}} \to \Delta^{\mathbf{n}} \to \mathrm{S}^n$. All maps $f\colon \Delta^{\mathbf{k}} \to \Delta^{\mathbf{n}}$ that are not surjective land inside the boundary of $\Delta^{\mathbf{n}}$ and therefore denote the same map $\Delta^{\mathbf{k}} \to \mathrm{S}^n$.

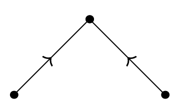
Definition 7.14. The *simplicial sphere* $\mathrm{S}^k \in \mathsf{sSet}$ of dimension k ($k \geq -1$) is defined as follows. If $k = -1$, we set $\mathrm{S}^{-1} = \emptyset$. Otherwise $k \geq 0$ and we set $\mathrm{S}^k_{\mathbf{m}} = \{*\} \sqcup \mathsf{Mor}_{\mathbf{\Delta}}(\mathbf{m}, \mathbf{k})_{\text{surjective}}$, where the subscript $_{\text{surjective}}$ means we only take the surjective maps of simplices. The simplicial structure map $\mathrm{S}^k_f\colon \mathrm{S}^k_{\mathbf{n}} \to \mathrm{S}^k_{\mathbf{m}}$ for a map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ is defined as follows: $\mathrm{S}^k_f(*) = *$ and for $\alpha \in \mathsf{Mor}_{\mathbf{\Delta}}(\mathbf{n}, \mathbf{k})_{\text{surjective}}$ we set

$$\mathrm{S}^k_f(\alpha) = \begin{cases} \alpha \circ f \in \mathsf{Mor}_{\mathbf{\Delta}}(\mathbf{m}, \mathbf{k}), & \alpha \circ f \text{ is a surjective map of simplices} \\ *, & \text{otherwise.} \end{cases}$$

Used in 7.13*, 7.14, 8.11, 9.7, 9.12, 9.13, 13.23, 17.11*.

This definition is quite verbose. Below we develop a much more efficient way to specify such simplicial sets.

Exercise 7.15. Formalize the following picture as a simplicial set X, i.e., give an explicit definition of sets $X_{\mathbf{m}}$ and simplicial structure maps X_f and prove that the functoriality properties in the definition of simplicial set are satisfied.



Used in 8.20.

8 Simplicial maps

Supplementary sources: [EIISS, §2, §3], [ICHT, §2, §3].

Definition 8.1. A *map of simplicial sets* (alias *morphism of simplicial sets* or *simplicial map*) $f\colon X \to Y$ is a family of maps of sets $f_{\mathbf{m}}\colon X_{\mathbf{m}} \to Y_{\mathbf{m}}$ (indexed by a simplex $\mathbf{m}$) such that the following *naturality property for simplicial maps* is satisfied: for any map of simplices $g\colon \mathbf{m} \to \mathbf{n}$ the following diagram commutes:

$$\begin{array}{ccc} X_{\mathbf{m}} & \xleftarrow{X_g} & X_{\mathbf{n}} \\ {\scriptstyle f_{\mathbf{m}}}\downarrow & & \downarrow{\scriptstyle f_{\mathbf{n}}} \\ Y_{\mathbf{m}} & \xleftarrow{Y_g} & Y_{\mathbf{n}}. \end{array}$$

Used in 6.14*, 7.0*, 7.3, 7.9, 7.13*, 8.2, 8.3, 8.5, 8.6, 8.7, 8.8, 8.10, 8.11, 8.12, 8.13, 8.14, 8.14*, 8.15, 8.17, 8.19, 8.21, 8.22, 9.3*, 10.3, 11.0*, 11.12, 11.13, 12.8, 13.7*, 13.12*, 13.19, 13.22*, 13.23, 14.4*, 14.13, 15.2*, 15.5, 15.13*, 15.14, 15.16, 15.18, 16.15, 17.3, 17.5, 18.1, 18.7, 20.12, 21.7*, 21.13*, 21.20, 21.22, 21.24, 21.24*, 21.25, 22.6, 25.2, 25.4, 25.8, 28.2, 28.3, 28.4*, 28.5*, 31.0*, 31.1, 32.1, 32.2, 34.0*, 34.2, 34.4, 34.5*, 35.1, 35.2, 35.2*, 35.3, 35.6*, 36.2, 36.8, 36.12, 39.4, 39.10, 39.10*, 39.13, 39.14, 39.20, 45.5*, 46.2, 46.3, 46.5, 46.7.

Definition 8.2. The set of all morphisms of simplicial sets $X \to Y$ is known as the hom-set (hom for homomorphism) and is denoted by $\mathsf{hom}(X,Y)$ (another notation: $\mathsf{Mor}(X,Y)$, where Mor stands for morphisms). If X is a simplicial set and $f\colon Y \to Z$ is a simplicial map, then $\mathsf{hom}(X,f)\colon \mathsf{hom}(X,Y) \to \mathsf{hom}(X,Z)$ ($g \mapsto f \circ g$) and $\mathsf{hom}(f,X)\colon \mathsf{hom}(Z,X) \to \mathsf{hom}(Y,X)$ ($g \mapsto g \circ f$) denote the maps of sets induced by composing a given element of the hom-set with the morphism f.

Warning 8.3. Simplicial maps should not be confused with maps of simplices. The former are between simplicial sets, the latter are between simplices.

Definition 8.4. The *identity map of a simplicial set* X is the map $\mathrm{id}_X\colon X \to X$ such that $(\mathrm{id}_X)_{\mathbf{m}} = \mathrm{id}_{X_{\mathbf{m}}}$. The *composition of simplicial maps* $f\colon X \to Y$ and $g\colon Y \to Z$ is the map $g \circ f\colon X \to Z$ such that $(g \circ f)_{\mathbf{m}} = g_{\mathbf{m}} \circ f_{\mathbf{m}}$. Used in 11.12.

Remark 8.5. The *associativity and unitality properties* are satisfied for compositions and identity maps: $\mathrm{id}_Y \circ f = f \circ \mathrm{id}_X = f$ and $(g \circ f) \circ e = g \circ (f \circ e)$ for all simplicial maps $e\colon W \to X$, $f\colon X \to Y$, $g\colon Y \to Z$. Used in 11.0*.

Example 8.6. A map of sets $f\colon S \to T$ induces a simplicial map $\mathrm{dis}\, f\colon \mathrm{dis}\, S \to \mathrm{dis}\, T$. Indeed,

$$(\mathrm{dis}\, f)_{\mathbf{m}}\colon (\mathrm{dis}\, S)_{\mathbf{m}} \to (\mathrm{dis}\, T)_{\mathbf{m}}$$

should be a map of sets of the form $S \to T$, for which we can simply take f. We have $\mathrm{dis}(g \circ f) = \mathrm{dis}\, g \circ \mathrm{dis}\, f$ and $\mathrm{dis}(\mathrm{id}_S) = \mathrm{id}_{\mathrm{dis}\, S}$.

Example 8.7. Recall the definitions of $\emptyset$ and 1 from Definition 7.8. From the definition of a simplicial map we immediately deduce that for any simplicial set X there is exactly one map $\emptyset \to X$ and exactly one map $X \to 1$. The proof boils down to observing that for any set A there is exactly one function $\emptyset \to A$ and exactly one function $A \to 1$ (here $\emptyset$ and 1 denote the empty respectively singleton set).

Definition 8.8. Given a map of simplices $f\colon \mathbf{m} \to \mathbf{n}$, we define a simplicial map $\Delta^f\colon \Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$ by setting $(\Delta^f)_{\mathbf{p}}\colon (\Delta^{\mathbf{m}})_{\mathbf{p}} \to (\Delta^{\mathbf{n}})_{\mathbf{p}}$ to the map of sets that sends an element $a\colon \mathbf{p} \to \mathbf{m}$ to the element $f \circ a\colon \mathbf{p} \to \mathbf{n}$.

Remark 8.9. The order of $\mathbf{m}$ and $\mathbf{n}$ in $\Delta^f\colon \Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$ is the same as in $f\colon \mathbf{m} \to \mathbf{n}$. We say that Δ^f depends *covariantly* on f. Later, we will formalize this using the notion of a covariant functor.

Exercise 8.10. Verify that the above formula indeed gives a simplicial map and show that $\Delta^{g \circ f} = \Delta^g \circ \Delta^f$ and $\Delta^{\mathrm{id}_{\mathbf{m}}} = \mathrm{id}_{\Delta^{\mathbf{m}}}$.

Example 8.11. We construct a simplicial map $\beta\colon \Delta^{\mathbf{m}} \to \mathrm{S}^{\mathbf{m}}$. Given a $\mathbf{k}$-simplex of $\Delta^{\mathbf{m}}$, i.e., a map of simplices $a\colon \mathbf{k} \to \mathbf{m}$, we must construct a $\mathbf{k}$-simplex of $\mathrm{S}^{\mathbf{m}}$, which according to Definition 7.14 can be either

$*$ or a surjective map of simplices $\mathbf{k} \to \mathbf{m}$. If a is surjective, we take $\beta(a) = a$. If a is not surjective, we take $\beta(a) = *$.

Exercise 8.12. Verify that the above formulas indeed give a simplicial map.

Example 8.13. Any simplicial map $f\colon \mathrm{S}^{\mathbf{1}} \to \mathrm{S}^{\mathbf{2}}$ must factor through the base point $b\colon \Delta^{\mathbf{0}} \to \mathrm{S}^{\mathbf{2}}$ of $\mathrm{S}^{\mathbf{2}}$, i.e., the following diagram must commute:

$$\begin{array}{ccc} \mathrm{S}^{\mathbf{1}} & \xrightarrow{f} & \mathrm{S}^{\mathbf{2}} \\ & {\scriptstyle c}\searrow \quad \nearrow{\scriptstyle b} & \\ & \Delta^{0} & \end{array},$$

where $c\colon \mathrm{S}^{\mathbf{1}} \to \Delta^{\mathbf{0}}$ denotes the unique map. Indeed, both 1-simplices (degenerate and not) of $\mathrm{S}^{\mathbf{1}}$ must map to the only (degenerate) 1-simplex of $\mathrm{S}^{\mathbf{2}}$, and the value of f on all higher-dimensional simplices is determined by its value on 1-simplices by the naturality property for simplicial maps. Used in 15.18.

Lemma 8.14. (The Yoneda lemma.) Consider a simplicial set X and a simplex $\mathbf{m}$. The canonical map of sets

$$y_{\mathbf{m}}\colon \mathsf{hom}(\Delta^{\mathbf{m}}, X) \to X_{\mathbf{m}}$$

that sends a map of simplicial sets $f\colon \Delta^{\mathbf{m}} \to X$ to $y_{\mathbf{m}}(f) := f_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}}) \in X_{\mathbf{m}}$ is an isomorphism. Used in 1.2*, 7.3, 8.16*, 8.17*, 8.20, 8.21, 9.3*, 9.4, 10.0*, 10.1, 30.10*, 34.0*, 39.9.

In other words, elements of $X_{\mathbf{m}}$ can be canonically identified with maps of simplicial sets $\Delta^{\mathbf{m}} \to X$, which yields a formal justification of Remark 7.3.

Proof. To establish injectivity, suppose that $f, g\colon \Delta^{\mathbf{m}} \to X$ are such that $y_{\mathbf{m}}(f) = f_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}}) = g_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}}) = y_{\mathbf{m}}(g)$. To show that $f = g$, we must demonstrate that $f_{\mathbf{k}} = g_{\mathbf{k}}\colon (\Delta^{\mathbf{m}})_{\mathbf{k}} \to X_{\mathbf{k}}$ for any simplex $\mathbf{k}$. To this end, we pick an arbitrary element $h \in (\Delta^{\mathbf{m}})_{\mathbf{k}}$, i.e., $h\colon \mathbf{k} \to \mathbf{m}$ and verify that $f_{\mathbf{k}}(h) = g_{\mathbf{k}}(h)$. By definition of $\Delta^{\mathbf{m}}$, we have $h = \mathrm{id}_{\mathbf{m}} \circ h = (\Delta^{\mathbf{m}})_h(\mathrm{id}_{\mathbf{m}})$, so $f_{\mathbf{k}}(h) = f_{\mathbf{k}}((\Delta^{\mathbf{m}})_h(\mathrm{id}_{\mathbf{m}}))$. The naturality property for simplicial maps applied to $\Delta^{\mathbf{m}} \to X$ says that for any map of simplices $r\colon \mathbf{p} \to \mathbf{q}$ the square

$$\begin{array}{ccc} (\Delta^{\mathbf{m}})_{\mathbf{p}} & \xleftarrow{(\Delta^{\mathbf{m}})_r} & (\Delta^{\mathbf{m}})_{\mathbf{q}} \\ {\scriptstyle f_{\mathbf{p}}}\downarrow & & \downarrow{\scriptstyle f_{\mathbf{q}}} \\ X_{\mathbf{p}} & \xleftarrow{X_r} & X_{\mathbf{q}} \end{array}$$

commutes. If we take $r = h$ above (so $\mathbf{p} = \mathbf{k}$ and $\mathbf{q} = \mathbf{m}$), then

$$f_{\mathbf{k}}((\Delta^{\mathbf{m}})_h(\mathrm{id}_{\mathbf{m}})) = X_h(f_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}})) = X_h(y_{\mathbf{m}}(f)).$$

Since $y_{\mathbf{m}}(f) = y_{\mathbf{m}}(g)$, we have (by the same argument with g instead of f)

$$X_h(y_{\mathbf{m}}(f)) = X_h(y_{\mathbf{m}}(g)) = X_h(g_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}})) = g_{\mathbf{k}}((\Delta^{\mathbf{m}})_h(\mathrm{id}_{\mathbf{m}})) = g_{\mathbf{k}}(h),$$

so $f_{\mathbf{k}}(h) = g_{\mathbf{k}}(h)$ as required.

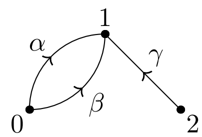
To establish surjectivity, suppose that $a \in X_{\mathbf{m}}$. We want to construct a map of simplicial sets $f\colon \Delta^{\mathbf{m}} \to X$ such that $y_{\mathbf{m}}(f) = a$. Thus, given a simplex $\mathbf{k}$, we must construct a maps of sets $f_{\mathbf{k}}\colon (\Delta^{\mathbf{m}})_{\mathbf{k}} \to X_{\mathbf{k}}$. By definition, $(\Delta^{\mathbf{m}})_{\mathbf{k}} = \mathsf{hom}(\mathbf{k}, \mathbf{m})$, so the map reads $f_{\mathbf{k}}\colon \mathsf{hom}(\mathbf{k}, \mathbf{m}) \to X_{\mathbf{k}}$. Given $h\colon \mathbf{k} \to \mathbf{m}$, we must construct $f_{\mathbf{k}}(h) \in X_{\mathbf{k}}$. We set $f_{\mathbf{k}}(h) = X_h(a)$, where $X_h\colon X_{\mathbf{m}} \to X_{\mathbf{k}}$ is a simplicial structure map of X. We have $f_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}}) = X_{\mathrm{id}_{\mathbf{m}}}(a) = a$, so the image of f under $y_{\mathbf{m}}$ is indeed a.

To show that the maps of sets $f_{\mathbf{k}}\colon (\Delta^{\mathbf{m}})_{\mathbf{k}} \to X_{\mathbf{k}}$ constructed above assemble into a simplicial map $f\colon \Delta^{\mathbf{m}} \to X$, we must verify the naturality property, exhibited by the above commutative square, for an arbitrary map of simplices $r\colon \mathbf{p} \to \mathbf{q}$. Expanding the definition of $\Delta^{\mathbf{m}}$, the diagram reads

$$\begin{array}{ccc} \mathsf{hom}(\mathbf{p}, \mathbf{m}) & \xleftarrow{\mathsf{hom}(r, \mathbf{m})} & \mathsf{hom}(\mathbf{q}, \mathbf{m}) \\ {\scriptstyle f_{\mathbf{p}}}\downarrow & & \downarrow{\scriptstyle f_{\mathbf{q}}} \\ X_{\mathbf{p}} & \xleftarrow{X_r} & X_{\mathbf{q}}. \end{array}$$

Given $b \in (\Delta^{\mathbf{m}})_{\mathbf{q}}$, i.e., $b\colon \mathbf{q} \to \mathbf{m}$, we evaluate both compositions on b and verify that the results are equal. Indeed, $f_{\mathbf{p}}((\Delta^{\mathbf{m}})_r(b)) = f_{\mathbf{p}}(b \circ r) = X_{b \circ r}(a)$ and $X_r(f_{\mathbf{q}}(b)) = X_r(X_b(a)) = X_{b \circ r}(a)$. ∎

Example 8.15. Consider the following simplicial set X (a *lasso*):



We have $X_{\mathbf{0}} = \{0, 1, 2\}$, and $X_{\mathbf{1}} = \{00, 11, 22, \alpha, \beta, \gamma\}$,

$$X_{\mathbf{2}} = \{\mathrm{s}_0\mathrm{s}_0(0) = 000, \mathrm{s}_0\mathrm{s}_0(1) = 111, \mathrm{s}_0\mathrm{s}_0(2) = 222, \mathrm{s}_0\alpha, \mathrm{s}_1\alpha, \mathrm{s}_0\beta, \mathrm{s}_1\beta, \mathrm{s}_0\gamma = 221, \mathrm{s}_1\gamma = 211\}.$$

The corresponding simplicial maps can be described as follows:

- For $X_{\mathbf{0}}$, the simplicial maps $\Delta^{\mathbf{0}} \to X$ send the only element of $\Delta^{\mathbf{0}}_{\mathbf{m}} = \{0 \ldots 0\}$ to $0 \ldots 0$, $1 \ldots 1$, or $2 \ldots 2$ respectively. In particular, evaluating at the only element of $\Delta^{\mathbf{0}}_{\mathbf{0}} = \{0\}$ gives back the original vertex 0, 1, or 2 respectively.
- For $X_{\mathbf{1}}$, the simplicial maps $\Delta^{\mathbf{1}} \to X$ corresponding to 00, 11, and 22 send all elements of $\Delta^{\mathbf{1}}_{\mathbf{m}}$ to $0 \ldots 0$, $1 \ldots 1$, or $2 \ldots 2$, respectively. In particular, evaluating at the element 01 of $\Delta^{\mathbf{1}}_{\mathbf{1}} = \{00, 11, 01\}$ gives back the original 1-simplex 00, 11, or 22 respectively. The simplicial map $\Delta^{\mathbf{1}} \to X$ corresponding to 21 sends an element $0 \ldots 01 \ldots 1$ of $\Delta^{\mathbf{1}}_{\mathbf{m}}$ to $2 \ldots 21 \ldots 1$, with the same lengths of digit strings. In particular, evaluating at the element 01 of $\Delta^{\mathbf{1}}_{\mathbf{1}} = \{00, 11, 01\}$ gives back the 1-simplex 21. The simplicial maps $\Delta^{\mathbf{1}} \to X$ corresponding to α and β send an element $0 \ldots 01 \ldots 1 = \mathrm{s}_0^k \mathrm{s}_1^l(01)$ (with k zeros and l ones, $k + l + 1 = m$) to the element $\mathrm{s}_0^k \mathrm{s}_1^l(\alpha)$ (respectively β) of $X_{\mathbf{m}}$. In particular, evaluating at the element 01 of $\Delta^{\mathbf{1}}_{\mathbf{1}}$ gives back the 1-simplex α respectively β.

Used in 15.13.

Corollary 8.16. Suppose $f\colon \mathbf{m} \to \mathbf{n}$ is a map of simplices and X is a simplicial set. The simplicial structure map $X_f\colon X_{\mathbf{n}} \to X_{\mathbf{m}}$ is isomorphic to the map

$$\mathsf{hom}(\Delta^f, X)\colon \mathsf{hom}(\Delta^{\mathbf{n}}, X) \to \mathsf{hom}(\Delta^{\mathbf{m}}, X)$$

given by precomposing with the simplicial map $\Delta^f\colon \Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$.

Proof. The following diagram commutes for any map of simplices $f\colon \mathbf{m} \to \mathbf{n}$:

$$\begin{array}{ccc} \mathsf{hom}(\Delta^{\mathbf{n}}, X) & \xrightarrow{\mathsf{hom}(\Delta^f, X)} & \mathsf{hom}(\Delta^{\mathbf{m}}, X) \\ {\scriptstyle f \mapsto f_{\mathbf{n}}(\mathrm{id}_{\mathbf{n}})}\downarrow & & \downarrow{\scriptstyle f \mapsto f_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}})} \\ X_{\mathbf{n}} & \xrightarrow{X_f} & X_{\mathbf{m}}. \end{array}$$

Indeed, if $\alpha\colon \Delta^{\mathbf{n}} \to X$ is an arbitrary element of the upper-left corner, then we have

$$\mathsf{hom}(\Delta^f, X)(\alpha) = \alpha \circ \Delta^f\colon \Delta^{\mathbf{m}} \to X,$$

$$y_{\mathbf{m}}(\alpha \circ \Delta^f) = (\alpha \circ \Delta^f)_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}}) = \alpha_{\mathbf{m}}(\Delta^f_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}})) = \alpha_{\mathbf{m}}(f \circ \mathrm{id}_{\mathbf{m}}) = \alpha_{\mathbf{m}}(f),$$

while

$$y_{\mathbf{n}}(\alpha) = \alpha_{\mathbf{n}}(\mathrm{id}_{\mathbf{n}}),$$

$$X_f(\alpha_{\mathbf{n}}(\mathrm{id}_{\mathbf{n}})) = \alpha_{\mathbf{m}}(\Delta^{\mathbf{n}_f}(\mathrm{id}_{\mathbf{n}})) = \alpha_{\mathbf{m}}(\mathrm{id}_{\mathbf{n}} \circ f) = \alpha_{\mathbf{m}}(f),$$

which proves that both compositions are equal. By the Yoneda lemma the vertical maps $y_{\mathbf{m}}$ and $y_{\mathbf{n}}$ are isomorphisms, which proves the claim. ▮

Corollary 8.17. Suppose $\mathbf{m}$ is simplex and $r\colon X \to Y$ is a simplicial map. The map of sets $r_{\mathbf{m}}\colon X_{\mathbf{m}} \to Y_{\mathbf{m}}$ is isomorphic to the map

$$\mathsf{hom}(\Delta^{\mathbf{m}}, r)\colon \mathsf{hom}(\Delta^{\mathbf{m}}, X) \to \mathsf{hom}(\Delta^{\mathbf{m}}, Y)$$

given by postcomposing with the simplicial map r.

Proof. The following diagram commutes:

$$\begin{array}{ccc}
\mathsf{hom}(\Delta^{\mathbf{m}}, X) & \xrightarrow{\mathsf{hom}(\Delta^{\mathbf{m}}, r)} & \mathsf{hom}(\Delta^{\mathbf{m}}, Y) \\
{\scriptstyle f \mapsto f_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}})}\downarrow & & \downarrow{\scriptstyle f \mapsto f_{\mathbf{m}}(\mathrm{id}_{\mathbf{m}})} \\
X_{\mathbf{m}} & \xrightarrow{r_{\mathbf{m}}} & Y_{\mathbf{m}}.
\end{array}$$

By the Yoneda lemma, the vertical maps are isomorphisms, which proves the claim. ▮

Exercise 8.18. Prove that the last diagram commutes.

Example 8.19. If S is a singleton, then $1 = \operatorname{dis} S$ is isomorphic to $\Delta^{\mathbf{0}}$. In general, the simplicial set $\operatorname{dis} S$ can be depicted by a collection of points, e.g., for $\operatorname{dis}\{0, 1, 2\}$ we would have

$$\overset{\bullet}{0} \qquad \overset{\bullet}{1} \qquad \overset{\bullet}{2}$$

A map of sets $f\colon S \to T$ yields a map of simplicial sets $\operatorname{dis} f\colon \operatorname{dis} S \to \operatorname{dis} T$. We have $\operatorname{dis} \mathrm{id}_{\mathbf{m}} = \mathrm{id}_{\operatorname{dis} \mathbf{m}}$ and $\operatorname{dis}(g \circ f) = \operatorname{dis} g \circ \operatorname{dis} f$.

Exercise 8.20. The simplicial set X in Exercise 7.15 has five 1-simplices, i.e., the set $X_{\mathbf{1}}$ has cardinality 5. For each of those five 1-simplices describe the map $\Delta^{\mathbf{1}} \to X$ produced by the Yoneda lemma explicitly, indicating where each simplex goes.

Summary 8.21. (The *Yoneda yoga*.) We summarize the yoga of the Yoneda lemma in the following table, where $f\colon \mathbf{m} \to \mathbf{n}$ is a map of simplices and $g\colon X \to Y$ is a simplicial map:

$\mathbf{m}$	$\Delta^{\mathbf{m}}$
$f\colon \mathbf{m} \to \mathbf{n}$	$\Delta^f\colon \Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$
$\alpha \in X_{\mathbf{n}}$	$\alpha\colon \Delta^{\mathbf{n}} \to X$
$X_f(\alpha) \in X_{\mathbf{m}}$	$\alpha \circ \Delta^f\colon \Delta^{\mathbf{m}} \to X$, i.e., the composition $\Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}} \to X$
$g_{\mathbf{m}}(\alpha) \in Y_{\mathbf{m}}$	$g \circ \alpha\colon \Delta^{\mathbf{n}} \to Y$, i.e., the composition $\Delta^{\mathbf{n}} \to X \to Y$
functoriality property for simplicial sets	
$X_{g \circ f} = X_f \circ X_g$	associativity of the composition $\Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}} \to \Delta^{\mathbf{p}} \to X$
$X_{\mathrm{id}_{\mathbf{m}}} = \mathrm{id}_{X_{\mathbf{m}}}$	unitality of the composition $\Delta^{\mathbf{m}} \to \Delta^{\mathbf{m}} \to X$
naturality property for simplicial maps	
$f_{\mathbf{m}} \circ X_g = Y_g \circ f_{\mathbf{n}}$	associativity of the composition $\Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}} \to X \to Y$

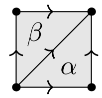
The left column indicates the state before moksha, whereas the right column is the enlightened version, where everything is translated in the abstract language (with no access to the internal structure of these objects) of simplicial sets, simplicial maps, and their properties of associativity and unitality. Used in 8.22.

Remark 8.22. The point of the Yoneda yoga is that one can for the most part forget about the internal structure of simplicial sets and simplicial maps as exhibited by the left column and use exclusively simplicial maps like $\Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}} \to X \to Y$ in an abstract fashion as exhibited by the right column. This also applies to simplices and maps of simplices: for the most part, one can just use $\Delta^{\mathbf{m}}$ and Δ^f and not mention $\mathbf{m}$ and f. Notice that maps of simplices $\mathbf{m} \to \mathbf{n}$ are in canonical bijection with simplicial maps $\Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$, so this does not create ambiguities. In particular, terminology, notation, and definitions we made for simplices and maps of simplices can be extended to their images under the Yoneda embedding. For instance, we could talk about degenerate maps $\Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$ (instead of $\mathbf{m} \to \mathbf{n}$), and likewise for face inclusions, etc.

9 Generators and relations for simplicial sets

An n-dimensional simplex has $2^{n+1}-1$ nondegenerate simplices and the number of its k-simplices (degenerate or not) grows exponentially with k. Spelling out the details of such constructions is cumbersome, especially if more than one simplex is involved. In this section we introduce a mechanism that allows us to specify simplicial sets by listing their nondegenerate simplices and how they glue together.

To illustrate this idea, consider the following picture:



Specifying the simplices of such a simplicial set directly would be cumbersome and error-prone. What we would like to say instead is that the above simplicial set is obtained by gluing two 2-simplices α and β along the diagonal 1-simplex, which happens to be the 1st face of both 2-simplices (i.e., the face opposite to the middle vertex). The following definition formalizes this idea.

Definition 9.1. A *system of generators and relations for a simplicial set* is specified as follows. For any simplex $\mathbf{m}$ we specify a set of *generating* $\mathbf{m}$*-simplices* $G_{\mathbf{m}}$. For any maps of simplices $f\colon \mathbf{m}\to\mathbf{n}$ and $g\colon \mathbf{m}\to\mathbf{p}$ we specify a subset $R_{f,g}\subset G_{\mathbf{n}}\times G_{\mathbf{p}}$. Used in 9.2, 13.21*.

Example 9.2. For the simplicial set depicted above, a system of generators and relations can be specified as follows: $G_{\mathbf{2}}=\{\alpha,\beta\}$ and $R_{\mathrm{d}^{[2],1},\mathrm{d}^{[2],1}}=\{(\alpha,\beta)\}$, whereas all the other sets are empty.

Remark 9.3. The pair (G,R) in the previous example means that the resulting simplicial set should have 2-simplices labeled α and β and they should satisfy $\mathrm{d}_1(\alpha)=\mathrm{d}_1(\beta)$, which refers to the diagonal 1-simplex. In practice, this type of description is used instead of the more formal description above.

Informally, the simplicial set X generated by (G,R) can be described as follows. Any element $x\in G_{\mathbf{m}}$ should yield a simplicial map $u(x)\colon \Delta^{\mathbf{m}}\to X$, equivalently by the Yoneda lemma, an element $u(x)\in X_{\mathbf{m}}$. The subset $R_{f,g}$ indicates pairs of simplices that should be identified. More precisely, if $(x,y)\in R_{f,g}$, then in the resulting simplicial set X the simplices $u(x)\circ\Delta^f$ and $u(y)\circ\Delta^g$ should be equal (both are maps of the form $\Delta^{\mathbf{m}}\to X$). We formalize this as follows (using the Yoneda lemma to identify simplicial maps $\Delta^{\mathbf{m}}\to X$ with elements of $X_{\mathbf{m}}$).

Definition 9.4. A pair (X,u), where $X\in\mathsf{sSet}$ and u is a family of maps of sets $u_{\mathbf{m}}\colon G_{\mathbf{m}}\to X_{\mathbf{m}}$ for every simplex $\mathbf{m}$, is a *solution for a system of generators and relations* (G,R) if for any $f\colon\mathbf{m}\to\mathbf{n}$, $s\in G_{\mathbf{n}}$, $g\colon\mathbf{m}\to\mathbf{p}$, $t\in G_{\mathbf{p}}$ such that $(s,t)\in R_{f,g}$ we have $X_f(u_{\mathbf{n}}(s))=X_g(u_{\mathbf{p}}(t))$, which is expressed by the following commutative square, where $u_{\mathbf{n}}(s)$ and $u_{\mathbf{p}}(t)$ were converted by the Yoneda lemma:

$$\begin{array}{ccc} \Delta^{\mathbf{m}} & \xrightarrow{\Delta^f} & \Delta^{\mathbf{n}} \\ {\scriptstyle\Delta^g}\downarrow & & \downarrow{\scriptstyle u_{\mathbf{n}}(s)} \\ \Delta^{\mathbf{p}} & \xrightarrow{u_{\mathbf{p}}(t)} & X. \end{array}$$

A *morphism of solutions* (or a *solution-preserving map*) $(X,u)\to(X',u')$ is a simplicial map $w\colon X\to X'$ such that for any simplex $\mathbf{m}$ and $x\in G_{\mathbf{m}}$ we have $w_{\mathbf{m}}(u_{\mathbf{m}}(x))=u'_{\mathbf{m}}(x)$. Using the Yoneda lemma, we can reformulate this relation as follows:

$$\begin{array}{ccc} & \Delta^{\mathbf{m}} & \\ {\scriptstyle u(x)}\swarrow & & \searrow{\scriptstyle u'(x)} \\ X & \xrightarrow[w]{} & X'. \end{array}$$

Used in 9.4*, 13.22*.

A solution for a system of generators and relations is highly nonunique. For instance, if (X,u) is a solution for (G,R), then we could add some junk to X using a disjoint union construction and get another solution. A related problem is that the solution can be trivial: for instance, for any (G,R) the pair $(\Delta^{\mathbf{0}},u)$, where u is the only possible map, is always a solution. To address this problem, we must ensure that X is not too big (e.g., does not have any additional junk in it like in the first example above) and not too small (e.g., does not collapse everything to a point like in the second example). The uniqueness condition in the definition below achieves the former and the existence achieves the latter.

Definition 9.5. The *simplicial set generated by a system of generators and relations* (G, R) is a solution (X, u) for (G, R) such that for any other solution (X', u') there is exactly one morphism of solutions $(X, u) \to (X', u')$. Used in 9.6*, 13.0*, 13.22, 13.24.

Below we will prove the existence of such a solution. On the other hand, uniqueness is almost trivial.

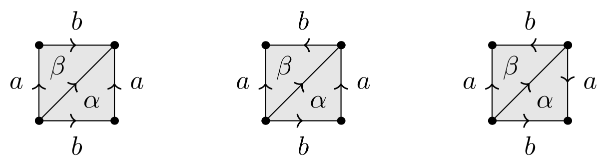
Lemma 9.6. If (X_1, u_1) and (X_2, u_2) both satisfy the above definition, then there is a unique isomorphism between them, i.e., there are unique morphisms of solutions $f\colon (X_1, u_1) \to (X_2, u_2)$ and $g\colon (X_2, u_2) \to (X_1, u_1)$, and, furthermore, $g \circ f = \mathrm{id}_{(X_1,u_1)}$ and $f \circ g = \mathrm{id}_{(X_2,u_2)}$. Used in 13.3*, 13.22.

Proof. The existence and uniqueness of f and g is a part of the definition of a simplicial set generated by a system of generators and relations. Observe now that $g \circ f$ and $\mathrm{id}_{(X_1,u_1)}$ are both morphisms of solutions $(X_1, u_1) \to (X_1, u_1)$. By the same property, there is exactly one such morphism, therefore $g \circ f = \mathrm{id}_{(X_1,u_1)}$ and likewise $f \circ g = \mathrm{id}_{(X_2,u_2)}$. ▮

We now illustrate this definition with several examples.

Example 9.7. The simplicial sphere S^n of dimension $n \geq -1$ that we defined in Definition 7.14 explicitly can be now given the following alternative definition: $\mathrm{S}^{-1} = \emptyset$ and for $n \geq 0$ the simplicial set S^n has a generating 0-simplex v and a generating n-simplex s. The relations are $\mathrm{d}_i(s) = \mathrm{s}_0^{n-1}(v)$ for all $i \in \mathsf{U}(\mathbf{n})$. Used in 13.23.

Example 9.8. The 2-dimensional *torus*, *Klein bottle*, and *real projective plane* are specified using the following schematic diagrams.



In all three cases we have a pair of 2-simplices α and β with some relations between them. In particular, we have:

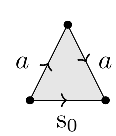
- Torus: $\mathrm{d}_1(\alpha) = \mathrm{d}_1(\beta)$ (the diagonal), $\mathrm{d}_0(\alpha) = \mathrm{d}_2(\beta)$ (a), $\mathrm{d}_2(\alpha) = \mathrm{d}_0(\beta)$ (b);
- Klein bottle: $\mathrm{d}_1(\alpha) = \mathrm{d}_2(\beta)$ (the diagonal), $\mathrm{d}_0(\alpha) = \mathrm{d}_1(\beta)$ (a), $\mathrm{d}_2(\alpha) = \mathrm{d}_1(\beta)$ (b).

Used in 9.10, 15.13, 32.16, 33.3, 33.9, 33.12.

Exercise 9.9. Write down the generators and relations of the real projective plane (the third picture above).

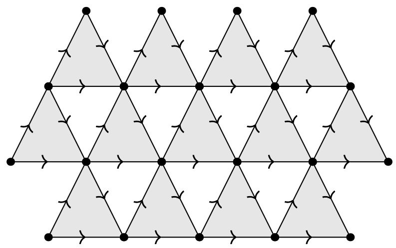
We now illustrate the point that a nondegenerate simplex can be identified with a degenerate simplex.

Example 9.10. Consider the simplicial set generated by a single 2-simplex α with relations $\mathrm{d}_0(\alpha) = \mathrm{d}_2(\alpha)$ and $\mathrm{d}_1(\alpha) = \mathrm{s}_0(\mathrm{d}_1(\mathrm{d}_1(\alpha)))$. The first relation identifies the 0th and 2nd edges of α, as depicted below by the letter a. The second relation collapses the bottom edge to a point: $\mathrm{d}_1(\alpha)$ is the bottom edge, $\mathrm{d}_1(\mathrm{d}_1(\alpha))$ is the bottom left vertex, and $\mathrm{s}_0(\mathrm{d}_1(\mathrm{d}_1(\alpha)))$ denotes the degenerate 1-simplex based on the bottom left vertex. The second relation therefore collapses the bottom edge to the bottom left vertex.



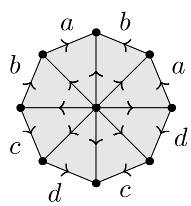
Once we define simplicial weak equivalences, this simplicial set will be shown to be weakly equivalent to the real projective plane defined in Example 9.8. This simplicial set only has a single nondegenerate 2-simplex, whereas the simplicial set defined in Example 9.8 has two nondegenerate 2-simplices.

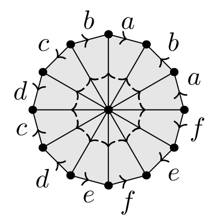
Exercise 9.11. Write down the generators and relations of the following simplicial set with infinitely many simplices, the *infinite grid* (the picture extends indefinitely in all directions, only the shaded triangles depict the simplicial set, whereas the unshaded triangles depict holes):



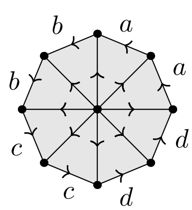
Used in 15.13.

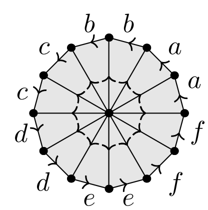
Exercise 9.12. Write down the generators and relations for the *orientable surface* of genus g. (For $g = 0$ one should take the simplicial sphere S^2, but here we assume $g \geq 1$.) It is given by a self-gluing of a polygon with $4g$ sides, which are identified as depicted below for $g = 2$ and $g = 3$. Used in 15.13, 38.6.





Exercise 9.13. Write down the generators and relations for the *nonorientable surface* with g crosscaps. (The case $g = 0$ is excluded, because it would give the simplicial sphere S^2 (which is orientable), so we assume $g \geq 1$.) It is given by a self-gluing of a polygon with $2g$ sides, which are identified as depicted below for $g = 4$ and $g = 6$. Used in 15.13, 38.7.





Exercise 9.14. Write down generators and relations for a triangular *sopapilla* and triangular *empanada*. A triangular sopapilla looks like a pair of triangles glued together along their boundary. A triangular empanada looks like a stuffed sopapilla, i.e., the space between two triangles is filled. Additional requirement: you may only use two generating simplices for the sopapilla and a single generating simplex for the empanada. Used in 15.13.

10 Simplices of a simplicial set

Supplementary sources: [EIISS, §3], [ICHT, §3].

Recall that a *simplex of a simplicial set* X is an element of $X_{\mathbf{n}}$ for some simplex $\mathbf{n}$, or, equivalently by Lemma 8.14, a simplicial map $\Delta^{\mathbf{n}} \to X$. If $\mathbf{n}$ is fixed, we talk about $\mathbf{n}$-simplices of X. We can also use a natural number n instead of a simplex $\mathbf{n}$, taking $\mathbf{n} = \{0 < 1 < 2 < \cdots < n\}$.

Warning 10.1. Simplices should not be confused with simplices of a simplicial set. The latter "live" in a given simplicial set X, whereas the former are "disembodied abstract simplices" and are homeless. A simplex $\mathbf{n}$ in the former sense yields a simplicial set $\Delta^{\mathbf{n}}$ and the Yoneda lemma tells us that maps $\Delta^{\mathbf{n}} \to X$ can be identified with $\mathbf{n}$-simplices of X, i.e., simplices in the latter sense. When we say "n-simplex of ..." or "$\mathbf{n}$-simplex of ..." we always use the latter meaning.

Definition 10.2. An $\mathbf{m}$-simplex $s\colon \Delta^{\mathbf{m}} \to X$ is *degenerate* if there is a surjective map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ that is not an isomorphism and an $\mathbf{n}$-simplex $t\colon \Delta^{\mathbf{n}} \to X$ such that $t \circ \Delta^f = s$.

$$\begin{array}{ccc} & \Delta^{\mathbf{n}} & \\ {\scriptstyle \Delta^f}\nearrow & & \searrow{\scriptstyle t} \\ \Delta^{\mathbf{m}} & \xrightarrow[s]{} & X \end{array}$$

Used in 9.0*, 9.9*, 10.3, 10.5, 15.7, 15.10, 15.14*, 17.11*, 20.7, 21.11, 22.5, 22.6, 22.9, 36.7, 39.9*.

Remark 10.3. The term "degenerate" is motivated by the fact that the "image" of a degenerate simplex $s\colon \Delta^{\mathbf{m}} \to X$ has dimension less than m. (This will be made completely precise later when we define images of simplicial maps.) For instance, a degenerate 1-simplex looks like a point (vertex) inside X, a degenerate 2-simplex may look like a point or 1-simplex inside X, and a degenerate 3-simplex may look like a point, 1-simplex, or 2-simplex inside X. Thus suggests the following: any degenerate simplex $s\colon \Delta^{\mathbf{m}} \to X$ has as its "image" some nondegenerate simplex $t\colon \Delta^{\mathbf{n}} \to X$, where $n < m$. Furthermore, $\Delta^{\mathbf{m}}$ should map to $\Delta^{\mathbf{n}}$ via some surjective map of simplices that may not be an isomorphism if s is degenerate. This idea is formalized by the following proposition.

The following proposition has a simple geometric interpretation: any $\mathbf{m}$-simplex of a simplicial set X has a well-defined "image", which is itself a simplex of X, of dimension (termed "rank" by Eilenberg and Zilber) at most the dimension of $\mathbf{m}$. Its proof, however, is surprisingly tricky.

Proposition 10.4. (Eilenberg and Zilber [SSCSH, 8.3].) Every $\mathbf{m}$-simplex is a unique degeneration of a unique nondegenerate simplex. In other words, for any simplicial set X and for any simplex $s\colon \Delta^{\mathbf{m}} \to X$ there is a surjective map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ and a nondegenerate simplex $t\colon \Delta^{\mathbf{n}} \to X$ such that $s = t \circ \Delta^f$:

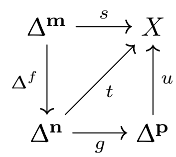
$$\begin{array}{ccc} \Delta^{\mathbf{m}} & \xrightarrow{s} & X \\ {\scriptstyle \Delta^f}\searrow & & \nearrow{\scriptstyle t} \\ & \Delta^{\mathbf{n}}. & \end{array}$$

The pair (f, t) is unique up to a unique isomorphism: if $(f'\colon \mathbf{m} \to \mathbf{n}', t'\colon \Delta^{\mathbf{n}'} \to X)$ is another such pair, then there is a unique isomorphism of simplices $h\colon \mathbf{n} \to \mathbf{n}'$ such that $h \circ f = f'$ and $t' \circ \Delta^h = t$:

$$\begin{array}{ccccc} \Delta^{\mathbf{m}} & & \xrightarrow{s} & & X \\ & {\scriptstyle \Delta^f}\searrow & & \nearrow{\scriptstyle t} & \\ {\scriptstyle \Delta^{f'}}\downarrow & & \Delta^{\mathbf{n}} & & \uparrow{\scriptstyle t'} \\ & & \downarrow{\scriptstyle \Delta^h} & & \\ & & \Delta^{\mathbf{n}'}. & & \end{array}$$

Used in 10.5, 39.9*.

Proof. We claim that any pair (f, t) for which $\dim \operatorname{codom} f$ is as small as possible is a pair for which t is nondegenerate. Indeed, if t is itself degenerate via some pair (g, u) (meaning $t = u \circ g$), then the pair $(g \circ f, u)$ would have the same properties and $\dim \operatorname{codom} g \circ f < \dim \operatorname{codom} f$, which contradicts minimality:



Suppose now that (f', t') is another pair with the same properties. By Exercise 6.8, the surjective map f respectively f' has a section $g: \Delta^{\mathbf{n}} \to \Delta^{\mathbf{m}}$ respectively $g': \Delta^{\mathbf{n}'} \to \Delta^{\mathbf{m}}$, i.e., $f \circ g = \mathrm{id}_{\Delta^{\mathbf{n}}}$ and $f' \circ g' = \mathrm{id}_{\Delta^{\mathbf{n}'}}$. We claim that $h = f' \circ g: \Delta^{\mathbf{n}} \to \Delta^{\mathbf{n}'}$ is the desired isomorphism. Indeed, $t \circ f = s = t' \circ f'$, so $(t' \circ f') \circ g = t \circ f \circ g = t$, i.e., $t' \circ h = t$. If we factor h into a surjection a followed by an injection b using Lemma 6.4, i.e., $h = b \circ a$, then a must be an isomorphism because otherwise t would be degenerate. Thus, h is injective, so $\dim \operatorname{dom} h \le \dim \operatorname{codom} h$, i.e., $\dim \mathbf{n} \le \dim \mathbf{n}'$. A symmetric argument yields $\dim \mathbf{n}' \le \dim \mathbf{n}$, so $\dim \mathbf{n} = \dim \mathbf{n}'$. Since h is injective, it must be an isomorphism.

The isomorphism h is unique because there is at most one isomorphism between any two simplices. We have $h \circ f = (f' \circ g) \circ f = f' \circ (g \circ f) = f'$, i.e., $h \circ f = f'$. To show that $h \circ f = f' \circ g \circ f = f'$, observe that g can be chosen to be arbitrary as long as $f \circ g = \mathrm{id}$. If $f'(g(f(i))) \ne f'(i)$ for some $i \in \mathbf{m}$, then choose g so that $i = g(f(i))$ and compute $f'(i) = f'(g(f(i))) \ne f'(i)$, a contradiction. ▮

Exercise 10.5. Which of the simplices of S^n are degenerate? (Give a complete proof of your claim.) For every degenerate simplex determine its Eilenberg–Zilber presentation, as in Proposition 10.4. Same question for $\Delta^{\mathbf{m}}$ and $\operatorname{dis} S$.

11 Categories

Supplementary sources: Lawvere and Rosebrugh [SETS], especially §1. Aluffi [ZERO, §I.3]. Also see [CATS, §4] for examples.

One cannot but observe a certain repetitiveness in the definitions of maps of simplices and simplicial maps: both Exercise 4.3 and Remark 8.5 say essentially the same thing, but in a slightly different context. Below we will see many more examples of this type, e.g., for chain maps, maps of groupoids, etc. Rather than repeat these properties ad nauseam, we bring out the underlying abstract notion.

Definition 11.1. (Eilenberg, MacLane, 1945.) A *category* C is specified by the following data and properties.

- A collection $\mathsf{Ob}(\mathsf{C})$ of *objects*. We write $X \in \mathsf{C}$ instead of $X \in \mathsf{Ob}(\mathsf{C})$.
- For any objects $X, Y \in \mathsf{Ob}(\mathsf{C})$ a set of *morphisms* (alias *hom-set*) $\mathsf{Mor}_{\mathsf{C}}(X, Y) = \mathsf{hom}_{\mathsf{C}}(X, Y)$, which can also be denoted by $\mathsf{C}(X, Y)$. We write $f: X \to Y$ instead of $f \in \mathsf{Mor}_{\mathsf{C}}(X, Y)$. We also write $\operatorname{dom} f = X$ (the *domain* of f) and $\operatorname{codom} f = Y$ (the *codomain* of f). Morphisms are also known as *maps* or *arrows*.
- For any objects $X, Y, Z \in \mathsf{Ob}(\mathsf{C})$ an operation of *composition*

$$\circ: \mathsf{Mor}_{\mathsf{C}}(Y, Z) \times \mathsf{Mor}_{\mathsf{C}}(X, Y) \to \mathsf{Mor}_{\mathsf{C}}(X, Z).$$

  We write $g \circ f$ instead of $\circ(g, f)$.
- For any object $X \in \mathsf{Ob}(\mathsf{C})$ an *identity morphism* $\mathrm{id}_X: X \to X$.
- Composition is *associative*: for any $f: W \to X$, $g: X \to Y$, $h: Y \to Z$ we have $(h \circ g) \circ f = h \circ (g \circ f)$, for which we may write $h \circ g \circ f$ instead.
- Composition is *unital*: for any $f: W \to X$ we have $\mathrm{id}_X \circ f = f \circ \mathrm{id}_W = f$.

Used in 1.2*, 1.3*, 2.0*, 4.0*, 4.1, 4.3, 4.4, 4.6*, 4.8, 5.2, 6.4*, 6.8, 6.14, 7.0*, 7.1, 7.7, 7.9, 7.12, 7.14, 8.2, 8.4, 8.5, 8.6, 8.10, 8.14, 8.14*, 8.16, 8.16*, 8.17, 8.17*, 8.19, 8.21, 9.6, 9.6*, 10.4*, 11.1, 11.2, 11.3, 11.5, 11.6, 11.7, 11.8, 11.9, 11.10, 11.11, 11.12, 11.13, 11.14, 11.14*, 11.15, 11.15*, 11.16, 11.17, 11.18, 11.19, 11.19*, 11.20, 11.21, 11.22, 11.23, 12.0*, 12.1, 12.2, 12.3, 12.4, 12.6, 12.8, 12.10, 12.15, 13.0*, 13.1, 13.4*, 13.7, 13.7*, 13.9, 13.13, 13.14, 13.20, 14.1, 14.1*, 14.2, 14.3, 14.4, 14.4*, 14.5, 14.6, 14.6*, 14.7, 14.8, 14.8*, 14.9, 14.10, 14.11, 14.15, 14.15*, 15.0*, 15.1, 15.4, 16.6*, 16.14*, 17.2, 17.4, 17.8, 17.8*, 17.13, 17.18, 17.19*, 17.20*, 17.21*, 20.10*, 20.14, 20.15, 21.0*, 21.1, 21.7, 21.7*, 21.9, 21.13, 21.14, 21.15, 21.20, 21.23, 21.28, 21.29, 22.8, 22.18*, 23.9, 25.6, 25.7, 25.7*, 25.8, 26.1, 26.2, 26.3, 26.5, 26.6, 26.7*, 26.9, 26.12, 26.13, 26.15, 26.16, 26.17, 26.17*, 26.19, 26.21, 26.23, 26.25, 26.25*, 26.27, 26.28, 26.29, 26.30, 26.30*, 26.31, 26.32, 26.33, 26.34, 26.42, 27.1, 27.2, 27.4*, 28.1, 28.2, 28.3, 28.4*, 28.7*, 29.2, 29.3, 29.8*, 29.13*, 29.19*, 29.22, 30.1, 30.9*, 30.10, 30.10*, 31.2, 31.4, 31.5, 33.1, 33.5, 34.0*, 34.1, 34.2, 34.4, 34.5*, 35.3, 35.5, 35.11, 39.2, 39.3, 39.5, 40.1, 40.11, 40.13*, 41.1, 41.2, 41.4, 42.1*, 43.4, 44.0*, 44.1, 46.2*, 46.4*, 48.3, 48.4, 50.1, 50.2, 50.4, 52.0*, 52.3, 59.1*, 59.3*, 59.4*.

Remark 11.2. A *collection* above refers to a set-like entity that can be too large to be a set. For instance, there is no set of all sets by Russell's paradox, but there is a collection of all sets. In the Zermelo–Fraenkel set theory such "large" sets are known as *classes*. A *proper class* is a class that is not a set. A *small category* is a category whose class of objects is a set (as opposed to a proper class), which implies that the class of all morphisms is also a set. A *finite category* is a category whose class of morphisms is a finite set, whcih implies that the class of objects is also a finite set. Used in 11.1, 11.2, 11.3, 11.5, 11.9, 11.10, 11.12, 11.14, 12.10, 13.9, 17.19*, 17.21, 26.1, 29.2, 30.5, 31.6, 42.1, 42.1*, 50.1, 50.2, 50.3, 50.4.

Remark 11.3. Occasionally, a variation of the above definition is used: instead of specifying the set of morphisms $\mathsf{Mor}_{\mathsf{C}}(X,Y)$ individually for all X and Y, we specify the collection of all morphisms, denoted by $\mathsf{Mor}(\mathsf{C})$, together with two maps of collections, the *domain map* $\operatorname{dom}\colon \mathsf{Mor}(\mathsf{C}) \to \mathsf{Ob}(\mathsf{C})$ and *codomain map* $\operatorname{codom}\colon \mathsf{Mor}(\mathsf{C}) \to \mathsf{Ob}(\mathsf{C})$. Composition is then defined for all pairs (g,f) such that $\operatorname{dom} g = \operatorname{codom} f$. In this case, $\operatorname{dom}(g\circ f) = \operatorname{dom} f$ and $\operatorname{codom}(g\circ f) = \operatorname{codom} g$. Likewise, $\operatorname{dom}(\mathrm{id}_X) = X = \operatorname{codom}(\mathrm{id}_X)$. In this definition, we must additionally require that $\mathsf{Mor}_{\mathsf{C}}(X,Y) := \{f \in \mathsf{Mor}(\mathsf{C}) \mid \operatorname{dom} f = X \wedge \operatorname{codom} f = Y\}$ is a set and not a proper class. Some authors do not impose this condition and refer to our variant as *locally small categories*.

Remark 11.4. Morphisms are composed from right to left: if $f\colon X \to Y$ and $g\colon Y \to Z$, then $g\circ f\colon X \to Z$. This is to accommodate Euler's notation $f(x)$ for the value of a map of sets f at an element x. Thus, $(g\circ f)(x) = g(f(x))$. If we wrote composition from left to right, we would have to write $(f\circ g)(x) = g(f(x))$, reversing the order of morphisms, which is error-prone.

Another way to see why morphisms should be composed from right to left is to fix a singleton set 1, e.g., $1 = \{\emptyset\}$. Maps of sets $1 \to X$ can be identified with elements of X: given a map $1 \to X$, its image is a singleton subset of X, i.e., an element of X; vice versa, an element $x \in X$ gives rise to a unique map $\bar{x}\colon 1 \to X$ whose image is a singleton subset $\{x\} \subset X$. We have $\overline{f(x)} = f\circ \bar{x}$, as long as morphisms are composed from right to left. Otherwise we would have $\overline{f(x)} = \bar{x}\circ f$, which is annoying.

Example 11.5. The primordial category is the *category of sets* Set.
- $\mathsf{Ob}(\mathsf{Set})$ is the class of all sets.
- $\mathsf{Mor}_{\mathsf{Set}}(X,Y)$ is the set of all functions from X to Y.
- The operation of composition is the standard composition of functions.
- The identity morphism of a set X is the identity function on X.
- As established in elementary set theory, the composition of functions is again a function, and the operation of composition is associative and unital.

Used in 11.5, 11.8, 11.14, 12.1*, 12.2, 12.3, 12.5, 12.8, 12.9, 12.15, 13.6, 13.7*, 13.18, 13.30, 14.5, 14.13, 14.15, 14.16, 14.17, 15.4*, 15.5, 15.14*, 17.2, 17.21*, 18.0*, 18.1, 21.6, 21.7*, 21.8, 21.18, 26.8, 26.19, 26.24, 26.26*, 26.37, 26.41, 30.1, 30.2, 30.3, 30.10, 32.4, 32.6, 32.7*, 32.8, 32.8*, 32.9, 33.4, 50.2, 61.8.

Warning 11.6. In the above example, the word "function" is used in the modern sense, which is synonymous with the word "map" (of sets). In particular, a function always "knows" not only its domain, but also its codomain, which stands in contrast to more archaic meanings of the word "function". Additionally, the composition $g\circ f$ only makes sense if $\operatorname{dom} g = \operatorname{codom} f$, which is once again very different from the archaic usage. See Remark 59.2 for more information.

Definition 11.7. A morphism $f\colon X \to Y$ in a category C is an *isomorphism* if there is a morphism $g\colon Y \to X$ in C such that $g\circ f = \mathrm{id}_X$ and $f\circ g = \mathrm{id}_Y$. Used in 4.4, 11.8, 14.11, 14.12, 21.13, 29.2.

Example 11.8. In the category Set isomorphisms are precisely bijective maps of sets.

Example 11.9. The *category of abelian groups* Ab is defined as follows.
- $\mathsf{Ob}(\mathsf{Ab})$ is the class of all abelian groups.
- $\mathsf{Mor}_{\mathsf{Ab}}(X,Y)$ is the set of all homomorphisms from X to Y.

- The operation of composition is given by the composition of underlying maps of sets.
- The identity morphism of a set X is the identity homomorphism on X.
- As established in elementary algebra, the composition of homomorphisms is again a homomorphism and the resulting operation is associative and unital.

Used in 11.9, 12.1*, 12.2, 12.3, 12.15, 14.16, 15.1, 15.4*, 15.14*, 16.5, 16.6, 16.8, 17.12, 17.23, 18.0*, 18.1, 18.2, 20.0*, 20.4, 20.6, 20.13, 20.16, 21.13, 22.6, 22.12, 22.13, 22.19, 23.5, 23.6*, 23.7, 25.8*, 26.8, 26.16, 26.19, 26.41, 29.25, 30.2, 33.1, 33.2, 33.4, 33.8*, 33.10, 61.8, 61.9*.

The *category of groups* Group and the *category of monoids* Monoid (groups without inverses) are defined analogously. Likewise for the *category of rings* Ring as well as the *category of modules* Mod_R over a ring R.

Example 11.10. The *category of simplices* $\mathsf{\Delta}$ is defined as follows.
- $\mathsf{Ob}(\mathsf{\Delta})$ is the class of all simplices.
- $\mathsf{Mor}_{\mathsf{\Delta}}(\mathbf{m}, \mathbf{n})$ is the set of all maps of simplices $\mathbf{m} \to \mathbf{n}$.
- The operation of composition is given by the composition of maps of simplices.
- The identity morphism of a simplex $\mathbf{m}$ is the identity map of $\mathbf{m}$.
- Associativity and unitality were shown in Exercise 4.3.

Used in 7.14, 11.10, 11.11, 12.5, 12.6, 12.7, 12.8, 12.15, 13.22*, 14.13, 14.16, 15.4*, 15.5, 17.8, 17.11, 17.20, 17.21, 18.1, 20.13, 26.8, 26.26*, 28.1, 28.2, 28.3, 28.4, 28.5, 28.5*, 28.6, 28.7, 28.7*, 28.8, 29.15, 29.16, 30.5, 31.1, 31.2, 31.3, 31.3*, 31.5, 31.6, 31.6*, 32.6, 32.7*, 33.10, 39.3.

Warning 11.11. The letter $\mathsf{\Delta}$, which denotes a category, should not be confused with the letter Δ, which denotes a functor (defined below), which sends an object $\mathbf{m}$ of $\mathsf{\Delta}$ to a simplicial set $\Delta^{\mathbf{m}}$. The typographic distinction between $\mathsf{\Delta}$ and Δ is admittedly subtle, but many sources make no distinction whatsoever. Goerss and Jardine use $\boldsymbol{\Delta}$ and Δ instead of $\mathsf{\Delta}$ and Δ, whereas we follow our established notational conventions for categories and functors respectively.

Example 11.12. The *category of simplicial sets* sSet is defined as follows.
- $\mathsf{Ob}(\mathsf{sSet})$ is the class of all simplicial sets.
- $\mathsf{Mor}_{\mathsf{sSet}}(X, Y)$ is the set of simplicial maps $X \to Y$, also denoted by $\mathsf{hom}(X, Y)$.
- The operation of composition is given by the composition of simplicial maps.
- The identity morphism of a simplicial set X is the identity simplicial map of X.
- Associativity and unitality were verified after Definition 8.4.

Used in 7.9, 7.14, 9.4, 11.12, 11.13, 12.7, 12.9, 12.15, 13.0*, 13.7, 13.7*, 13.19, 13.20, 13.30, 14.16, 15.14, 15.14*, 15.15, 16.0*, 16.4, 16.8, 17.3, 17.5, 17.11, 17.12, 17.23, 18.1, 18.2, 18.6, 20.5, 20.6, 20.13, 20.16, 21.0*, 21.7, 21.7*, 21.12, 21.13, 21.19, 21.20, 21.21, 21.30, 22.5, 22.6, 22.12, 22.13, 22.19, 23.5, 23.7, 25.8*, 26.8, 26.26, 26.26*, 26.39, 28.3, 28.4, 28.5, 28.5*, 28.6, 28.7*, 28.8, 28.9, 29.16, 29.26, 30.3, 30.5, 31.1, 31.2, 31.3, 31.5, 31.6*, 32.1, 32.6, 32.7*, 32.9, 32.10, 33.6, 34.0*, 34.1, 34.3, 34.4, 34.6, 35.7*, 39.2, 39.3, 39.5, 39.16, 40.13*, 41.2, 42.3, 42.4, 46.2, 46.7, 51.1.

Remark 11.13. In many typical examples the definition of composition and identity morphisms, as well as the verification of associativity and unitality properties is a fairly routine task (as can be seen from the above examples), and is often omitted. Accordingly, one often specifies categories by saying what their objects and morphisms are. For instance, one could say that sSet is the category of simplicial sets and simplicial maps. Sometimes the definition of morphisms is also clear from the context, and in this case one simply specifies the objects. For instance, one could say that sSet is the category of simplicial sets. However, one must keep in mind that one can encounter in practice categories with the same collection of objects, but different morphisms. For instance, one has three very different notions of a morphism between metric spaces:
- contractive maps: $f\colon X \to Y$ is contractive if $\mathrm{d}(f(x), f(x')) \le \mathrm{d}(x, x')$ for any points $x, x' \in X$.
- uniformly continuous maps: $f\colon X \to Y$ is uniformly continuous if for any $\epsilon > 0$ there is $\delta > 0$ such that $\mathrm{d}(x, x') < \delta$ implies $\mathrm{d}(f(x), f(x')) < \epsilon$.
- continuous maps: $f\colon X \to Y$ is continuous if for any $x \in X$ and $\epsilon > 0$ there is $\delta > 0$ such that $\mathrm{d}(x, x') < \delta$ implies $\mathrm{d}(f(x), f(x')) < \epsilon$.

These three types of maps give rise to three different categories of metric spaces:
- the category of metric spaces and contractive maps;
- the category of metric spaces and uniformly continuous maps;
- the category of metric spaces and continuous maps.

Example 11.14. The *category of graphs* Graph is defined as follows.

- $\mathsf{Ob}(\mathsf{Graph})$ is the class of quadruples (V, E, s, t), where V and E are sets (of vertices respectively edges) and $s: E \to V$ and $t: E \to V$ are maps of sets (the source and target map respectively).
- $\mathsf{Mor}_{\mathsf{Graph}}((V, E, s, t), (V', E', s', t'))$ is the set of pairs (v, e), where $v: V \to V'$ and $e: E \to E'$ are maps of sets such that $t' \circ e = v \circ t$ and $s' \circ e = v \circ s$.
- Composition is pairwise: $(v', e') \circ (v, e) = (v' \circ v, e' \circ e)$.
- The identity morphism of a graph (V, E, s, t) is $(\mathrm{id}_V, \mathrm{id}_E)$.
- Associativity and unitality of composition follow from the same properties of Set.

Used in 11.14, 11.15.

For example, discarding the data of composition and identity morphisms from a category C produces the *underlying graph* of C, denoted by $\mathsf{U}(\mathsf{C})$. Another example of a graph is a path of length n: $[n] := (\{0, \ldots, n\}, \{0, \ldots, n-1\}, s, t)$, where $s(n) = n$ and $t(n) = n + 1$. This graph can be depicted by $0 \to 1 \to 2 \to \cdots \to n$. Given a morphism of graphs $p: [n] \to \mathsf{U}(\mathsf{C})$, its composition can be defined in the obvious way, resulting in a morphism $p_0 \to p_n$ in C, where p_0 and p_n denotes the images of $0 \in V$ and $n \in V$ under p. The composition is well-defined by associativity and unitality properties.

Definition 11.15. (Eduard Study, 1891.) A *simple diagram* in a category C is a morphism of graphs $D: G \to \mathsf{U}(\mathsf{C})$, where G is a graph. A *commutative diagram* is a simple diagram $D: G \to \mathsf{U}(\mathsf{C})$ such that for any $p: [m] \to G$ and $q: [n] \to G$ with $p_0 = q_0$ and $p_m = q_n$ the compositions of $D \circ p$ and $D \circ q$ coincide. Here $\mathsf{U}(\mathsf{C})$ denotes the underlying graph of C defined above. Used in 11.15, 11.15*, 13.7*, 28.4*, 46.2*.

Informally, a simple diagram is commutative if composing any two paths between the same pair of objects results in identical morphisms. We have seen multiple examples of commutative diagrams above, e.g., when we defined chain maps in Definition 15.3.

We conclude this section by giving some examples that should dispel the idea that objects in a category are "sets with structures" and morphisms are "functions that preserve structures" (as one could guess from the above examples).

Our first example explores the familiar notion of a measurable space from real analysis, while carefully incorporating the notion of equality almost everywhere, which is omnipresent in measure theory.

Example 11.16. The category EMS of *enhanced measurable spaces* is defined as follows. Objects are triples (X, M, N), where X is a set, M is a σ-algebra on X (a collection of subsets of X closed under complements and countable unions), and $N \subset M$ is a σ-ideal of X (a collection of subsets of X closed under passage to subsets and countable unions). Morphisms $(X, M, N) \to (X', M', N')$ are equivalence classes of *measurable maps of sets* $f: X \to X'$, which are defined as maps of sets such that for any $m \in M'$ we have $f^{-1}(m) \in M$ and for any $n \in N'$ we have $f^{-1}(n) \in N$. Two measurable maps of sets $f, g: X \to X'$ are equivalent if $\{x \in X \mid f(x) \neq g(x)\} \in N$. Composition of measurable maps of sets respects this equivalence relation, therefore we can talk about equivalence classes of measurable maps of sets and their compositions. These equivalence classes are also known as *maps of measurable spaces* or simply *measurable maps*, not to be confused with measurable maps of sets, which are mere representatives of these equivalence classes. A measurable map $t: (X, M, N) \to (X', M', N')$ is not a map of sets: given a point $x \in X$, there is no way to "evaluate" t on x: if we choose some representative f of t and compute $f(x)$, the result depends on the choice of f. Used in 11.16, 12.12, 12.13, 17.9.

The next two examples come from algebra.

Example 11.17. The category Poset has posets $(S, \leq)$ as objects, whereas morphisms $(S, \leq) \to (T, \leq)$ are maps of sets $f: S \to T$ such that $s \leq s'$ implies $f(s) \leq f(s')$ for all $s, s' \in S$. The categories Order and $\mathsf{Preorder}$ of ordered sets and preordered sets are defined analogously. Used in 17.20.

Example 11.18. Suppose $(P, \leq)$ is a poset. We construct a category C as follows: $\mathsf{Ob}(\mathsf{C}) = P$ and if $x, y \in P$ then $\mathsf{Mor}_{\mathsf{C}}(x, y)$ is a singleton respectively empty set if $x \leq y$ respectively $x \not\leq y$. There is exactly one way to define compositions and identity morphisms. The resulting category is very special: there is at most one morphism between any pair of objects. Such categories are known as *thin categories*. In fact, every thin category is induced by the above construction from a preordered set, defined in the same way as a poset, but without the antisymmetry condition. Used in 11.18, 17.20.

Example 11.19. Suppose G is a group or a monoid (defined like a group, but without inverses). We define the *delooping category* $\mathsf{B}G$ as follows: $\mathsf{Ob}(\mathsf{C}) = \{*\}$ is a singleton set and $\mathsf{Mor}_{\mathsf{C}}(*,*) = \mathsf{U}(G)$. The operation of composition is given by multiplication: $\mathsf{Mor}(*,*) \times \mathsf{Mor}(*,*) = G \times G \to G = \mathsf{Mor}(*,*)$. The identity morphism of $*$ is given by the identity element of G. Used in 26.23, 29.4.

Notice how in the last example G can be extracted from $\mathsf{B}G$ as $\mathsf{Mor}(*,*)$ with the operation of composition.

Example 11.20. The *empty category* $\emptyset$ has $\mathsf{Ob}(\mathsf{C}) = \emptyset$. (There is no other data left to specify.)

Our final example is abstract and may be particularly difficult to comprehend for newbies.

Example 11.21. Suppose C is a category. The *opposite category* of C is denoted by C^{op} and is defined as follows: the set $\mathsf{Ob}(\mathsf{C}^{\mathsf{op}})$ equals $\mathsf{Ob}(\mathsf{C})$ (via the identity map, denoted by $X \mapsto X^{\mathsf{op}}$) and $\mathsf{Mor}_{\mathsf{C}^{\mathsf{op}}}(X^{\mathsf{op}}, Y^{\mathsf{op}})$ equals $\mathsf{Mor}_{\mathsf{C}}(Y, X)$ (via the identity map denoted by $f \mapsto f^{\mathsf{op}}$). The composition is defined by $g^{\mathsf{op}} \circ f^{\mathsf{op}} = (f \circ g)^{\mathsf{op}}$ and $\mathrm{id}_{X^{\mathsf{op}}} = \mathrm{id}_X^{\mathsf{op}}$. (We use op to denote both the opposite category C^{op} as well as objects and morphisms in it.) Used in 11.21, 11.22, 11.24, 12.8, 12.11, 12.13, 12.15, 14.13, 14.16, 15.4*, 15.5, 17.16, 18.1, 20.0*, 20.1, 20.3, 20.5, 20.6, 20.13, 20.16, 21.4*, 23.5, 23.7, 24.1, 26.26*, 26.30*, 30.1, 30.9*, 30.10, 34.4, 34.6, 39.16, 40.13*, 40.14, 43.6, 50.1, 50.4, 51.1, 51.2.

Remark 11.22. The category $(\mathsf{C}^{\mathsf{op}})^{\mathsf{op}}$ equals the category C. (Reversing the direction of morphisms twice amounts to not doing anything.) Thus, $(X^{\mathsf{op}})^{\mathsf{op}} = X$ and $(f^{\mathsf{op}})^{\mathsf{op}} = f$.

Exercise 11.23. For each of the sets of data given below, determine whether the missing elements (e.g., composition, identity morphisms) can be specified as to yield a category, or prove that such an extension is impossible. The data is listed in the following order: objects, morphisms, composition (if given), identity morphisms (if given).

- Sets, injective maps of sets, the standard composition.
- Sets, maps of sets that are not surjective, the standard composition.
- Given a pair (S, R), where S is a set and $R \subset S \times S$ is a reflexive transitive relation on S, objects are elements of S, morphisms from s to s' are pairs $(s, s') \in R$.
- Objects are sets, morphisms from S to S' are elements in the intersection $S \cap S'$.
- Objects are sets, morphisms from S to S' are elements in the union $S \cup S'$.
- Objects are sets, morphisms from S to S' are maps $f\colon S \to 2^{S'}$ that send different elements of S to disjoint subsets of S' and such that $\bigcup_{s \in S} f(s) = S'$.

Exercise 11.24. Given a monoid G, is $(\mathsf{B}G)^{\mathsf{op}} = \mathsf{B}H$ for some monoid H?

12 Functors

Supplementary sources: Lawvere and Rosebrugh [SETS], especially §10.2. Aluffi [ZERO, §VIII.1]. Also see [CATS, §6] for examples.

We have already encountered many constructions that preserve composition of morphisms and identity morphisms. Rather than to continue repeating these properties indefinitely, we elect to formalize them using the previously defined notion of categories.

Definition 12.1. (Eilenberg, MacLane, 1945.) Suppose C and D are categories. A *functor* F from C to D, denoted by $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$, is specified by the following data and properties.

- A map of collections $\mathsf{Ob}(\mathsf{F})\colon \mathsf{Ob}(\mathsf{C}) \to \mathsf{Ob}(\mathsf{D})$. (We write $\mathsf{F}(X)$ or $\mathsf{F}X$ instead of $\mathsf{Ob}(\mathsf{F})(X)$.)
- A collection of maps of sets $\mathsf{Mor}_\mathsf{F}(X,Y)\colon \mathsf{Mor}_\mathsf{C}(X,Y) \to \mathsf{Mor}_\mathsf{D}(\mathsf{F}(X),\mathsf{F}(Y))$, one for each pair of objects $X, Y \in \mathsf{C}$. (We write $\mathsf{F}(f)$ or $\mathsf{F}f$ instead of $\mathsf{Mor}_\mathsf{F}(X,Y)(f)$, where $f\colon X \to Y$ is a morphism in C.)
- Composition is preserved: if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms in C, then $\mathsf{F}(g \circ f) = \mathsf{F}(g) \circ \mathsf{F}(f)$.
- Identity morphisms are preserved: if $X \in \mathsf{C}$, then $\mathsf{F}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{F}(X)}$.

The last two properties taken together are known as *functoriality properties.* Used in 2.0*, 4.5, 5.1, 7.0*, 7.1, 7.2, 7.12, 7.15, 8.6, 8.9, 8.10, 8.19, 11.11, 12.3*, 12.5, 12.6, 12.7, 12.8, 12.9, 12.10, 12.12, 12.13, 12.15, 13.7*, 14.1*, 14.2, 14.3, 14.4*, 14.7, 14.8, 14.8*, 14.9, 14.10, 14.12, 14.14, 14.15, 14.15*, 14.17, 15.4*, 16.6, 16.6*, 16.8, 16.9, 17.3, 17.5, 17.11, 17.12, 17.19*, 17.20, 17.23, 18.2, 20.0*, 20.1, 20.3, 20.6, 21.7*, 25.7*, 25.8*, 26.1, 26.5, 26.40, 26.41, 26.42, 27.3, 27.4, 28.1, 28.3, 28.5*, 28.6, 28.7, 28.9, 29.16, 29.17*, 30.1, 31.1, 31.4, 31.5, 31.6, 33.1, 33.4, 39.2, 39.6, 39.6*, 39.16, 40.1, 42.1.

Our first two examples of categories were Set and Ab, so we start by exhibiting some functors between them.

Example 12.2. The *forgetful functor* $\mathsf{U}\colon \mathsf{Ab} \to \mathsf{Set}$ is defined as follows.

- For an abelian group $A = (S, +, -, 0)$ we set $\mathsf{U}(A) = S$, the underlying set of A.
- A homomorphism of abelian groups $f\colon A = (S, +_A, -_A, 0_A) \to B = (T, +_B, -_B, 0_B)$ is by definition a map of sets $g\colon S \to T$ that satisfies some additional properties. We set $\mathsf{U}(f) = g$, the underlying map of sets of f.
- Composition is preserved because the composition of two homomorphisms of abelian groups is by definition the composition of the underlying maps of sets.
- Identity morphisms are preserved for the same reason.

Example 12.3. The *free abelian group functor* $\mathsf{Free} = \mathbf{Z}[-]\colon \mathsf{Set} \to \mathsf{Ab}$ is defined as follows.

- For a set S we set $\mathsf{Free}(S) = \mathbf{Z}[S] = \{c\colon S \to \mathbf{Z} \mid \#\{s \in S \mid c(s) \neq 0\} < \infty\}$, i.e., the abelian group of finitely supported functions $S \to \mathbf{Z}$ equipped with the pointwise operations induced from the abelian group $\mathbf{Z}$.
- For a map of sets $f\colon S \to T$ we set $\mathsf{Free}(f) = \mathbf{Z}[f]\colon \mathbf{Z}[S] \to \mathbf{Z}[T]$ to the homomorphism of abelian groups that sends any $c\colon S \to \mathbf{Z}$ to the map $T \to \mathbf{Z}$ that sends $t \mapsto \sum_{s \in f^*(\{t\})} c(s)$.
- As shown in elementary algebra, composition and identity morphisms are preserved by $\mathbf{Z}[-]$.

Used in 12.3, 15.4*, 18.0*, 18.1, 30.2.

We will use the functor $\mathbf{Z}[-]$ when we define simplicial chains using certain quotient groups of $\mathbf{Z}[X_\mathbf{n}]$ in Definition 15.5.

Example 12.4. The *empty functor* $\mathrm{id}_\emptyset\colon \emptyset \to \emptyset$ is defined by $\mathsf{Ob}(\mathrm{id}_\emptyset) = \mathrm{id}_\emptyset\colon \emptyset \to \emptyset$. (Recall that maps of sets $\emptyset \to \emptyset$ can be identified with subsets of $\emptyset \times \emptyset$ possessing a certain property. The only subset of $\emptyset \times \emptyset = \emptyset$ is the empty set $\emptyset$, which has this property.)

Example 12.5. Definition 4.5 is nothing else than a definition of a *forgetful functor* $\mathsf{U}\colon \mathsf{\Delta} \to \mathsf{Set}$. (Any functor that "forgets" structure like abelian group operations or a total ordering can be referred to as a forgetful functor and denoted by U.)

Example 12.6. Definition 5.1 combined with Remark 5.3 defines a functor $|-|\colon \mathsf{\Delta} \to \mathsf{Space}$, where Space denotes any of the categories of "geometric spaces" mentioned in Definition 17.8. The functoriality properties are verified in Remark 5.2. Used in 12.6, 17.8, 17.11, 17.12, 17.13, 17.18, 20.13, 20.14, 20.15.

Example 12.7. The Yoneda embedding of Definition 7.10 is a functor $\Delta\colon \mathsf{\Delta} \to \mathsf{sSet}$. (Here Δ and $\mathsf{\Delta}$ are typeset in different fonts, so denote different entities: a functor and a category respectively.)

Example 12.8. A simplicial set is nothing else than a functor $\mathsf{\Delta}^{\mathsf{op}} \to \mathsf{Set}$. Here the superscript op denotes the opposite category of $\mathsf{\Delta}$ defined in Example 11.21. Indeed, expanding the definitions, we see that such a functor X assigns a set $X_{\mathbf{m}}$ to any simplex $\mathbf{m}$, a map of sets $X_{\mathbf{n}} \to X_{\mathbf{m}}$ to any map of simplices $\mathbf{m} \to \mathbf{n}$ (the direction of the map is reversed because of the opposite category), and the functoriality conditions demand that $X_{\mathrm{id}_{\mathbf{m}}} = \mathrm{id}_{X_{\mathbf{m}}}$ for any simplex $\mathbf{m}$ and $X_{g \circ f} = X_f \circ X_g$ for any maps of simplices $f\colon \mathbf{m} \to \mathbf{n}$ and $g\colon \mathbf{n} \to \mathbf{p}$. This is precisely Definition 7.1. (One may wonder how simplicial maps fit into this picture. These turn out to be precisely natural transformations of functors, to be defined later.) Used in 14.13, 15.4*.

Example 12.9. The discrete simplicial set (Definition 7.7) construction is a functor $\mathrm{dis}\colon \mathsf{Set} \to \mathsf{sSet}$.

Example 12.10. Categories and functors themselves can be organized into a category, the *category of categories*, commonly denoted by Cat or CAT. (These two choices correspond to requiring the collection of all objects in a category to form a set respectively a class in the Zermelo–Fraenkel set theory, a technical issue that can be ignored for the time being.) More precisely,

- $\mathsf{Ob}(\mathsf{Cat})$ is the collection of all categories whose collection of objects is a set.
- $\mathsf{Mor}_{\mathsf{Cat}}(\mathsf{X}, \mathsf{Y})$ is the set of functors $\mathsf{F}\colon \mathsf{X} \to \mathsf{Y}$.
- Composition is defined as follows: $(\mathsf{G} \circ \mathsf{F})(A) = \mathsf{G}(\mathsf{F}(A))$ for any object $A \in \operatorname{dom} \mathsf{X}$; $(\mathsf{G} \circ \mathsf{F})(f) = \mathsf{G}(\mathsf{F}(f))$ for any morphism f in $\operatorname{dom} \mathsf{X}$.
- The identity morphism of a category X is the *identity functor*: $\mathrm{id}_{\mathsf{X}}(A) = A$ for any object $A \in \mathsf{X}$; $\mathrm{id}_{\mathsf{X}}(f) = f$ for any morphism f in X.
- Composition of functors is associative and unital because composition of maps of sets is associative and unital.

Used in 12.10, 14.15*, 17.19*, 17.20, 17.21, 29.2, 29.9, 30.5, 30.6, 30.7, 31.6, 31.6*, 41.5, 50.1, 50.2, 50.3.

Remark 12.11. A functor of the form $\mathsf{C}^{\mathsf{op}} \to \mathsf{D}$ is sometimes referred to as a *contravariant functor* $\mathsf{C} \to \mathsf{D}$ (without op). The adjective "contravariant" refers to the following: if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms in C, then $F(g^{\mathsf{op}} \circ f^{\mathsf{op}}) = F(f^{\mathsf{op}}) \circ F(g^{\mathsf{op}})$, i.e., contravariant functors exchange the order of composition. "Traditional" functors are then referred to as *covariant functors*. Used in 12.13.

We illustrate the important difference between covariant and contravariant functors by a familiar example of L^p-spaces from analysis.

Example 12.12. We specify a (covariant) functor $\mathsf{L}^1\colon \mathsf{EMS} \to \mathsf{Banach}$ as follows. Banach is the category of (complex) Banach spaces and continuous linear maps. A map of sets $\mu\colon M \to \mathbf{C}$ is *countably additive* if for any countable family $\{m_i\}_{i \in I}$ of elements of M such that $m_i \cap m_j = \emptyset$ we have $\mu\left(\bigcup_{i \in I} m_i\right) = \sum_{i \in I} \mu(m_i)$. We set $\mathsf{L}^1(X, M, N)$ to the set of all countably additive maps of sets $\mu\colon M \to \mathbf{C}$ such that $\mu|_N = 0$. This set can be equipped with a structure of a Banach space: the vector space operations are induced from $\mathbf{C}$ and the norm is $\mu \mapsto \|\mu\| = \sup_{|f| \le 1} \int f \mathrm{d}\mu$, where $\int$ denotes the Lebesgue integral. Given a morphism $f\colon (X, M, N) \to (X', M', N')$, we define $\mathsf{L}^1(f)\colon \mathsf{L}^1(X, M, N) \to \mathsf{L}^1(X', M', N')$ by sending $\mu \in \mathsf{L}^1(X, M, N)$ to $f_*(\mu) \in \mathsf{L}^1(X', M', N')$ such that $f_*(\mu)(m') = \mu(g^{-1}(m'))$. Here g is a representative of f. A different choice of g gives the same answer because f^{-1} sends elements of N' to N. This assignment is functorial: $\mathsf{L}^1(g \circ f)(\mu)(m'') = \mu((g \circ f)^{-1}(m'')) = \mu(f^{-1}(g^{-1}(m''))) = (\mathsf{L}^1(f)(\mu))(g^{-1}(m'')) = \mathsf{L}^1(g)(\mathsf{L}^1(f))(m'')$, so $\mathsf{L}^1(g \circ f) = \mathsf{L}^1(g) \circ \mathsf{L}^1(f)$. Used in 12.12, 12.13.

Example 12.13. We specify a functor $\mathsf{L}^\infty\colon \mathsf{EMS}^{\mathsf{op}} \to \mathsf{Banach}$ (i.e., a contravariant functor $\mathsf{EMS} \to \mathsf{Banach}$) as follows. We set $\mathsf{L}^\infty(X, M, N)$ to the set of morphisms $\{s\colon (X, M, N) \to (\mathbf{C}, \mathbf{C}_{\mathrm{Borel}}, \{\emptyset\})$ (here $\mathbf{C}_{\mathrm{Borel}}$ denotes the Borel σ-algebra of $\mathbf{C}$) that are bounded, i.e., one of their representatives factors through a bounded subset of $\mathbf{C}$. Given a morphism $f\colon (X, M, N) \to (X', M', N')$, we define $\mathsf{L}^\infty(f)\colon \mathsf{L}^\infty(X', M', N') \to \mathsf{L}^\infty(X, M, N)$ by sending $s\colon (X', M', N') \to (\mathbf{C}, \mathbf{C}_{\mathrm{Borel}}, \{\emptyset\})$ to $s \circ f$. This assignment is functorial: $\mathsf{L}^\infty(g \circ f)(s) = s \circ (g \circ f) = (s \circ g) \circ f = \mathsf{L}^\infty(f)(s \circ g) = \mathsf{L}^\infty(f)(\mathsf{L}^\infty(g)(s))$, so $\mathsf{L}^\infty(g \circ f) = \mathsf{L}^\infty(f) \circ \mathsf{L}^\infty(g)$. Used in 12.13.

Exercise 12.14. Are there categories C and D such that there are no functors $\mathsf{C} \to \mathsf{D}$? Are there categories C and D such that there are no functors $\mathsf{C} \to \mathsf{D}$ and no functors $\mathsf{D} \to \mathsf{C}$?

Exercise 12.15. For each of the sets of data given below, determine whether the missing elements (e.g., the values on objects or morphisms) can be specified as to yield a functor, or prove that such an extension is impossible. (If there are no missing elements, you must prove or disprove that the data specifies a functor.)

The data is listed in the following order: source category, target category, values on objects (if given), values on morphisms (if given).

- Source $\mathsf{B}G$, target $\mathsf{B}H$, sends a morphism $g{:}\,* \to *$ in $\mathsf{B}G$ to the morphism $f(g){:}\,* \to *$ in $\mathsf{B}H$, where $f{:}\,G \to H$ is a homomorphism of monoids.
- Source $(\mathsf{B}G)^{\mathsf{op}}$, target $\mathsf{B}G$, a morphism $g{:}\,* \to *$ is sent to the morphism $g^{-1}{:}\,* \to *$. Here G is an arbitrary group.
- Source $\mathsf{B}G$, target Set, the object $*$ is sent to the set $\mathsf{U}(G)$, a morphism $g{:}\,* \to *$ is sent to the map of sets $\mathsf{U}(G) \to \mathsf{U}(G)$ such that $h \mapsto hg$. Here G is an arbitrary group.
- Same setting as the previous item, but $h \mapsto gh$.
- Source Set, target Group, a set S is sent to the symmetric group Σ_S, i.e., the group of bijections $S \to S$ with the operation of composition.
- Source sSet, target Set, a simplicial set X is sent to the disjoint union of $X_{\mathbf{m}}$ for all standard simplices $\mathbf{m} = \{0 < 1 < 2 < \cdots < m\}$.
- Source Δ, target Δ, a simplex $\mathbf{m} = (V, \le)$ is sent to the simplex $(V, \le^{\mathsf{op}})$, where $v_1 \le^{\mathsf{op}} v_2$ if $v_2 \le v_1$, i.e., the order is reversed.

13 Coproducts and coequalizers of simplicial sets

Supplementary sources: Lawvere and Rosebrugh [SETS, §2.1, §4.4]. Aluffi [ZERO, §I.5]. Also see [CATS, §12, §14] for examples.

The aim of this section is to show that a simplicial set generated by a system of generators and relations exists and is unique. This gives us a convenient reason to introduce a few constructions with simplicial sets: coproducts and coequalizers.

The first notion, coproduct of simplicial sets, has a very simple geometric interpretation: we assemble two pictures side by side, without intersections, like a disjoint union of sets.

We define coproducts in an arbitrary category C and then instantiate to $\mathsf{C} = \mathsf{sSet}$.

Definition 13.1. The *coproduct* of objects X and Y in a category C (if it exists) is a triple $(X \sqcup Y, \iota_X{:}\,X \to X \sqcup Y, \iota_Y{:}\,Y \to X \sqcup Y)$, where $X \sqcup Y \in \mathsf{C}$ and ι_X, ι_Y are morphisms in C (the *injection maps* or injection morphisms) such that the following *universal property of coproducts* is satisfied: for any $Z \in \mathsf{C}$ the map $(\mathsf{hom}(\iota_X, Z), \mathsf{hom}(\iota_Y, Z)){:}\,\mathsf{hom}(X \sqcup Y, Z) \to \mathsf{hom}(X, Z) \times \mathsf{hom}(Y, Z)$ that sends $f{:}\,X \sqcup Y \to Z$ to $(f \circ \iota_X, f \circ \iota_Y)$ is a bijection. Used in 13.0*, 13.7, 13.7*, 13.8, 13.9, 13.11, 13.12, 13.22*, 18.5*, 26.1*, 26.2*, 26.11, 26.18, 26.30, 39.9, 42.3, 43.4, 43.6, 43.7.

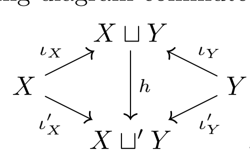
Notation 13.2. Given $f{:}\,X \to Z$ and $g{:}\,Y \to Z$, the inverse image of (f, g) under the above map is known as the *copairing* of f and g and is denoted by $[f, g]{:}\,X \sqcup Y \to Z$.

Notation 13.3. Given $f_1{:}\,X_1 \to Y_1$ and $f_2{:}\,X_2 \to Y_2$, we define the map $f_1 \sqcup f_2{:}\,X_1 \sqcup X_2 \to Y_1 \sqcup Y_2$ as $[\iota_{Y_1} \circ f_1, \iota_{Y_2} \circ f_2]$.

Informally, we say that a map $h{:}\,X \sqcup Y \to Z$ is the "same thing" as a pair of maps $f{:}\,X \to Z$ and $g{:}\,Y \to Z$. Given h, we recover f and g as $f = h \circ \iota_X$ and $g = h \circ \iota_Y$. Given f and g, we recover h as $h = [f, g]$.

The following uniqueness lemma is entirely analogous to Lemma 9.6.

Lemma 13.4. If X and Y are objects in a category C and $(X \sqcup Y, \iota_X{:}\,X \to X \sqcup Y, \iota_Y{:}\,Y \to X \sqcup Y)$, $(X \sqcup' Y, \iota'_X{:}\,X \to X \sqcup' Y, \iota'_Y{:}\,Y \to X \sqcup' Y)$ are coproducts of X and Y, then there is a unique map $h{:}\,X \sqcup Y \to X \sqcup' Y$ that makes the following diagram commute:



.

Furthermore, h is an isomorphism. Used in 13.16, 21.4*.

Proof. The set of morphisms $h{:}\,X \sqcup Y \to X \sqcup' Y$ can be identified with the set of pairs of morphisms $X \to X \sqcup' Y$, $Y \to X \sqcup' Y$ by sending $h \mapsto (h \circ \iota_X, h \circ \iota_Y)$. The commutativity of the diagram requires that the latter pair equals (ι'_X, ι'_Y). This shows that h exists and is unique.

The above argument can be used with two triples exchanged, yielding a unique morphism $h'\colon X\sqcup' Y \to X \sqcup Y$ that makes the two triangles with the same morphisms as above commute.

The morphism $h'\circ h\colon X\sqcup Y \to X\sqcup Y$ as well as the morphism $\mathrm{id}_{X\sqcup Y}\colon X\sqcup Y\to X\sqcup Y$ both make the above diagram commute when the second triple equals the first triple. By the uniqueness statement in the universal property of coproducts, $h'\circ h = \mathrm{id}_{X\sqcup Y}$. Likewise, $h\circ h' = \mathrm{id}_{X\sqcup' Y}$. Thus, h and h' are isomorphisms. ▮

Remark 13.5. Although coproducts are always unique, their existence depends on a particular choice of C.

Example 13.6. Recall from §59 that coproducts in the category $\mathsf{C} = \mathsf{Set}$ are characterized by the property that ι_X and ι_Y are injective maps with disjoint images whose union is $X\sqcup Y$. Thus, the coproduct of X and Y is simply the disjoint union of X and Y, with ι_X and ι_Y the two injection maps.

Proposition 13.7. In the category sSet, the coproduct of X and Y exists. Used in 21.6*, 42.3.

Proof. Define $(X\sqcup Y)_{\mathbf{m}} = X_{\mathbf{m}}\sqcup Y_{\mathbf{m}}$ and $(X\sqcup Y)_f = X_f\sqcup Y_f\colon X_{\mathbf{n}}\sqcup Y_{\mathbf{n}}\to X_{\mathbf{m}}\sqcup Y_{\mathbf{m}}$. We now verify the functoriality property. We have

$$(X\sqcup Y)_{\mathrm{id}_{\mathbf{m}}} = X_{\mathrm{id}_{\mathbf{m}}}\sqcup Y_{\mathrm{id}_{\mathbf{m}}} = \mathrm{id}_{X_{\mathbf{m}}\sqcup Y_{\mathbf{m}}}\,.$$

Likewise,

$$(X\sqcup Y)_{g\circ f} = X_{g\circ f}\sqcup Y_{g\circ f} = X_f\circ X_g\sqcup Y_f\circ Y_g = (X_f\sqcup Y_f)\circ(X_g\sqcup Y_g) = (X\sqcup Y)_f\circ(X\sqcup Y)_g,$$

which completes the construction of $X\sqcup Y$.

We construct the simplicial maps $\iota_X\colon X\to X\sqcup Y$ and $\iota_Y\colon Y\to X\sqcup Y$ as follows. Set $(\iota_X)_{\mathbf{m}}$ to $\iota_{X_{\mathbf{m}}}\colon X_{\mathbf{m}}\to X_{\mathbf{m}}\sqcup Y_{\mathbf{m}} = (X\sqcup Y)_{\mathbf{m}}$ and likewise for Y. The naturality property of simplicial maps is verified by the following commutative diagram for any map of simplices $f\colon\mathbf{m}\to\mathbf{n}$:

$$\begin{array}{ccc}
X_{\mathbf{m}} & \xrightarrow{\iota_{X_{\mathbf{m}}}} & X_{\mathbf{m}}\sqcup Y_{\mathbf{m}} \\
{\scriptstyle X_f}\big\uparrow & & \big\uparrow{\scriptstyle X_f\sqcup Y_f} \\
X_{\mathbf{n}} & \xrightarrow{\iota_{X_{\mathbf{n}}}} & X_{\mathbf{n}}\sqcup Y_{\mathbf{n}}.
\end{array}$$

It remains to show the universal property of coproducts. If $Z\in\mathsf{sSet}$ and $f\colon X\to Z$, $g\colon Y\to Z$ are simplicial maps, we must show that there is a unique $h\colon X\sqcup Y\to Z$ such that $h\circ\iota_X = f$ and $h\circ\iota_Y = g$. Pick an arbitrary simplex $\mathbf{m}$ and consider the corresponding components of the above simplicial maps: $h_{\mathbf{m}}\circ(\iota_X)_{\mathbf{m}} = f_{\mathbf{m}}$ and $h_{\mathbf{m}}\circ(\iota_Y)_{\mathbf{m}} = g_{\mathbf{m}}$. By definition, $(\iota_X)_{\mathbf{m}} = \iota_{X_{\mathbf{m}}}\colon X_{\mathbf{m}}\to X_{\mathbf{m}}\sqcup Y_{\mathbf{m}}$ and likewise for $(\iota_Y)_{\mathbf{m}}$, so by the universal property of coproducts in the category Set, we see that $h_{\mathbf{m}}\colon X_{\mathbf{m}}\sqcup Y_{\mathbf{m}}\to Z_{\mathbf{m}}$ is forced to be equal to $[f_{\mathbf{m}}, g_{\mathbf{m}}]$. Furthermore, such choice of $h_{\mathbf{m}}$ indeed defines a simplicial map $h\colon X\sqcup Y\to Z$, as one sees by substituting into the naturality property of simplicial maps the definition of $X\sqcup Y$, obtaining the following commutative diagram for any map of simplices $e\colon\mathbf{m}\to\mathbf{n}$:

$$\begin{array}{ccc}
X_{\mathbf{n}}\sqcup Y_{\mathbf{n}} & \xrightarrow{X_e\sqcup Y_e} & X_{\mathbf{m}}\sqcup Y_{\mathbf{m}} \\
\big\downarrow{\scriptstyle h_{\mathbf{n}}} & & \big\downarrow{\scriptstyle h_{\mathbf{m}}} \\
Z_{\mathbf{n}} & \xrightarrow{Z_e} & Z_{\mathbf{m}}.
\end{array}$$

Indeed, the top-right composition equals $[f_{\mathbf{m}}\circ X_e, g_{\mathbf{m}}\circ Y_e]$ and the bottom-left composition equals $[Z_e\circ f_{\mathbf{m}}, Z_e\circ g_{\mathbf{m}}]$. The two maps coincide by the naturality property of simplicial maps f and g. ▮

Example 13.8. The simplicial set $\mathrm{dis}\{0,1\}$ is the coproduct of $\mathrm{dis}\{0\}$ and $\mathrm{dis}\{1\}$, both of which are isomorphic to $\Delta^{\mathbf{0}}$.

Remark 13.9. The coproduct of an arbitrary family $\{X_i\}_{i\in I}$ of objects in C is defined in a completely analogous way, yielding an object $\coprod_{i\in I} X_i$ together with morphisms $\iota_i\colon X_i\to\coprod_{i\in I} X_i$. If I is a set (as opposed to a proper class), we talk about *small coproducts.* Used in 50.2.

Definition 13.10. A simplicial set is *connected* if it is not empty (i.e., not isomorphic to the empty simplicial set $\operatorname{dis}\emptyset$) and is not isomorphic to $A \sqcup B$ for any simplicial sets A and B that are not empty. Used in 13.27*, 18.6, 29.18, 29.19, 32.9, 32.10, 32.14, 32.15, 33.7, 33.8, 36.11.

Exercise 13.11. Prove that any simplicial set can be presented as the coproduct of a (possibly infinite) family of connected simplicial sets. What is the indexing set of this coproduct? Formulate an appropriate notion of uniqueness for such a decomposition and prove it.

Exercise 13.12. Demonstrate that it is misleading to think of coproducts exclusively as disjoint unions by proving that the coproduct of abelian groups A and B is isomorphic to their direct sum $A \oplus B$. In particular, show that the underlying set $\mathsf{U}(A \oplus B)$ of the coproduct of A and B is isomorphic to the *product* of underlying sets $\mathsf{U}(A) \times \mathsf{U}(B)$.

The second notion, coequalizer of simplicial sets, is a typical quotient construction that occurs often in mathematics. Given two simplicial maps $f, g\colon X \to Y$, one should think of the coequalizer of f and g as a quotient of Y, more precisely, we identify some simplices of Y: for any simplex $s \in X_{\mathbf{m}}$ the two simplices $f(s)$ and $g(s)$ in $Y_{\mathbf{m}}$ must be identified. This is what allows us to implement various gluing constructions.

Definition 13.13. A *coequalizer fork* of morphisms $f, g\colon X \to Y$ in a category C is a morphism $q\colon Y \to Q$ such that $q \circ f = q \circ g$:

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\;q\;} Q.$$

If q and q' are coequalizer forks of f and g, then a *morphism of coequalizer forks* is a morphism $r\colon Q \to Q'$ such that the following diagram commutes:

$$\begin{array}{ccc} & & Q \\ & \overset{q}{\nearrow} & \\ Y & & \downarrow r \\ & \underset{q'}{\searrow} & \\ & & Q' \end{array}$$

Used in 13.13, 13.14, 26.17.

Definition 13.14. The *coequalizer* of morphisms $f, g\colon X \to Y$ in a category C is a coequalizer fork $q\colon Y \to Q$ such that the following *universal property of coequalizers* holds: for any coequalizer fork $q'\colon Y \to Q'$ there is a unique morphism of coequalizer forks $q \to q'$. Used in 13.0*, 13.12*, 13.16, 13.18, 13.19, 13.19*, 13.20, 13.21, 13.22*, 13.24, 26.1*, 26.2*, 26.12, 26.17, 26.18.

Notation 13.15. We denote Q by $\operatorname{coeq}(f, g)$. By abuse of notation, Q is often denoted by Y/X, especially if the map g is "canonical" or "implied".

Informally, we say that a map $Q \to Z$ is the "same thing" as a map $Y \to Z$ such that the compositions $X \to Y \to Z$ (for both choices of the map $X \to Y$) are equal.

Exercise 13.16. Formulate and prove a uniqueness result for coequalizers in analogy with Lemma 13.4.

Remark 13.17. Once again, although coequalizers are always unique, they need not exist and existence must be proved separately.

Example 13.18. Recall from §59 that for $\mathsf{C} = \mathsf{Set}$ the coequalizer of f and g exists and can be computed as the quotient of Y by the equivalence relation R on Y generated by all pairs of the form $(f(x), g(x))$, where $x \in X$. In particular, if $X \in \mathsf{Set}$ and $R \subset X \times X$ is an equivalence relation on X, then the coequalizer of two projection maps $R \to X$ is the quotient map $X \to X/R$.

Proposition 13.19. In the category sSet, the coequalizer of any simplicial maps $f\colon X \to Y$ and $g\colon X \to Y$ exists. Used in 13.24, 21.19.

Proof. We define the map $q_{\mathbf{m}}\colon Y_{\mathbf{m}} \to Q_{\mathbf{m}}$ as the coequalizer of the maps $f_{\mathbf{m}}, g_{\mathbf{m}}\colon X_{\mathbf{m}} \to Y_{\mathbf{m}}$. Given $f\colon \mathbf{m} \to \mathbf{n}$, the simplicial structure map $Q_f\colon Q_{\mathbf{n}} \to Q_{\mathbf{m}}$ is constructed using the universal property of coequalizers of sets: a map $Q_{\mathbf{n}} \to Q_{\mathbf{m}}$ is induced by a map $Y_{\mathbf{n}} \to Q_{\mathbf{m}}$ such that the two compositions $X_{\mathbf{n}} \to Y_{\mathbf{n}} \to Q_{\mathbf{m}}$ are equal. The map $Y_{\mathbf{n}} \to Q_{\mathbf{m}}$ is constructed as the composition $Y_{\mathbf{n}} \to Y_{\mathbf{m}} \to Q_{\mathbf{m}}$. ∎

Exercise 13.20. Complete the proof by proving the functoriality property for simplicial sets for Q, the naturality property for q, and the universal property of coequalizers for (Q, q) in the category sSet.

Exercise 13.21. Draw a picture (with an explanation) of the coequalizer of two maps

$$[\iota_0 \circ \mathrm{d}^0, \iota_1 \circ \mathrm{d}^0], [\iota_1 \circ \mathrm{d}^1, \iota_0 \circ \mathrm{d}^1] \colon \mathbf{\Delta^0} \sqcup \mathbf{\Delta^0} \rightrightarrows \mathbf{\Delta^1} \sqcup \mathbf{\Delta^1}.$$

With these tools now available to us, we can now easily prove the existence of simplicial sets generated by a system of generators and relations.

Proposition 13.22. The simplicial set generated by a system of generators and relations exists (and is unique by Lemma 9.6). Used in 13.23, 13.24, 15.16.

Proof. Given (G, R), we construct the simplicial set X generated by (G, R) as the coequalizer q of two simplicial maps:

$$\coprod_{\substack{\mathbf{m},\mathbf{n},\mathbf{p}\in\mathbf{\Delta} \\ f\colon\mathbf{m}\to\mathbf{n},g\colon\mathbf{m}\to\mathbf{p},r\in R_{f,g}}} \mathbf{\Delta^m} \rightrightarrows \coprod_{\mathbf{k},g\in G_{\mathbf{k}}} \mathbf{\Delta^k} \xrightarrow{q} X.$$

The two simplicial maps are constructed using the universal property of coproducts. Given $\mathbf{m}$, $\mathbf{n}$, $\mathbf{p}$, $f\colon \mathbf{m} \to \mathbf{n}$, $g\colon \mathbf{m} \to \mathbf{p}$, and $r = (s, t) \in R_{f,g}$, we must construct two simplicial maps $\mathbf{\Delta^m} \to \coprod_{k,g\in G_{\mathbf{k}}} \mathbf{\Delta^k}$. For the first map we take $\iota_{\mathbf{n},s} \circ \Delta^f$ and for the second map we take $\iota_{\mathbf{p},t} \circ \Delta^g$. This yields a solution (X, u) for (G, R), where for any $\mathbf{m}$ and $g \in G_{\mathbf{m}}$ the map $u_g\colon \mathbf{\Delta^m} \to X$ is the composition $q \circ \iota_{\mathbf{m},g}$.

We now show that if (X', u') is another solution for (G, R), then there is a unique morphism of solutions $(X, u) \to (X', u')$. By the universal property of coequalizers, a simplicial map $X \to X'$ can be rewritten as simplicial map $\coprod_{\mathbf{k},g\in G_{\mathbf{k}}} \mathbf{\Delta^k} \to X'$ that makes the two compositions equal. By the universal property of coproducts, the latter map can be rewritten as a family of maps $\mathbf{\Delta^k} \to X'$ for each $\mathbf{k}$ and $g \in G_{\mathbf{k}}$. By the definition of a morphism of solutions, the latter maps must be equal to u'_g. This establishes uniqueness of morphisms $(X, u) \to (X', u')$. For existence, observe that the two compositions are indeed equal because (X', u') is a solution. Thus, we constructed a simplicial map $X \to X'$, and this simplicial map is a morphism of solutions by construction. ∎

Example 13.23. We explain how to apply Proposition 13.22 to construct simplicial maps. Consider, for instance, S^n, the simplicial sphere of dimension $n \geq 0$. According to Example 9.7, it has a generating 0-simplex v and a generating n-simplex s with relations $\mathrm{d}_i(s) = \mathrm{s}_0^{n-1}(v)$ for all i. A simplicial map $\mathrm{S}^n \to X$ for any simplicial set X can be identified with the following data: a 0-simplex $v' \in X_0$ and an n-simplex $s' \in X_n$ such that $\mathrm{d}_i(s') = \mathrm{s}_0^{n-1}(v')$ for all i. Typically, it is much easier to construct v and s than to construct the entire totality of the data associated with a simplicial map $\mathrm{S}^n \to X$, i.e., the components $\mathrm{S}^n_{\mathbf{p}} \to X_{\mathbf{p}}$ for any simplex $\mathbf{p}$. Proposition 13.22 justifies this by showing that simplicial maps $\mathrm{S}^n \to X$ are in bijection with pairs (v, s) that satisfy the above property.

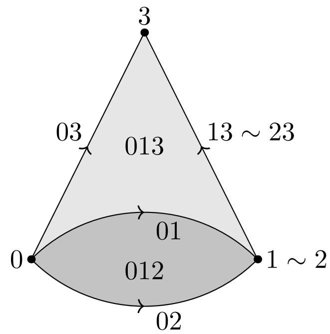
Example 13.24. The proof of Proposition 13.22 shows how to explicitly compute the simplicial set generated by a system of generators and relations, meaning that we compute the sets of n-simplices for all n and values of simplicial structure maps on these simplices. Indeed, the proof of Proposition 13.22 computes the simplicial set as a coequalizer, and Proposition 13.19 gives an explicit way to compute coequalizers of simplicial sets. We illustrate this by the following example: take a single generating 3-simplex α and impose a single relation, $\mathrm{d}_0(\alpha) = \mathrm{s}_0(\mathrm{d}_1(\mathrm{d}_0(\alpha)))$. Recall from Example 7.11 that an n-simplex of α can be represented by a string of $n+1$ digits 0, 1, 2, 3, in this order, which enumerates the vertices of α touched by this n-simplex. The ith face map d_i applied to such a simplex removes the ith digit, whereas the ith degeneracy map duplicates the ith digit. So $\alpha = 0123$, $\mathrm{d}_0(\alpha) = 123$, $\mathrm{d}_0(\alpha) = 123$, $\mathrm{d}_1(\mathrm{d}_0(\alpha)) = 13$, $\mathrm{s}_0(\mathrm{d}_1(\mathrm{d}_0(\alpha))) = 113$. The relation $\mathrm{d}_0(\alpha) = \mathrm{s}_0(\mathrm{d}_1(\mathrm{d}_0(\alpha)))$ forces us to identify the 2-simplices 123 and 113. This means, in particular, that (say) the vertex $\mathrm{d}_0(\mathrm{d}_2(123)) = 2$ should be identified with the vertex $\mathrm{d}_0(\mathrm{d}_2(113)) = 1$. We emphasize that identifying the vertices 1 and 2 does not force us to identify the 1-simplices (say) 01 and 02 (as one could naively assume by looking at their representations). Indeed, there is no way to identify 01 and 02 the vertex 0 cannot appear after applying face or degeneracy maps to 123 or 113. With this remark in mind, we write down the identifications performed on simplices, with degenerate simplices listed first:

- 0-simplices: 0, $1 \sim 2$, 3;

- 1-simplices: 00, 11 $\sim$ 12 $\sim$ 22, 33; 01, 02, 03, 13 $\sim$ 23;
- 2-simplices: 000, 111 $\sim$ 112 $\sim$ 122 $\sim$ 222, 333, 001, 011, 002, 022, 003, 033, 113 $\sim$ 123, 133, 223, 233; 012, 013, 023;
- 3-simplices: degenerations of 2-simplices; 0123.
- n-simplices for $n > 3$: all are degenerate.

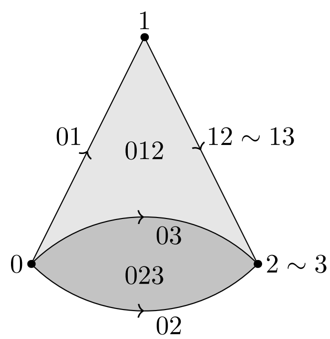
Notice how a simplex that was previously nondegenerate (like 12) can become degenerate after identification.

The resulting object is a “trihedron”:



As described above, it has three 0-simplices, four nondegenerate 1-simplices, three nondegenerate 2-simplices (the back face 023 is obscured from our view), and a single nondegenerate 3-simplex 0123 that corresponds to the interior.

Example 13.25. It is instructive to see what happens when in the previous example we replace s_0 with s_1. The relation now reads $\mathrm{d}_0(\alpha) = \mathrm{s}_1(\mathrm{d}_1(\mathrm{d}_0(\alpha)))$. As before, $\alpha = 0123$, $\mathrm{d}_0(\alpha) = 123$, $\mathrm{d}_0(\alpha) = 123$, $\mathrm{d}_1(\mathrm{d}_0(\alpha)) = 13$. Now we have $\mathrm{s}_1(\mathrm{d}_1(\mathrm{d}_0(\alpha))) = 133$. Thus, 123 is identified with 133. This exchanges the role of 1 and 3 in the above example:



The back face 013 is obscured from our view.

We illustrate some of the ideas behind coproducts and coequalizers by developing the notion of connected components of a simplicial set.

Definition 13.26. The *set of connected components* of a simplicial set X *connected component* is a set $\pi_0(X)$ equipped with a map $q\colon X \to \operatorname{dis} \pi_0(X)$ such that for set S equipped with a map $r\colon X \to \operatorname{dis} S$ there is a unique map of sets $s\colon \pi_0(X) \to S$ such that $(\operatorname{dis} s) \circ q = r$. Used in 13.25*, 13.26, 13.26*, 13.27, 13.28, 13.29, 13.30, 29.12, 29.13*, 29.16, 29.17*, 29.18, 29.19, 29.19*, 29.22, 29.25, 30.3, 30.4, 31.6, 31.6*, 32.2, 32.3, 32.4, 32.6, 32.7*, 32.8, 32.8*, 32.9, 32.10, 32.12, 32.12*, 32.13, 32.13*, 32.14*, 32.15, 32.19, 32.19*, 33.1, 33.2, 33.4, 33.8, 33.8*, 33.10, 39.11, 39.23, 40.12, 42.4.

The idea behind the map $X \to \operatorname{dis} \pi_0(X)$ is that it collapses every connected component of X to a single point in $\operatorname{dis} \pi_0(X)$. The universal property guarantees that different components are collapsed to different points.

Exercise 13.27. Show that $\pi_0(X)$ can be computed as the coequalizer of $X_1 \rightrightarrows X_0$, where the two maps are d_1 and d_0. (Geometrically, we identify those vertices of X that are connected by a chain of 1-simplices going in any direction.)

Recall the definition of a connected simplicial set from Definition 13.10.

Exercise 13.28. Prove that a simplicial set X is connected if and only if $\pi_0(X)$ is a singleton set.

Example 13.29. We compute π_0 for some simple simplicial sets.
- $\pi_0(\operatorname{dis} S) = S$, so $\operatorname{dis} S$ is connected if and only if the set S is a singleton.
- $\pi_0(\boldsymbol{\Delta}^{\mathbf{m}}) = \{*\}$, so $\boldsymbol{\Delta}^{\mathbf{m}}$ is always connected.

Exercise 13.30. Define $\pi_0(f)$ for a simplicial map $f\colon X \to Y$. Prove that this yields a functor $\pi_0\colon \mathsf{sSet} \to \mathsf{Set}$.

14 Natural transformations

Example 14.1. Consider the category 1, with a single object 0 and a single morphism $0 \to 0$, which is the identity morphism on 0. There is only one way to define composition. Given an arbitrary category C, there is a canonical bijection between the class of functors of the form $1 \to \mathsf{C}$ and the class of objects of C. This bijection is constructed as follows. Given a functor $\mathsf{F}\colon 1 \to \mathsf{C}$, we send it to the object $\mathsf{F}(0)$. To define a map going in the opposite direction, given an object $X \in \mathsf{C}$, we send it to the functor $\mathsf{F}\colon 1 \to \mathsf{C}$ such that $\mathsf{F}(0) = X$ and $\mathsf{F}(\mathrm{id}_0) = \mathrm{id}_X$, which preserves composition.

This example demonstrates that it would be unreasonable to talk about *equal* functors in the same way that it is unreasonable to talk about equal groups in algebra, where we recognize that *isomorphism*, not equality, is the reasonable notion for groups. Thus, we expect that functors may turn out to be *isomorphic*, but typically not equal. Moreover, we expect noninvertible morphisms between functors in the same way that we have noninvertible morphisms between objects in a category. In other words, functors $\mathsf{C} \to \mathsf{D}$ should themselves form a category, which we denote $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ or simply D^{C} in analogy with sets (see below). To figure out the correct definition of a morphism of functors, we examine another example.

Example 14.2. Consider the category 2, with objects 0 and 1 and a single nonidentity morphism $\alpha\colon 0 \to 1$. There is only one way to define composition and identities. Given an arbitrary category C, there is a canonical bijection between the class of functors of the form $2 \to \mathsf{C}$ and the class of all morphisms of C. This bijection is constructed as follows. Given a functor $\mathsf{F}\colon 2 \to \mathsf{C}$, we send it to the morphism $\mathsf{F}(\alpha)$. To define a map going in the opposite direction, given a morphism $f\colon X \to Y$ (where $X, Y \in \mathsf{C}$), we send it to the functor $\mathsf{F}\colon 2 \to \mathsf{C}$ such that $\mathsf{F}(0) = X$, $\mathsf{F}(1) = Y$, and $\mathsf{F}(\alpha) = f$, which preserves composition.

Example 14.3. Continuing the previous example, we explain how to recover sources and targets of morphisms. If $2 \to \mathsf{C}$ represents some morphism f in C, then the two compositions $1 \rightrightarrows 2 \to \mathsf{C}$ yield two functors $1 \to \mathsf{C}$, which represent the domain and codomain of f. Here $\iota_0, \iota_1\colon 1 \rightrightarrows 2$ denote the two functors $1 \to 2$ that send the only object of the category 1 to the object 0 respectively 1 of the category 2.

Example 14.4. Continuing the previous example, we explain how to recover identity morphisms and compositions. If $1 \to \mathsf{C}$ represents some object X in C, then the composition $2 \to 1 \to \mathsf{C}$ represents the morphism id_X in C. If $a\colon 2 \to \mathsf{C}$ and $b\colon 2 \to \mathsf{C}$ represent some morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in C (so that $a \circ \iota_1 = b \circ \iota_0$), then the data in a and b can be combined into a functor $c\colon 3 \to \mathsf{C}$, where the category 3 has objects $\{0, 1, 2\}$ and three nonidentity morphisms, namely, $0 \to 1$, $1 \to 2$, and their composition $0 \to 2$, which by the above example correspond to three functors $2 \to 3$, denoted by ι_{01}, ι_{12}, and ι_{02}. Compositions and identities of 3 are unquely defined. The functor $c\colon 3 \to \mathsf{C}$ is unquely determined by the conditions $c \circ \iota_{01} = a$ and $c \circ \iota_{12} = b$. Then the functor $c \circ \iota_{02}\colon 2 \to \mathsf{C}$ represents the morphism $g \circ f$, i.e., the composition of f and g.

To summarize, looking at the sets of functors of the form $1 \to \mathsf{C}$, $2 \to \mathsf{C}$, and $3 \to \mathsf{C}$ as well as the maps between these sets given by compositions with various functors between 1, 2, and 3 allows us to completely reconstruct the category C without ever looking at its individual objects or morphisms. (This should remind you of simplicial sets: a simplicial set X can be completely reconstructed by looking at the sets of simplicial maps $\Delta^{\mathbf{m}} \to X$ for any simplex $\mathbf{m}$ as well as the maps between these sets given by compositions with various simplicial maps $\Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$.) This will become handy when we take $\mathsf{C} = \mathsf{E}^{\mathsf{D}}$, where it is easier to say what the above functors from 1, 2, and 3 are thanks to the following analogy with sets.

Example 14.5. If $X, Y, Z \in \mathsf{Set}$, then $X \times Y$ and $Z^Y = \{f\colon Y \to Z\} = \mathsf{hom}_{\mathsf{Set}}(Y, Z)$ are also sets. Maps of sets $X \times Y \to Z$ are in canonical bijective correspondence with maps of sets $X \to Z^Y$, i.e., we have an isomorphism $\mathsf{hom}_{\mathsf{Set}}(X \times Y, Z) \to \mathsf{hom}_{\mathsf{Set}}(X, Z^Y)$. Indeed, given $f\colon X \times Y \to Z$, we construct $g\colon X \to Z^Y$ as follows. For any $x \in X$ we have to define $g(x) \in Z^Y$, i.e., $g(x)\colon Y \to Z$. This means that for any $y \in Y$ we have to define $g(x)(y) \in Z$. We set $g(x)(y) = f(x, y)$. Vice versa, given $g\colon X \to Z^Y$, we construct $f\colon X \times Y \to Z$ by setting $f(x, y) = g(x)(y)$.

Definition 14.6. If C and D are categories, then $\mathsf{C}\times\mathsf{D}$ is another category, defined as follows: $\mathsf{Ob}(\mathsf{C}\times\mathsf{D}) = \mathsf{Ob}(\mathsf{C})\times\mathsf{Ob}(\mathsf{D})$ and $\mathsf{Mor}_{\mathsf{C}\times\mathsf{D}}((X,Y),(X',Y')) = \mathsf{Mor}_{\mathsf{C}}(X,X')\times\mathsf{Mor}_{\mathsf{D}}(Y,Y')$. Composition and identities are defined pairwise.

For any category C we have a canonical functor $1\times\mathsf{C}\to\mathsf{C}$, which is an isomorphism.

We now act by analogy with sets. Suppose D and E are categories. If there is a category of functors $\mathsf{Fun}(\mathsf{D},\mathsf{E}) = \mathsf{E}^{\mathsf{D}}$, then by the above example it makes sense to ask that the set of functors $\mathsf{C}\to\mathsf{E}^{\mathsf{D}}$ is in bijection with the set of functors $\mathsf{C}\times\mathsf{D}\to\mathsf{E}$ for any category C. Substituting $\mathsf{C}=2$, functors $2\to\mathsf{E}^{\mathsf{D}}$, i.e., morphisms of the category E^{D}, should be in bijection with functors $2\times\mathsf{D}\to\mathsf{E}$.

Definition 14.7. If C and D are categories and $\mathsf{F},\mathsf{G}\colon\mathsf{C}\to\mathsf{D}$ are functors, then a *morphism of functors* $t\colon F\to G$ is a functor $t\colon 2\times\mathsf{C}\to\mathsf{D}$ such that the two compositions $\mathsf{C}\to 1\times\mathsf{C}\rightrightarrows 2\times\mathsf{C}\to\mathsf{D}$ are F and G respectively. Used in 14.1*, 14.8*.

Definition 14.8. If C and D are categories, then the category of functors $\mathsf{Fun}(\mathsf{C},\mathsf{D}) = \mathsf{D}^{\mathsf{C}}$ is defined as follows. Its objects are functors $\mathsf{C}\to\mathsf{D}$. Morphisms were defined above. The composition of morphisms $t\colon F\to G$ and $u\colon G\to H$ given by two functors $t,u\colon 2\times\mathsf{C}\to\mathsf{D}$ is defined by assembling them into a functor $v\colon 3\times\mathsf{C}\to\mathsf{D}$ such that $v\circ(\iota_{01}\times\mathrm{id}_{\mathsf{C}}) = F$ and $v\circ(\iota_{12}\times\mathrm{id}_{\mathsf{C}}) = G$. Then $u\circ t$ is defined as the composition $v\circ(\iota_{02}\times\mathrm{id}_{\mathsf{C}})$. Finally, the identity morphism on $F\colon\mathsf{C}\to\mathsf{D}$ is the composition $2\times\mathsf{C}\to 1\times\mathsf{C}\to\mathsf{D}$.

While the above definition of morphism of functors is fairly well-motivated, it is also rather cumbersome to use in practice, because specifying a functor $2\times\mathsf{C}\to\mathsf{D}$ involves a lot of additional work. As it turns out, almost all information encoded in a functor $t\colon 2\times\mathsf{C}\to\mathsf{D}$ is predetermined by the definition of a morphism of functors. What remains can be extracted by evaluating t on morphisms of the form $(0\to 1,\mathrm{id}_X)$ for all objects $X\in\mathsf{C}$, which yields a morphism $F(X)\to G(X)$, commonly denoted by t_X. Indeed, the value of t on objects of $2\times\mathsf{C}$ as well as morphisms of the form $(\mathrm{id}_i, f\colon X\to Y)$ is prescribed by the conditions $t\circ(\iota_0\times\mathrm{id}_{\mathsf{C}}) = F$ and $t\circ(\iota_1\times\mathrm{id}_{\mathsf{D}}) = G$. Any morphism $(\alpha\colon i\to j, f\colon X\to Y)$ of $2\times\mathsf{C}$ is the composition of morphisms (id_i, f) and (α,id_Y), by definition of $2\times\mathsf{C}$, which shows that knowing t_X suffices to recover the entire functor t. This allows us to restate the definition of morphism of functors in a much more concise form, which is commonly referred to as a *natural transformation*.

Definition 14.9. If C and D are categories and $\mathsf{F},\mathsf{G}\colon\mathsf{C}\to\mathsf{D}$ are functors, then a *natural transformation* $t\colon F\to G$ is a family $\{t_X\}_{X\in\mathsf{C}}$ of morphisms in C such that the following *naturality property* is satisfied for all morphisms $f\colon X\to Y$ in C: the square

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle t_X}\big\downarrow & & \big\downarrow{\scriptstyle t_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y). \end{array}$$

commutes. Used in 2.0*, 8.1, 8.14*, 12.8, 13.20, 14.10, 14.11, 14.12, 14.13, 14.14, 14.15, 14.15*, 14.17, 18.1, 26.5, 26.10, 26.22, 26.25*, 26.33, 28.5*, 31.1, 31.2, 31.4, 35.2*, 35.10, 39.3, 39.5, 42.1, 50.3.

We restate the definition of the category of functors in the new language.

Definition 14.10. If C and D are categories, then the *category of functors* $\mathsf{Fun}(\mathsf{C},\mathsf{D}) = \mathsf{D}^{\mathsf{C}}$ is defined as follows. Its objects are functors $\mathsf{C}\to\mathsf{D}$. Morphisms from $F\colon\mathsf{C}\to\mathsf{D}$ to $G\colon\mathsf{C}\to\mathsf{D}$ are natural transformations $t\colon F\to G$. The composition of natural transformations $t\colon F\to G$ and $u\colon G\to H$ is defined by setting $(u\circ t)_X = u_X\circ t_X$ for any $X\in\mathsf{C}$. Finally, the identity morphism on $F\colon\mathsf{C}\to\mathsf{D}$ is the natural transformation id_F such that $(\mathrm{id}_F)_X = \mathrm{id}_{F(X)}$. Used in 14.1*, 14.6*, 14.8, 14.9*, 14.11, 14.15*, 14.16, 17.21, 26.26*, 28.5, 30.5, 30.8, 30.10, 31.1, 31.2, 31.3, 31.5, 32.4, 32.6, 32.8, 32.9, 33.4, 33.8*, 42.1*, 44.0*, 50.1, 50.4.

Notation 14.11. A *natural map* is a morphism in some category of functors, i.e., a natural transformation. More precisely, a natural transformation $F\to G$ of functors $\mathsf{C}\to\mathsf{D}$ induces a natural map $F(X)\to G(X)$ for an object $X\in\mathsf{C}$. A *natural isomorphism* is an isomorphism in some category of functors, i.e., an invertible natural transformation. This amounts to saying that a natural isomorphism is a natural map $F(X)\to G(X)$ that is an isomorphism. Used in 14.11, 14.14, 26.42, 34.5, 34.5*.

Exercise 14.12. Prove or disprove: a natural transformation $t\colon F \to G$ is an isomorphism (in the category of functors $\mathsf{C} \to \mathsf{D}$) if and only if t_X is an isomorphism in C for any $X \in \mathsf{C}$, where $F, G\colon \mathsf{C} \to \mathsf{D}$ are arbitrary functors.

Example 14.13. In Example 12.8 we saw that simplicial sets are nothing else than functors $\Delta^{\mathsf{op}} \to \mathsf{Set}$. Comparing the definition of a natural transformation and simplicial map, we see that simplicial maps are nothing else than natural transformations of functors $\Delta^{\mathsf{op}} \to \mathsf{Set}$.

Exercise 14.14. Previously we discussed a concrete interpretation of functors of the form $1 \to \mathsf{C}$ and $2 \to \mathsf{C}$ as objects respectively morphisms of C. Explain, in similar concrete terms, what it means to give a natural transformation of such functors. Give a simple criterion for such a natural transformation to be a natural isomorphism.

Exercise 14.15. Given a group G, recall the category $\mathrm{B}G$ with one object $*$ and $\mathsf{Mor}(*, *) = G$. Show that functors $\mathrm{B}G \to \mathsf{Set}$ coincide with a familiar concept from elementary algebra. Show that natural transformations of such functors coincide with a certain related concept. (Prove your claims.) Used in 18.1.

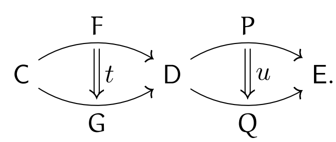
We conclude this section by discussing how to compose natural transformations with functors. In Example 12.10 we discussed how to compose two functors, i.e., defined a map of sets

$$\mathrm{hom}_{\mathsf{Cat}}(\mathsf{D}, \mathsf{E}) \times \mathrm{hom}_{\mathsf{Cat}}(\mathsf{C}, \mathsf{D}) \to \mathrm{hom}_{\mathsf{Cat}}(\mathsf{C}, \mathsf{E}).$$

Recall that $\mathrm{hom}_{\mathsf{Cat}}(\mathsf{A}, \mathsf{B})$ denotes the set of functors of the form $\mathsf{A} \to \mathsf{B}$. Now that $\mathrm{hom}_{\mathsf{Cat}}(\mathsf{A}, \mathsf{B})$ became a better version of itself (no doubt by following one of the many self-improvement books available on the market), it is only natural to ask for a functor

$$\mathsf{Fun}(\mathsf{D}, \mathsf{E}) \times \mathsf{Fun}(\mathsf{C}, \mathsf{D}) \to \mathsf{Fun}(\mathsf{C}, \mathsf{E}).$$

Since $\mathsf{Ob}(\mathsf{Fun}(\mathsf{A}, \mathsf{B})) = \mathrm{hom}_{\mathsf{Cat}}(\mathsf{A}, \mathsf{B})$, on objects this functor should be given by the first displayed map above. The interesting new part is what to do with morphisms, which in our case are pairs (u, t) of natural transformations $u\colon P \to Q$ and $t\colon F \to G$, where $F, G\colon \mathsf{C} \to \mathsf{D}$ and $P, Q\colon \mathsf{D} \to \mathsf{E}$ are functors. The result should be a natural transformation $u \diamond t\colon P \circ F \to Q \circ G$, where both functors are of the form $\mathsf{C} \to \mathsf{E}$, as illustrated by the following diagram (double arrows denote natural transformations):



That is, for every $X \in \mathsf{C}$ we should have a morphism in E of the form $P(F(X)) \to Q(G(X))$. Such a morphism can produced by taking either composition in the following diagram:

$$\begin{array}{ccc} P(F(X)) & \xrightarrow{P(t_X)} & P(G(X)) \\ {\scriptstyle u_{F(X)}}\big\downarrow & & \big\downarrow{\scriptstyle u_{G(X)}} \\ Q(F(X)) & \xrightarrow{Q(t_X)} & Q(G(X)). \end{array}$$

This diagram is obtained from the naturality property of natural transformation u applied to the morphism $t_X\colon F(X) \to G(X)$, and therefore is commutative by definition of natural transformation.

Of particular importance are the two special cases when either u or t is the identity natural transformation. If $u = \mathrm{id}_P$, we write $P \diamond t$ instead of $\mathrm{id}_P \diamond t$. This is known as the *whiskering* of P and t. Likewise, we write $u \diamond F$ instead of $u \diamond \mathrm{id}_F$.

Example 14.16. The category sSet is the category $\mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{Set})$. Previously we constructed a functor $\mathbf{Z}[-]\colon \mathsf{Set} \to \mathsf{Ab}$, i.e., an object in $\mathsf{Fun}(\mathsf{Set}, \mathsf{Ab})$. Substituting this object into

$$\mathsf{Fun}(\mathsf{Set}, \mathsf{Ab}) \times \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{Set}) \to \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{Ab}) = \mathsf{sAb},$$

we get a functor (denoted by the same notation)

$$\mathbf{Z}[-] \diamond -\colon \mathsf{sSet} \to \mathsf{sAb},$$

which is again denoted by $\mathbf{Z}[-]$. The category sAb is known as the *category of simplicial abelian groups.* Used in 14.16, 15.14*.

Exercise 14.17. Continuing the previous exercise, suppose $f\colon G \to H$ is a homomorphism of groups and $\mathsf{B}f\colon \mathsf{B}G \to \mathsf{B}H$ is the induced functor. Given a functor $\mathsf{F}\colon \mathsf{B}H \to \mathsf{Set}$, how can we describe the functor $\mathsf{F} \circ \mathsf{B}f$ in familiar terms? If $t\colon \mathsf{F} \to \mathsf{G}$ is a natural transformation of such functors, how can we describe the natural transformation $t \diamond \mathsf{B}f$ in familiar terms?

Simplicial homology and cohomology

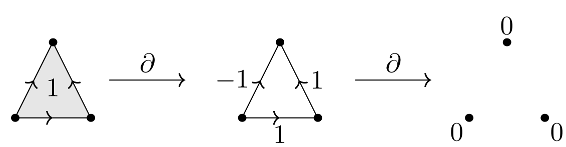
Homology and cohomology groups are one of the simplest, yet very powerful, invariants of simplicial sets.

15 Simplicial chains

Informally, a simplicial chain of dimension n (or simply an n-chain) on a simplicial set X can be thought of as a map $K \to X$, where K is an n-dimensional "shape". This map need not be injective or surjective, and the dimension of X need not have any relation to n. Furthermore, this map can "cover" some parts of X several times. In particular, n-chains can be added: if $K \to X$ and $L \to X$ are n-chains, then so is $K \sqcup L \to X$. This operations turns the set of n-chains into an abelian group. The additive inverse of an n-chain can be thought of as the same n-chain, but "traversed" in the opposite direction.

An n-chain has a boundary, which is an $(n-1)$-chain. Informally, we restrict a given map $K \to X$ along $\partial K \to K$ (the boundary of K), obtaining an $(n-1)$-chain $\partial K \to X$. For instance, the boundary of a 1-chain $e\colon \Delta^{\mathbf{1}} \to X$ is a formal difference of two vertices: $\partial e = \mathrm{d}_0 e - \mathrm{d}_1 e$, so that the boundary of an embedded circle $e\colon \mathrm{S}^1 \to X$ is empty: $\partial e = 0$.

Additionally, we expect that the boundary of a boundary is empty: $\partial(\partial(c)) = 0$ for all chains c. For example, the boundary $\partial\Delta^{\mathbf{2}}$ of $\Delta^{\mathbf{2}}$ is a formal sum $\mathrm{d}_0(\alpha) - \mathrm{d}_1(\alpha) + \mathrm{d}_2(\alpha)$ of three edges of the 2-simplex $\alpha = \mathrm{id}_{\mathbf{2}} \in \Delta^{\mathbf{2}}_{\mathbf{2}}$, and the boundary of $\partial\Delta^{\mathbf{2}}$ is $\partial\partial\Delta^{\mathbf{2}} = (v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) = 0$, where $v_i \in \Delta^{\mathbf{2}}_{\mathbf{0}}$ are the three vertices of $\Delta^{\mathbf{2}}$. This is illustrated by the following picture:



To summarize, we expect the following structure: for each $n \geq 0$ we have an abelian group C_n of n-chains and a homomorphism of abelian groups $\partial_n\colon C_n \to C_{n-1}$ such that $\partial_{n-1}\partial_n = 0\colon C_n \to C_{n-2}$.

Definition 15.1. (Mayer, 1929.) A *chain complex* (of abelian groups) is a pair (C, ∂), where $C\colon \mathbf{Z} \to \mathsf{Ob}(\mathsf{Ab})$ is a sequence of abelian groups and $\partial\colon \mathbf{Z} \to \mathsf{Mor}(\mathsf{Ab})$ is a sequence of homomorphisms of abelian groups (*differentials*):

$$\partial_n\colon C_n \to C_{n-1}$$

for all $n \in \mathbf{Z}$. We require that the map $\partial_{n-1} \circ \partial_n\colon C_n \to C_{n-2}$ is the zero homomorphism for all $n \in \mathbf{Z}$. A chain complex is *nonnegatively graded* if C_n is the zero abelian group for all $n < 0$. In this case we often suppress the mention of C_n for $n < 0$ and ∂_n for $n \leq 0$ altogether. The number n is known as the *chain degree.* Used in 15.1*, 15.2, 15.3, 15.3*, 15.4, 15.5, 15.8, 15.9, 15.10, 16.7, 18.5, 20.0*, 20.1, 20.1*, 20.2*, 20.3, 22.1, 22.3, 22.8, 22.10*, 23.4, 24.0*, 24.2, 24.3*, 33.10, 33.10*, 35.7*, 40.4, 40.6, 40.7, 40.14.

The data of a nonnegatively graded chain complex is often written from right to left as follows:

$$C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} \cdots$$

Warning 15.2. The word "complex" here has a different meaning than in "simplicial complex". As we will see later, the two notions are closely related: the Dold–Kan correspondence establishes an equivalence between nonnegatively graded chain complexes and simplicial abelian groups, which are defined in the same way as simplicial sets, but using abelian groups instead of sets.

In our informal framework for chains, if $X \to Y$ is a simplicial map, then an n-chain $K \to X$ can be composed with $X \to Y$, yielding an n-chain $K \to Y$ in Y. Furthermore, this type of construction is compatible with boundary maps: the boundary of $K \to Y$ is the composition $\partial K \to K \to Y$, which is also the image of the chain $K \to X \to Y$. We formalize these observations in the following definition.

Definition 15.3. Suppose C and D are chain complexes. A *chain map* $f\colon C \to D$ is a sequence f of homomorphisms of abelian groups $f_n\colon C_n \to D_n$ such that the following square commutes for all n:

$$\begin{array}{ccc} C_{n-1} & \xleftarrow{\partial_n^C} & C_n \\ {\scriptstyle f_{n-1}}\downarrow & & \downarrow{\scriptstyle f_n} \\ D_{n-1} & \xleftarrow{\partial_n^D} & D_n. \end{array}$$

Used in 11.0*, 11.15*, 15.3*, 15.4, 15.13*, 15.14, 15.15, 15.16, 15.17, 16.6*, 16.14, 20.3, 22.1, 22.6, 22.10, 22.12, 22.14, 22.17, 22.21, 24.1, 24.2, 24.6, 35.7*, 35.8, 35.9, 35.10*, 35.11, 38.5, 40.4.

The data of a chain map of nonnegatively graded chain complexes is often written as follows:

$$\begin{array}{ccccccc} C_0 & \xleftarrow{\partial_1^C} & C_1 & \xleftarrow{\partial_2^C} & C_2 & \xleftarrow{\partial_3^C} & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \vdots \\ D_0 & \xleftarrow{\partial_1^D} & D_1 & \xleftarrow{\partial_2^D} & D_2 & \xleftarrow{\partial_3^D} & \cdots \end{array}$$

Exercise 15.4. Define the *composition of chain maps* and prove that chain complexes and chain maps form a category, denoted by Ch. Used in 15.4*, 15.14, 15.14*, 15.15, 16.1, 16.2, 16.3, 16.6, 16.6*, 18.1, 18.2, 18.5, 20.1, 20.3, 22.1, 22.21, 23.4, 24.1, 35.7*, 40.14, 51.2.

The *category of nonnegatively graded chain complexes* $\mathsf{Ch}_{\geq 0}$ is defined analogously.

Recall that in Example 12.8 we discovered that simplicial sets are nothing else than functors $\Delta^{\mathsf{op}} \to \mathsf{Set}$. Recall that in Example 12.3 we constructed a functor $\mathsf{Free} = \mathbf{Z}[-]\colon \mathsf{Set} \to \mathsf{Ab}$ that sends a set S to the free abelian group $\mathbf{Z}[S]$ on S, whose elements are finitely supported functions $S \to \mathbf{Z}$ and group operations are defined pointwise. Thus, any simplicial set $X\colon \Delta^{\mathsf{op}} \to \mathsf{Set}$ gives rise to a functor $\mathbf{Z}[X]\colon \Delta^{\mathsf{op}} \to \mathsf{Ab}$. In concrete terms, $\mathbf{Z}[X]_{\mathbf{m}}$ is the free abelian group on the set $X_{\mathbf{m}}$ and $\mathbf{Z}[X]_f\colon \mathbf{Z}[X]_{\mathbf{n}} \to \mathbf{Z}[X]_{\mathbf{m}}$ is the homomorphism of free abelian groups induced by the map of sets $X_f\colon X_{\mathbf{n}} \to X_{\mathbf{m}}$. Such a homomorphism sends a basis element of $\mathbf{Z}[X]_{\mathbf{n}}$ corresponding to some $\alpha \in X_{\mathbf{n}}$ to the basis element of $\mathbf{Z}[X]_{\mathbf{m}}$ corresponding to $X_f(\alpha) \in X_{\mathbf{m}}$.

Definition 15.5. Given a simplicial set $X\colon \Delta^{\mathsf{op}} \to \mathsf{Set}$, its chain complex $\mathsf{C}(X)$ of (normalized) *simplicial chains* (or simply *chains*) is defined as follows. The abelian group $\mathsf{C}_n(X)$ is the quotient of the free abelian group on $X_{\mathbf{n}}$ by the subgroup generated by degenerate n-simplices. (Equivalently, one could take the free abelian group on the set of nondegenerate n-simplices of X.) The differentials $\partial_n\colon \mathsf{C}_n(X) \to \mathsf{C}_{n-1}(X)$ are induced by the universal property of quotients of abelian groups from the map of sets $X_{\mathbf{n}} \to \mathsf{U}(\mathsf{C}_{n-1}(X))$ that sends an n-simplex $\sigma \in X_{\mathbf{n}}$ (equivalently, a simplicial map $\sigma\colon \Delta^{\mathbf{n}} \to X$) to the alternating sum $\sum_{0\leq i\leq n}(-1)^i \mathrm{d}_i(\sigma)$, where $\mathrm{d}_i(\sigma) \in \mathsf{C}_{n-1}(X)$ via the map $X_{\mathbf{n}\setminus\{i\}} \cong X_{n-1} \to \mathbf{Z}[X_{n-1}] \to \mathsf{C}_{n-1}(X)$. Used in 12.3*, 15.0*, 15.7, 15.9, 15.10, 15.11, 15.13, 15.13*, 16.0*, 16.7, 16.10, 16.12, 18.0*, 18.6*, 20.7*, 22.0*, 22.8, 22.13, 22.15, 22.22, 23.0*, 24.2, 24.5, 24.8, 24.9*, 25.3, 35.7*, 35.13*, 36.12, 37.1, 38.4.

Remark 15.6. The above definition should be adjusted in a subtle way: instead of taking $X_{\mathbf{n}}$ for a fixed $\mathbf{n}$ (typically the standard simplex $\{0 < 1 < \cdots < n\}$), we should take $\coprod_{\mathbf{n}} X_{\mathbf{n}}$ for *all* $\mathbf{n}$ of some fixed dimension $n \geq 0$ and quotient by the equivalence relation that identifies two simplices $\sigma \in X_{\mathbf{n}}$ and $\sigma' \in X_{\mathbf{n}'}$ if $X_f(\sigma) = \sigma'$, where $f\colon \mathbf{n}' \to \mathbf{n}$ is the unique isomorphism of simplices. This is used implicitly when we say that $\mathrm{d}_i(\sigma) \in \mathsf{C}_{n-1}(X)$ because $\mathrm{d}_i(\sigma) \in X_{\mathbf{m}}$, where $\mathbf{m} = \mathbf{n} \setminus \{i\}$ is obtained from $\mathbf{n}$ by removing the ith vertex.

Remark 15.7. The adjective "normalized" refers to the fact that degenerate simplices are modded out. One could also look at the nonnormalized simplicial chains, defined without such modding out. Later, we will show that the nonnormalized chains are chain homotopy equivalent (to be defined later) to the normalized chains. However, the normalized chains are far more convenient because many simplicial sets have finitely many nondegenerate simplices, but infinitely many degenerate ones.

Lemma 15.8. (*The boundary map is a chain differential.*) For any simplicial set X and $n \geq 2$ we have $\partial_{n-1} \circ \partial_n = 0$, so the above definition indeed defines a (nonnegatively graded) chain complex $\mathsf{C}(X)$. Used in 33.10*.

We give two proofs: one is conceptual, whereas the other one just blindly applies the definitions. Both proofs ultimately explore the same idea.

Conceptual proof. Fix a simplex $\mathbf{n}$ and some $\mathbf{n}$-chain $c \in \mathsf{C}_{\dim \mathbf{n}}(X)$. Also fix some simplices $\mathbf{m}$ and $\mathbf{l}$ such that $\dim \mathbf{m} = \dim \mathbf{n} - 1$ and $\dim \mathbf{l} = \dim \mathbf{m} - 1$. The coefficient of $\rho \in X_{\mathbf{m}}$ in the expression for $\partial_n(c)$ is the sum of $(-1)^i c(\sigma)$ over all injections $\alpha\colon \mathbf{m} \to \mathbf{n}$ and $\mathbf{n}$-simplices $\sigma \in X_{\mathbf{n}}$ such that $X_\alpha(\sigma) = \rho$, where i is the index of the only element of $\mathbf{n} \setminus \alpha(\mathbf{m})$ inside $\mathbf{n}$ (the indexing starts from 0). Likewise, the coefficient of $\pi \in X_{\mathbf{l}}$ in the expression for $\partial_{n-1}(\partial_n(c))$ is the sum of $(-1)^j(-1)^i c(\sigma)$ over all injections $\beta\colon \mathbf{l} \to \mathbf{m}$ and $\alpha\colon \mathbf{m} \to \mathbf{n}$ and $\mathbf{n}$-simplices $\sigma \in X_{\mathbf{n}}$ such that $X_{\alpha\circ\beta}(\sigma) = \pi$, where i is the index of the only element of $\mathbf{n} \setminus \alpha(\mathbf{m})$ inside $\mathbf{n}$ and j is the index of the only element of $\mathbf{m} \setminus \beta(\mathbf{l})$ inside $\mathbf{m}$. We group together pairs (α, β) that have the same set $\mathbf{n} \setminus \alpha(\beta(\mathbf{l}))$ (consisting of 2 elements). Each group contains exactly 2 pairs, which add these 2 elements in the different orders. The terms $(-1)^j(-1)^i$ have opposite signs because removing the smaller elements decreases the index of the larger element by 1, and this happens for exactly one of the two pairs. ∎

Computational proof. By the universal property of free abelian groups, it suffices to verify this identity on a simplex $\sigma \in X_{\mathbf{n}}$. We have

$$\begin{aligned}
\partial_{n-1}(\partial_n(\sigma)) &\overset{1}{=} \partial_{n-1}\left(\sum_{0\leq i\leq n} (-1)^i \mathrm{d}_i(\sigma)\right) \\
&\overset{2}{=} \sum_{0\leq i\leq n} (-1)^i \partial_{n-1}(\mathrm{d}_i(\sigma)) \\
&\overset{3}{=} \sum_{0\leq i\leq n} (-1)^i \sum_{0\leq j\leq n-1} (-1)^j \mathrm{d}_j(\mathrm{d}_i(\sigma)) \\
&\overset{4}{=} \sum_{0\leq i\leq j\leq n-1} (-1)^{i+j} \mathrm{d}_j(\mathrm{d}_i(\sigma)) + \sum_{0\leq j<i\leq n} (-1)^{i+j} \mathrm{d}_j(\mathrm{d}_i(\sigma)) \\
&\overset{5}{=} \sum_{0\leq i\leq j\leq n-1} (-1)^{i+j} \mathrm{d}_{i,j+1}(\sigma) + \sum_{0\leq j<i\leq n} (-1)^{i+j} \mathrm{d}_{j,i}(\sigma) \\
&\overset{6}{=} \sum_{0\leq i<j\leq n} (-1)^{i+j-1} \mathrm{d}_{i,j}(\sigma) + \sum_{0\leq i<j\leq n} (-1)^{i+j} \mathrm{d}_{i,j}(\sigma) \\
&\overset{7}{=} \sum_{0\leq i<j\leq n} ((-1)^{i+j-1} \mathrm{d}_{i,j}(\sigma) + (-1)^{i+j} \mathrm{d}_{i,j}(\sigma)) = 0.
\end{aligned}$$

The first equality expanded $\partial_n(\sigma)$ using the definition of ∂_n. The second equality used the fact that ∂_{n-1} is a homomorphism of abelian groups. The third equality expanded $\partial_{n-1}(\mathrm{d}_i(\sigma))$ using the definition of ∂_{n-1}. The fourth equality split the resulting double sum into two sums with $i \leq j$ and $i > j$ respectively. The fifth equality used the cosimplicial identities of Exercise 6.14. The sixth equality replaced j by $j-1$ in the first sum and exchanged i and j in the second sum. The seventh equality observed that both sums are now indexed in the same way and the summation terms differ only by a sign, so their sum is zero. ∎

Example 15.9. We compute the simplicial chain complexes of $\Delta^{\mathbf{m}}$ when $\dim \mathbf{m} \in [0, 2]$.

- For $\Delta^{\mathbf{0}}$ we get $\mathsf{C}(\Delta^{\mathbf{0}})_0 = \mathbf{Z}$ and $\mathsf{C}(\Delta^{\mathbf{0}})_n = 0$ for $n \neq 0$. This chain complex is denoted by $\mathbf{Z}[0]$.
- For $\Delta^{\mathbf{1}}$ we get

$$\mathbf{Z}_{\langle 0\rangle} \oplus \mathbf{Z}_{\langle 1\rangle} \xleftarrow{-1\oplus 1} \mathbf{Z}_{\langle 01\rangle}.$$

  The angle brackets denote the vertices corresponding to the given copy of $\mathbf{Z}$.
- For $\Delta^{\mathbf{2}}$ we get

$$\mathbf{Z}_{\langle 0\rangle} \oplus \mathbf{Z}_{\langle 1\rangle} \oplus \mathbf{Z}_{\langle 2\rangle} \xleftarrow{\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}} \mathbf{Z}_{\langle 01\rangle} \oplus \mathbf{Z}_{\langle 02\rangle} \oplus \mathbf{Z}_{\langle 12\rangle} \xleftarrow{1\oplus -1\oplus 1} \mathbf{Z}_{\langle 012\rangle}.$$

Used in 16.10.

Example 15.10. We compute the simplicial chain complexes of $\mathrm{S}^{\mathbf{m}}$ when $\dim \mathbf{m} \in [0, 2]$. Recall that $\mathrm{S}^{\mathbf{m}}$ has exactly two nondegenerate simplices, in degrees 0 and $\dim \mathbf{m}$. This, $\mathsf{C}(\mathrm{S}^{\mathbf{m}})$ has exactly two copies of $\mathbf{Z}$, in chain degrees 0 (denoted by $*$) and $\dim \mathbf{m}$.

- For $\mathrm{S}^{\mathbf{0}}$ we get

$$\mathbf{Z}_{\langle *\rangle} \oplus \mathbf{Z}.$$

- For $\mathrm{S}^{\mathbf{1}}$ we get

$$\mathbf{Z}_{\langle *\rangle} \xleftarrow{0} \mathbf{Z}.$$

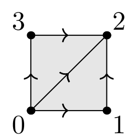
- For $\mathrm{S}^{\mathbf{2}}$ we get

$$\mathbf{Z}_{\langle *\rangle} \xleftarrow{0} 0 \xleftarrow{0} \mathbf{Z}.$$

- For $\mathrm{S}^{\mathbf{m}}$, $\dim \mathbf{m} > 2$, we get a similar chain complex with $k - 1$ zeros between $\mathbf{Z}_{\langle *\rangle}$ and $\mathbf{Z}$.

Used in 16.11.

Example 15.11. We compute the simplicial chain complex of simplicial square

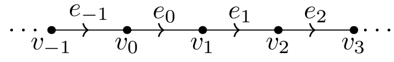


as

$$\mathbf{Z}_{\langle 0\rangle} \oplus \mathbf{Z}_{\langle 1\rangle} \oplus \mathbf{Z}_{\langle 2\rangle} \oplus \mathbf{Z}_{\langle 3\rangle} \xleftarrow{\begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}} \mathbf{Z}_{\langle 01\rangle} \oplus \mathbf{Z}_{\langle 02\rangle} \oplus \mathbf{Z}_{\langle 03\rangle} \oplus \mathbf{Z}_{\langle 12\rangle} \oplus \mathbf{Z}_{\langle 32\rangle} \xleftarrow{\begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} \mathbf{Z}_{\langle 012\rangle} \oplus \mathbf{Z}_{\langle 032\rangle}.$$

Used in 16.12, 22.22.

Example 15.12. Consider the following simplicial set X:



The set of 0-simplices $\{\ldots, v_{-1}, v_0, v_1, \ldots\}$ is isomorphic to $\mathbf{Z}$, the set of 1-simplices $\{\ldots, e_{-1}, e_0, e_1, \ldots\}$ is also isomorphic to $\mathbf{Z}$. The boundary map sends e_i to $v_{i+1} - v_i$. We compute $\mathsf{B}_0(X) = \operatorname{im} \partial_1$. That is, we must find all $\sum_i \alpha_i v_i \in \mathsf{C}_0(X)$ such that there is $\sum_i \beta_i e_i \in \mathsf{C}_1(X)$ so that

$$\sum_i \alpha_i v_i = \partial\left(\sum_i \beta_i e_i\right) = \sum_i \beta_i \partial e_i = \sum_i \beta_i (v_{i+1} - v_i) = \sum_i (\beta_{i-1} - \beta_i) v_i.$$

The coefficients before v_i on both sides must be equal, so we get

$$\alpha_i = \beta_{i-1} - \beta_i.$$

Thus,

$$\sum_i \alpha_i = \sum_i (\beta_{i-1} - \beta_i) = 0.$$

(Recall that all sums are finite because an element of a free abelian group is a finite linear combination of generators.) In other words, the subgroup $\mathsf{B}_0(X) \subset \mathsf{C}_0(X)$ is the kernel of the map $\mathsf{C}_0(X) \to \mathbf{Z}$ that sends $\sum_i \alpha_i v_i \mapsto \sum_i \alpha_i$. The latter map is surjective, therefore it is the quotient map for the inclusion $\mathsf{B}_0(X) \to \mathsf{C}_0(X)$. Thus $\mathsf{H}_0(X) \cong \mathbf{Z}$.

We now compute $\mathsf{H}_1(X) \cong \mathsf{Z}_1(X) = \ker \partial_1$. As computed above,

$$\partial\left(\sum_i \beta_i e_i\right) = \sum_i (\beta_{i-1} - \beta_i) v_i.$$

Thus, $\sum_i \beta_i e_i \in \ker \partial_1$ if and only if $\beta_{i-1} = \beta_i$ for all i, i.e., $\beta_i = \gamma$ for some $\gamma \in \mathbf{Z}$. Since only finitely many coefficients can be nonzero in a simplicial chain, we have $\gamma = 0$ and $\mathsf{H}_1(X) \cong 0$.

Exercise 15.13. Compute the simplicial chain complexes for the following simplicial sets. (a) The real projective plane. (b) The infinite grid of Exercise 9.11. (c) The orientable surface of genus g and (d) the nonorientable surface with g crosscaps. (e) The empanada and (f) the sopapilla of Exercise 9.14. (g) The lasso of Example 8.15. Used in 16.13, 18.7, 19.5, 20.12, 22.23, 23.11, 24.12, 29.24.

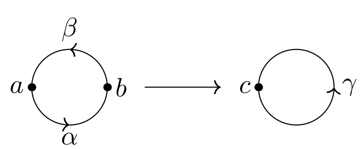
As it turns out, a simplicial map induces a chain map between simplicial chains.

Proposition 15.14. Any simplicial map $f\colon X \to Y$ induces a chain map $\mathsf{C}(f)\colon \mathsf{C}(X) \to \mathsf{C}(Y)$. These maps organize into a *simplicial chain functor* $\mathsf{C}\colon \mathsf{sSet} \to \mathsf{Ch}_{\geq 0}$. Used in 35.7*, 35.12.

Proof. Below, we will give a conceptual proof by turning the normalization construction into a functor $\mathsf{N}\colon \mathsf{sAb} \to \mathsf{Ch}$, so the functor $\mathsf{C}\colon \mathsf{sSet} \to \mathsf{Ch}$ can be defined as the composition of $\mathbf{Z}[-]\colon \mathsf{sSet} \to \mathsf{sAb}$ and $\mathsf{N}\colon \mathsf{sAb} \to \mathsf{Ch}$. Right now we give a simple hands-on description. We start by constructing homomorphisms of abelian groups $\mathsf{C}(f)_\mathbf{m}\colon \mathsf{C}(X)_\mathbf{m} \to \mathsf{C}(Y)_\mathbf{m}$. The functor $\mathbf{Z}[-]\colon \mathsf{Set} \to \mathsf{Ab}$ sends the map of sets $f_\mathbf{m}\colon X_\mathbf{m} \to Y_\mathbf{m}$ to the homomorphism $\mathbf{Z}[f_\mathbf{m}]\colon \mathbf{Z}[X_\mathbf{m}] \to \mathbf{Z}[Y_\mathbf{m}]$, which then descends to the quotients by the abelian subgroup generated by degenerate simplices because the map of sets $X_\mathbf{m} \to Y_\mathbf{m}$ preserves degenerate simplices. ∎

Exercise 15.15. Prove that this construction defines a chain map. Prove that C is a functor $\mathsf{C}\colon \mathsf{sSet} \to \mathsf{Ch}_{\geq 0}$.

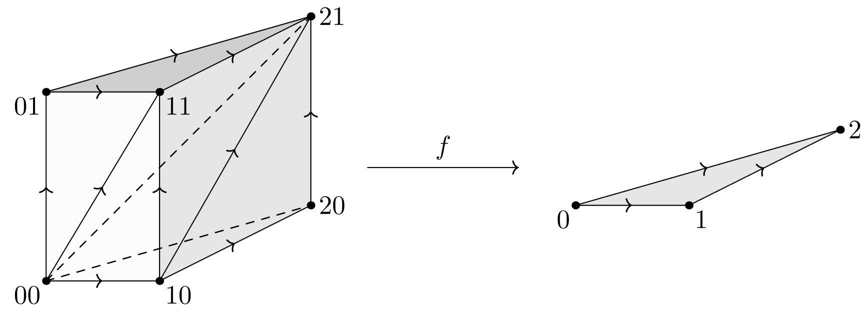
Example 15.16. Consider the following simplicial map:



We map $a, b \mapsto c$ and $\alpha, \beta \mapsto \gamma$ so that the source circle wraps around the target circle twice. By Proposition 13.22, this indeed defines a simplicial map because the source is a simplicial set defined using generators $\{a, b\}$ in degree 0 and $\{\alpha, \beta\}$ in degree 1 and relations $a = \mathrm{d}_1(\alpha) = \mathrm{d}_0(\beta)$ and $b = \mathrm{d}_0(\alpha) = \mathrm{d}_1(\beta)$. After mapping to the target these relations become $c = \mathrm{d}_1(\gamma) = \mathrm{d}_0(\gamma)$ and $c = \mathrm{d}_0(\gamma) = \mathrm{d}_1(\gamma)$, which indeed hold in the target. We now compute the induced chain map as follows. Used in 16.14, 20.10.

$$\begin{array}{ccccccc}
\mathbf{Z}\oplus\mathbf{Z} & \xleftarrow{\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}} & \mathbf{Z}\oplus\mathbf{Z} & \xleftarrow{0} & 0 & \xleftarrow{0} & \cdots \\
\big\downarrow{\scriptstyle (1\ 1)} & & \big\downarrow{\scriptstyle (1\ 1)} & & \big\downarrow{\scriptstyle 0} & & \vdots \\
\mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \xleftarrow{0} & 0 & \xleftarrow{0} & \cdots
\end{array}$$

Example 15.17. Consider the projection of a prism onto its top/bottom face:



We now compute the induced chain map as follows.

$$\begin{array}{ccccccccc} \mathbf{Z}^6 & \longleftarrow & \mathbf{Z}^{12} & \longleftarrow & \mathbf{Z}^{10} & \longleftarrow & \mathbf{Z}^3 & \xleftarrow{0} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 0} & & \vdots \\ \mathbf{Z}^3 & \longleftarrow & \mathbf{Z}^3 & \longleftarrow & \mathbf{Z} & \xleftarrow{0} & 0 & \xleftarrow{0} & \cdots \end{array}$$

Since we do not want to write down matrices of size 3×12, we will describe the induced chain map more conceptually.

On 0-chains, the map $\mathbf{Z}^6 = \mathbf{Z}^3 \oplus \mathbf{Z}^3 \to \mathbf{Z}^3$ is given by the identity map $\mathbf{Z}^3 \to \mathbf{Z}^3$ on each of the two direct summands corresponding to the three vertices in the bottom respectively top face.

On 1-chains, the map $\mathbf{Z}^{12} = \mathbf{Z}^3 \oplus \mathbf{Z}^3 \oplus \mathbf{Z}^3 \oplus \mathbf{Z}^3 \to \mathbf{Z}^3$ is given by the identity map $\mathbf{Z}^3 \to \mathbf{Z}^3$ on each of the first three direct summands, which correspond to the edges of the bottom face, the top face, and the diagonal edges on each of the three square faces. On the fourth direct summand, which corresponds to the three vertical edges, the map is zero because the vertical edges project to degenerate 1-simplices.

On 2-chains, the map $\mathbf{Z}^{10} = \mathbf{Z}^4 \oplus \mathbf{Z}^6 \to \mathbf{Z}$ is given by the identity map $\mathbf{Z} \to \mathbf{Z}$ on each of the four summands $\mathbf{Z}$ of the first direct summand $\mathbf{Z}^4$, which corresponds to the top and bottom faces and the two interior triangles $00 \to 10 \to 21$ and $00 \to 11 \to 21$. On the second direct summand $\mathbf{Z}^6$ the map vanishes because each of the six triangles located on one of the square sides projects to a degenerate 1-simplex.

Exercise 15.18. Compute the induced chain maps for each of the three simplicial maps in Example 8.13.
Used in 16.15, 18.7, 20.12.

16 Homology

Supplementary sources: Boltyanskiĭ and Efremovich [ICT], Fomenko [VGT].

Simplicial sets that we can consider to be the "same" (to be formalized later using the notion of simplicial weak equivalence) can have different (i.e., nonisomorphic) complexes of simplicial chains.

However, we can extract a graded abelian group that is invariant under simplicial weak equivalences.

In the following definition one should think of $C = \mathsf{C}(X)$ for some $X \in \mathsf{sSet}$.

Definition 16.1. (Poincaré, 1899.) Suppose $C \in \mathsf{Ch}$ and $c \in C_n$ for some $n \in \mathbf{Z}$.

- The n-chain c is a *cycle* if $\partial c = 0$. Cycles form an abelian group $\ker \partial_n$, denoted by $\mathsf{Z}_n(C)$, where Z stands for the German word Zykel.
- The n-chain c is a *boundary* if there is $b \in \mathsf{C}_{n+1}(C)$ such that $\partial b = c$. Boundaries form an abelian group $\operatorname{im} \partial_{n+1}$, denoted by $\mathsf{B}_n(C)$.

Used in 35.13*.

Lemma 16.2. For any $C \in \mathsf{Ch}$ and $n \in \mathbf{Z}$ the group $\mathsf{B}_n(C)$ is a subgroup of $\mathsf{Z}_n(C)$.

Proof. If $c \in \mathsf{B}_n(C)$, then there is $b \in \mathsf{C}_{n+1}(C)$ such that $\partial b = c$, so $\partial c = \partial(\partial(b)) = 0$, i.e., $c \in \mathsf{Z}_n(C)$. ∎

Definition 16.3. (Mayer, 1929.) The nth *homology group* of $C \in \mathsf{Ch}$ is the quotient group $\mathsf{H}_n(C) = \mathsf{Z}_n(C)/\mathsf{B}_n(C)$. Elements of $\mathsf{H}_n(C)$ are known as *homology classes* in degree n. Used in 16.7, 16.10, 16.11, 24.6, 24.10.

Definition 16.4. Suppose $X \in \mathsf{sSet}$. We define $\mathsf{Z}_n(X) = \mathsf{Z}_n(\mathsf{C}(X))$, $\mathsf{B}_n(X) = \mathsf{B}_n(\mathsf{C}(X))$, $\mathsf{H}_n(X) = \mathsf{H}_n(\mathsf{C}(X))$. Elements of these groups are referred to as *simplicial cycles*, *simplicial boundaries*, and *simplicial homology classes*.

Sometimes it makes sense to manipulate the entire collection of the above groups for all n as a single whole. This can be formalized as follows.

Definition 16.5. Suppose I is a set. An *I-graded abelian group* is a family of abelian groups indexed by I. If A and B are I-graded abelian groups, then a homomorphism from A to B is a family of homomorphisms of abelian groups $A_i \to B_i$ for all $i \in I$. Thus, I-graded abelian groups form a category Ab^I. Used in 16.7, 23.4, 23.8*, 40.4.

Proposition 16.6. We have functors $\mathsf{Z}\colon \mathsf{Ch} \to \mathsf{Ab}^{\mathbf{Z}}$, $\mathsf{B}\colon \mathsf{Ch} \to \mathsf{Ab}^{\mathbf{Z}}$, $\mathsf{H}\colon \mathsf{Ch} \to \mathsf{Ab}^{\mathbf{Z}}$. The functor H is known as the *homology functor*. Used in 20.4, 35.14.

Proof. We defined these functors on objects of Ch above. It remains to define them on morphisms and verify the functoriality properties. Given a chain map $f\colon C \to D$, observe that $f(\mathsf{Z}_n(C)) \subset \mathsf{Z}_n(D)$ because $\partial_D(f(c)) = f(\partial_C(c)) = f(0) = 0$ for any $c \in \mathsf{Z}_n(C)$. Likewise, $f(\mathsf{B}_n(C)) \subset \mathsf{B}_n(D)$ because $f(\partial_C(c)) = \partial_D(f(c))$ for any $c \in C_{n+1}$. Thus $\mathsf{Z}(f)$ and $\mathsf{B}(f)$ can be defined as the restriction/corestriction of $\mathsf{C}(f)$ to the appropriate domains/codomains. The functoriality properties are satisfied for Z and B because restriction and corestriction operations are compatible with compositions and identity maps.

To define $\mathsf{H}_n(f)\colon \mathsf{H}_n(C) \to \mathsf{H}_n(D)$, we use the universal property of quotient groups for $\mathsf{H}_n(C) = \mathsf{Z}_n(C)/\mathsf{B}_n(C)$: homomorphisms of the form $\mathsf{H}_n(C) \to \mathsf{H}_n(D)$ are in bijective correspondence with homomorphisms $\mathsf{Z}_n(C) \to \mathsf{H}_n(D)$ whose restriction to $\mathsf{B}_n(C)$ vanishes. Take the homomorphism $\mathsf{Z}_n(C) \to \mathsf{H}_n(D)$ given by the composition of $\mathsf{Z}_n(f)\colon \mathsf{Z}_n(C) \to \mathsf{Z}_n(D)$ and the quotient map $q\colon \mathsf{Z}_n(D) \to \mathsf{H}_n(D)$. This composition vanishes on $\mathsf{B}_n(C)$ because $\mathsf{Z}_n(f)(\mathsf{B}_n(C)) \subset \mathsf{B}_n(D)$ and q vanishes on $\mathsf{B}_n(D)$. Thus, we defined a map $\mathsf{H}_n(f)\colon \mathsf{H}_n(C) \to \mathsf{H}_n(D)$.

The functoriality properties for H_n is satisfied: given $f\colon C \to D$ and $g\colon D \to E$, the morphisms $\mathsf{H}_n(g \circ f)$ and $\mathsf{H}_n(g) \circ \mathsf{H}_n(f)$ are both induced from maps $\mathsf{Z}_n(C) \to \mathsf{H}_n(E)$ that vanish on $\mathsf{B}_n(C)$ by construction. Thus, it remains to show that

$$\mathsf{Z}_n(C) \xrightarrow{\mathsf{Z}_n(g \circ f)} \mathsf{Z}_n(E) \to \mathsf{H}_n(E)$$

and

$$\mathsf{Z}_n(C) \xrightarrow{\mathsf{Z}_n(f)} \mathsf{Z}_n(D) \to \mathsf{H}_n(D) \xrightarrow{\mathsf{H}_n(g)} \mathsf{H}_n(E)$$

compose to the same map. Indeed, by the universal property of quotient groups the latter composition can be rewritten as

$$\mathsf{Z}_n(C) \xrightarrow{\mathsf{Z}_n(f)} \mathsf{Z}_n(D) \xrightarrow{\mathsf{Z}_n(g)} \mathsf{Z}_n(E) \to \mathsf{H}_n(E). \quad ∎$$

Remark 16.7. The word "*homology*", when used in isolation and not as a part of "homology group", typically refers to the graded abelian group defined above. Sometimes it is also used informally to refer to the chain complex of simplicial chains. Used in 1.0*, S.0*, 16.9, 17.24, 20.4, 22.0*, 24.10, 33.11, 33.12, 40.4.

Definition 16.8. (Riemann 1857; Betti, 1871; Poincaré, 1895, 1899, 1900; E. Noether, 1925.) The *simplicial homology* is the functor $\mathsf{H} \circ \mathsf{C} \colon \mathsf{sSet} \to \mathsf{Ab}^{\mathbf{Z}}$. Abusing notation, we denote this functor also by H. Used in 16.10, 16.11, 17.18, 17.26, 35.14.

Remark 16.9. Below we will introduce several other homology functors and all of them will be denoted by H. The particular choice of a functor must be deduced from the type of its arguments.

Example 16.10. We compute the simplicial homology groups of $\Delta^{\mathbf{m}}$ when $\dim \mathbf{m} \in [0,2]$ using simplicial chains computed in Example 15.9.

- For $\Delta^{\mathbf{0}}$ we get $\mathsf{H}_0(\Delta^{\mathbf{0}}) \cong \mathbf{Z}$ and $\mathsf{H}_n(\Delta^{\mathbf{0}}) = 0$ for $n \neq 0$.
- For $\Delta^{\mathbf{1}}$ we get the simplicial chain complex

$$\mathsf{C}(\Delta^{\mathbf{1}}) = \mathbf{Z}_{\langle 0\rangle} \oplus \mathbf{Z}_{\langle 1\rangle} \xleftarrow{-1\oplus 1} \mathbf{Z}_{\langle 01\rangle}.$$

  The groups $\mathsf{H}_0(\Delta^{\mathbf{1}})$ and $\mathsf{H}_1(\Delta^{\mathbf{1}})$ are given by the cokernel and kernel of $\partial_1 \colon \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z}$. The latter is 0, whereas the former is the codomain of the quotient map $\mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z}$ that sends $a \oplus b \in \mathbf{Z} \oplus \mathbf{Z}$ to $a + b \in \mathbf{Z}$. Indeed, this map is surjective and its kernel is the image of ∂_1, which guarantees that the map is the cokernel of ∂_1. Thus $\mathsf{H}_0(\Delta^{\mathbf{1}}) \cong \mathbf{Z}$ and the other homology groups are zero.
- For $\Delta^{\mathbf{2}}$ we get the simplicial chain complex

$$\mathbf{Z}_{\langle 0\rangle} \oplus \mathbf{Z}_{\langle 1\rangle} \oplus \mathbf{Z}_{\langle 2\rangle} \xleftarrow{\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}} \mathbf{Z}_{\langle 01\rangle} \oplus \mathbf{Z}_{\langle 02\rangle} \oplus \mathbf{Z}_{\langle 12\rangle} \xleftarrow{1\oplus -1 \oplus 1} \mathbf{Z}_{\langle 012\rangle}.$$

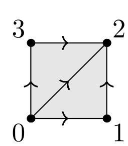
  We compute $\mathsf{H}_0(\Delta^{\mathbf{2}})$. The image of ∂_1 does not change under elementary column operations. Column-reducing the matrix produces

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

  The quotient map is $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z}$ ($a \oplus b \oplus c \mapsto a + b + c$). It is surjective and its kernel is precisely the image of the above matrix, which is $\{-a \oplus a - b \oplus b \mid a, b, c \in \mathbf{Z}\}$. By inspection, the kernel of ∂_1 coincides with the image of ∂_2, so $\mathsf{H}_2(\Delta^{\mathbf{2}}) \cong 0$.

Example 16.11. We compute the simplicial homology of $\mathrm{S}^{\mathbf{m}}$. All boundary maps vanish by Example 15.10. Thus, for $m > 0$ we have $\mathsf{H}_k(\mathrm{S}^{\mathbf{m}}) \cong \mathbf{Z}$ if $k = 0$ or $k = m$, and $\mathsf{H}_k(\mathrm{S}^{\mathbf{m}}) \cong 0$ otherwise. We also have $\mathsf{H}_0(\mathrm{S}^{\mathbf{0}}) \cong 0$, with all other homology groups vanishing.

Example 16.12. In Example 15.11 we computed the simplicial chain complex of a square



as

$$\mathbf{Z}_{\langle 0\rangle} \oplus \mathbf{Z}_{\langle 1\rangle} \oplus \mathbf{Z}_{\langle 2\rangle} \oplus \mathbf{Z}_{\langle 3\rangle} \xleftarrow{\begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}} \mathbf{Z}_{\langle 01\rangle} \oplus \mathbf{Z}_{\langle 02\rangle} \oplus \mathbf{Z}_{\langle 03\rangle} \oplus \mathbf{Z}_{\langle 12\rangle} \oplus \mathbf{Z}_{\langle 32\rangle} \xleftarrow{\begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} \mathbf{Z}_{\langle 012\rangle} \oplus \mathbf{Z}_{\langle 032\rangle}.$$

We now compute H_0 as the cokernel of ∂_1. Column-reducing the corresponding matrix yields

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

The three nonzero columns constitute a basis of $\mathsf{B}_0 \cong \mathbf{Z}^3$. We claim that the quotient map is $\mathbf{Z}^4 \to \mathbf{Z}$ $(a \oplus b \oplus c \oplus d \mapsto a + b + c + d)$. Indeed, its composition with $\partial_1 \colon \mathbf{Z}^5 \to \mathbf{Z}^4$ vanishes and its kernel coincides with the image of ∂_1, namely, $\{-a \oplus -b \oplus -c \oplus (a + b + c) \mid a, b, c \in \mathbf{Z}\}$.

We now compute Z_1 as the kernel of ∂_1. Row-reducing the corresponding matrix yields

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

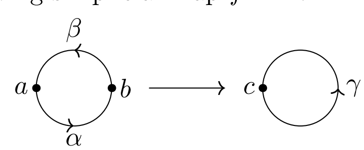
The three nonzero rows allow us to read off a basis of $\mathsf{Z}_1 \cong \mathbf{Z}^2$:

$$\begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This coincides with the matrix of ∂_2, so $\mathsf{H}_1 \cong 0$. Finally, $\mathsf{Z}_2 \cong 0$, so $\mathsf{H}_2 \cong 0$.

Exercise 16.13. For each of the simplicial sets listed in Exercise 15.13, compute its homology.

Example 16.14. Consider the following simplicial map $f \colon A \to B$ from Example 15.16:



We map $a, b \mapsto c$ and $\alpha, \beta \mapsto \gamma$ so that the source circle wraps around the target circle twice. We computed the induced chain map as follows:

$$\begin{array}{ccccccc}
\mathbf{Z} \oplus \mathbf{Z} & \xleftarrow{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}} & \mathbf{Z} \oplus \mathbf{Z} & \xleftarrow{\ 0\ } & 0 & \xleftarrow{\ 0\ } & \cdots \\
\downarrow{\scriptstyle (1\ 1)} & & \downarrow{\scriptstyle (1\ 1)} & & \downarrow{\scriptstyle 0} & & \vdots \\
\mathbf{Z} & \xleftarrow{\ 0\ } & \mathbf{Z} & \xleftarrow{\ 0\ } & 0 & \xleftarrow{\ 0\ } & \cdots
\end{array}$$

Its homology is computed as follows. We have $\mathsf{Z}_0(A) = \mathsf{C}_0(A)$ and $\mathsf{B}_0(A) = \{a \oplus -a \mid a \in \mathbf{Z}\}$. The surjective map $\mathsf{Z}_0(A) \to \mathbf{Z}$ $(a \oplus b \mapsto a + b)$ has $\mathsf{B}_0(A)$ as its kernel, therefore $\mathsf{H}_0(A) \cong \mathbf{Z}$. Likewise, $\mathsf{Z}_1(A) = \{a \oplus a \mid a \in \mathbf{Z}\}$ and $\mathsf{B}_1(A) = 0$, so $\mathsf{H}_1(A) \cong \mathsf{Z}_1(A) \cong \mathbf{Z}$. The computation for B is easy: $\mathsf{H}_0(A) = \mathsf{Z}_0(B)/\mathsf{B}_0(B) = \mathbf{Z}/0 \cong \mathbf{Z}$, $\mathsf{H}_1(B) = \mathsf{Z}_1(B)/\mathsf{B}_1(B) = \mathbf{Z}/0 \cong \mathbf{Z}$.

The map $\mathsf{C}_0(f) \colon \mathsf{C}_0(A) \to \mathsf{C}_0(B)$ sends $a \oplus b \mapsto a + b$. To compute $\mathsf{H}_0(f)$, we choose a section of $\mathsf{U}(\mathsf{Z}_0(A)) = \mathbf{Z} \times \mathbf{Z} \to \mathsf{U}(\mathsf{H}_0(A)) \cong \mathbf{Z}$, e.g., $\mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ $(a \mapsto (a, 0))$ and postcompose it with $\mathsf{U}(\mathsf{Z}_0(f)) \colon \mathsf{U}(\mathsf{Z}_0(A)) \to \mathsf{U}(\mathsf{Z}_0(B))$, which yields the map $a \mapsto a + 0 = a$, i.e., the identity map $\mathbf{Z} \to \mathbf{Z}$. Thus, $\mathsf{H}_0(f) \cong \mathrm{id}_{\mathbf{Z}}$.

Likewise, $\mathsf{C}_1(f) \colon \mathsf{C}_1(A) \to \mathsf{C}_1(B)$ sends $a \oplus b \mapsto a + b$. Since $\mathsf{B}_1(f) = 0$, we have $\mathsf{H}_1(f) \cong \mathsf{Z}_1(f)$, which in its turn is $\{a \oplus a \mid a \in \mathbf{Z}\} \mapsto \mathbf{Z}$ $(a \oplus a \mapsto a + a = 2a)$, which is isomorphic to $\mathbf{Z} \to \mathbf{Z}$ $(a \mapsto 2a)$. We summarize this as follows.

$$\begin{array}{cccc}
\mathbf{Z} & \mathbf{Z} & 0 & \cdots \\
\downarrow{\scriptstyle 1} & \downarrow{\scriptstyle 2} & \downarrow{\scriptstyle 0} & \vdots \\
\mathbf{Z} & \mathbf{Z} & 0 & \cdots
\end{array}$$

Exercise 16.15. For each of the simplicial maps listed in Exercise 15.18, compute the induced map on homology.

17 Examples of simplicial sets in mathematics

17.1. ***Simplicial complexes*** Used in 7.5.

Simplicial complexes are a formalism very closely related to simplicial sets. There are two primary distinctions between simplicial complexes and simplicial sets:

- Individual simplices of a simplicial complex are no longer ordered;
- There can be at most one simplex with a given set of vertices.

Given the vastly superior formal properties of simplicial sets, today simplicial complexes are used mostly for historical reasons. However, it is important to be acquainted with them in order to be able to use the vast body of literature that uses simplicial complexes.

Definition 17.2. A *simplicial complex* is a pair (V, S), where $V \in \mathsf{Set}$ is a set of vertices and $S \subset 2^V$ is a set of simplices. We require that $\emptyset \notin S$, all singleton subsets belong to S, and S is closed under passage to nonempty subsets. A *map of simplicial complexes* $(V, S) \to (V', S')$ is a function $f\colon V \to V'$ such that for any $s \in S$ we have $f_*(s) \in S$. The category of simplicial complexes is denoted by $\mathsf{SimpComp}$. Used in 2.0*, 7.5, 17.1*, 17.2, 17.2*, 17.3, 17.4, 17.6, 17.6*.

We now discuss two ways to turn a simplicial complex into a simplicial set.

Definition 17.3. We define a functor $\mathsf{F}\colon \mathsf{SimpComp} \to \mathsf{sSet}$ as follows. Given a simplicial complex (V, S), we define a simplicial set $\mathsf{F}(V, S) = X$ by setting $X_{\mathbf{m}}$ to the set of maps $s\colon \mathsf{U}(\mathbf{m}) \to V$ such that $\operatorname{im} s \in S$, whereas the simplicial structure maps X_f send s to $s \circ \mathsf{U}(f)$. Likewise, a map of simplicial complexes $g\colon (V, S) \to (V', S')$ yields a simplicial map $\mathsf{F}(V, S) \to \mathsf{F}(V', S')$ by sending s to $g \circ s$.

The resulting simplicial set is much bigger than the original simplicial complex (V, S): for any simplex $s \in S$ we have many simplices in $\mathsf{F}(V, S)$, obtained by ordering the vertices of s in all possible ways. In many cases we can do better: the vertices of simplices often already possess a canonical ordering.

Definition 17.4. A *locally ordered simplicial complex* is a pair $(V, S, \le)$, where (V, S) is a simplicial complex and $\le\colon S \to 2^{V\times V}$ sends each $s \in S$ to a subset of $V \times V$ that is contained in $s \times s$ and defines an ordering on s. Furthermore, we require that if $s \subset t$, then the ordering of s is induced by the ordering of t. A *map of locally ordered simplicial complexes* $(V, S, \le) \to (V', S', \le)$ is a map of simplicial complexes $f\colon V \to V'$ such that if we restrict f to any $s \in S$ and corestrict it to $f_*(s)$, the resulting map is order-preserving. The category of locally ordered simplicial complexes is denoted by $\mathsf{LocOrdSimpComp}$. Used in 17.4, 17.5, 17.6.

Definition 17.5. We define a functor $\mathsf{F}\colon \mathsf{LocOrdSimpComp} \to \mathsf{sSet}$ as follows. Given a locally ordered simplicial complex $(V, S, \le)$, we define a simplicial set $\mathsf{F}(V, S) = X$ by setting $X_{\mathbf{m}}$ to the set of maps $s\colon \mathsf{U}(\mathbf{m}) \to V$ such that $\operatorname{im} s \in S$ and the induced map $\mathsf{U}(\mathbf{m}) \to \operatorname{im} s$ is order-preserving, whereas the simplicial structure maps X_f send s to $s \circ \mathsf{U}(f)$. Likewise, a map of locally ordered simplicial complexes $f\colon (V, S, \le) \to (V', S', \le)$ yields a simplicial map $\mathsf{F}(V, S) \to \mathsf{F}(V', S')$ by sending s to $f \circ s$.

Example 17.6. An *ordered simplicial complex* is a triple $(V, S, \le)$, where (V, S) is a simplicial complex and $(V, \le)$ is an ordered set. Such a data canonically induces a locally ordered simplicial complex by inducing an ordering of each simplex from the given global ordering on V.

We finish this section with a comparison of simplicial sets and simplicial complexes. Jungerman and Ringel proved that the minimal number of 2-simplices in a simplicial complex that represents an orientable surface of genus g is precisely $4g - 4 + 2\lfloor(7 + (1 + 48g)^{1/2})/2\rfloor$ if $g \ne 2$ and 24 if $g = 2$. For the nonorientable surface with g crosscaps it is $2g - 4 + 2\lfloor(7 + (1 + 24g)^{1/2})/2\rfloor$ if $g \notin \{2, 3\}$, 16 if $g = 2$, and 20 if $g = 3$. For simplicial sets we gave generic presentations with $4g$ respectively $2g$ 2-simplices if $g > 0$ and a single 2-simplex if $g = 0$, and also gave smaller presentations when $g \le 2$. This makes a large difference for small g: a Klein bottle needs 16 2-simplices if it is encoded as a simplicial complex, but only 2 if it is encoded as a simplicial set.

As another example of efficiency of simplicial sets, we cite the fact that an n-sphere needs at least $n+2$ n-simplices to be encoded as a simplicial complex, but only a single n-simplex if encoded as a simplicial set.

Additionally, simplicial sets have much better theoretical properties: they form a cartesian combinatorial model category, which provides for a large number of standard tools, whereas simplicial complexes require ad hoc constructions. Some constructions, like products, are much easier to implement in simplicial sets than in simplicial complexes.

17.7. *Singular simplicial sets*

Supplementary sources: [EIISS, §3].

Definition 17.8. A *category of geometric spaces*, denoted by Space, is a category equipped with a functor $|-|\colon \Delta \to \mathsf{Space}$. Its objects and morphisms will be referred to as *geometric spaces* and *geometric map*. Used in 12.6, 17.8*, 17.9, 17.10, 17.11, 17.11*.

Thus, $|\mathbf{m}|$ is a geometric space for any simplex $\mathbf{m}$, $|f|$ is a geometric map for any map of simplices $f\colon \mathbf{m} \to \mathbf{n}$, for any maps of simplices f and g we have $|g \circ f| = |g| \circ |f|$, and for any simplex $\mathbf{m}$ we have $|\mathrm{id}_{\mathbf{m}}| = \mathrm{id}_{|\mathbf{m}|}$.

Example 17.9. Each of the following constitutes a category of geometric spaces.

- metric spaces and contractive maps;
- metric spaces and continuous maps;
- topological spaces and continuous maps;
- uniform spaces and uniform maps;
- smooth manifolds and smooth maps;
- real analytic spaces and real analytic maps;
- enhanced measurable spaces and measurable maps;
- algebraic varieties over a field k and regular maps;

For all of the above examples, the functor $|-|$ is constructed using essentially the same construction as for the geometric realization of a simplex. We recall it briefly here. First, we fix $|\mathbf{1}|$; in the first four examples we can take the real interval $[0, 1]$ with the appropriate structure; for smooth manifolds, real analytic spaces, and enhanced measurable spaces we take $|\mathbf{1}| = \mathbf{R}$ equipped with the appropriate structure; and for algebraic varieties we take $|\mathbf{1}| = \mathbf{A}^1_k$. The we set $|\mathbf{m}|$ to the subspace of $|\mathbf{1}|^{m+1}$ (i.e., the product of $m+1$ copies of $|\mathbf{1}|$) cut out by the equation $x_0 + \cdots + x_m = 1$, where x_i denotes the projection map to the ith factor. Finally, given a map of simplices $f\colon \mathbf{m} \to \mathbf{n}$, we set $|f|\colon |\mathbf{m}| \to |\mathbf{n}|$ to the map induced by the map $|\mathbf{1}|^{m+1} \to |\mathbf{1}|^{n+1}$ whose jth component ($0 \le j \le n$) is the map that takes the sum of all coordinates with indices in $f^{-1}(j)$.

Definition 17.10. The *singular simplicial set* $\mathrm{Sing}(M)$ of a geometric space M is defined by setting $\mathrm{Sing}(M)_{\mathbf{m}}$ to the set of geometric maps $|\mathbf{m}| \to M$ (known as *singular simplices*) and

$$\mathrm{Sing}(M)_f\colon \mathrm{Sing}(M)_{\mathbf{n}} \to \mathrm{Sing}(M)_{\mathbf{m}}$$

to the map of sets that sends $a\colon |\mathbf{n}| \to M$ to $a \circ f\colon |\mathbf{m}| \to M$. Given a geometric map $r\colon M \to N$, the induced map $\mathrm{Sing}(r)\colon \mathrm{Sing}(M) \to \mathrm{Sing}(N)$ is defined by setting $\mathrm{Sing}(r)_{\mathbf{m}}\colon \mathrm{Sing}(M)_{\mathbf{m}} \to \mathrm{Sing}(N)_{\mathbf{m}}$ to the map of sets that sends $a\colon |\mathbf{m}| \to M$ to $r \circ a\colon |\mathbf{m}| \to N$. Used in 2.0*, 17.10, 17.11, 17.11*, 17.12, 17.17*, 17.21, 20.13, 25.2, 25.4, 35.6, 35.6*, 35.7, 39.12, 40.8.

Exercise 17.11. Verify that the singular simplicial set construction is a functor $\mathrm{Sing}\colon \mathsf{Space} \to \mathsf{sSet}$, where Space is one of the categories of geometric spaces of Definition 17.8. (The nature of geometric spaces is irrelevant here, only the fact that Space is a category equipped with the functor $|-|\colon \Delta \to \mathsf{Space}$ from Example 12.6 matters.)

The idea behind the singular simplicial set is that we "probe" a geometric space M by mapping all possible geometric realizations of simplices into it, and record the resulting information in a simplicial set.

Singular simplicial sets are important in theoretical considerations, but direct computations with them are impractical due to the huge number of simplices involved. For instance, if $M = \{(x, y) \mid x^2 + y^2 = 1\}$, i.e., a circle, then $\mathrm{Sing}(M)$ has uncountably many simplices in every dimension, e.g., one vertex for every point, and even more higher-dimensional simplices. On the other hand, the simplicial circle of Definition 7.14 has a single nondegenerate simplex in dimensions 0 and 1 and is much easier to work with in practice.

Definition 17.12. (Eilenberg, 1944.) The *singular homology* is the functor $\mathsf{H}\colon \mathsf{Space} \to \mathsf{Ab}^{\mathbf{Z}}$ given by the composition of $\mathrm{Sing}\colon \mathsf{Space} \to \mathsf{sSet}$ and $\mathsf{H}\colon \mathsf{sSet} \to \mathsf{Ab}^{\mathbf{Z}}$. Used in 1.0*, 17.13, 20.12*, 35.14, 38.4.

Exercise 17.13. Suppose Space is the category of metric spaces and continuous maps. Compute the singular homology groups of the metric space $\mathbf{R}^n$ for every $n \ge 0$.

17.14. ***Nerves of covers and Vietoris simplicial sets***

In this section we introduce two classical constructions of simplicial sets: nerves and Vietoris complexes. Both were introduced in 1927, the former by Paul Alexandroff [Approx, §13] and the latter by Leopold Vietoris [ZH].

The input data to both of these constructions is a triple (X, Y, R), where X and Y are sets and $R \subset X \times Y$ is a relation from X to Y.

In typical applications X is the underlying set of some space, whereas Y is a cover of that space, i.e., a family of subspaces of that space, whose union is X. We define $(x, y) \in R$ if $x \in y$, i.e., a point x belongs to the element y of the cover.

Definition 17.15. (Paul Alexandroff, 1927.) The *nerve of a cover* of (X, Y, R) is the simplicial set $\mathrm{N}(X, Y, R)$ defined by setting $\mathrm{N}(X, Y, R)_{\mathbf{m}}$ to the set of maps $f\colon \mathsf{U}(\mathbf{m}) \to Y$ for which there is an element $x \in X$ such that $(x, y) \in R$ for any $y \in \operatorname{im} f$. For a map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ the structure map

$$\mathrm{N}(X, Y, R)_f\colon \mathrm{N}(X, Y, R)_{\mathbf{n}} \to \mathrm{N}(X, Y, R)_{\mathbf{m}}$$

sends $f\colon \mathsf{U}(\mathbf{n}) \to Y$ to $f \circ \mathsf{U}(f)$. Used in 17.17*.

Definition 17.16. The *Vietoris complex* $\mathrm{V}(X, Y, R)$ of (X, Y, R) is defined as $\mathrm{N}(Y, X, R^{\mathsf{op}})$, where R^{op} is the image of R under the isomorphism $X \times Y \to Y \times X$. Used in 17.16*, 17.17, 17.17*.

Thus, an $\mathbf{m}$-simplex of the Vietoris complex is a family of elements of X indexed by the vertices of $\mathbf{m}$, for which there is an element $y \in Y$ such that $(x, y) \in R$ for all x in the family.

Remark 17.17. If the set Y is ordered, we may also consider the ordered variant of $\mathrm{N}(X, Y, R)$, which requires the map f to be order-preserving. This also applies to the Vietoris complex.

It is useful to interpret the above definitions when X is the underlying set of some metric or topological space, Y is an open cover of that space, and $R(x, y)$ holds if and only if $x \in y$. In this case, we write $\mathrm{N}(X, Y)$ and $\mathrm{V}(X, Y)$. An $\mathbf{m}$-simplex of the nerve of a cover is a family of $\dim \mathbf{m} + 1$ elements of the open cover that have a nonempty intersection. An $\mathbf{m}$-simplex of the Vietoris complex is a family of $\dim \mathbf{m} + 1$ points in X that together form a subset of some element of the open cover.

A remarkable theorem due to Dowker [HGR] shows that the simplicial sets $\mathrm{V}(X, Y, R)$ and $\mathrm{N}(X, Y, R)$ are weakly equivalent (to be defined later).

Later, we will prove the following very important result, known as the *nerve theorem*: if any finite interesection of elements of an open cover of a topological space X is empty or contractible (to be defined later), then the nerve of this open cover is weakly equivalent (to be defined later) to the singular simplicial set of X. This result will allow us to reduce problems about topological spaces to problems about simplicial sets. For now, we illustrate this idea with a simple exercise.

Exercise 17.18. Suppose Space is the category of metric spaces and continuous maps. Consider the open cover of the 2-dimensional sphere $S^2 = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ by the six hemispheres centered at each of the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. Compute the simplicial homology of the nerve of this open cover.

17.19. ***Nerves of categories and classifying simplicial sets of groups and monoids***

Supplementary sources: [EIISS, §6].

We now move on to very different examples that do not have any obvious geometric underpinnings at all.

Recall that Cat denotes the category of small categories and functors, and a small category is a category whose objects form a set as opposed to a proper class.

Definition 17.20. The functor $[-]\colon \Delta \to \mathsf{Cat}$ is the composition of the forgetful functor $\Delta \to \mathsf{Poset}$ and the functor $\mathsf{Poset} \to \mathsf{Cat}$ constructed in Example 11.18.

For example, if $n \geq 0$, then $[\mathbf{n}]$ is the category with objects $\{0, 1, \ldots, n\}$ and the set of morphisms $i \to j$ is empty if $i > j$ and is a singleton set if $i \leq j$. There is exactly one way to define composition and identity morphisms.

Definition 17.21. The *nerve* of a small category I is the simplicial set $\mathrm{N}I$, defined by applying Definition 17.10 to the functor $[-]\colon \Delta \to \mathsf{Cat}$. Thus,

$$(\mathrm{N}I)_n = \mathsf{Fun}([n], I)$$

and the simplicial structure map for $f\colon \mathbf{m} \to \mathbf{n}$ is

$$(\mathrm{N}I)_f = \mathsf{Fun}([f], I),$$

where

$$[f]\colon [m] \to [n]$$

is induced by f. Used in 17.22, 25.2, 28.1.

Recall that a *monoid* is a set equipped with an associative operation. Formally, a monoid is a triple $(S, \cdot, 1)$, where $S \in \mathsf{Set}$, $\cdot\colon S \times S \to S$ is a binary operation, $1 \in S$ is the identity element, and $x\cdot(y\cdot z) = (x\cdot y)\cdot z$ and $1 \cdot x = x \cdot 1 = x$ for all $x, y, z \in S$. In particular, any group is a monoid. Other simple examples of monoids include $(\mathbf{N}, +, 0)$ (natural numbers with addition and zero) and $(\mathbf{N}, \cdot, 1)$ (natural numbers with multiplication and one). The multiplication operation need not be commutative. For instance, given a set X we can consider the noncommutative monoid $(S^S, \circ, \mathrm{id}_S)$, whose elements are maps $S \to S$, multiplication is given by the composition of maps, and the identity element is given by the identity map.

Definition 17.22. The *classifying simplicial set of a monoid M* is the nerve of the category $\mathrm{B}M$. By abuse of notation, it is also denoted by $\mathrm{B}M$. Used in 17.23, 17.24.

Thus, $\mathrm{B}M$ is the simplicial set that sends a simplex $\mathbf{m}$ to $\mathsf{U}(M)^{\dim \mathbf{m}}$ and a map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ to the map $\mathsf{U}(M)^{\dim \mathbf{n}} \to \mathsf{U}(M)^{\dim \mathbf{m}}$ whose ith component ($0 \le i < \dim \mathbf{m}$) is the product of components with indices in $[f(i), f(i+1))$.

Exercise 17.23. Verify that the classifying simplicial set construction is a functor $\mathrm{B}\colon \mathsf{Monoid} \to \mathsf{sSet}$.

Many important invariants of groups and other algebraic structures are defined in terms of $\mathrm{B}M$. For instance, the homology of a group G is defined as the homology of $\mathrm{B}M$. Likewise for cohomology of groups.

Definition 17.24. (Hurewicz, 1936.) The *homology of a group G* is the homology of the classifying simplicial set $\mathrm{B}G$. Used in 17.23*, 17.25, 20.12*.

Exercise 17.25. Compute the homology of the group $\mathbf{Z}/2$.

Exercise 17.26. Compute the simplicial homology of the classifying simplicial set of the monoid with two elements 1 and e, where 1 is the unit and $e^2 = e$.

Exercise 17.27. The word "nerve" is used for two different constructions: one with open covers, another one with categories. Find a nontrivial connection between these two notions. Formulate a precise claim and prove it.

18 Homology with coefficients

Recall that the construction of simplicial chains crucially relied on a free abelian group functor

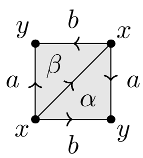
$$\mathsf{Free}\colon \mathsf{Set} \to \mathsf{Ab}.$$

Concretely, we have $\mathsf{Free}(S) = \coprod_S \mathbf{Z} = \bigoplus_S \mathbf{Z}$, whereas $\mathsf{Free}(f, A)\colon \mathsf{Free}(S, A) \to \mathsf{Free}(T, A)$ for a map of sets $f\colon S \to T$ sends a basis element s of $\mathsf{Free}(S)$ to the basis element $f(s)$ of $\mathsf{Free}(T)$. This construction can be generalized as follows: given an abelian group A, we have a functor $\mathsf{Free}(-, A)\colon \mathsf{Set} \to \mathsf{Ab}$ such that $\mathsf{Free}(S, A) = \coprod_S A$ and $\mathsf{Free}(f, A)\colon \mathsf{Free}(S, A) \to \mathsf{Free}(T, A)$ sends direct summands to direct summands as prescribed by f.

Definition 18.1. Given an abelian group A (henceforth the *abelian group of coefficients*), we define a functor $\mathsf{C}(-, A)\colon \mathsf{sSet} \to \mathsf{Ch}$, known as the (normalized) *simplicial chains with coefficients* in A, by sending a simplicial set X, i.e., a functor $X\colon \Delta^{\mathsf{op}} \to \mathsf{Set}$, to the composition $\mathsf{Free}(-, A) \circ X\colon \Delta^{\mathsf{op}} \to \mathsf{Ab}$, and a simplicial map $f\colon X \to Y$, i.e., a natural transformation of functors, to the whiskering $\mathsf{Free}(-, A) \diamond f$. Used in 18.3, 20.8.

Definition 18.2. The functors $\mathsf{Z}(-, A), \mathsf{B}(-, A), \mathsf{H}(-, A)\colon \mathsf{sSet} \to \mathsf{Ab}^{\mathbf{Z}}$ are defined by composing the functors $\mathsf{Z}, \mathsf{B}, \mathsf{H}\colon \mathsf{Ch} \to \mathsf{Ab}^{\mathbf{Z}}$ with the functor $\mathsf{C}(-, A)\colon \mathsf{sSet} \to \mathsf{Ch}$. The functor $\mathsf{H}(-, A)$ is known as *simplicial homology with coefficients* in A. Used in 18.3.

Example 18.3. We compute the simplicial homology with coefficients of a real projective plane represented via two 2-simplices:



The simplicial chains with coefficients in A are

$$A_{\langle x\rangle} \oplus A_{\langle y\rangle} \xleftarrow{\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}} A_{\langle a\rangle} \oplus A_{\langle b\rangle} \oplus A_{\langle d\rangle} \xleftarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}} A_{\langle \alpha\rangle} \oplus A_{\langle \beta\rangle}.$$

The matrix reduction algorithm works as before. We have $\mathsf{B}_0 = \{a \oplus -a \mid a \in A\}$. The quotient map $\mathsf{Z}_0 \to \mathsf{Z}_0/\mathsf{B}_0 = A$ can be taken to be $x \oplus y \mapsto x + y$, so $\mathsf{H}_0 \cong A$.

We get $\mathsf{Z}_1 = \{a \oplus b \oplus d \mid a + b = 0\}$. Likewise, B_1 is the image of the column-reduced matrix

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 2 \end{pmatrix}.$$

The quotient map $\mathsf{Z} = \mathsf{Z}_1/\mathsf{B}_1 \cong A/2A$ sends $a \oplus b \oplus d \mapsto [d - a]$. Indeed, the composition $\mathsf{B}_1 \to \mathsf{Z}_1 \to A/2A$ sends $\alpha \oplus \beta \mapsto \alpha \oplus -\alpha \oplus (\alpha + 2\beta) \mapsto [2\beta] = 0$. The kernel of the quotient map is a subgroup of $\mathsf{Z}_1 = \{a \oplus b \oplus d \mid a + b = 0\}$ defined by $[d - a] = 0$, i.e., $d - a \in 2A$, or $d - a = 2\beta$ for some $\beta \in A$. But this means that $a \oplus b \oplus d = a \oplus -a \oplus (a \oplus 2\beta) \in \operatorname{im} \partial_2$.

Finally, we get $\mathsf{Z}_2 = \{\alpha \oplus \beta \mid \alpha + \beta = 0 \wedge \alpha - \beta = 0\} \cong \{\alpha \in A \mid 2\alpha = 0\}$, a group that is known as the 2-torsion of A, denoted by $\mathsf{Tor}(\mathbf{Z}/2, A)$. Thus, $\mathsf{H}_2 \cong \mathsf{Tor}(\mathbf{Z}/2, A)$.

Thus, the sequence of homology groups is A, $A/2A \cong \mathbf{Z}/2 \otimes A$, $\mathsf{Tor}(\mathbf{Z}/2, A)$. If $A = \mathbf{Z}$, we recover the previously computed $\mathbf{Z}$, $\mathbf{Z}/2$, and 0. However, other groups yield different results. For instance, if we take $A = \mathbf{Z}/2$, we get $\mathbf{Z}/2$, $\mathbf{Z}/2$, and $\mathbf{Z}/2$, which makes the second homology group nonzero, i.e., $A = \mathbf{Z}/2$ can see something that $A = \mathbf{Z}$ cannot. For $A = \mathbf{Q}$, we get $\mathbf{Q}$, 0, 0, i.e., $A = \mathbf{Q}$ sees less than $A = \mathbf{Z}$.

Proposition 18.4. For any family of simplicial sets $\{X_i\}_{i \in I}$ (I is an arbitrary set) we have an isomorphism

$$\bigoplus_{i \in I} \mathsf{H}(X_i, A) \to \mathsf{H}\left(\coprod_{i \in I} X_i, A\right).$$

Used in 18.6*, 23.8*.

Proof. We observe that $\mathsf{C}(\coprod_{i\in I} X_i, A) \cong \bigoplus_{i\in I} \mathsf{C}(X_i, A)$. Indeed, $\mathsf{C}(\coprod_{i\in I} X_i, A)_n$ is the free abelian group on the set of nondegenerate n-simplices of $\coprod_{i\in I} X_i$, which is the disjoint union of the sets of nondegenerate n-simplices of X_i for all $i \in I$. The free abelian group functors sends disjoint unions of sets to direct sums of abelian groups.

Lemma 18.5 completes the proof. ∎

Lemma 18.5. If $C_i \in \mathsf{Ch}$ is a chain complex for all $i \in I$, then the canonical map

$$\bigoplus_{i\in I} \mathsf{H}(C_i) \to \mathsf{H}\left(\bigoplus_{i\in I} C_i\right)$$

is an isomorphism. Used in 18.4*.

Proof. The canonical map is constructed using the universal property of coproducts from individual maps

$$\mathsf{H}(\iota_i)\colon \mathsf{H}(C_i) \to \mathsf{H}\left(\bigoplus_{i\in I} C_i\right),$$

where $\iota_i\colon C_i \to \coprod_{i\in I} C_i$ is the injection map for a coproduct.

We construct a map in the opposite direction:

$$\mathsf{H}\left(\bigoplus_{i\in I} C_i\right) \to \bigoplus_{i\in I} \mathsf{H}(C_i),$$

which is induced by the map

$$\mathsf{Z}\left(\bigoplus_{i\in I} C_i\right) \to \bigoplus_{i\in I} \mathsf{Z}(C_i).$$

Indeed, $c \in \mathsf{Z}_n\left(\bigoplus_{i\in I} C_i\right)$ is a collection of elements of $\mathsf{C}_n(C_i)$ that are nonzero for finitely many $i \in I$ and whose differentials vanish. This is by definition an element of the right side. In the same way, we have an induced map

$$\mathsf{B}\left(\bigoplus_{i\in I} C_i\right) \to \bigoplus_{i\in I} \mathsf{B}(C_i).$$

These two maps induce a quotient map

$$\mathsf{H}\left(\bigoplus_{i\in I} C_i\right) \to \bigoplus_{i\in I} \mathsf{H}(C_i).$$

By construction, the maps in both directions are inverse to each other. ∎

Proposition 18.6. If $X \in \mathsf{sSet}$ is connected, then the canonical map

$$\mathsf{H}_0(X, A) \to A$$

is an isomorphism. For an arbitrary X, the canonical map

$$\mathsf{H}_0(X, A) \to \bigoplus_{\pi_0(X)} A$$

is an isomorphism. Used in 20.11.

Proof. The second claim follows from the first claim and Proposition 18.4. The canonical map

$$\mathsf{H}_0(X, A) = \mathsf{C}_0(X, A)/\mathsf{B}_0(X, A) \to A$$

is induced by the homomorphism

$$\sum\colon \mathsf{C}_0(X,A)\to A$$

that takes the sum of coefficients of all 0-simplices. It remains to verify that the kernel of this map equals $\mathsf{B}_0(X,A)$. Indeed, $\sum$ vanishes on $\mathsf{B}_0(X,A)$, so it suffices to show that any simplicial chain c in the kernel of Σ is a boundary. We prove this by induction on the number n of nonzero coefficients in c. If $n=0$, then $c=0$ is a boundary. If $n>0$, pick any vertex $v\in X_0$ whose coefficient $c_v\in A$ in c is nonzero. Since $\sum_w c_w=0$, there is another vertex $u\neq v$ with a nonzero coefficient $c_u\in A$.

If there is a 1-chain L such that $\partial L=c_v u-c_v v$, the 0-chain $c+\partial L$ has $n-1$ or $n-2$ nonvanishing coefficients because the coefficient of v is now zero and the coefficient of u may also vanish. By induction, there is a 1-chain W such that $c+\partial L=\partial W$. Hence, $c=\partial(W-L)$, which proves the statement.

In order to prove that such a 1-chain exists, recall that X is connected, i.e., $\pi_0(X)$ is a singleton. This means that any two elements of X_0 (such as u and v in our case) can be connected by a chain of vertices such that every consecutive pair forms the endpoints of some 1-simplex, in either order. The chain L will now be assembled of all 1-simplices that occur in such a chain, taken with coefficient c_v or $-c_v$ depending on its orientation. The boundary L will see all interior vertices annihilated because of our choice of coefficients. Only the endpoints of L survive, so $\partial L=c_v u-c_v v$. ▮

Exercise 18.7. For each of the simplicial sets listed in Exercise 15.13, compute its homology with coefficients in an arbitrary abelian group A. For each of the simplicial maps listed in Exercise 15.18, compute the induced map on homology with coefficients in an aribtrary abelian group A.

19 The Euler characteristic

Definition 19.1. The *rank* of an abelian group A, denoted by $\operatorname{rank} A$, is the cardinality of a maximal linear independent subset of A, i.e., a subset $S\subset A$ such that $\sum_{s\in S}\alpha_s\cdot s=0$ implies $\alpha_s=0$ for all $s\in S$, where $\alpha\colon S\to\mathbf{Z}$ is a system of integer coefficients with finite support. The cardinality can be finite or infinite (in which case it is a cardinal number). Equivalently, one can compute $\operatorname{rank} A=\dim(A\otimes_{\mathbf{Z}}\mathbf{Q})$. Used in 19.1, 19.2, 19.3.

Definition 19.2. The *Euler characteristic* of a simplicial set X that satisfies the finiteness condition given below is defined as

$$\chi(X)=\sum_{n\geq 0}(-1)^n\operatorname{rank}\mathsf{H}_n(X).$$

The finiteness condition requires that all ranks are finite and only finitely many terms in this sum are nonzero.

Lemma 19.3. We have $\operatorname{rank}\mathsf{H}_n(X)=\dim\mathsf{H}_n(X,\mathbf{Q})$, so

$$\chi(X)=\sum_{n\geq 0}(-1)^n\dim\mathsf{H}_n(X,\mathbf{Q}).$$

Proof. A linearly independent subset of $\mathsf{H}_n(X)$ is equivalently a subset S of $\mathsf{Z}_n(X)$ such that $\sum_{s\in S}\alpha_s\cdot s\in\mathsf{B}_n(X)$ implies $\alpha_s=0$ for all $s\in S$. Furthermore, if $\alpha_s\in\mathbf{Q}$, we can multiply them by their common denominator, which makes $\alpha_S\in\mathbf{Z}$. Thus a subset of $\mathsf{Z}_n(X)$ that is linearly independent over $\mathbf{Z}$ remains linearly independent over $\mathbf{Q}$ once we embed it using $\mathsf{Z}_n(X,\mathbf{Z})\to\mathsf{Z}_n(X,\mathbf{Q})$.

Vice versa, if we have a subset of $\mathsf{Z}_n(X,\mathbf{Q})$ linearly independent over $\mathbf{Q}$, we can multiply each of its elements by the product of the denominators of coefficients that occur in any of the chains under consideration. Thus a linearly independent subset of $\mathsf{Z}_n(X,\mathbf{Q})$ gives rise to a linearly independent subset of $\mathsf{Z}_n(X,\mathbf{Z})$ of the same cardinality. ▮

Proposition 19.4. If a simplicial set X has only finitely many nondegenerate simplices, then $\chi(X)=\sum_{n\geq 0}(-1)^n\#X'_n$, where X'_n denotes the set of nondegenerate n-simplices of X.

Proof. In linear algebra, this statement is known as the "rank-nullity theorem" or the "first isomorphism theorem". We have

$$\begin{aligned}\chi(X) &= \sum_{n\geq 0}(-1)^n \dim \mathsf{H}_n(X,\mathbf{Q}) \\ &= \sum_{n\geq 0}(-1)^n(\dim \mathsf{Z}_n(X,\mathbf{Q}) - \dim \mathsf{B}_n(X,\mathbf{Q}) \\ &= \sum_{n\geq 0}(-1)^n(\dim \mathsf{Z}_n(X,\mathbf{Q}) - (\dim \mathsf{C}_{n+1}(X,\mathbf{Q}) - \dim \mathsf{Z}_{n+1}(X,\mathbf{Q}))) \\ &= \sum_{n\geq 0}(-1)^n(\dim \mathsf{Z}_n(X,\mathbf{Q}) + \dim \mathsf{Z}_{n+1}(X,\mathbf{Q}) - \dim \mathsf{C}_{n+1}(X,\mathbf{Q})) \\ &= \dim \mathsf{Z}_0(X,\mathbf{Q}) + \sum_{n\geq 0}(-1)^n(-\dim \mathsf{C}_{n+1}(X,\mathbf{Q})) \\ &= \sum_{n\geq 0}(-1)^n \dim \mathsf{C}_n(X,\mathbf{Q}). \quad \blacksquare\end{aligned}$$

Later we will show how the Euler characteristic behave under operations such as gluing (homotopy pushouts).

Exercise 19.5. For each of the simplicial sets listed in Exercise 15.13, compute its Euler characteristic or prove that it is undefined.

Exercise 19.6. If $\chi(X)$ exists (and is finite), does this imply that X has only finitely many nondegenerate simplices?

20 Cohomology

Recall for any abelian group A there is a functor $\mathsf{Hom}(-,A)\colon \mathsf{Ab}^{\mathsf{op}} \to \mathsf{Ab}$ (the *internal hom of abelian groups*) that sends an abelian group X to $\mathsf{Hom}(X,A)$, the abelian group whose elements are homomorphisms of abelian groups $X \to A$ and operations are defined pointwise. The functor $\mathsf{Hom}(-,A)$ sends a homomorphism of abelian groups $f\colon X \to Y$ to the homomorphism $\mathsf{Hom}(f,A)\colon \mathsf{Hom}(Y,A) \to \mathsf{Hom}(X,A)$ that sends a homomorphism $Y \to A$ to its precomposition with f. This construction can be generalized to chain complexes.

Definition 20.1. Given an abelian group A, we define a functor $\mathsf{Hom}(-,A)\colon \mathsf{Ch}^{\mathsf{op}} \to \mathsf{Ch}$ by sending a chain complex X to the chain complex $\mathsf{Hom}(X,A)$ such that $\mathsf{Hom}(X,A)_n = \mathsf{Hom}(X_{-n},A)$ and the differential $\partial^n\colon \mathsf{Hom}(X,A)_n \to \mathsf{Hom}(X,A)_{n-1}$ is the map $\mathsf{Hom}(\partial_{n+1},A)\colon \mathsf{Hom}(X_n,A) \to \mathsf{Hom}(X_{n+1},A)$.

In practice, if X is nonnegatively graded, then $\mathsf{Hom}(X,A)$ will be concentrated in nonpositive degrees, which is inconvenient. To mitigate this, we introduce a dual notion of cochain complex.

Definition 20.2. The category coCh of *cochain complexes* is defined as follows. Objects are families $\{X_i\}_{i\in\mathbf{Z}}$ of abelian groups indexed by integer numbers equipped with a family $\{\partial^i\}_{i\in\mathbf{Z}}$ of differentials $\partial^i\colon X_i \to X_{i+1}$ (homomorphisms of abelian groups) such that $\partial^{i+1}\circ\partial^i = 0$ for all $i \in \mathbf{Z}$. Morphisms $X \to Y$ are *cochain maps*, i.e., families $\{f_i\}_{i\in\mathbf{Z}}$ of homomorphisms $f_i\colon X_i \to Y_i$ such that $f_{i+1}\circ\partial^i_X = \partial^i_Y \circ f_i$ for all $i \in \mathbf{Z}$. Used in 20.1*, 20.2*, 20.3, 20.4, 20.5, 20.6, 20.10, 22.21, 23.3, 23.4, 23.6*, 23.8*, 24.5.

Thus, the only difference between chain complexes and cochain complexes is the direction of differentials: chain differentials decrease the degree by 1, whereas cochain differentials increase the degree by 1. We now redefine the hom-functor to land in cochain complexes, using the same notation.

Definition 20.3. Given an abelian group A, we define a functor $\mathsf{Hom}(-,A)\colon \mathsf{Ch}^{\mathsf{op}} \to \mathsf{coCh}$ by sending a chain complex X to the chain complex $\mathsf{Hom}(X,A)$ such that $\mathsf{Hom}(X,A)_n = \mathsf{Hom}(X_n,A)$ and the differential $\partial^n\colon \mathsf{Hom}(X,A)_n \to \mathsf{Hom}(X,A)_{n+1}$ is the map $\mathsf{Hom}(\partial_{n+1},A)\colon \mathsf{Hom}(X_n,A) \to \mathsf{Hom}(X_{n+1},A)$. A chain map $f\colon X \to Y$ is sent to the cochain map $\mathsf{Hom}(f, A)\colon \mathsf{Hom}(Y, A) \to \mathsf{Hom}(X, A)$ with components $\mathsf{Hom}(f_n, A)\colon \mathsf{Hom}(Y_n, A) \to \mathsf{Hom}(X_n, A)$.

Definition 20.4. The functors $\mathsf{Z}^*, \mathsf{B}^*, \mathsf{H}^*\colon \mathsf{coCh} \to \mathsf{Ab}^{\mathbf{Z}}$ (*cocycles*, *coboundaries*, and *cohomology*) send a cochain complex X to the graded abelian groups whose components in degree n are $\mathsf{Z}^n(X) = \ker \partial^n$, $\mathsf{B}^n(X) = \operatorname{im} \partial^{n-1}$, and $\mathsf{H}^n(X) = \mathsf{Z}^n(X)/\mathsf{B}^n(X)$. The latter groups are known as the *cohomology groups* of X and their elements are known as *cohomology classes*. The values on morphisms are defined analogously to the case of homology in Proposition 16.6. Used in 1.0*, S.0*, 22.21, 23.0*, 23.6, 23.6*, 23.7, 23.9, 24.10.

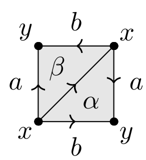
Definition 20.5. Given an abelian group of coefficients A, the functor $\mathsf{C}^*(-, A)\colon \mathsf{sSet}^{\mathsf{op}} \to \mathsf{coCh}$ (known as the (normalized) *simplicial cochains with coefficients* in A, or simply as *simplicial cochains* if $A = \mathbf{Z}$, and sometimes simply as *cochains*) is defined as $(\mathsf{Hom}(-, A) \circ \mathsf{C}(-))^{\mathsf{op}}$. In other words, $C^*(X, A) = \mathsf{Hom}(\mathsf{C}(X), A)$ and $f\colon X \to Y$ is sent to $\mathsf{C}^*(f, A) = \mathsf{Hom}(\mathsf{C}(f), A)\colon \mathsf{C}^*(Y, A) \to \mathsf{C}^*(X, A)$. Used in 20.7, 20.7*, 20.8, 23.0*, 23.1, 23.2, 23.3*, 23.9, 24.2, 24.8, 24.8*, 36.12, 37.1.

Definition 20.6. The functors $\mathsf{Z}^*(-, A), \mathsf{B}^*(-, A), \mathsf{H}^*(-, A)\colon \mathsf{sSet}^{\mathsf{op}} \to \mathsf{Ab}^{\mathbf{Z}}$ are defined by composing the functors $\mathsf{Z}^*, \mathsf{B}^*, \mathsf{H}^*\colon \mathsf{coCh} \to \mathsf{Ab}^{\mathbf{Z}}$ with the functor $\mathsf{C}^*(-, A)\colon \mathsf{sSet}^{\mathsf{op}} \to \mathsf{coCh}$. The functor $\mathsf{H}(-, A)$ is known as the *simplicial cohomology with coefficients* in A. Used in 20.8, 20.13, 20.15, 23.0*, 23.10.

Lemma 20.7. Given a simplicial set X, abelian group A, cochain degree $n \geq 0$, and a simplicial cochain $u \in \mathsf{C}^n(X, A)$, its coboundary $\partial u \in \mathsf{C}^{n+1}(X, A)$ can be computed as $\sigma \mapsto \sum_{0 \leq i \leq n+1} (-1)^i u(\mathrm{d}_i \sigma)$, where σ is a nondegenerate n-simplex of X.

Proof. This follows immediately from the definition of simplicial cochains in Definition 20.5 as the dual of simplicial chains and the definition of the boundary map for simplicial chains in Definition 15.5. ∎

Example 20.8. We compute the simplicial cohomology with coefficients of a real projective plane represented via two 2-simplices:



The simplicial cochains with coefficients in A are, by definition, obtained from simplicial chains with coefficients in A by replacing direct sums with direct products and transposing all matrices of differentials:

$$A_{\langle x \rangle} \times A_{\langle y \rangle} \xrightarrow[\partial^0]{\begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}} A_{\langle a \rangle} \times A_{\langle b \rangle} \times A_{\langle d \rangle} \xrightarrow[\partial^1]{\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}} A_{\langle \alpha \rangle} \times A_{\langle \beta \rangle}.$$

Recall that finite direct sums of abelian groups are isomorphic to finite direct products, so the cochain groups turn out to be isomorphic to chain groups in this case. This is false for infinite simplicial sets. Thus, below we use the notation with direct sums, since in our case all products are finite.

After row-reducing the matrix of ∂^0 we have $\mathsf{H}^0 \cong \mathsf{Z}^0 = \{a \oplus a \mid a \in A\} \cong A$.

Column-reducing the same matrix yields $\mathsf{B}^1 = \{a \oplus a \oplus 0 \mid a \in A\} \cong A$. Row-reducing the matrix of ∂^1 produces $\mathsf{Z}^1 = \{(b - d) \oplus b \oplus d \mid 2d = 0\} \cong A \oplus \mathsf{Tor}(\mathbf{Z}/2, A)$. The quotient map $\mathsf{Z}^1/\mathsf{B}^1 \to \mathsf{Tor}(\mathbf{Z}/2, A)$ sends $(b - d) \oplus b \oplus d \mapsto d$. Indeed, this homomorphism is clearly surjective. Furthermore, its kernel is $\{(b - d) \oplus b \oplus d \mid 2d = 0 \wedge d = 0\} = \{b \oplus b \oplus 0\} = \mathsf{B}^1$, as required. Thus, $\mathsf{H}^1 \cong \mathsf{Tor}(\mathbf{Z}/2, A)$.

Finally, column-reducing the matrix of ∂^1 produces $\mathsf{B}^2 = \{(a + 2d) \oplus (-a) \mid a, d \in A\}$. We have $\mathsf{Z}^2 = \mathsf{C}^2$ and the quotient map is $\mathsf{Z}^2 \to A/2A \cong \mathbf{Z}/2 \otimes_{\mathbf{Z}} A$ ($\alpha \oplus \beta \mapsto [\alpha + \beta] \in A/2A$). Indeed, this map is clearly surjective and its kernel is $\{\alpha \oplus \beta \mid \alpha + \beta \in 2A\} = \{\alpha \oplus \beta \mid \alpha + \beta = 2d\} = \{(2d - \beta) \oplus \beta\} = \mathsf{B}^2$. Thus, $\mathsf{H}^2 \cong A/2A$.

Altogether, $\mathsf{H}^0 \cong A$, $\mathsf{H}^1 \cong \mathsf{Tor}(\mathbf{Z}/2, A)$, $\mathsf{H}^2 \cong \mathbf{Z}/2 \otimes_{\mathbf{Z}} A \cong A/2A$. This looks quite similar to the sequence of homology groups for the same simplicial set that we previously computed: $\mathsf{H}_0 \cong A$, $\mathsf{H}_1 \cong A/2A \cong \mathbf{Z}/2 \otimes_{\mathbf{Z}} A$, $\mathsf{H}_2 \cong \mathsf{Tor}(\mathbf{Z}/2, A)$. Notice how torsion groups moved one degree up, whereas tensor products moved one degree down. That something like this should happen can be easily seen as follows. Consider the chain complex concentrated in degrees 0 and 1 with the differential that multiplies by 2:

$$A \xleftarrow{2} A.$$

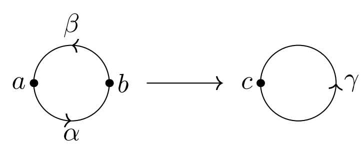
Its homology groups are $A/2A$ and $\mathsf{Tor}(\mathbf{Z}/2, A)$ in degrees 0 and 1. Consider now the dual cochain complex, concentrated in cochain degrees 0 and 1, with the dual differential that also multiplies by 2:

$$A \xrightarrow{2} A.$$

Its cohomology groups are $\mathsf{Tor}(\mathbf{Z}/2, A)$ and $A/2A$ in cochain degrees 0 and 1. This explains the phenomenon of groups moving up and down that we observed above. The universal coefficient theorem, which we will study later, will express this more precisely and allow us to compute cohomology using homology as an input. Used in 23.10.

Remark 20.9. The above example may mislead one into thinking that there is not much benefit to studying cohomology since it seems to compute similar invariants. However, cohomology enjoys vastly superior theoretical properties and admits a much richer set of tools. In particular, the cup product, studied below, is defined in cohomology, not homology. Although there is a formal dual analog in homology, the coproduct of a chain, it is far more esoteric and difficult to study for the same reason that coalgebras are more esoteric than algebras and rings.

Example 20.10. Consider the following simplicial map $f\colon S \to T$ from Example 15.16:



We map $a, b \mapsto c$ and $\alpha, \beta \mapsto \gamma$ so that the source circle wraps around the target circle twice. We compute the induced cochain map as follows:

$$\begin{array}{ccccccc}
A \oplus A & \xrightarrow{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}} & A \oplus A & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots \\
\uparrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} & & \uparrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} & & \uparrow 0 & & \vdots \\
A & \xrightarrow{0} & A & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots
\end{array}$$

Its cohomology is computed as follows. We have $\mathsf{H}^0(S) \cong \mathsf{Z}^0(S) = \{a \oplus a \mid a \in A\} \cong A$. Also $\mathsf{Z}^1(S) \cong \mathsf{C}^1(S)$ and $\mathsf{B}^1(S) = \{\alpha \oplus -\alpha \mid \alpha \in A\}$. The quotient map $\mathsf{Z}^1(S) \to A$ sends $\alpha \oplus \beta \mapsto \alpha + \beta$. Indeed, it is surjective and its kernel is precisely $\mathsf{B}^1(S)$. Thus, $\mathsf{H}^1(S) \cong A$.

We read off the cohomology of T as $\mathsf{H}^0(T) \cong \mathsf{H}^1(T) \cong A$.

The map $\mathsf{C}^0(f)\colon \mathsf{C}^0(T) \to \mathsf{C}^0(S)$ sends $a \mapsto a \oplus a$. The map $\mathsf{Z}^0(f)\colon \mathsf{Z}^0(T) \to \mathsf{Z}^0(S)$ is a restriction of this map, and under the above identifications it becomes $\mathrm{id}_A\colon A \to A$.

To compute $\mathsf{H}^1(f)$, we compose $\mathsf{Z}^1(T) \to \mathsf{Z}^1(S) \to \mathsf{H}^1(S)$, which yields the map $A \to A$ that sends $a \mapsto a + a$, i.e., $a \mapsto 2a$. We summarize this as follows.

$$\begin{array}{cccc}
A & A & 0 & \cdots \\
\uparrow 1 & \uparrow 2 & \uparrow 0 & \vdots \\
A & A & 0 & \cdots
\end{array}$$

Exercise 20.11. Prove that for any family of simplicial sets $\{X_i\}_{i \in I}$ (I is an arbitrary set) we have an isomorphism

$$\mathsf{H}^*\left(\coprod_{i \in I} X_i, A\right) \to \prod_{i \in I} \mathsf{H}^*(X_i, A).$$

Prove that the canonical map

$$A^{\pi_0(X)} \to \mathsf{H}^0(X, A)$$

is an isomorphism. (See Proposition 18.6 for an analogous statement in homology.)

Exercise 20.12. For each of the simplicial sets listed in Exercise 15.13, compute its cohomology with coefficients in an arbitrary abelian group A. For each of the simplicial maps listed in Exercise 15.18, compute the induced map on cohomology with coefficients in an aribtrary abelian group A.

We conclude this section by extending the notions of singular homology and group homology to cohomology.

Definition 20.13. The *singular cohomology* is the composition of the singular simplicial set functor

$$\mathcal{Sing}^{\mathsf{op}} \colon \mathsf{Space}^{\mathsf{op}} \to \mathsf{sSet}^{\mathsf{op}}$$

and the simplicial cohomology functor $\mathsf{H}^*(-, A) \colon \mathsf{sSet}^{\mathsf{op}} \to \mathsf{Ab}^{\mathbf{Z}}$. Thus, singular cohomology is a functor $\mathsf{Space}^{\mathsf{op}} \to \mathsf{Ab}^{\mathbf{Z}}$. As usual, Space can mean any category equipped with a functor $\Delta \to \mathsf{Space}$, but most commonly the category of topological spaces and continuous maps is used. Other important cases include smooth manifolds and smooth maps, as well as more abstract examples, such as the category of small categories and functors, which is important for the nerve construction. Used in 20.13, 20.14, 20.14*.

Except for some cases, singular cohomology is very hard to compute directly.

Exercise 20.14. Suppose Space is the category of metric spaces and continuous maps. Compute the singular cohomology groups of the metric space $\mathbf{R}^n$ for every $n \geq 0$.

Later we will develop powerful tools such as the Mayer–Vietoris sequence and nerve theorem, which will allow us to compute singular cohomology efficiently. As a preview of things to come, we indicate how one could compute the singular cohomology of a 2-sphere using the nerve theorem that we prove below.

Exercise 20.15. Suppose Space is the category of metric spaces and continuous maps. Consider the open cover of the 2-dimensional sphere $S^2 = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ by the six hemispheres centered at each of the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. Compute the simplicial cohomology of the nerve of this open cover.

The following definition lies at the core of modern number theory (Galois cohomology is nothing else than the study of group cohomology of Galois groups).

Definition 20.16. The *cohomology of a group G* is defined as the composition of the functors $\mathsf{B}^{\mathsf{op}} \colon \mathsf{Group}^{\mathsf{op}} \to \mathsf{sSet}^{\mathsf{op}}$ and $\mathsf{H}^* \colon \mathsf{sSet}^{\mathsf{op}} \to \mathsf{Ab}^{\mathbf{Z}}$, applied to the group G. Thus, *group cohomology* is a functor of the form $\mathsf{Group}^{\mathsf{op}} \to \mathsf{Ab}^{\mathbf{Z}}$. Used in 17.23*, 20.15*, 20.17.

Exercise 20.17. Compute the cohomology of the group $\mathbf{Z}/2$.

21 Products and equalizers of simplicial sets

Supplementary sources: Lawvere and Rosebrugh [SETS, §3.3, §3.4]. Aluffi [ZERO, §I.4]. Also see [CATS, §11, §13] for examples.

This sections introduce products and equalizers of simplicial sets. The development is parallel to that of coproducts and coequalizers. This is not a coincidence: products are coproducts in the opposite category.

The first notion, product of simplicial sets, has a very simple geometric interpretation: we assemble two pictures side by side, without intersections, like a disjoint union of sets.

We define products in an arbitrary category C and then instantiate to $\mathsf{C} = \mathsf{sSet}$.

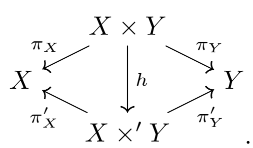
Definition 21.1. The *product* of objects X and Y in a category C (if it exists) is a triple $(X \times Y, \pi_X \colon X \times Y \to X, \pi_Y \colon X \times Y \to Y)$, where $X \times Y \in \mathsf{C}$ and π_X, π_Y are morphisms in C such that the following *universal property of products* is satisfied: for any $Z \in \mathsf{C}$ the map $(\mathtt{hom}(Z, \pi_X), \mathtt{hom}(Z, \pi_Y)) \colon \mathtt{hom}(Z, X \times Y) \to \mathtt{hom}(Z, X) \times \mathtt{hom}(Z, Y)$ that sends $f \colon Z \to X \times Y$ to $(\pi_X \circ f, \pi_Y \circ f)$ is a bijection. Used in 21.0*, 21.7, 21.7*, 21.9, 21.20, 21.21, 26.35, 42.3, 43.7.

Notation 21.2. Given $f \colon Z \to X$ and $g \colon Z \to Y$, the inverse image of (f, g) under the above map is known as the *pairing* of f and g and is denoted by $(f, g) \colon Z \to X \times Y$.

Notation 21.3. Given $f_1 \colon X_1 \to Y_1$ and $f_2 \colon X_2 \to Y_2$, we define the map $f_1 \times f_2 \colon X_1 \times X_2 \to Y_1 \times Y_2$ as $(g_1 \circ \pi_{X_1}, g_2 \circ \pi_{X_2})$.

Informally, we say that a map $h \colon Z \to X \times Y$ is the "same thing" as a pair of maps $f \colon Z \to X$ and $g \colon Z \to Y$. Given h, we recover f and g as $f = \pi_X \circ h$ and $g = \pi_Y \circ h$. Given f and g, we recover h as $h = (f, g)$.

Lemma 21.4. If X and Y are objects in a category C and $(X \times Y, \pi_X \colon X \times Y \to X, \pi_Y \colon X \times Y \to Y)$, $(X \times' Y, \pi'_X \colon X \times' Y \to X, \pi'_Y \colon X \times' Y \to Y)$ are products of X and Y, then there is a unique map $h \colon X \times Y \to X \times' Y$ that makes the following diagram commute:



Furthermore, h is an isomorphism.

Proof. Apply Lemma 13.4 to the category C^{op}. ▮

Remark 21.5. Although products are always unique, their existence depends on a particular choice of C.

Example 21.6. Recall from §59 that products in the category $\mathsf{C} = \mathsf{Set}$ are characterized by the property that $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ are maps of sets such that for any $x \in X$ and $y \in Y$ the set

$$\pi_X^{-1}(\{x\}) \cap \pi_Y^{-1}(\{y\})$$

is a singleton set. Thus, the product of X and Y is simply the set of ordered pairs (x, y), where $x \in X$ and $y \in Y$. The maps π_X and π_Y extract the first respectively second component.

The following proposition is analogous to Proposition 13.7.

Proposition 21.7. In the category sSet, the product of X and Y exists. Used in 21.19, 42.3.

Proof. Define $(X \times Y)_{\mathbf{m}} = X_{\mathbf{m}} \times Y_{\mathbf{m}}$ and $(X \times Y)_f = X_f \times Y_f \colon X_{\mathbf{n}} \times Y_{\mathbf{n}} \to X_{\mathbf{m}} \times Y_{\mathbf{m}}$. We now verify the functoriality property. We have

$$(X \times Y)_{\mathrm{id}_{\mathbf{m}}} = X_{\mathrm{id}_{\mathbf{m}}} \times Y_{\mathrm{id}_{\mathbf{m}}} = \mathrm{id}_{X_{\mathbf{m}} \times Y_{\mathbf{m}}}.$$

Likewise,

$$(X \times Y)_{g \circ f} = X_{g \circ f} \times Y_{g \circ f} = (X_f \circ X_g) \times (Y_f \circ Y_g) = (X_f \times Y_f) \circ (X_g \times Y_g) = (X \times Y)_f \circ (X \times Y)_g,$$

which completes the construction of $X \times Y$.

We construct the simplicial maps $\pi_X: X \times Y \to X$ and $\pi_Y: X \times Y \to Y$ as follows. Set $(\pi_X)_{\mathbf{m}}$ to $\pi_{X_{\mathbf{m}}}: (X \times Y)_{\mathbf{m}} = X_{\mathbf{m}} \times Y_{\mathbf{m}} \to X_{\mathbf{m}}$ and likewise for Y. The naturality property of simplicial maps is verified by the following commutative diagram for any map of simplices $f: \mathbf{m} \to \mathbf{n}$:

$$\begin{array}{ccc} X_{\mathbf{m}} & \xleftarrow{\pi_{X_{\mathbf{m}}}} & X_{\mathbf{m}} \times Y_{\mathbf{m}} \\ {\scriptstyle X_f}\uparrow & & \uparrow{\scriptstyle X_f \times Y_f} \\ X_{\mathbf{n}} & \xleftarrow{\pi_{X_{\mathbf{n}}}} & X_{\mathbf{n}} \times Y_{\mathbf{n}}. \end{array}$$

It remains to show the universal property of products. If $Z \in \mathsf{sSet}$ and $f: X \to Z$, $g: Y \to Z$ are simplicial maps, we must show that there is a unique $h: Z \to X \times Y$ such that $\pi_X \circ h = f$ and $\pi_Y \circ h = g$.

Pick an arbitrary simplex $\mathbf{m}$ and consider the component $\mathbf{m}$ of the above simplicial maps: $(\pi_X)_{\mathbf{m}} \circ h_{\mathbf{m}} = f_{\mathbf{m}}$ and $(\pi_Y)_{\mathbf{m}} \circ h_{\mathbf{m}} = g_{\mathbf{m}}$. By definition, $(\pi_X)_{\mathbf{m}} = \pi_{X_{\mathbf{m}}}: X_{\mathbf{m}} \times Y_{\mathbf{m}} \to X_{\mathbf{m}}$ and likewise for $(\pi_Y)_{\mathbf{m}}$, so by the universal property of products in the category Set, we see that $h_{\mathbf{m}}: Z_{\mathbf{m}} \to X_{\mathbf{m}} \times Y_{\mathbf{m}}$ is forced to be equal to $(f_{\mathbf{m}}, g_{\mathbf{m}})$. Furthermore, such choice of $h_{\mathbf{m}}$ indeed defines a simplicial map $h: Z \to X \times Y$, as one sees by substituting into the naturality property of simplicial maps the definition of $X \times Y$, obtaining the following commutative diagram for any map of simplices $e: \mathbf{m} \to \mathbf{n}$:

$$\begin{array}{ccc} X_{\mathbf{n}} \times Y_{\mathbf{n}} & \xrightarrow{X_e \times Y_e} & X_{\mathbf{m}} \times Y_{\mathbf{m}} \\ \uparrow{\scriptstyle h_{\mathbf{n}}} & & \uparrow{\scriptstyle h_{\mathbf{m}}} \\ Z_{\mathbf{n}} & \xrightarrow{Z_e} & Z_{\mathbf{m}}. \end{array}$$

Indeed, the top-left composition equals $[X_e \circ f_{\mathbf{m}}, Y_e \circ g_{\mathbf{m}}]$ and the bottom-right composition equals $[f_{\mathbf{m}} \circ Z_e, g_{\mathbf{m}} \circ Z_e]$. The two maps coincide by the naturality property of simplicial maps f and g. ▮

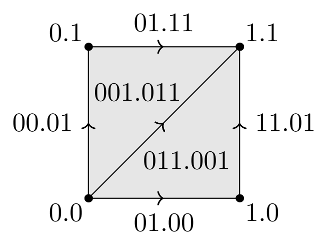
Example 21.8. If $S, T \in \mathsf{Set}$, then $\mathrm{dis}(S) \times \mathrm{dis}(T) \cong \mathrm{dis}(S \times T)$.

Remark 21.9. The product of an arbitrary family $\{X_i\}_{i \in I}$ of objects in C is defined in a completely analogous way, yielding an object $\prod_{i \in I} X_i$ together with morphisms $\pi_i: \prod_{i \in I} X_i \to X_i$.

Example 21.10. Consider the simplicial set $X \times Y$, where $X = Y = \Delta^{\mathbf{1}}$. We have

- $(X \times Y)_0 = X_0 \times Y_0 = \{0, 1\} \times \{0, 1\} = \{0.0, 0.1, 1.0, 1.1\}$;
- $(X \times Y)_1 = X_1 \times Y_1 = \{00, 01, 11\}^2 = \{00.00, 00.01, 00.11, 01.00, 01.01, 01.11, 11.00, 11.01, 11.11\}$;
- $(X \times Y)_2 = X_2 \times Y_2 = \{000, 001, 011, 111\}^2 = \{001.011, 011.001, \ldots\}$.

Each $\mathbf{n}$-simplex is a pair of $\mathbf{n}$-simplices of $\Delta^{\mathbf{1}}$, separated by a period. The simplex before the period is the horizontal projection, whereas the simplex after the period is the vertical projection. The face and degeneracy maps acts on both parts simultaneously. Thus 01.00 denotes the 1-simplex whose vertices are 0.0 and 1.0. In dimension 1, five 1-simplices are nondegenerate and are depicted below, whereas the other four 1-simplices are degenerate and correspond to the four vertices of the square. In dimension 2, two 2-simplices are nondegenerate and are depicted below as triangles, ten 2-simplices are degenerations of five nondegenerate 1-simplices (two different degenerations for each), and four 2-simplices are double degenerations of four vertices. In dimension 3 and higher, all simplices are degenerate. We depict the resulting simplicial set as follows.



Used in 22.8.

Exercise 21.11. Compute the number of nondegenerate simplices in every dimension and draw pictures of the simplicial sets $\Delta^{\mathbf{1}} \times \Delta^{\mathbf{2}}$ and $\Delta^{\mathbf{1}} \times \Delta^{\mathbf{1}} \times \Delta^{\mathbf{1}}$.

Exercise 21.12. Suppose $\alpha = (x, y) \in (X \times Y)_{\mathbf{m}} = X_{\mathbf{m}} \times Y_{\mathbf{m}}$ is an m-simplex of $X \times Y$, where $X, Y \in \mathsf{sSet}$. Prove or disprove:

- If x and y are degenerate, then so is α.
- If x or y is nondegenerate, then so is α.
- If x or y is degenerate, then so is α.

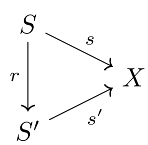
Exercise 21.13. Given two objects $X, Y \in \mathsf{C}$ in an arbitrary category C, construct a canonical morphism $X \sqcup Y \to X \times Y$ using the universal properties of products and coproducts. For the cases $\mathsf{C} = \mathsf{sSet}$ and $\mathsf{C} = \mathsf{Ab}$, prove or disprove that this map is an isomorphism for any $X, Y \in \mathsf{C}$.

The second notion, equalizer of simplicial sets, allows one to "solve equations" with simplicial maps. Given two simplicial maps $f, g\colon X \to Y$, one should think of the equalizer of f and g as a subobject of X where f and g coincide.

Definition 21.14. An *equalizer fork* of morphisms $f, g\colon X \to Y$ in a category C is a morphism $s\colon S \to X$ such that $f \circ s = g \circ s$:

$$S \xrightarrow{\ s\ } X \underset{g}{\overset{f}{\rightrightarrows}} Y.$$

If s and s' are equalizer forks of f and g, then a *morphism of equalizer forks* is a morphism $r\colon S \to S'$ such that the following diagram commutes:



Used in 21.14, 21.15, 26.34.

Definition 21.15. The *equalizer* of morphisms $f, g\colon X \to Y$ in a category C is an equalizer fork $s\colon S \to X$ such that the following *universal property of equalizers* holds: for any equalizer fork $s'\colon S' \to X$ there is a unique morphism of equalizer forks $s' \to s$. Used in 21.0*, 21.13*, 21.18, 21.19, 21.25, 26.34, 26.35.

Notation 21.16. We denote S by $\mathrm{eq}(f, g)$. By abuse of notation, the object S is often used instead of the pair (S, s), especially if the map s is "canonical" or "implied".

Informally, we say that a map $Z \to S$ is the "same thing" as a map $Z \to X$ such that the compositions $Z \to X \to Y$ (for both choices of the map $X \to Y$) are equal.

Remark 21.17. Once again, although equalizers are always unique, they need not exist and existence must be proved separately.

Example 21.18. Recall from §59 that for $\mathsf{C} = \mathsf{Set}$ the equalizer of f and g exists and can be computed as the subset $\{x \in X \mid f(x) = g(x)\}$ of X, with the map $s\colon S \to X$ being the inclusion of sets.

Exercise 21.19. Formulate and prove an existence result for equalizers in the category sSet in analogy with Proposition 21.7 and Proposition 13.19.

Exercise 21.20. For any category C and objects $X, Y \in \mathsf{C}$, construct a canonical map $p_{X,Y}\colon X \times Y \to Y \times X$ using the universal property of products. For the case $\mathsf{C} = \mathsf{sSet}$, compute the equalizer of $p_{X,X}$ and $\mathrm{id}_{X \times X}$, where $X = \mathrm{S}^{\mathbf{1}}$. Draw a picture that illustrates the simplicial sets and simplicial maps involved in the equalizer.

Exercise 21.21. For any category C and object $X \in \mathsf{C}$, construct a canonical map $d_X\colon X \to X \times X$ using the universal property of products. For the case $\mathsf{C} = \mathsf{sSet}$, draw a picture of d_X, where $X = \mathrm{S}^{\mathbf{1}}$.

We finish this section with a definition of a concept related to coequalizers: simplicial subsets.

Definition 21.22. A *simplicial subset* of a simplicial set X is a simplicial set Y such that $Y_{\mathbf{m}} \subset X_{\mathbf{m}}$ and these inclusions form a simplicial map $Y \to X$. Used in 21.21*, 21.24, 21.25, 21.26, 21.27, 21.30, 29.18, 36.1, 36.5, 36.13, 38.1, 38.6, 39.8, 39.10*.

Definition 21.23. A *monomorphism* in a category C is a morphism $f: X \to Y$ such that for any $g, h: W \to X$ with $f \circ g = f \circ h$ we have $g = h$. Used in 21.24, 21.24*, 39.6, 39.6*, 45.1, 45.1*, 45.3, 45.3*.

Proposition 21.24. A simplicial map $f: Y \to X$ arises from a simplicial subset of X if and only if f is a monomorphism.

Proof. It suffices to show that monomorphisms of simplicial sets coincide with degreewise injections. Since equality of simplicial maps can be checked degreewise, degreewise injections are monomorphisms. Vice versa, given a monomorphism $f: X \to Y$, if $f \circ \sigma_1 = f \circ \sigma_2$ for some $\sigma_1, \sigma_2: \Delta^{\mathbf{n}} \to X$, then $\sigma_1 = \sigma_2$, which means that f is a degreewise injection. ▮

Example 21.25. The equalizer $s: S \to X$ of any pair of simplicial maps $f, g: X \to Y$ exhibits S as simplicial subset of X.

Exercise 21.26. Prove that the union and intersection of an arbitrary family of simplicial subsets of a simplicial set X is again a simplicial subset of X.

Definition 21.27. Suppose X is a simplicial set and S is a collection of simplices of X of any dimension. Then the simplicial set $X \setminus S$ obtained by removing all simplices in S from X is defined as the union of all simplicial subsets of X that do not contain any simplices from S.

Example 21.28. For any simplex $\mathbf{m}$, the boundary of $\Delta^{\mathbf{m}}$, denoted by $\partial\Delta^{\mathbf{m}}$, is defined as $\Delta^{\mathbf{m}} \setminus \{\mathrm{id}_{\mathbf{m}}\}$, where $\mathrm{id}_{\mathbf{m}}$ denotes the nondegenerate $\mathbf{m}$-simplex of $\Delta^{\mathbf{m}}$. Used in 15.0*, 21.29, 22.6, 28.9, 36.2, 36.4, 36.9*, 36.10, 39.3, 39.9, 39.9*, 39.10, 39.10*, 43.3, 45.1, 45.5*, 46.1, 46.2*, 46.4*.

Example 21.29. For any simplex $\mathbf{m}$ and any vertex $k \in \mathbf{m}$, the *simplicial horn* $\Lambda^{\mathbf{m}}_k$ is defined as $\partial\Delta^{\mathbf{m}} \setminus \{\mathrm{d}_k(\mathrm{id}_{\mathbf{m}})\}$, i.e., removing the codimension 1 face opposite of vertex k from the simplicial set $\partial\Delta^{\mathbf{m}}$. Used in 39.1, 39.7*, 43.3, 45.1, 45.5*.

Exercise 21.30. Prove or disprove: any simplicial subset $S \subset X$ occurs as the equalizer of some pair of maps $f, g: X \to Y$, where $Y \in \mathsf{sSet}$ can be aribtrary.

22 The Eilenberg–Zilber and Alexander–Whitney maps

Supplementary sources: Mac Lane [Homol, §VIII.8], Dold [LAT, §VII.2], tom Dieck [ATd, §9.7], Spanier [ATs, §5.3].

If $f: X \to \mathbf{R}$ and $g: Y \to \mathbf{R}$ are two real functions on spaces X and Y, then $f \times g: X \times Y \to \mathbf{R}$ defined as $(f \times g)(x, y) = f(x)g(y)$ is a real function on $X \times Y$. If we denote the vector space of real functions on X by $C(X)$, then the above map is a bilinear map $C(X), C(Y) \to C(X \times Y)$, or, equivalently, a linear map $C(X) \otimes C(Y) \to C(X \times Y)$. Similarly, if μ is a measure on X and ν is a measure on Y, then $\mu \times \nu$ is a measure on $X \times Y$, where $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$. Simplicial chains are the homotopy-theoretic analog of measures in analysis, so we can expect to have a linear map of the form $\mathsf{C}(X) \otimes \mathsf{C}(Y) \to \mathsf{C}(X \times Y)$ for any simplicial sets X and Y. Such a map indeed exists and is known as the Eilenberg–Zilber map for simplicial chains.

In analysis, if we complete the tensor product, then there is also a map going in the opposite direction. This analogy extends to homotopy theory, resulting in a map $\mathsf{C}(X \times Y) \to \mathsf{C}(X) \otimes \mathsf{C}(Y)$, known as the Alexander–Whitney map for simplicial chains. Continuing the analogy, the composition of both maps $\mathsf{C}(X) \otimes \mathsf{C}(Y) \to \mathsf{C}(X \times Y) \to \mathsf{C}(X) \otimes \mathsf{C}(Y)$ equals the identity map. The other composition, $\mathsf{C}(X \times Y) \to \mathsf{C}(X) \otimes \mathsf{C}(Y) \to \mathsf{C}(X \times Y)$ is not equal to the identity map, but its homology is the identity map $\mathsf{H}(X \times Y) \to \mathsf{H}(X \times Y)$, which is sufficient for the purposes of homotopy theory.

Definition 22.1. If $C, D \in \mathsf{Ch}$ are chain complexes, then $C \otimes D \in \mathsf{Ch}$ is another chain complex that has a universal property with respect to bichain maps: we have a universal bichain map $C, D \to C \otimes D$ such that composing it with any chain map $C \otimes D \to E$ establishes a bijection between chain maps $C \otimes D \to E$ and bichain maps $C, D \to E$. A *bichain map* $f: C, D \to E$ is a collection of bilinear maps $f_{m,n}: C_m, D_n \to E_{m+n}$ such that $\mathrm{d} f_{m,n}(a, b) = f_{m+1,n}(\mathrm{d}a, b) + (-1)^m f_{m,n+1}(a, \mathrm{d}b)$. Used in 22.1, 22.17, 23.5*, 24.1*, 24.2, 24.3, 24.9*.

Remark 22.2. The factor of $(-1)^m$ is necessary for the differential on the tensor product to square to zero. We will provide extensive additional motivation later. This sign first appeared in the work of Grassmann in the 19th century, and various names are associated with it, such as the *Koszul sign convention.*

Proposition 22.3. The tensor product $C \otimes D$ of any pair of chain complexes exists.

Proof. We exhibit an explicit construction. Set $(C \otimes D)_k = \bigoplus_{m,n:m+n=k} C_m \otimes D_n$. Set $\mathrm{d}(c \otimes d) = (\mathrm{d}c) \otimes d + (-1)^m c \otimes (\mathrm{d}d)$ for any $c \in C_m$ and $d \in D_n$. We verify that d is indeed a differential:

$$\begin{aligned} \mathrm{d}(\mathrm{d}(c \otimes d)) &= \mathrm{d}((\mathrm{d}c) \otimes d + (-1)^m c \otimes (\mathrm{d}d)) \\ &= \mathrm{d}(\mathrm{d}c) \otimes d + (-1)^{m+1} \mathrm{d}c \otimes \mathrm{d}d + (-1)^m \mathrm{d}c \otimes \mathrm{d}d + (-1)^m (-1)^m c \otimes \mathrm{d}(\mathrm{d}d) \\ &= 0 - (-1)^m \mathrm{d}c \otimes \mathrm{d}d + (-1)^m \mathrm{d}c \otimes \mathrm{d}d + 0 = 0. \ \blacksquare \end{aligned}$$

Exercise 22.4. Complete the proof by verifying the universal property.

Example 22.5. Consider the tensor product $\mathsf{C}(X) \otimes \mathsf{C}(Y)$, where $X, Y \in \mathsf{sSet}$. By the above formula, we have

$$(\mathsf{C}(X) \otimes \mathsf{C}(Y))_k = \bigoplus_{m+n=k} \mathsf{C}(X)_m \otimes \mathsf{C}(Y)_n.$$

The tensor product of the free abelian groups on sets A and B is the free abelian group on the set $A \times B$. Since $\mathsf{C}(X)_m$ is the free abelian group on the set of nondegenerate m-simplices of X, and likewise for $\mathsf{C}(Y)_n$, the abelian group $\mathsf{C}(X)_m \otimes \mathsf{C}(Y)_n$ is the free abelian group on the set of pairs (α, β), where $\alpha \in X_m$ and $\beta \in Y_n$ are nondegenerate simplices. Used in 22.6.

Definition 22.6. If $X, Y \in \mathsf{sSet}$ and $A \in \mathsf{Ab}$, then the *Eilenberg–Zilber map for simplicial chains* is a chain map

$$\nabla: \mathsf{C}(X) \otimes \mathsf{C}(Y) \to \mathsf{C}(X \times Y)$$

defined as follows. Recall from Example 22.5 that $\mathsf{C}(X) \otimes \mathsf{C}(Y)$ in degree k is the free abelian group on the set of all pairs (α, β), where $\alpha \in X_m$ and $\beta \in Y_n$ are nondegenerate simplices such that $m + n = k$. The pair (α, β), i.e., simplicial maps $\alpha: \mathbf{\Delta^m} \to X$ and $\beta: \mathbf{\Delta^n} \to Y$, yields a map $\alpha \times \beta: \mathbf{\Delta^m} \times \mathbf{\Delta^n} \to X \times Y$. The value of ∇ on a generator $\alpha \otimes \beta$ is an $(m+n)$-chain $\nabla(\alpha \otimes \beta)$ on $X \times Y$ that we construct by applying the map

$$\mathsf{C}_{m+n}(\alpha \times \beta): \mathsf{C}_{m+n}(\mathbf{\Delta^m} \times \mathbf{\Delta^n}) \to \mathsf{C}_{m+n}(X \times Y)$$

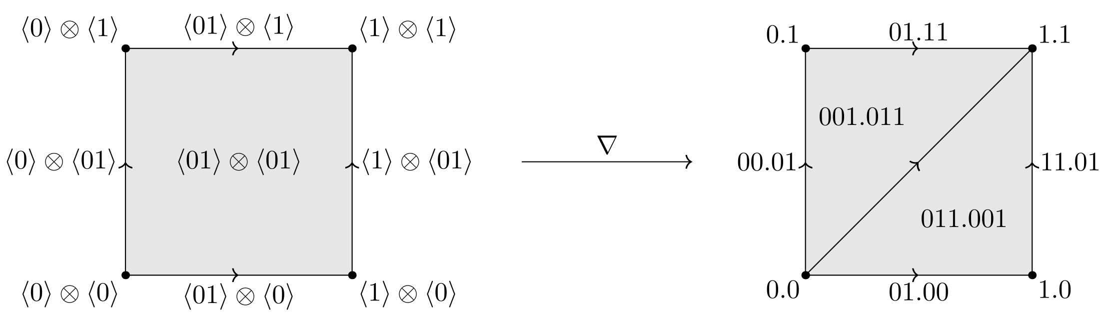
to a certain element σ of $\mathsf{C}(\Delta^{\mathbf{m}} \times \Delta^{\mathbf{n}})$, which we refer to as the *fundamental chain.* The fundamental chain σ is an $(m+n)$-chain on $\Delta^{\mathbf{m}} \times \Delta^{\mathbf{n}}$, and as such it is a formal linear combination of nondegenerate $(m+n)$-simplices of $\Delta^{\mathbf{m}} \times \Delta^{\mathbf{n}}$. In our case, all coefficients are going to be either 1 or -1. The choice of signs is determined almost uniquely by the requirement that $\partial\sigma$ lies in the image of the map

$$\mathsf{C}_{m+n-1}(\partial\Delta^{\mathbf{m}} \times \Delta^{\mathbf{n}} \cup \Delta^{\mathbf{m}} \times \partial\Delta^{\mathbf{n}}) \to \mathsf{C}_{m+n-1}(X \times Y).$$

Geometrically speaking, we require that $\partial\sigma$ has nonzero coefficients only for those $(m+n-1)$-simplices of $\Delta^{\mathbf{m}} \times \Delta^{\mathbf{n}}$ that are contained in the outer boundary, as opposed to the interior. This condition determines σ uniquely up to a sign, which turns out to be uniquely determined by the requirement that ∇ is a chain map. We fix the sign by forcing the coefficient of the simplex (ρ, ρ') to be equal to 1 for $\rho\colon \Delta^{m+n} \to \Delta^{\mathbf{m}}$ $(i \mapsto \min(i, m))$ and $\rho'\colon \Delta^{m+n} \to \Delta^{\mathbf{n}}$ $(j \mapsto \max(0, j-m))$. Geometrically, (ρ, ρ') is adjacent to the bottom face $\Delta^{\mathbf{m}} \times 0$ of $\Delta^{\mathbf{m}} \times \Delta^{\mathbf{n}}$, where $0 \in \Delta^{\mathbf{n}}$ is the initial (bottom) vertex of $\Delta^{\mathbf{n}}$. Used in 22.0*, 22.8, 22.9, 22.12, 22.12*, 22.16, 22.17*, 24.5, 35.7*.

Exercise 22.7. Prove that the fundamental chain σ is uniquely determined by the condition on $\partial\sigma$ and the choice of sign in the above definition. Prove that the coefficient of the fundamental chain $\sigma \in \mathsf{C}_{m+n}(\Delta^{\mathbf{m}} \times \Delta^{\mathbf{n}})$ on an $(m+n)$-simplex $(\tau, \tau') \in (\Delta^{\mathbf{m}} \times \Delta^{\mathbf{n}})_{m+n} = \Delta^{\mathbf{m}}_{m+n} \times \Delta^{\mathbf{n}}_{m+n}$ equals the sign of the shuffle permutation of (τ, τ'), i.e., the permutation of $\{1, \ldots, m+n\}$ whose first m terms enumerate positions i such that $\tau(i-1) < \tau(i)$ and $\tau'(i-1) = \tau'(i)$, whereas the last n terms enumerate positions i such that $\tau(i-1) = \tau(i)$ and $\tau'(i-1) < \tau'(i)$. Recall that the sign of a permutation p of the set $\{1, \ldots, k\}$ equals $(-1)^I$, where I is the number of inversions in p, i.e., the number of pairs (i, j) such that $1 \le i < j \le k$ and $p(i) > p(j)$.

Example 22.8. Consider the case $X = Y = \Delta^{\mathbf{1}}$. The Eilenberg–Zilber map for simplicial chains $\nabla\colon \mathsf{C}(\Delta^{\mathbf{1}}) \otimes \mathsf{C}(\Delta^{\mathbf{1}}) \to \mathsf{C}(\Delta^{\mathbf{1}} \times \Delta^{\mathbf{1}})$ can be visualized as follows.



Here the left side is not a simplicial set. Rather, it is a visualization of the tensor product of the chain complex of simplicial chains of $\Delta^{\mathbf{1}}$ with itself. Recall that $\mathsf{C}(\Delta^{\mathbf{1}})$ is the chain complex

$$\mathbf{Z}_{\langle 0\rangle} \oplus \mathbf{Z}_{\langle 1\rangle} \xleftarrow{-1\oplus 1} \mathbf{Z}_{\langle 01\rangle}.$$

The subscripts denote the nondegenerate simplices in $\Delta^{\mathbf{1}}$ that correspond to the given generators. Thus, $\mathsf{C}(\Delta^{\mathbf{1}}) \otimes \mathsf{C}(\Delta^{\mathbf{1}})$ is the chain complex

$$\mathbf{Z}_{\langle 0\rangle\otimes\langle 0\rangle} \oplus \mathbf{Z}_{\langle 0\rangle\otimes\langle 1\rangle} \oplus \mathbf{Z}_{\langle 1\rangle\otimes\langle 0\rangle} \oplus \mathbf{Z}_{\langle 1\rangle\otimes\langle 1\rangle} \longleftarrow \mathbf{Z}_{\langle 01\rangle\otimes\langle 0\rangle} \oplus \mathbf{Z}_{\langle 01\rangle\otimes\langle 1\rangle} \oplus \mathbf{Z}_{\langle 0\rangle\otimes\langle 01\rangle} \oplus \mathbf{Z}_{\langle 1\rangle\otimes\langle 01\rangle} \longleftarrow \mathbf{Z}_{\langle 01\rangle\otimes\langle 01\rangle}.$$

Here the subscripts indicate pairs of nondegenerate simplices of $\Delta^{\mathbf{1}}$, which geometrically correspond to the horizontal and vertical projections. The four vertices are generators in chain degree 0, the four edges are generators in chain degree 1, and the square is a generator in chain degree 2.

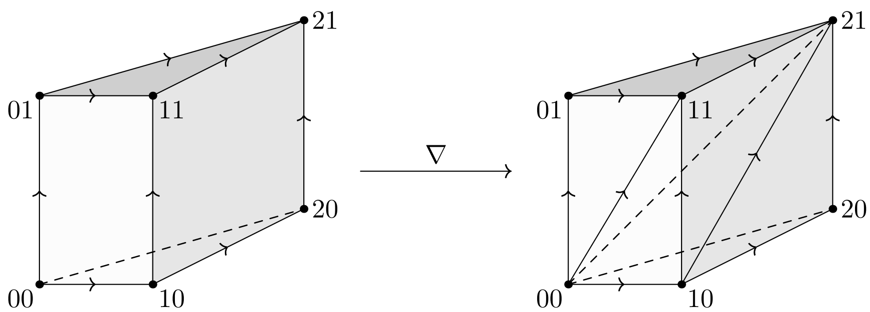
The right side depicts simplicial chains of $\Delta^{\mathbf{1}} \times \Delta^{\mathbf{1}}$. Recall (Example 21.10) that a k-simplex of $\Delta^{\mathbf{1}} \times \Delta^{\mathbf{1}}$ is a pair of k-simplices of $\Delta^{\mathbf{1}}$, which themselves are strings of $k+1$ vertices of $\Delta^{\mathbf{1}}$, i.e., 0 or 1. A pair of such simplices is separated by a period.

We now explain how the map works. We start with the chain degree 0, so $m = n = 0$. All four generators work similarly, so we pick one of them, namely, $\langle 1\rangle \otimes \langle 0\rangle$. Thus $\alpha\colon \Delta^{\mathbf{0}} \to \Delta^{\mathbf{1}}$ picks the vertex 1 and $\beta\colon \Delta^{\mathbf{0}} \to \Delta^{\mathbf{1}}$ picks the vertex 0. The map $\alpha \times \beta\colon \Delta^{\mathbf{0}} \times \Delta^{\mathbf{0}} \cong \Delta^{\mathbf{0}} \to \Delta^{\mathbf{1}} \times \Delta^{\mathbf{1}}$ picks the vertex 1.0. The fundamental chain σ is a 0-chain on $\Delta^{\mathbf{0}} \times \Delta^{\mathbf{0}} \cong \Delta^{\mathbf{0}}$. We have $(\tau, \tau') = (0, 0)$ and the shuffle permutation is the identity permutation, which has sign 1. Thus, σ is the sole vertex of $\Delta^{\mathbf{0}}$ taken with coefficient 1. The map $\alpha \times \beta$ sends this vertex to the vertex 1.0. Thus, the map $\mathsf{C}(\alpha \times \beta)\colon \mathsf{C}(\Delta^{\mathbf{0}}) \to \mathsf{C}(\Delta^{\mathbf{1}} \times \Delta^{\mathbf{1}})$ sends σ to the 0-chain of $\Delta^{\mathbf{1}} \times \Delta^{\mathbf{1}}$ given by the vertex 1.0 taken with coefficient 1. Thus, each generator in degree 0 of the form $\langle i \rangle \otimes \langle j \rangle$ is mapped by ∇ to the generator $i.j$.

We proceed to the chain degree 1. Again, all four cases work similarly, so we take the generator $\langle 1 \rangle \otimes \langle 01 \rangle$. Thus $m = 0$, $n = 1$, $\alpha\colon \Delta^{\mathbf{0}} \to \Delta^{\mathbf{1}}$ picks the vertex 1, and $\beta\colon \Delta^{\mathbf{1}} \to \Delta^{\mathbf{1}}$ picks the 1-simplex 01. The fundamental chain σ is a 1-chain on $\Delta^{\mathbf{0}} \times \Delta^{\mathbf{1}} \cong \Delta^{\mathbf{1}}$. We have $(\tau, \tau') = (00, 01)$ and the shuffle permutation is the identity permutation, which has sign 1. Thus, σ is the sole nondegenerate 1-simplex of $\Delta^{\mathbf{1}}$ taken with coefficient 1. The map $\alpha \times \beta$ sends this 1-simplex to the 1-simplex 11.01. Thus, the map $\mathsf{C}(\alpha \times \beta)\colon \mathsf{C}(\Delta^{\mathbf{1}}) \to \mathsf{C}(\Delta^{\mathbf{1}} \times \Delta^{\mathbf{1}})$ sends σ to the 1-chain of $\Delta^{\mathbf{1}} \times \Delta^{\mathbf{1}}$ given by the 1-simplex 11.01 taken with coefficient 1. Thus, each generator in degree 0 of the form $\langle i \rangle \otimes \langle jk \rangle$ is mapped by ∇ to the generator $ii.jk$ and $\langle ij \rangle \otimes \langle k \rangle$ is mapped to $ij.kk$.

The remaining chain degree is 2. Here $m = n = 1$, $\alpha = \beta = \mathrm{id}\colon \Delta^{\mathbf{1}} \to \Delta^{\mathbf{1}}$. The fundamental chain σ is a 2-chain on $\Delta^{\mathbf{1}} \times \Delta^{\mathbf{1}}$. The 2-simplex (τ, τ') is either $(011, 001)$ or $(001, 011)$. The shuffle permutations are identity and the transposition respectively. Their signs are 1 and -1 respectively. Thus, σ is the 2-chain $011.001 - 001.011$. We have $\alpha \times \beta = \mathrm{id}$, so applying $\mathsf{C}(\alpha \times \beta)$ does nothing. Thus, $\langle 01 \rangle \otimes \langle 01 \rangle$ is mapped by ∇ to the 2-chain $011.001 - 001.011$. An important observation to make here for the future is that the boundary of the latter 2-chain is $(11.01 - 01.01 + 01.00) - (01.11 - 01.01 + 00.01) = 11.01 + 01.00 - 01.11 - 00.01$, in particular, the diagonal 1-simplex annihilates itself. Thus, the boundary of this 2-chain is the outer square.
Used in 22.15.

Example 22.9. Consider the case $X = \Delta^{\mathbf{2}}$, $Y = \Delta^{\mathbf{1}}$. The Eilenberg–Zilber map for simplicial chains $\nabla\colon \mathsf{C}(\Delta^{\mathbf{2}}) \otimes \mathsf{C}(\Delta^{\mathbf{1}}) \to \mathsf{C}(\Delta^{\mathbf{2}} \times \Delta^{\mathbf{1}})$ can be visualized as follows.



On the left, vertices, edges, square and triangle faces, and the prism itself denote various generators of $\mathsf{C}^*(\Delta^{\mathbf{2}}) \otimes \mathsf{C}^*(\Delta^{\mathbf{1}})$. On the right, vertices, edges, triangles, and tetrahedra denote various generators of $\mathsf{C}^*(\Delta^{\mathbf{2}} \times \Delta^{\mathbf{1}})$. The Eilenberg–Zilber map for simplicial chains maps each vertex and edge on the left to the same vertex or edge on the right. It maps each triangle face to the same face on the right, and each square face is mapped to the difference of two triangles inside, as explained in the previous example. Finally, the prism is mapped to the alternating sum of the three nondegenerate 3-simplices of $\Delta^{\mathbf{2}} \times \Delta^{\mathbf{1}}$, namely, 0122.0001, 0112.0011, and 0012.0111. The shuffle permutations for these tetrahedra are $(1, 2, 3)$, $(1, 3, 2)$, and $(2, 3, 1)$ respectively, and their signs are 1, -1, and 1.

Lemma 22.10. The map ∇ satisfies $\nabla d = d \nabla$, i.e., it is indeed a chain map.

Proof. It suffices to verify this identity separately in each degree. As observed in the definition, the chain complex $\mathsf{C}(X) \otimes \mathsf{C}(Y)$ is a free graded abelian group on pairs of nondegenerate simplices of X and Y. Thus, it suffices to verify the identity individually on each such pair $\alpha \otimes \beta$ of simplices in bidegree (m, n). We have $\nabla(d(\alpha \otimes \beta)) = \nabla(\partial\alpha \otimes \beta + (-1)^m \alpha \otimes \partial\beta)$ and $d\nabla(\alpha \otimes \beta)$ can be computed by expanding the definition of simplicial boundary maps. As can be seen from the above examples, the coefficients for simplices in the interior of $\Delta^{\mathbf{m}} \times \Delta^{\mathbf{n}}$ will vanish because of a cancellation that arises from our choice of signs for the coefficients of the fundamental chain. ▮

Exercise 22.11. Complete the proof by verifying that the coefficients on both sides are equal.

We apply the above definition to define Eilenberg–Zilber maps for simplicial chains with coefficients.

Definition 22.12. If $X, Y \in \mathsf{sSet}$ and $A \in \mathsf{Ab}$, then the *Eilenberg–Zilber map for simplicial chains with coefficients* in A and B is the chain map

$$\mathsf{C}(X, A) \otimes \mathsf{C}(Y, B) \to \mathsf{C}(X \times Y, A \otimes B)$$

obtained by tensoring the Eilenberg–Zilber map for simplicial chains with $A \otimes B$ on both sides and using the associativity, commutativity, and distributivity laws on the left side to make it isomorphic to $\mathsf{C}(X, A) \otimes \mathsf{C}(Y, B)$. Used in 22.11*.

We now define a map going in the opposite direction, the Alexander–Whitney map for simplicial chains. This map will be used to define the cup product in the next section. Its definition is somewhat less intuitive than that of the Eilenberg–Zilber map for simplicial chains.

Definition 22.13. If $X, Y \in \mathsf{sSet}$ and $A \in \mathsf{Ab}$, then the *Alexander–Whitney map for simplicial chains* is the map

$$\Delta\colon \mathsf{C}(X \times Y) \to \mathsf{C}(X) \otimes \mathsf{C}(Y)$$

defined as follows. The generators of $\mathsf{C}(X \times Y)$ in degree $\mathbf{m}$ are pairs of simplices $(x \in X_{\mathbf{m}}, y \in Y_{\mathbf{m}})$. The map Δ sends such a pair to

$$\sum_{0 \le i \le m} (\mathrm{d}_{i+1} \cdots \mathrm{d}_m(x)) \otimes (\mathrm{d}_0 \cdots \mathrm{d}_{i-1}(y)).$$

(Some simplices in the latter formula may be degenerate. Our simplicial chains are normalized, which means that the corresponding terms vanish.) Used in 22.0*, 22.12*, 22.16, 22.17, 22.17*, 24.5, 24.8*.

Proposition 22.14. The map Δ is a chain map, i.e., $\partial \circ \Delta = \Delta \circ \partial$.

Proof. Given $c \in \mathsf{C}(X \times Y)$, which we can assume to be given by a single simplex $c = (x, y) \in (X \times Y)_m$, we evaluate both sides

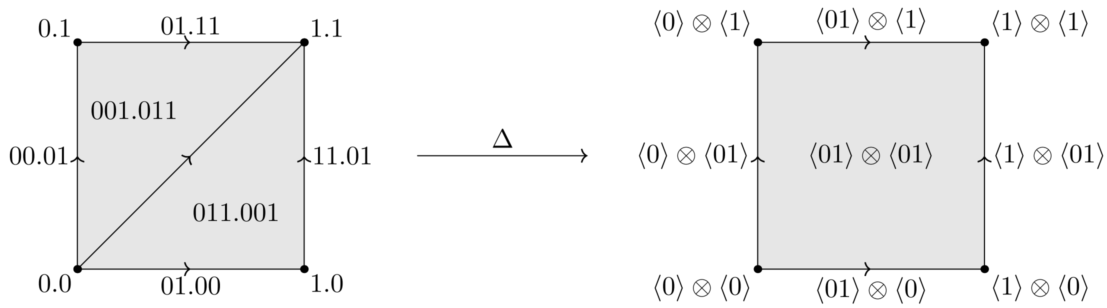
$$\begin{aligned}
\partial(\Delta(c)) &= \partial\left(\sum_{0 \le i \le m} (\mathrm{d}_{i+1} \cdots \mathrm{d}_m(x)) \otimes (\mathrm{d}_0 \cdots \mathrm{d}_{i-1}(y))\right) \\
&= \sum_{0 \le i \le m} \partial((\mathrm{d}_{i+1} \cdots \mathrm{d}_m(x)) \otimes (\mathrm{d}_0 \cdots \mathrm{d}_{i-1}(y))) \\
&= \sum_{0 \le i \le m} \partial(\mathrm{d}_{i+1} \cdots \mathrm{d}_m(x)) \otimes (\mathrm{d}_0 \cdots \mathrm{d}_{i-1}(y)) + (-1)^i (\mathrm{d}_{i+1} \cdots \mathrm{d}_m(x)) \otimes \partial(\mathrm{d}_0 \cdots \mathrm{d}_{i-1}(y)) \\
&= \sum_{0 \le i \le m} \left(\sum_{0 \le j \le i} (-1)^j \mathrm{d}_j \mathrm{d}_{i+1} \cdots \mathrm{d}_m(x)\right) \otimes (\mathrm{d}_0 \cdots \mathrm{d}_{i-1}(y)) \\
&\qquad + (-1)^i (\mathrm{d}_{i+1} \cdots \mathrm{d}_m(x)) \otimes \left(\sum_{0 \le k \le j-i} (-1)^k \mathrm{d}_k \mathrm{d}_0 \cdots \mathrm{d}_{i-1}(y)\right)
\end{aligned}$$

and

$$\begin{aligned}
\Delta(\partial c) &= \Delta\left(\sum_{0 \le l \le m} (-1)^l \mathrm{d}_l c\right) \\
&= \sum_{0 \le i \le m-1} (\mathrm{d}_{i+1} \cdots \mathrm{d}_{m-1}(\partial x)) \otimes (\mathrm{d}_0 \cdots \mathrm{d}_{i-1}(y)) + \sum_{0 \le i \le m-1} (\mathrm{d}_{i+1} \cdots \mathrm{d}_{m-1}(x)) \otimes (\mathrm{d}_0 \cdots \mathrm{d}_{i-1}(\partial y)) \\
&= \sum_{0 \le i \le m-1} \mathrm{d}_{i+1} \cdots \mathrm{d}_{m-1} \left(\sum_{0 \le j \le i} (-1)^j \mathrm{d}_j x\right) \otimes (\mathrm{d}_0 \cdots \mathrm{d}_{i-1}(y)) \\
&\qquad + \sum_{0 \le i \le m-1} (\mathrm{d}_{i+1} \cdots \mathrm{d}_{m-1}(x)) \otimes \mathrm{d}_0 \cdots \mathrm{d}_{i-1} \left(\sum_{0 \le k \le j-i} (-1)^k \mathrm{d}_k y\right)
\end{aligned}$$

Comparing the coefficients on both sides completes the proof. ∎

Example 22.15. Consider the case $X = Y = \Delta^{\mathbf{1}}$, $m = 2$ (see Example 22.8 for details):



We compute

$$\begin{aligned}\Delta(\alpha\cdot 011.001+\beta\cdot 001.011) &= \alpha\cdot\sum_{0\le i\le 2}(\mathrm{d}_{i+1}\cdots\mathrm{d}_2(011))\otimes(\mathrm{d}_0\cdots\mathrm{d}_{i-1}(001))\\ &\quad+\beta\cdot\sum_{0\le i\le 2}(\mathrm{d}_{i+1}\cdots\mathrm{d}_2(001))\otimes(\mathrm{d}_0\cdots\mathrm{d}_{i-1}(011))\\ &=\alpha\cdot(\mathrm{d}_1\mathrm{d}_2(011)\otimes 001+\mathrm{d}_2(011)\otimes\mathrm{d}_0(001)+011\otimes\mathrm{d}_0\mathrm{d}_1(001))\\ &\quad+\beta\cdot(\mathrm{d}_1\mathrm{d}_2(001)\otimes 011+\mathrm{d}_2(001)\otimes\mathrm{d}_0(011)+001\otimes\mathrm{d}_0\mathrm{d}_1(011))\\ &=\alpha\cdot(0\otimes 001+01\otimes 01+011\otimes 1)+\beta\cdot(0\otimes 011+00\otimes 11+001\otimes 1)\\ &=\alpha\cdot(0+01\otimes 01+0)+\beta\cdot(0+0+0)=\alpha\cdot(01\otimes 01).\end{aligned}$$

Most terms vanish because simplicial chains are normalized, so degenerate simplices (represented by strings of digits where two or more consecutive digits repeat) vanish.

Remark 22.16. An astute reader has noticed already that the Alexander–Whitney map for simplicial chains is asymmetric: the expression $\alpha\cdot(\tau\otimes\tau)$ contains α, but does not contain β. This is not a coincidence: one can prove that it is impossible to define a symmetric map with all the desired properties. In contrast, the Eilenberg–Zilber map for simplicial chains is symmetric with respect to the permutation of its arguments, which can be seen directly from the definition.

Definition 22.17. The *Alexander–Whitney map for simplicial cochains* with coefficients in A is a map

$$\mathsf{C}^*(X,A)\otimes\mathsf{C}^*(Y,B)\to\mathsf{C}^*(X\times Y,A\otimes B)$$

obtained by applying the functor $\mathsf{Hom}(-,A\otimes B)$ to the Alexander–Whitney map for simplicial chains, resulting in a map

$$\mathsf{Hom}(\mathsf{C}(X)\otimes\mathsf{C}(Y),A\otimes B)\to\mathsf{C}^*(X\times Y,A\otimes B)$$

and composing it with the map

$$\mathsf{Hom}(\mathsf{C}(X),A)\otimes\mathsf{Hom}(\mathsf{C}(Y),B)\to\mathsf{Hom}(\mathsf{C}(X)\otimes\mathsf{C}(Y),A\otimes B).$$

The latter map is defined using the universal property of tensor products of cochain complexes, with the associated bichain map sending $\varphi\in\mathsf{Hom}(\mathsf{C}(X),A)$ and $\psi\in\mathsf{Hom}(\mathsf{C}(Y),B)$ to the chain map $\mathsf{C}(X)\otimes\mathsf{C}(Y)\to A\otimes B$ whose associated bichain map sends $u\in\mathsf{C}(X)$ and $v\in\mathsf{C}(Y)$ to $(-1)^{|\psi|\cdot|u|}\varphi(u)\otimes\psi(v)$. Used in 22.19, 23.1.

We now examine the interaction between Eilenberg–Zilber maps and Alexander–Whitney maps.

Exercise 22.18. Prove that the composition $\Delta\circ\nabla\colon\mathsf{C}(X)\otimes\mathsf{C}(Y)\to\mathsf{C}(X)\otimes\mathsf{C}(Y)$ equals the identity map.

The other composition, $\nabla\circ\Delta$, is not equal to the identity map. However, as we shall see later, the homology of this map is equal to the identity, which is hardly worse from the viewpoint of homotopy theory. This will follow from the fact that $\nabla\circ\Delta$ is chain homotopic (to be defined later) to id.

We finish this section by explaining how the above maps induce maps on cohomology classes.

Definition 22.19. (Lefschetz, 1942.) Given $X, Y \in \mathsf{sSet}$ and $A \in \mathsf{Ab}$, the *cross product in cohomology* is a collection of homomorphisms of abelian groups

$$\times\colon \mathsf{H}^m(X, A) \otimes \mathsf{H}^n(Y, B) \to \mathsf{H}^{m+n}(X \times Y, A \otimes B)$$

(one for each $m, n \in \mathbf{Z}$) induced by the Alexander–Whitney map for simplicial cochains. Used in 23.7*.

Lemma 22.20. Cross-product is well-defined.

Proof. We apply Lemma 22.21 to the map $\times$. ∎

Lemma 22.21. If $C, D, E \in \mathsf{Ch}$, then any chain map $f\colon C \otimes D \to E$ induces a map

$$\mathsf{H}(C) \otimes \mathsf{H}(D) \to \mathsf{H}(E).$$

The same is true for cochain complexes and cohomology. Used in 22.20*, 23.6*, 24.6*, 24.7, 24.9.

Proof. The desired map is the composition of the map

$$\mathsf{H}(f)\colon \mathsf{H}(C \otimes D) \to \mathsf{H}(E)$$

with the map

$$\mathsf{H}(C) \otimes \mathsf{H}(D) \to \mathsf{H}(C \otimes D)$$

induced by the graded bilinear map

$$\mathsf{H}(C), \mathsf{H}(D) \to \mathsf{H}(C \otimes D)$$

constructed as follows. Given $c \in \mathsf{Z}_m(C)$ and $d \in \mathsf{Z}_n(D)$, the tensor product $c \otimes d \in C_m \otimes D_n$ is a cycle in $C \otimes D$ because $\partial(c \otimes d) = \partial c \otimes d + (-1)^c c \otimes \partial d = 0 \otimes d + (-1)^c c \otimes 0 = 0$. It remains to show that the above map on cycles factors through the quotient map to homology groups. This means that if u and v are replaced by homologous cycles u' and v', then their tensor product is also replaced by a homologous cycle, i.e., $u' \otimes v' - u \otimes v$ is a boundary. Indeed,

$$\begin{aligned} u' \otimes v' &= (u + (u' - u)) \otimes (v + (v' - v)) \\ &= (u + \partial x) \otimes (v + \partial y) \\ &= u \otimes v + u \otimes \partial y + \partial x \otimes v + \partial x \otimes \partial y \\ &= u \otimes v + \partial((-1)^{|u|} u \otimes y + x \otimes v + x \otimes \partial y). \end{aligned}$$

Thus, $u' \otimes v' - u \otimes v$ is a boundary, as required. ∎

Example 22.22. Conside $X = Y = \mathrm{S}^1$. Recall that $\mathsf{C}^*(\mathrm{S}^1) = \mathbf{Z} \xleftarrow{0} \mathbf{Z}$. Thus we have

$$\mathsf{C}^*(\mathrm{S}^1) \otimes \mathsf{C}^*(\mathrm{S}^1) = \mathbf{Z} \xleftarrow{0} \mathbf{Z} \oplus \mathbf{Z} \xleftarrow{0} \mathbf{Z}.$$

Next, in Example 15.11 we compute the simplicial chains of $\boldsymbol{\Delta}^1 \times \boldsymbol{\Delta}^1$, so the cochains We compute the cross product on the cohomology of a torus, denoted by X. Due to the bilinearity property of cross product, it suffices to compute the cross product on some set of generators of cohomology groups. Recall that $\mathsf{H}^0(X) \cong \mathbf{Z}$, $\mathsf{H}^1(X) \cong \mathbf{Z} \oplus \mathbf{Z}$, and $\mathsf{H}^2(X) \cong \mathbf{Z}$.

Exercise 22.23. For each of the simplicial sets listed in Exercise 15.13, compute the cross product on cohomology with coefficients in $\mathbf{Z}$. More precisely, if X is a simplicial set, compute the maps $\mathsf{H}^m(X) \otimes \mathsf{H}^n(X) \to \mathsf{H}^{m+n}(X \times X)$ by determining their values on some set of generators of cohomology groups.

23 Cup product

Supplementary sources: Hatcher [ATh, §3.2, §3.B], tom Dieck [ATd, §17.6].

We have the following table of analogies:

topology	calculus
simplicial set	smooth manifold
simplicial 0-chains	densities (or measures)
simplicial n-chains	n-currents with values in densities
simplicial 0-cochains	real-valued functions
simplicial n-cochains	differential n-forms
pairing of 0-chains and 0-cochains	integration of functions with respect to measures
pairing of n-chains and n-cochains	integration of forms with respect to currents
coboundary	de Rham differential
n-cocycles	closed n-forms
n-coboundaries	exact n-forms
nth cohomology group	nth de Rham cohomology group
cup product of 0-cochains	product of real-valued functions
cup product of cochains	exterior product of differential forms

Our goal in this section is to explain the bottom entry, the cup product.

Definition 23.1. (Alexander, 1935; Kolmogoroff, 1936; Čech, 1936; Whitney, 1938.) Given a simplicial set X, the *cup product* on simplicial cochains of X is the chain map given by the composition

$$\cup\colon \mathsf{C}^*(X)\otimes\mathsf{C}^*(X)\xrightarrow{\Delta^*_{X,X}}\mathsf{C}^*(X\times X)\xrightarrow{\mathsf{C}^*(d)}\mathsf{C}^*(X),$$

where $\Delta^*_{X,X}$ is the Alexander–Whitney map for simplicial cochains and $d\colon X\to X\times X$ is the diagonal map. More generally, suppose A is a ring, i.e., an abelian group equpped with a bilinear operation of multiplication $A\otimes A\to A$ and a unit element $1\in A$ that are associatve and unital, i.e., $x\cdot(y\cdot z)=(x\cdot y)\cdot z$ and $1\cdot x=x\cdot 1=x$. Then we can define *cup product with coefficients* in A as the composition

$$\cup\colon \mathsf{C}^*(X,A)\otimes\mathsf{C}^*(X,A)\xrightarrow{\Delta^*_{X,X}}\mathsf{C}^*(X\times X,A\otimes A)\xrightarrow{\mathsf{C}^*(d,A\otimes A)}\mathsf{C}^*(X,A\otimes A)\xrightarrow{\mathsf{C}^*(X,\mu)}\mathsf{C}^*(X,A),$$

where the last map is induced by the homomorphism of abelian groups $\mu\colon A\otimes A\to A$ given by the multiplication map. We recover the original cup product when $A=\mathbf{Z}$ is the ring of integer numbers. Used in 20.9, 22.12*, 23.0*, 23.2, 23.2*, 23.3*, 23.5, 23.7, 23.9, 23.10, 24.11.

Lemma 23.2. Suppose X is a simplicial set and $u\in\mathsf{C}^m(X)$ and $v\in\mathsf{C}^n(X)$ are simplicial cochains on X. The cup product $u\cup v\in\mathsf{C}^{m+n}(X)$ is a simplicial cochain on X whose value on an $(m+n)$-simplex $\alpha\in X_{m+n}$ can be computed as follows:

$$(u\cup v)(\alpha)=u(\alpha_{0,\ldots m})v(\alpha_{m,\ldots,m+n})=u(\mathrm{d}_{m+1}\cdots\mathrm{d}_{m+n}\alpha)v(\mathrm{d}_0\cdots\mathrm{d}_{m-1}\alpha).$$

Used in 23.3*, 23.8*, 23.9.

Proof. By definition of the cup product,

$$\begin{aligned}(u\cup v)(\alpha)&=(\mathsf{C}^*(X,\mu)(\mathsf{C}^*(d,A\otimes A)(\Delta^*_{X,X}(u\otimes v))))(\alpha)\\&=\mu(\mathsf{C}^*(d,A\otimes A)(\Delta^*_{X,X}(u\otimes v))(\alpha))\\&=\mu((\Delta^*_{X,X}(u\otimes v))(\mathsf{C}(d)(\alpha)))\\&=\mu((\Delta^*_{X,X}(u\otimes v))(\alpha,\alpha))\\&=\mu((u\otimes v)(\Delta_{X,X}(\alpha\otimes\alpha)))\\&=\mu(u(\mathrm{d}_{m+1}\cdots\mathrm{d}_{m+n}(\alpha))\otimes v(\mathrm{d}_0\cdots\mathrm{d}_{m-1}(\alpha)))\\&=u(\mathrm{d}_{m+1}\cdots\mathrm{d}_{m+n}(\alpha))v(\mathrm{d}_0\cdots\mathrm{d}_{m-1}(\alpha)).\ \blacksquare\end{aligned}$$

Proposition 23.3. The structure of a graded A-module on $\mathsf{H}^*(X, A)$ induced by the cochain map

$$A \otimes \mathsf{C}^*(X, A) \cong \mathsf{C}^*(X, A \otimes A) \xrightarrow{\mathsf{C}^*(X,\mu_A)} \mathsf{C}^*(X, A)$$

coincides with the structure of a graded A-module induced by the cochain map

$$A \otimes \mathsf{C}^*(X, A) \to \mathsf{C}^0(X, A) \otimes \mathsf{C}^*(X, A) \xrightarrow{\cup} \mathsf{C}^*(X, A),$$

where the *constant 0-cochain* map $A \cong \mathsf{C}^0(\Delta^{\mathbf{0}}, A) \to \mathsf{C}^0(X, A)$ is induced by the terminal map $X \to \Delta^{\mathbf{0}}$. Used in 23.5, 23.5*.

Proof. The image of $a \in A$ unded the map $A \to \mathsf{C}^0(X, A)$ is the constant simplicial cochain that maps every 0-simplex of X to a. Using Lemma 23.2, we compute (with $m = 0$) $(u \cup v)(\alpha) = u(\mathrm{d}_1 \cdots \mathrm{d}_n \alpha)v(\alpha) = av(\alpha)$, so $u \cup v = av$, as desired. ▮

We axiomatize the properties of cup products in the following definition.

Definition 23.4. A *differential graded ring* is a triple (R, μ, u), where $R \in \mathsf{Ch}$ is a cochain complex (or chain complex), $\mu\colon R \otimes R \to R$ is the multiplication map, and $u\colon \mathbf{Z}[0] \to R$ is the unit map such that multiplication is associative

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \qquad x \in R_p, y \in R_q, z \in R_r$$

and unital

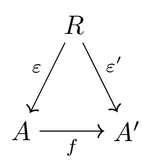
$$1 \cdot x = x \cdot 1 = x, \qquad x \in R_p.$$

A *morphism of differential graded rings* $f\colon (R, \mu, u) \to (R', \mu', u')$ is a cochain map $f\colon R \to R'$ that preserves multiplication and units:

$$f(1) = 1, \qquad f(x \cdot y) = f(x) \cdot f(y), \qquad x \in R_p, y \in R_q.$$

The *category of differential graded rings* DGR has objects and morphisms as above. The *category of graded rings* GR is defined like DGR, but with graded abelian groups instead of cochain complexes.

A *differential graded algebra* A over a ring R is a morphism of differential graded rings $\varepsilon\colon R[0] \to A$ (known as the unit map), where $R[0]$ is the differential graded ring that has R in degree 0 and zero groups in all other degrees. Abusing notation, differential graded algebras are often denoted by A, with the morphism $R[0] \to A$ being implied. A *morphism of differential graded algebras* $f\colon A \to A'$ is a morphism of differential graded rings $f\colon A \to A'$ that commutes with the unit maps:



The *category of differential graded algebras* DGA_R over a ring R has objects and morphisms as above. The *category of graded algebras* GA_R is defined like DGA_R, but with graded abelian groups instead of cochain complexes. Used in 23.4, 23.5, 23.6, 23.6*, 23.7.

Proposition 23.5. For any simplicial set X and ring A, the cup product turns $\mathsf{C}^*(X, A)$ into a differential graded algebra over the ring A with the multiplication map given by the cup product and the unit map given by the constant 0-cochain. Furthermore, this construction yields a functor

$$\mathsf{C}^*\colon \mathsf{sSet}^{\mathsf{op}} \times \mathsf{Ring} \to \mathsf{DGR}$$

that sends a simplicial set X and a ring A to the differential graded ring $\mathsf{C}^*(X, A)$.

Proof. This means that we have a multiplication map

$$\cup\colon \mathsf{C}^*(X, A) \otimes \mathsf{C}^*(X, A) \to \mathsf{C}^*(X, A)$$

and a unit map

$$1\colon A \to \mathsf{C}^0(X, A)$$

(abusing notation, we will often write a instead of $1(a)$) such that the multiplication is associative

$$(u \cup v) \cup w = u \cup (v \cup w)$$

and unital

$$1(a) \cup u = au, \qquad u \cup 1(a) = ua.$$

Furthermore, the Leibniz identity is satisfied

$$\mathrm{d}(u \cup v) = (\mathrm{d}u) \cup v + (-1)^u u \cup \mathrm{d}v$$

and

$$\mathrm{d}(1(a)) = 0.$$

The unit map $1\colon A \to \mathsf{C}^0(X, A)$ sends $a \in A$ to the constant 0-cochain on X, as in Proposition 23.3, where we prove its properties. For associativity observe that both $(u \cup v) \cup w$ and $\cup(v \cup w)$ attain the same value on an given simplex α, namely,

$$u(\mathrm{d}_{|u|+1} \cdots \mathrm{d}_{|u|+|v|+|w|}\alpha)v(\mathrm{d}_0 \cdots \mathrm{d}_{|u|-1}\mathrm{d}_{|u|+|v|+1} \cdots \mathrm{d}_{|u|+|v|+|w|}\alpha)w(\mathrm{d}_0 \cdots \mathrm{d}_{|u|+|v|-1}\alpha).$$

The Leibniz rule follows imediately from the formula for the differential on a tensor product of cochain complexes. ▮

Proposition 23.6. We have induced cohomology functors

$$\mathsf{H}^*\colon \mathsf{DGR} \to \mathsf{GR}$$

and

$$\mathsf{H}^*\colon \mathsf{DGA}_A \to \mathsf{GA}_A.$$

Thus, the cohomology of a differential graded ring is a graded ring and likewise for algebras.

Proof. Given $A \in \mathsf{DGR}$, Lemma 22.21 shows that the multiplication map

$$A \otimes A \to A$$

induces

$$\mathsf{H}^*(A) \otimes \mathsf{H}^*(A) \to \mathsf{H}^*(A).$$

As usual, denote the quotient map by

$$[-]\colon A \to \mathsf{H}^*(A).$$

By definition of the multiplication on $\mathsf{H}^*(A)$, we have $[a][b] = [ab]$, so

$$([a][b])[c] = [ab][c] = [(ab)c] = [a(bc)] = [a][bc] = [a]([b][c]).$$

Likewise,

$$[a][1] = [a1] = [a] = [1a] = [1][a],$$

so the multiplication on $\mathsf{H}^*(A)$ is associative and unital. Definition 20.4 shows that H^* yields a functor $\mathsf{coCh} \to \mathsf{Ab}^{\mathbf{Z}}$. It remains to show that this construction induces a functor $\mathsf{DGR} \to \mathsf{GR}$, which boils down to showing that $\mathsf{H}^*(f)$ preserves multiplication and units for any morphism $f\colon A \to A'$ in DGR. By definition, $\mathsf{H}^*(f)([a]) = [f(a)]$. Thus,

$$f([a][b]) = f([ab]) = [f(ab)] = [f(a)f(b)] = [f(a)][f(b)]$$

and

$$f([1]) = [f(1)] = [1]. \blacksquare$$

Proposition 23.7. Suppose A is an aribtrary ring, such as $A = \mathbf{Z}$. The cup product descends to cohomology, yielding a morphism of graded abelian groups

$$\cup\colon \mathsf{H}^*(X, A) \otimes \mathsf{H}^*(X, A) \to \mathsf{H}^*(X, A).$$

This operation turns $\mathsf{H}^*(X, A)$ into a graded ring, known as the *cohomology ring* of X with coefficients in A. As usual, if $A = \mathbf{Z}$, we omit the coefficients. Furthermore, we have a functor

$$\mathsf{H}^*\colon \mathsf{sSet}^{\mathsf{op}} \times \mathsf{Ring} \to \mathsf{DGR}$$

that sends a simplicial set X and a ring A to the cohomology ring $\mathsf{H}^*(X, A)$. Used in 23.7, 23.9.

Proof. The desired map is the composition of the cross product in cohomology

$$\times\colon \mathsf{H}^*(X, A) \otimes \mathsf{H}^*(X, A) \to \mathsf{H}^*(X \times X, A \otimes A),$$

the map on cohomology induced by the diagonal map $d\colon X \to X \times X$:

$$d^*\colon \mathsf{H}^*(X \times X, A \otimes A) \to \mathsf{H}^*(X, A \otimes A),$$

and the map

$$\mathsf{H}^*(X, A \otimes A) \to \mathsf{H}^*(X, A)$$

induced by the multiplication homomorphism $A \otimes A \to A$. $\blacksquare$

Proposition 23.8. The canonical maps

$$\mathsf{C}^*\left(\coprod_{i \in I} X_i, A\right) \to \prod_{i \in I} \mathsf{C}^*(X_i, A)$$

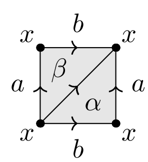
and

$$\mathsf{H}^*\left(\coprod_{i \in I} X_i, A\right) \to \prod_{i \in I} \mathsf{H}^*(X_i, A)$$

are isomorphisms of differential graded rings respectively graded rings.

Proof. Proposition 18.4 and its analog for cohomology already establish isomorphisms of cochain complexes respectively graded abelian groups. It remains to show that these isomorphisms preserve multiplications, which follows from Lemma 23.2: the simplices used on the right side belong to the same connected component as the simplex α. $\blacksquare$

Example 23.9. We compute the *cohomology ring of a torus*:



The cohomology groups with coefficients in a ring A are $\mathsf{H}^0 \cong A$, $\mathsf{H}^1 \cong A \oplus A$, $\mathsf{H}^2 \cong A$. Recall that a cohomology class is an equivalence class of simplicial cochains, modulo the equivalence relation that identifies two cochains when their difference is a coboundary. In order to compute the cup product of two cohomology classes, we choose cochain representatives for cohomology classes, multiply them using the above explicit formula, and then take the cohomology class of the resulting cochain. As we proved above, the result is independent of any choices of representatives that we made.

In order to choose such representatives, we have to construct sections of quotient maps $q^n\colon \mathsf{Z}^n \to \mathsf{H}^n$, i.e., we have to construct maps of sets $s^n\colon \mathsf{U}(\mathsf{H}^n) \to \mathsf{U}(\mathsf{Z}^n)$ such that $\mathsf{U}(q^n) \circ s^n = \mathrm{id}_{\mathsf{H}^n}$. In many cases s^n will be a homomorphism of groups, but in some examples (like the real projective plane) s^n cannot be a homomorphism.

The quotient map $q^0\colon \mathsf{Z}^0 \to \mathsf{H}^0$ is the identity map $A \to A$, so $s^0\colon \mathsf{H}^0 \to \mathsf{Z}^0$ must also be the identity map $A \to A$. This map sends $u \in A$ to the simplicial cochain $u \cdot x^* \in \mathsf{Z}^0$ (i.e., the cochain whose value on x equals u).

We have $\mathsf{B}^1 = 0$, so Z^1 is isomorphic to H^1, and the quotient map (in this case, an isomorphism) $q^1\colon \mathsf{Z}^1 \to \mathsf{H}^1$ is the map $\{v_a \oplus v_b \oplus v_d \in A \oplus A \oplus A \mid v_d = v_a + v_b\} \to A \oplus A$ that sends $v_a \oplus v_b \oplus v_d \mapsto v_a \oplus v_b$. Its inverse is the map $s^1\colon \mathsf{H}^1 \to \mathsf{Z}^1$ that sends $v_a \oplus v_b \mapsto v_a \oplus v_b \oplus (v_a + v_b) = v_a \cdot a^* + v_b \cdot b^* + (v_a + v_b) \cdot d^*$, the simplicial cochain that takes values v_a, v_b, and $v_a + v_b$ on a, b, and d respectively.

Finally, $\mathsf{Z}^2 = \mathsf{C}^2 = A \oplus A$ and $\mathsf{B}^2 = \{v_\alpha \oplus v_\beta \in A \oplus A \mid v_\alpha = v_\beta\}$. The quotient map $q^2\colon \mathsf{Z}^2 \to \mathsf{H}^2 = A$ sends $v_\alpha \oplus v_\beta \mapsto v_\alpha - v_\beta$. Its inverse map $s^2\colon \mathsf{H}^2 \to \mathsf{Z}^2$ $(w \mapsto s_\alpha(w) \oplus s_\beta(w))$ must satisfy $w = q^2(s^2(w)) = q^2(s^2_\alpha(w) \oplus s^2_\beta(w)) = s^2_\alpha(w) - s^2_\beta(w)$. Thus, $s^2_\alpha(w) = s^2_\beta(w) + w$, and any pair (s^2_α, s^2_β) that satisfies this condition will give us a section. We take $s^2_\beta(w) = 0$, so $s^2_\alpha(w) = w$ and $s^2(w) = w \oplus 0 = w \cdot \alpha^*$.

Recall now the formula for cup products form Lemma 23.2:

$$(u \cup v)(\gamma) = u(\mathrm{d}_{m+1} \cdots \mathrm{d}_{m+n}\gamma)v(\mathrm{d}_0 \cdots \mathrm{d}_{m-1}\gamma).$$

If p_i and p'_i denotes a cochain of degree i and r_i denotes a nondegenerate simplex of dimension i, then by specializing the above formula we get the following formulas for cup products:

$$(p_0 \cup p'_0)(r_0) = p_0(r_0)p'_0(r_0),$$

$$(p_0 \cup p'_1)(r_1) = p_0(\mathrm{d}_1 r_1)p'_1(r_1), \qquad (p_1 \cup p'_0)(r_1) = p_1(r_1)p'_0(\mathrm{d}_0 r_1),$$
$$(p_0 \cup p'_2)(r_2) = p_0(\mathrm{d}_1\mathrm{d}_2 r_2)p'_2(r_2), \qquad (p_2 \cup p'_0)(r_2) = p_2(r_2)p'_0(\mathrm{d}_0\mathrm{d}_1 r_2),$$
$$(p_1 \cup p'_1)(r_2) = p_1(\mathrm{d}_2 r_2)p'_1(\mathrm{d}_0 r_2).$$

(The other cup products will take values in cochains of degree 3 or higher, which are all zero.) The values of various simplicial operators on nondegenerate 1- and 2-simplices are as follows. First, there is a single vertex, so we automatically have

$$\mathrm{d}_0 r_1 = \mathrm{d}_1 r_1 = x, \qquad \mathrm{d}_1\mathrm{d}_2 r_2 = \mathrm{d}_0\mathrm{d}_1 r_2 = x$$

for any choice of r_1 and r_2. The remaining values are as follows:

$$\mathrm{d}_2\alpha = b, \qquad \mathrm{d}_0\alpha = a, \qquad \mathrm{d}_2\beta = a, \qquad \mathrm{d}_0\beta = b.$$

Substituting these values into the above formulas, we get

$$(p_0 \cup p'_0)(x) = p_0(x)p'_0(x),$$

$$(p_0 \cup p'_1)(a) = p_0(x)p'_1(a), \qquad (p_1 \cup p'_0)(a) = p_1(a)p'_0(x),$$
$$(p_0 \cup p'_1)(b) = p_0(x)p'_1(b), \qquad (p_1 \cup p'_0)(b) = p_1(b)p'_0(x),$$
$$(p_0 \cup p'_1)(d) = p_0(x)p'_1(d), \qquad (p_1 \cup p'_0)(d) = p_1(d)p'_0(x),$$
$$(p_0 \cup p'_2)(\alpha) = p_0(x)p'_2(\alpha), \qquad (p_2 \cup p'_0)(\alpha) = p_2(\alpha)p'_0(x),$$
$$(p_0 \cup p'_2)(\beta) = p_0(x)p'_2(\beta), \qquad (p_2 \cup p'_0)(\beta) = p_2(\beta)p'_0(x),$$
$$(p_1 \cup p'_1)(\alpha) = p_1(b)p'_1(a).$$
$$(p_1 \cup p'_1)(\beta) = p_1(a)p'_1(b).$$

By specializing the above formulas to the chosen representatives of cohomology classes, we compute their cup products. First, if one of the arguments has degree 0, then the cup product simply multiplies coefficients:

$$\begin{aligned} (u\cdot x^*)\cup(u'\cdot x^*) &= uu'\cdot x^*,\\ (u\cdot x^*)\cup(v_a\cdot a^*+v_b\cdot b^*+(v_a+v_b)\cdot d^*) &= (uv_a\cdot a^*+uv_b\cdot b^*+u(v_a+v_b)\cdot d^*),\\ (v_a\cdot a^*+v_b\cdot b^*+(v_a+v_b)\cdot d^*)\cup(u\cdot x^*) &= (v_a u\cdot a^*+v_b u\cdot b^*+(v_a+v_b)u\cdot d^*),\\ (u\cdot x^*)\cup(w\cdot\alpha^*) &= uw\cdot\alpha^*,\\ (w\cdot\alpha^*)\cup(u\cdot x^*) &= wu\cdot\alpha^*. \end{aligned}$$

Otherwise, if one of the arguments has degree 2, then the total degree will be greater than 2, hence the cup product vanishes. Finally, if both arguments have degree 1, we get

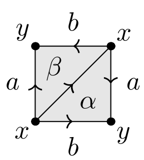
$$(v_a\cdot a^*+v_b\cdot b^*+(v_a+v_b)\cdot d^*)\cdot(v'_a\cdot a^*+v'_b\cdot b^*+(v'_a+v'_b)\cdot d^*) = v_b v'_a\cdot\alpha^*+v_a v'_b\cdot\beta^*.$$

Passing to generating cohomology classes, we compute

$$\begin{array}{ll} [x^*]\cup[x^*]=[x^*], & \\ [x^*]\cup[a^*+d^*]=[a^*+d^*], & [x^*]\cup[b^*+d^*]=[b^*+d^*],\\ [a^*+d^*]\cup[x^*]=[a^*+d^*], & [b^*+d^*]\cup[x^*]=[b^*+d^*],\\ [x^*]\cup[\alpha^*]=[\alpha^*], & [\alpha^*]\cup[x^*]=[\alpha^*],\\ [a^*+d^*]\cup[a^*+d^*]=0, & [a^*+d^*]\cup[b^*+d^*]=[\beta^*]=-[\alpha^*],\\ [b^*+d^*]\cup[a^*+d^*]=[\alpha^*], & [b^*+d^*]\cup[b^*+d^*]=0. \end{array}$$

We can express the result of these computations concisely by saying that the cohomology ring $\mathsf{H}^*(X,A)$ is $A[s,t]/(s^2=0,t^2=0,st=-ts)$. Here $s=[a^*+d^*]$, $t=[b^*+d^*]$, and $st=-[\alpha^*]$. Used in 24.10.

Example 23.10. Recall from Example 20.8 the simplicial cohomology with coefficients in A (assumed to be a ring here) of a real projective plane:



We have $\mathsf{H}^0\cong A$, $\mathsf{H}^1\cong\mathsf{Tor}(\mathbf{Z}/2,A)$, $\mathsf{H}^2\cong\mathbf{Z}/2\otimes A\cong A/2A$. The map $A\to\mathsf{Z}^0$ sends $u\in A$ to $u\cdot(x^*+y^*)$. The map $\mathsf{Tor}(\mathbf{Z}/2,A)\to\mathsf{Z}^1$ sends $v\in\mathsf{Tor}(\mathbf{Z}/2,A)\subset A$ to $v\cdot(a^*+d^*)$. The map $A/2A\to\mathsf{Z}^2$ sends $w\in A/2A$ to $w'\cdot\alpha^*$, where $w'\in A$ is any element such that $[w']=w$ in $A/2A$. The last map is not a homomorphism of groups, which is to be expected: in general, it is not possible to construct a homomorphism $\mathsf{H}^n\to\mathsf{Z}^n$ such that the composition $\mathsf{H}^n\to\mathsf{Z}^n\to\mathsf{H}^n$ is equal to the identity map, i.e., we cannot choose representatives for cohomology or homology classes in a linear way. We compute the cup products of all generators. As before, if one of the generators has degree 0, then we simply multiply the coefficients of the other cochain by the corresponding element of A. For dimension reasons, the only remaining nonvanishing cup product is in bidegree $(1,1)$. If $f,g\in\mathsf{C}^1$, then $f\cup g=f(d)g(a)\cdot\alpha^*+f(d)g(b)\cdot\beta^*$. Thus, $v\cdot(a^*+d^*)\cup v'\cdot(a^*+d^*)=vv'\cdot\alpha^*+(v\cdot 0)\cdot\beta^*=vv'\cdot\alpha^*$. Thus, the cohomology ring is $A[0]\oplus\mathsf{Tor}(\mathbf{Z}/2,A)[1]\oplus A/2A[2]$, where the products that involve A use the multiplication on A, the product in bidegree $(1,1)$ is induced by the multiplication on A, and the other products are zero. Here $B[m]$ denotes the graded abelian group whose component in degree m is B and all other components are zero. In particular, specializing to the ring $A=\mathbf{Z}/2\mathbf{Z}$, we get $\mathsf{Tor}(\mathbf{Z}/2,A)=\mathbf{Z}/2$ and $A/2A=\mathbf{Z}/2$, so the entire ring can be expressed concisely as $\mathbf{Z}/2[c]/(c^3)$, where $c=[a^*+d^*]$.

Exercise 23.11. For each of the simplicial sets listed in Exercise 15.13, compute the cohomology ring with coefficients in $\mathbf{Z}$. More precisely, compute the cohomology groups, choose some set of generators, and compute the cup product of each pair of generators, expressing it as a linear combination of generators.

24 Cap product

Supplementary sources: tom Dieck [ATd, §18.1], Hatcher [ATh, §3.B].

Cap products are the analog of fiberwise integration of functions. Before we define cap products, we pause briefly to define the adjoint counterpart of the tensor product functor for chain complexes.

Proposition 24.1. Up to an isomorphism, there is a unique functor $\mathsf{Hom}\colon \mathsf{Ch}^{\mathsf{op}} \times \mathsf{Ch} \to \mathsf{Ch}$ equipped for each $Y \in \mathsf{Ch}$ with a natural map $\mathsf{Hom}(Y, Z) \otimes Y \to Z$ (the *evaluation map*) with the following universal property: chain maps of the form $X \to \mathsf{Hom}(Y, Z)$ are in a natural bijection with chain maps of the form $X \otimes Y \to Z$, constructed as follows. Given a chain map $X \to \mathsf{Hom}(Y, Z)$, we tensor it with Y, obtaining a map $X \otimes Y \to \mathsf{Hom}(Y, Z) \otimes Y$, which we compose with the evaluation map $\mathsf{Hom}(Y, Z) \otimes Y \to Z$.

Proof. We set $\mathsf{Hom}(Y, Z)_n = \prod_k \mathsf{Hom}(Y_k, Z_{k+n})$, where the right side uses the internal hom of abelian groups. The differential $\mathrm{d}_n\colon \mathsf{Hom}(Y, Z)_n \to \mathsf{Hom}(Y, Z)_{n-1}$ sends $f \in \mathsf{Hom}(Y, Z)_n$ to $\mathrm{d} \circ f - (-1)^n f \circ \mathrm{d}$. We verify that $\mathrm{d}_{n-1} \circ \mathrm{d}_n = 0$. Indeed,

$$\mathrm{d}(\mathrm{d}f) = \mathrm{d}(\mathrm{d} \circ f - (-1)^n f \circ \mathrm{d}) = \mathrm{d} \circ \mathrm{d} \circ f - (-1)^f \mathrm{d} \circ f \circ \mathrm{d} + (-1)^f \mathrm{d} \circ f \circ \mathrm{d} - f \circ \mathrm{d} \circ \mathrm{d} = 0.$$

The evaluation map

$$\mathsf{Hom}(Y, Z) \otimes Y \to Z$$

is induced by the bichain map

$$\mathsf{Hom}(Y, Z), Y \to Z$$

that sends f, y to $f(y)$ for any $f \in \mathsf{Hom}(Y, Z)_m$ and $y \in Y_n$. The universal property now follows from the same universal property for tensor products and internal homs of abelian groups. ∎

Definition 24.2. The *Kronecker pairing* of simplicial cochains and simplicial chains with coefficients in a ring A on a simplicial set X is the chain map

$$\langle -, - \rangle\colon \mathsf{C}^*(X, A) \otimes \mathsf{C}(X, A) \to A$$

such that the induced bichain map sends f, x to $f(x)$ if f and x have the same degree, and to 0 otherwise. Here we turn simplicial cochains into a chain complex by replacing all degrees with their additive inverses, so that $\mathsf{C}^*(X, A)$ lives in nonpositive chain degrees. Used in 24.8*.

Lemma 24.3. The above map is indeed a bichain map.

Proof. The target is a chain complex concentrated in degree 0. For degree reasons, it suffices to show that 0-boundaries in $\mathsf{C}^*(X, A) \otimes \mathsf{C}(X, A)$ are sent to zero by the pairing. We have

$$\langle \mathrm{d}(f \otimes x) \rangle = \langle (\partial^* f) \otimes x + (-1)^f f \otimes \partial x \rangle = (\partial^* f)(x) + (-1)^f f(\partial x) = -(-1)^f f(\partial x) + (-1)^f f(\partial x) = 0.$$ ∎

Proposition 24.4. The pairing between cochains and chains descends to the level of cohomology and homology, producing a pairing

$$\mathsf{H}^*(X, A) \otimes \mathsf{H}(X, A) \to A,$$

i.e., a collection of pairings

$$\mathsf{H}^m(X, A) \otimes \mathsf{H}_n(X, A) \to A,$$

which vanish if $m \neq n$.

Proof. Suppose $f \in \mathsf{Z}^m(X, A)$ and $x \in \mathsf{Z}_n(X, A)$. We have to prove that $f(x) = f(x + \partial y) = (f + \partial^* g)(x)$ for any $y \in \mathsf{C}_{n+1}(X, A)$ and $g \in \mathsf{C}^{m-1}(X, A)$. Indeed,

$$f(x + \partial y) = f(x) + f(\partial y) = f(x) + (\partial^* f)(y) = f(x) + 0 = f(x)$$

because $f \in \mathsf{Z}^m(X, A)$, i.e., $\partial^* f = 0$. Likewise,

$$(f + \partial^* g)(x) = f(x) + g(\partial x) = f(x) + g(0) = f(x).$$ ∎

Definition 24.5. (Steenrod, 1953.) Given a coefficient ring A and simplicial sets X and Y, we define the *slant product for simplicial cochains* as the composition

$$\backslash\colon \mathsf{C}^*(X,A)\otimes\mathsf{C}(Y\times X,A)\to\mathsf{C}^*(X,A)\otimes\mathsf{C}(Y,A)\otimes\mathsf{C}(X,A)\to\mathsf{C}(Y,A),$$

where the first arrow uses the Alexander–Whitney map for simplicial chains and the second arrow uses the pairing on simplicial chains and cochains of X. In analysis, the analog of this construction produces a measure $\mu=(p_Y)_*(\nu\cdot(f\circ p_X))$ on Y from a function f on X and a measure ν on $Y\times X$, where $p_X\colon Y\times X\to X$ and $p_Y\colon Y\times X\to Y$ are projections.

The *slant product for simplicial chains* is the cochain map

$$/\colon \mathsf{C}^*(X\times Y,A)\otimes\mathsf{C}(X,A)\to\mathsf{C}^*(Y,A),$$

defined as the composition

$$\begin{aligned}\mathsf{C}^*(X\times Y,A)\otimes\mathsf{C}(X,A)&=\mathsf{Hom}(\mathsf{C}(X\times Y),A)\otimes\mathsf{C}(X)\otimes A\\&\to\mathsf{Hom}(\mathsf{C}(X)\otimes\mathsf{C}(Y),A)\otimes\mathsf{C}(X)\otimes A\\&\to\mathsf{Hom}(\mathsf{C}(Y),A)\otimes A\to\mathsf{C}^*(Y,A),\end{aligned}$$

where the first arrow uses the Eilenberg–Zilber map for simplicial chains, the second arrow is adjoint to the evaluation map, and the last arrows uses the multiplication on A. This product is analogous to the fiberwise integration of a function on $X\times Y$ with respect to a measure on X, the result being a function on Y. Used in 24.6, 24.7.

Proposition 24.6. Given a coefficient ring A and simplicial sets X and Y, the slant product for simplicial cochains is a chain map

$$\backslash\colon \mathsf{C}^*(X,A)\otimes\mathsf{C}(Y\times X,A)\to\mathsf{C}(Y,A),$$

that induces a map on homology groups:

$$\backslash\colon \mathsf{H}^m(X,A)\otimes\mathsf{H}_n(Y\times X,A)\to\mathsf{H}_{n-m}(Y,A).$$

Proof. We apply Lemma 22.21 to the map \\. ∎

Definition 24.7. (Čech, 1936; Whitney, 1938.) Given a coefficient ring A and a simplicial sets X, the *cap product* with coefficients in a ring A on a simplicial set X is the map

$$\cap\colon \mathsf{C}^m(X,A)\otimes\mathsf{C}_n(X,A)\to\mathsf{C}_{n-m}(X,A)$$

defined as the composition

$$\mathsf{C}^m(X,A)\otimes\mathsf{C}_n(X,A)\to\mathsf{C}^m(X,A)\otimes\mathsf{C}_n(X\times X,A)\to\mathsf{C}_{n-m}(X,A).$$

The first map is $\mathsf{C}^m(X,A)\otimes\mathsf{C}_n(d,A)$, where $d\colon X\to X\times X$ is the diagonal map. The second map is the slant product for simplicial cochains with $X=Y$. As usual, by Lemma 22.21, we have an induced cap product on (co)homology groups:

$$\cap\colon \mathsf{H}^m(X,A)\otimes\mathsf{H}_n(X,A)\to\mathsf{H}_{n-m}(X,A)$$

Used in 24.0*, 24.7, 24.8, 24.9, 24.10, 24.12.

Lemma 24.8. Suppose X is a simplicial set and $u\in\mathsf{C}^m(X)$ and $v\in\mathsf{C}_n(X)$ are a simplicial cochain and simplicial chain on X, where (abusing notation) $v\in X_n$ is a single simplex in X. The cap product $u\cap v\in\mathsf{C}_{n-m}(X)$ is the simplicial chain

$$u(v_{n-m,\dots,n})v_{0,\dots,n-m},$$

where $v_{0,\ldots,n-m}$ and $v_{n-m,\ldots,n}$ denote the $n-m$- and m-simplices of X given by the first $n-m$ and the last m vertices of v. Used in 24.9*.

Proof. The simplicial cochains of the diagonal map $d\colon X \to X \times X$ send the singleton chain $v \in X_n$ on X to the singleton chain $(v,v) \in (X \times X)_n = X_n \times X_n$ on $X \times X$. The Alexander–Whitney map for simplicial chains sends the singleton chain (v,v) to

$$\sum_{0\le i\le n} (\mathrm{d}_{i+1}\cdots \mathrm{d}_n(v)) \otimes (\mathrm{d}_0 \cdots \mathrm{d}_{i-1}(v)).$$

The composition of the above two maps sends $u \otimes v$ to

$$\sum_{0\le i\le n} u \otimes (\mathrm{d}_{i+1}\cdots \mathrm{d}_n(v)) \otimes (\mathrm{d}_0 \cdots \mathrm{d}_{i-1}(v)).$$

The map K induced by the Kronecker pairing now pairs the first and the third factor. If their dimensions are different, i.e., $m \ne n-i$, then the pairing is zero. Otherwise (if $m = n-i$) the pairing simply evaluates the cochain on the chain. Thus only a single term in the sum (namely, the one with $i = n-m$) is nonvanishing. We get

$$\begin{aligned} K\left(\sum_{0\le i\le n} u \otimes (\mathrm{d}_{i+1}\cdots \mathrm{d}_n(v)) \otimes (\mathrm{d}_0 \cdots \mathrm{d}_{i-1}(v))\right) &= K(u \otimes (\mathrm{d}_{n-m+1}\cdots \mathrm{d}_n(v)) \otimes (\mathrm{d}_0 \cdots \mathrm{d}_{n-m-1}(v))) \\ &= (\mathrm{d}_{n-m+1}\cdots \mathrm{d}_n(v)) \cdot u(\mathrm{d}_0 \cdots \mathrm{d}_{n-m-1}(v)). \end{aligned}$$

The last expression is the same as the one in the statement: $\mathrm{d}_{n-m+1}\cdots \mathrm{d}_n$ removes the last m vertices, leaving the first $n-m$ vertices of v, and $\mathrm{d}_0 \cdots \mathrm{d}_{n-m-1}$ leaves the first $n-m$ vertices of v. ∎

Proposition 24.9. For any simplicial set X and ring A, the cap product turns $\mathsf{C}(X,A)$ into a differential graded module over the differential graded ring $\mathsf{C}^*(X,A)$. This means that we have a multiplication map

$$\cap\colon \mathsf{C}^*(X,A) \otimes \mathsf{C}(X,A) \to \mathsf{C}(X,A)$$

such that the multiplication is associative

$$(u \cup v) \cap w = u \cap (v \cap w)$$

and unital

$$1 \cap u = u.$$

Furthermore, the Leibniz identity is satisfied

$$\mathrm{d}(u \cap v) = (\mathrm{d}u) \cap v + (-1)^u u \cap \mathrm{d}v$$

and

$$\mathrm{d}1 = 0.$$

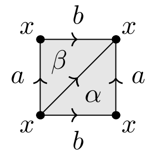
Thus, $\mathsf{H}(X,A)$ is a graded module over the graded ring $\mathsf{H}^*(X,A)$ by Lemma 22.21.

Proof. To verify associativity, we can assume that w is a singleton chain and use Lemma 24.8. Thus, $(u \cup v) \cap w$ and $u \cap (v \cap w)$ are the same simplicial chain, namely,

$$u(w_{|w|-|v|-|u|,\ldots,|w|-|v|})v(w_{|w|-|v|,\ldots,|w|})w_{0,\ldots,|w|-|v|-|u|}.$$

Unitality follows from the definition of the unit cochain and Lemma 24.8. The Leibniz rule follows imediately from the formula for the differential on a tensor product of chain complexes. ∎

Example 24.10. Consider the homology and cohomology of a torus:



The homology groups with coefficients in a ring A are $\mathsf{H}_0 \cong A$, $\mathsf{H}_1 \cong A \oplus A$, $\mathsf{H}_2 \cong A$, with representatives $[x]$; $[a]$, $[b]$; $[\alpha - \beta]$. The cohomology groups with coefficients in a ring A are $\mathsf{H}^0 \cong A$, $\mathsf{H}^1 \cong A \oplus A$, $\mathsf{H}^2 \cong A$, with representatives $[x^*]$; $[a^* + d^*]$, $[-b^* - d^*]$; $[\alpha^*]$. For any chain c we have $[x^*] \cap c = c$ because $m = 0$ and $\mathrm{d}_1 \cdots \mathrm{d}_n(v) = x$ in the formula for cap product. Next,

$$[a^* + d^*] \cap [a] = [x], \qquad [a^* + d^*] \cap [b] = 0, \qquad [-b^* - d^*] \cap [a] = 0, \qquad [-b^* - d^*] \cap [b] = -[x].$$

Also,

$$[a^* + d^*] \cap [\alpha - \beta] = [b], \qquad [-b^* - d^*] \cap [\alpha - \beta] = [a].$$

Finally,

$$[\alpha^*] \cap [\alpha - \beta] = [x].$$

The cohomology ring of a torus was computed in Example 23.9 as $A[s,t]/(s^2 = 0, t^2 = 0, st = -ts)$, where $s = [a^* + d^*]$, $t = [-b^* - d^*]$. The above computation identifies the structure of a module over this ring on the homology of a torus. Namely, H is the free graded module on a single generator λ in degree 2. Here λ is the fundamental class $[\alpha - \beta]$ (to be defined later).

Observation 24.11. If we fix the second argument of the cup product in the previous example to $[\alpha - \beta]$, we get maps $\mathsf{H}^i \to \mathsf{H}_{2-i}$ that act as follows:

$$[x^*] \mapsto [\alpha - \beta], \qquad [a^* + d^*] \mapsto [b], \qquad [-b^* - d^*] \mapsto [a], \qquad [\alpha^*] \mapsto [x].$$

Thus, all these maps are isomorphisms. This is an instance of Poincaré duality. The special chain $[\alpha - \beta]$ is known as the fundamental class. In the above example, we formulated the Poincaré duality by stating that $\mathsf{H}(X, A)$ is a free graded module over the graded ring $\mathsf{H}^*(X, A)$ on a single element in degree 2 (namely, the fundamental class).

Exercise 24.12. For each of the simplicial sets listed in Exercise 15.13, compute the homology module over the cohomology ring with coefficients in an arbitrary ring A. More precisely, compute the homology and cohomology groups, choose some set of generators for both, compute the cup product of each pair of cohomological generators, expressing it as a linear combination of cohomological generators, and then compute the cap product of each cohomological and homological generator, expressing it as a linear combination of homological generators.

25 Brouwer fixed point theorem and degrees of maps

We will need a result whose validity will be established later using machinery that we have not developed yet.

Definition 25.1. A metric space (or a topological space) X is *contractible* if there is a point $b \in X$ such that there is a continuous map $h\colon \mathbf{R}\times X \to X$ (or $h\colon [0,1]\times X \to X$) with $h(0,x)=x$ and $h(1,x)=b$ for all points $x \in X$. Used in 25.2.

Theorem 25.2. (The nerve theorem.) Suppose $\{U_i\}$ is an open cover of a metric or topological space X such that every finite intersection $U_{i_0}\cap\cdots\cap U_{i_k}$ (for any $k \ge 0$) is either empty or a contractible space. Here I is a totally ordered set. Then there is a simplicial map $T_{X,U}$ from the nerve of the open cover U to the singular simplicial set of X:

$$T_{X,U}\colon \mathrm{N}(X,U) \to \mathsf{Sing}(X)$$

that is a simplicial weak equivalence (to be defined later). In particular, applying the functors H or H^* to this map produces isomorphisms.

Example 25.3. Consider the sphere $S^n = \{x \in \mathbf{R}^{n+1} \mid \|x\| = 1\}$. The open cover $\{U_i^+, U_i^-\}$ is defined as $U_i^+ = \{x \in S^n \mid x_i > 0\}$ and $U_i^- = \{x \in S^n \mid x_i < 0\}$. Since $U_i^+ \cap U_i^- = \emptyset$, we never have to consider intersections that contain both U_i^+ and U_i^-. Accordingly, a nondegenerate k-simplex of $\mathrm{N}(S^n,U)$ is an injective map of simplices $\mathbf{k}\to\mathbf{n}$ (whose image consists of those i for which one of $U_i^\pm$ is in the intersection) together with a map of sets $\mathbf{k}\to\{+,-\}$, indicating whether U_i^+ or U_i^- is taken. In particular, for a nondegenerate n-simplex the map $\mathbf{n}\to\mathbf{n}$ must be an isomorphism, hence there are exactly 2^{n+1} nondegenerate n-simplices corresponding to the 2^{n+1} maps $\mathbf{n}\to\{+,-\}$. For example, if $n=2$, we have eight nondegenerate 2-simplices $+++$, $++-$, $+-+$, $+--$, $-++$, $-+-$, $--+$, $---$. Likewise, there are $(n+1)2^n$ nondegenerate $(n-1)$-simplices, because there are $n+1$ ways to choose an injective map $[n-1]\to[n]$, and 2^n ways to choose a map $[n-1]\to\{+,-\}$. We represent nondegenerate n-simplices by a sequence of $n+1$ elements of $\{+,-,*\}$, with exactly one "$*$". For example, if $n=2$, we have twelve nondegenerate 1-simplices $*++$, $*+-$, $*-+$, $*--$, $+*+$, $+*-$, $-*+$, $-*-$, $++*$, $+-*$, $-+*$, $--*$. Observe now that in the simplicial chains of the resulting simplicial set the nth boundary map is a map of the form $A^{2^{n+1}} \to A^{(n+1)2^n}$. The differential is easy to describe: the domain is a direct sum of copies of A indexed by a sequence of $n+1$ signs; the ith face map replaces the ith sign by "$*$". Thus $\partial(+-+) = (*-+) - (+*+) + (+-*)$, for example. Accordingly, the coefficient indexed by a nondegenerate $(n-1)$-simplex given by a sequence of $n+1$ elements of $\{+,-,*\}$ with exactly one "$*$" will be a linear combination (with identical signs) of the coefficients of two nondegenerate n-simplices given by replacing "$*$" with either "$+$" or "$-$". For example, $(\partial c)_{-*+} = -(c_{-++}+c_{--+})$. Thus, an n-cycle is a chain c such that $c_\alpha = -c_\beta$, where α and β differ in exactly one position. For example, $c_{+++} = -c_{++-} = c_{+--} = -c_{+-+} = c_{--+} = -c_{---} = c_{-+-} = -c_{-++}$. Thus, $c_\alpha = (-1)^{\alpha_+}a$, where α_+ denotes the number of "$+$" in α and $a \in A$ is some element of A. Accordingly, $\mathsf{H}_n(S^n) \cong \mathsf{Z}_n(S^n) \cong A$.

Theorem 25.4. (The nerve theorem, relative version.) Suppose (X,U) is a space with an open cover as in the previous theorem, and (Y,V) is another pair with the same property. Suppose $f\colon X\to Y$ is a continuous map and $g\colon I\to J$ is a map of indexing sets with the following property: $f(U_i)\subset V_{g(i)}$ for any $i\in I$. Then g induces a simplicial map $\mathrm{N}(f,g)\colon \mathrm{N}(X,U)\to\mathrm{N}(Y,V)$ and the simplicial maps $T_{X,U}\colon \mathrm{N}(X,U)\to\mathsf{Sing}(X)$ and $T_{Y,V}\colon \mathrm{N}(Y,V)\to\mathsf{Sing}(Y)$ can be chosen in such a way that the following square commutes:

$$\begin{array}{ccc} \mathrm{N}(X,U) & \xrightarrow{\mathrm{N}(f,g)} & \mathrm{N}(Y,V) \\ \big\downarrow{\scriptstyle T_{X,U}} & & \big\downarrow{\scriptstyle T_{Y,V}} \\ \mathsf{Sing}(X) & \xrightarrow{\mathsf{Sing}(f)} & \mathsf{Sing}(Y) \end{array}$$

Example 25.5. Consider the inclusion $\iota\colon S^n\to D^{n+1}$, where $S^n = \{x\in\mathbf{R}^{n+1}\mid \|x\|=1\}$ and $D^{n+1} = \{x\in\mathbf{R}^{n+1}\mid \|x\|\le 1\}$. We cover S^n by $\{U_i^\pm\}$ defined in the pervious example. We cover D^{n+1} by a singleton open cover consisting of D^{n+1} itself, which is contractible. The nerve $\mathrm{N}(D^{n+1},\{D^{n+1}\})\cong\Delta^{\mathbf{0}}$ is a single vertex. The map $\mathrm{N}(\iota,g)$ (where $g\colon[n]\times\{+,-\}\to\{*\}$ is the unique map of indexing sets) is the terminal map $\mathrm{N}(S^n,U)\to\Delta^{\mathbf{0}}$. Its nth homology $\mathsf{H}_n(\mathrm{N}(S^n,U),A)\cong A\to\mathsf{H}_n(\mathrm{N}(D^{n+1},\{D^{n+1}\}),A)\cong 0$ is the same map as the map $\mathsf{H}_n(\iota, A)\colon \mathsf{H}_n(S^n, A) \to \mathsf{H}_n(D^{n+1}, A)$ (here H denotes the singular homology functor), hence the latter map is the zero map $A \to 0$.

Definition 25.6. Suppose C is a category and $X \in \mathsf{C}$. We say that a morphism $s\colon X \to Y$ admits a *retraction* if there is a morphism $r\colon Y \to X$ such that $r \circ s = \mathrm{id}_X$. Used in 25.7.

Lemma 25.7. If a morphism s in a category C admits a retraction, then so does $F(s)$, where $F\colon \mathsf{C} \to \mathsf{D}$ is an arbitrary functor.

Proof. If $r \circ s = \mathrm{id}_X$, then by functoriality, $F(r) \circ F(s) = F(r \circ s) = F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$. ▮

Corollary 25.8. Suppose $f\colon X \to Y$ is a simplicial map such that for some $n \geq 0$ the homomorphism of abelian groups $\mathsf{H}_n(f)\colon \mathsf{H}_n(X) \to \mathsf{H}_n(Y)$ does not admit a retraction, i.e., there is no homomorphism $h\colon \mathsf{H}_n(Y) \to \mathsf{H}_n(X)$ such that $h \circ \mathsf{H}_n(f) = \mathrm{id}_{\mathsf{H}_n(X)}$. Then f does not admit a retraction either, i.e., there is no map $g\colon Y \to X$ such that $g \circ f = \mathrm{id}_X$.

Proof. Holds by the previous lemma because $\mathsf{H}_n\colon \mathsf{sSet} \to \mathsf{Ab}$ is a functor. ▮

This result is typically applied when $\mathsf{H}_n(Y) \cong 0$, while $\mathsf{H}_n(X)$ is nontrivial. The most important example is $\mathsf{H}_n(X) = \mathbf{Z}$, which occurs for the inclusion map $S^n \to D^{n+1}$, as established above.

Theorem 25.9. (L. E. J. Brouwer, 1912.) Any continuous map $f\colon D^n \to D^n$ has a fixed point, i.e., there is $x \in D^n$ such that $f(x) = x$.

Proof. Suppose not. Consider the map $g\colon D^n \to S^{n-1}$ that sends a point $x \in D^n$ to the point $y \in S^{n-1}$ given by the intersection of S^{n-1} with the open ray that originates at $f(x)$ and passes through $x \neq f(x)$. This map is continuous because $f(x) \neq x$ for all $x \in D^n$. The restriction of g to S^{n-1} is the identity map by construction. Thus, g is a retraction of $S^{n-1} \to D^n$, which is impossible. ▮

Exercise 25.10. Consider the sphere $S^2 = \{x \in \mathbf{R}^3 \mid \|x\| = 1\}$. We identify x with $-x$ for any $x \in S^2$. Use the nerve theorem to compute the singular homology of the resulting space Q. Use the relative nerve theorem to compute the map on singular homology induced by the quotient map $S^2 \to Q$. Does the map $S^2 \to Q$ admit a retraction? Bonus question: what is Q?

Exercise 25.11. Consider the disk $D^3 = \{x \in \mathbf{R}^3 \mid \|x\| \leq 1\}$. We identify x with $-x$ for any $x \in S^2 \subset D^3$. Use the nerve theorem to compute the singular homology of the resulting space Q. Use the relative nerve theorem to compute the map on singular homology induced by the quotient map $D^3 \to Q$. Bonus question: what is Q?

Exercise 25.12. Suppose $R \subset D^n \times D^n$ is a closed subset such that for any $x \in D^n$ the set $S_x = \{y \in D^n \mid (x, y) \in R\}$ is a nonempty convex subset of $\mathbf{R}^n$ (meaning that for any two points in this subset, the line segment between these points is contained in the subset). Prove or disprove: there is $x \in D^n$ such that $(x, x) \in R$ (equivalently, $x \in S_x$).

We conclude this section by showing that dimension is a well-defined invariant of metric or topological spaces.

Proposition 25.13. (Brouwer's invariance of dimension theorem, 1912.) If S^m is homeomorphic to S^n, then $m = n$. If $\mathbf{R}^m$ is homeomorphic to $\mathbf{R}^n$, then $m = n$.

Proof. If $f\colon S^m \to S^n$ is a homeomorphism, then $f'\colon S^m \setminus \{*\} \to S^n \setminus \{f(*)\}$ is also a homeomorphism. Thus, $f''\colon S^m \setminus \{*, **\} \to S^n \setminus \{f(*), f(**)\}$ is also a homeomorphism. As shown above, the singular homology of this map can be computed using nervesA as the zero map if $m \neq n$, which makes it impossible for f to be a homeomorphism. Thus, $m = n$. ▮

The fundamental groupoid

26 Limits and colimits of simplicial sets

First, we extend the notion of a diagram to accommodate more complicated examples:

Definition 26.1. A *diagram* in a category C is a functor $D: I \to \mathsf{C}$. The category I is known as the *indexing category* of the diagram D. A *small diagram* is a diagram whose indexing category I is a small category. A *finite diagram* is a diagram whose indexing category I is a finite category. Used in 13.4, 13.4*, 26.0*, 26.1, 26.2, 26.4, 26.5, 26.9, 26.11, 26.12, 26.13, 26.14, 26.16, 26.17, 26.20, 26.21, 26.23, 26.25, 26.25*, 26.26*, 26.27, 26.30, 26.31, 26.33, 26.34, 26.38, 29.8*, 30.9*.

We now generalize coproducts and coequalizers. First, we formalize the data used in the definition of coproducts and coequalizers.

Definition 26.2. A *cocone* under a diagram $D: I \to \mathsf{C}$ is an object $A \in \mathsf{C}$ together with a family of *injection morphisms* $c_P: D(P) \to A$ for any $P \in I$ such that the following triangle commutes for an arbitrary morphism $f: P \to P'$ in I:

$$\begin{array}{ccc} D(P) & \xrightarrow{D(f)} & D(P') \\ & {\scriptstyle c_P}\searrow \quad \swarrow{\scriptstyle c_{P'}} & \\ & A & \end{array}$$

Used in 13.1, 26.4, 26.10, 26.11, 26.12, 26.16, 26.17*, 26.20, 26.25*, 26.27, 26.30, 28.3, 28.4*, 29.8*.

Next, we formalize the compatibility property used in the universal property of coproducts and coequalizers.

Definition 26.3. A *morphism of cocones* $(A, c) \to (A', c')$ under a diagram $D: I \to \mathsf{C}$ is a morphism $g: A \to A'$ in the category C such that the following triangle commutes for an arbitrary object $P \in I$:

$$\begin{array}{ccc} & D(P) & \\ & {\scriptstyle c_P}\swarrow \quad \searrow{\scriptstyle c'_P} & \\ A & \xrightarrow[g]{} & A' \end{array}$$

Used in 26.4, 26.16, 28.4*.

Definition 26.4. The *category of cocones* under a diagram $D: I \to \mathsf{C}$ has cocones under D as objects and morphisms of cocones under D as morphisms. Used in 26.5, 26.9, 28.4.

Remark 26.5. The category of cocones under a diagram $D: I \to \mathsf{C}$ can be defined in a much more concise way as the category whose objects are pairs (A, c), where $A \in \mathsf{C}$ and c is a natural transformation $D \to \operatorname{const} A$, and morphisms $(A, c) \to (A', c')$ are morphisms $g: A \to A'$ in C such that the following triangle of functors and natural transformations commutes:

$$\begin{array}{ccc} & D & \\ & {\scriptstyle c}\swarrow \quad \searrow{\scriptstyle c'} & \\ \operatorname{const} A & \xrightarrow[g]{} & \operatorname{const} A' \end{array}$$

Here $\operatorname{const} A: I \to \mathsf{C}$ denotes the *constant functor* $I \to \mathsf{C}$, defined by $(\operatorname{const} A)(P) = A$ and $(\operatorname{const} A)(f) = \mathrm{id}_A$ for any $P \in I$ and $f: P \to P'$. Used in 26.5.

Finally, we formulate the universal property of colimits. Before we do this, we isolate and study this property in a more simple context.

Definition 26.6. Suppose A is an object of a category C. We say that A is *initial* if for any object $B \in \mathsf{C}$ there is exactly one morphism $A \to B$. Used in 7.9, 26.7, 26.7*, 26.8, 26.9, 26.11, 26.12, 26.19, 28.4, 44.8.

Lemma 26.7. If $A, A' \in \mathsf{C}$ are initial objects, then there is a unique isomorphism $A \to A'$. Thus, if C has an initial object, then it is unique up to a unique isomorphism. Used in 26.10*.

Proof. Since A is initial, there is a unique morphism $u: A \to A'$. Since A' is initial, there is a unique morphism $v: A' \to A$. Since A is initial, there is a unique morphism $A \to A$, and since $\mathrm{id}_A: A \to A$ is such a morphism, the morphism $v \circ u \colon A \to A$ must be equal to id_A. Likewise, $u \circ v$ must be equal to $\mathrm{id}_{A'}$. Thus, u is an isomorphism. ▮

Example 26.8. The following categories have initial objects as indicated.

- Set: the empty set;
- Ab, Group: the group with one element;
- Ring: the ring of integers;
- Δ: does not exist;
- sSet: the empty simplicial set.

Definition 26.9. The *colimit* of a diagram $D \colon I \to \mathsf{C}$ is an initial object in the category of cocones under D. If a category C admits colimits of all small diagrams, then we say that C is *cocomplete*. Likewise, if C admits colimits of all finite diagrams, then we say that C is *finitely cocomplete*. Used in 26.9, 26.10, 26.10*, 26.11, 26.12, 26.15, 26.17, 26.18, 26.19, 26.20, 26.20*, 26.21, 26.23, 26.25, 26.26, 26.26*, 26.30, 26.30*, 26.43, 28.5, 28.5*, 29.8, 30.9, 31.2, 39.5, 42.1*, 42.5, 43.7, 44.5, 44.6, 45.3*.

Remark 26.10. Abusing language, the object A is often referred to as the colimit of D. We can say "*colimit cocone*" if we want to emphasize the injection maps. The *universal property of colimits* says that there is a natural bijective correspondence between morphisms

$$\mathtt{colim} D \to A$$

and families of maps

$$c_P \colon D(P) \to A$$

that form a cocone under D. Used in 26.16, 26.21, 26.25*, 26.30, 26.40, 28.4, 29.17*, 31.3*, 32.8*, 43.4.

We automatically infer from Lemma 26.7 that if a diagram admits a colimit, then it is unique up to a unique isomorphism.

Example 26.11. Consider the indexing category $I = \{0, 1\}$, with no nonidentity morphisms. A diagram $D \colon I \to \mathsf{C}$ is a pair of objects $D_0, D_1 \in \mathsf{C}$, a cocone c under D is a triple $(A, D_0 \to A, D_1 \to A)$. Such a cocone is initial if this triple satisfied the universal property of coproducts. Thus, colimits over $\{0, 1\}$ are precisely coproducts.

Example 26.12. Consider the indexing category $I = \{0 \rightrightarrows 1\}$, with exactly two nonidentity morphisms. A diagram $D \colon I \to \mathsf{C}$ is a pair of objects $D_0, D_1 \in \mathsf{C}$ together with two morphisms $f, g \colon D_0 \to D_1$, a cocone c under D is a morphism $c \colon D_1 \to A$ such that $c \circ f = c \circ g$. Such a cocone is initial if this triple satisfied the universal property of coequalizers. Thus, colimits over $\{0 \rightrightarrows 1\}$ are precisely coequalizers.

Example 26.13. Consider the empty indexing category $I = \emptyset$. For any category C there is a unique empty diagram $D \colon \emptyset \to \mathsf{C}$. A colimit of such a diagram is precisely an initial object of C (if it exists).

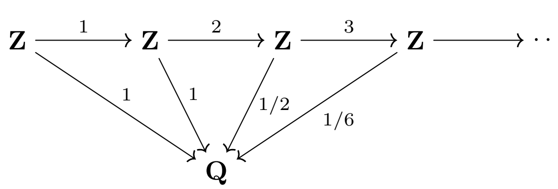
Example 26.14. Consider the indexing category $I = 1$ consisting of a single object and the identity morphism. A diagram $D \colon I \to \mathsf{C}$ is an object X of C. The colimit of D is X.

Example 26.15. Consider the category $I = \{0 \to 1 \to 2 \to \cdots\}$ whose objects are natural numbers and $\mathtt{hom}(i, j)$ is empty if $i > j$ or consists of a single element if $i \le j$. I-indexed colimits are known as *sequential colimits*. An I-diagram D is a collection of objects $D(i)$ for each $i \ge 0$ together with morphisms $D(i) \to D(i+1)$ for all $i \ge 0$. Used in 26.26*.

Example 26.16. Consider the following diagram in the category Ab:

$$\mathbf{Z} \xrightarrow{1} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{3} \mathbf{Z} \xrightarrow{4} \cdots$$

We claim that its colimit cocone is $\mathbf{Q}$ equipped with maps $\iota_n \colon \mathbf{Z} \to \mathbf{Q}$ that multiply by $1/n!$:



Indeed, suppose $(A, \{\rho_n\}_n)$ is another cocone of the same type. By definition of a morphism of cocones, the triangle

$$\begin{array}{ccc} & \mathbf{Z} & \\ {\scriptstyle \iota_n}\swarrow & & \searrow{\scriptstyle \rho_n} \\ \mathbf{Q} & \xrightarrow[f]{} & A \end{array}$$

must commute for any $n \geq 0$. Thus, $\rho_n(m) = (f \circ \iota_n)(m) = f(\iota_n(m)) = f(m/n!)$, so $\rho_n(m(n-1)!) = f(m/n)$ for any $m \in \mathbf{Z}$ and $n > 0$. Hence, there is at most one such morphism of cocones.

To show existence, for any $r \in \mathbf{Q}$ we define $f(r) = \rho_n(rn!)$, where $n \in \mathbf{Z}$ is such that $rn! \in \mathbf{Z}$. The right side is independent of the choice of n because $\rho_n(nk) = \rho_{n-1}(k)$ for any $n > 0$ and $k \in \mathbf{Z}$. Furthermore, f is a homomorphism of abelian groups because $f(r + r') = \rho_n(rn!) + \rho_{n'}(rn'!) = \rho_m(rm!) + \rho_m(r'm!) = \rho_m((r + r')m!) = f(r + r')$, where $m \geq n$ and $m \geq n'$.

Proposition 26.17. Suppose a category C admits colimits of small diagrams. If $D\colon I \to \mathsf{C}$ is a diagram in C, then we can establish the following canonical bijective correspondence between cocones under D and coequalizer forks of the following pair of morphisms:

$$\coprod_{f\colon P \to P'} D(P) \underset{v}{\overset{u}{\rightrightarrows}} \coprod_{Q \in I} D(Q),$$

where the top morphism u has components

$$D(P) \xrightarrow{\iota_P} \coprod_{Q \in I} D(Q)$$

given by the injection morphisms ι_Q, whereas the bottom morphism v has components

$$D(P) \xrightarrow{D(f)} D(P') \xrightarrow{\iota_{P'}} \coprod_{Q \in I} D(Q).$$

As a consequence, the colimit of D can be computed as the coequalizer of u and v.

Proof. Suppose $q\colon \coprod_{Q \in I} D(Q) \to Z$ coequalizes u and v. We claim that Z and the family $\{c_Q = q \circ \iota_Q\colon D(Q) \to Z\}_i$ is a cocone under D. Indeed, the triangle

$$\begin{array}{ccc} D(P) & \xrightarrow{f} & D(P') \\ {\scriptstyle c_P}\searrow & & \swarrow{\scriptstyle c_{P'}} \\ & Z & \end{array}$$

commutes because $u \circ \iota_f = \iota_P$ and $v \circ \iota_f = \iota_{P'} \circ D(f)$, so $q \circ u \circ \iota_f = q \circ \iota_P = c_P$ and $q \circ v \circ \iota_f = q \circ \iota_{P'} \circ D(f) = c_{P'} \circ D(f)$. Running this argument in the opposite direction shows that any cocone $(Z, \{c_P\})$ under D produces a morphism $q\colon \coprod_{Q \in I} D(Q) \to Z$ (i.e., $q \circ \iota_Q = c_Q$) that coequalizes u and v. ∎

Corollary 26.18. If a category C admits coequalizers and small coproducts, then it is cocomplete.

Example 26.19. The following categories are cocomplete: Set, Ab, Group, Mod_R, Ring. The category of fields does not have an initial object and so is not cocomplete.

Exercise 26.20. Suppose the indexing category I has a terminal object $1 \in I$, as defined in Definition 26.32. If $D\colon I \to \mathsf{C}$ is a diagram, prove that $D(1)$ is the colimit of D. What are the injection morphisms?

We examine another example of colimits due to its importance.

Example 26.21. Suppose $I = \{1 \leftarrow 0 \rightarrow 2\}$ is a category with three objects and two nonidentity morphisms as depicted. A diagram $D: I \to \mathsf{C}$ is a pair of morphisms $B \leftarrow A \to C$ in C. The *pushout* of $D: I \to \mathsf{C}$ is defined as its colimit. The colimit cocone $(Q, \iota_A: A \to Q, \iota_B: B \to Q, \iota_C: C \to Q)$ is depicted by the following diagram:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & Q. \end{array}$$

An arbitrary commutative square is *cocartesian* if it arises as the pushout of top and left arrows. We also characterize this situation by saying that the right morphism $B \to Q$ is the *cobase change* of the left morphism $A \to C$ along the top morphism $A \to B$. Notice how the symmetry of pushouts is broken in this formulation. Informally, cobase change attaches C to B along the image of A in B. Used in 26.21, 39.9*, 43.7, 45.5*, 46.2*.

Remark 26.22. It is useful to explicitly formulate the *universal property of pushouts*: morphisms $B \sqcup_A C \to Q$ are in a natural bijective correspondence with pairs of morphisms $b: B \to Q$ and $c: C \to Q$ such that the above square commutes. Used in 46.2*.

Example 26.23. Given a group G, consider the delooping category $\mathsf{B}G$ from Example 11.19, which has a single object whose endomorphisms form a group isomorphic to G. A diagram $D: \mathsf{B}G \to \mathsf{C}$ picks an object $X \in \mathsf{C}$ and equips it with an action of G. The colimit of D, if it exists, is known as the *coinvariant object* of X and is denoted by X_G. Used in 26.24.

Exercise 26.24. Prove that in the case $\mathsf{C} = \mathsf{Set}$ the coinvariant object of a G-set X is naturally isomorphic to the set of orbits of the action of G on X.

Proposition 26.25. Suppose C is a cocomplete category and I is an indexing category. Then there is a *colimit functor*

$$\mathsf{colim}: \mathsf{C}^I \to \mathsf{C}$$

whose value on objects of C^I, i.e., I-indexed diagrams, is given by the colimit. Used in 26.10, 26.25*, 26.26*, 28.5*, 29.8*, 30.8, 30.9*, 31.2, 32.6, 32.7*, 34.6, 36.8, 42.1*, 42.2, 43.6.

Proof. We have to define colim on morphisms of diagrams, i.e., natural transformations. If $D, D': I \to \mathsf{C}$ are diagrams and $t: D \to D'$ is a natural transformation, then

$$\mathsf{colim}\, t: \mathsf{colim} D \to \mathsf{colim} D'$$

is defined using the universal property of colimits as the collection of morphisms of the form

$$D(P) \longrightarrow \mathsf{colim} D',$$

namely, the composition

$$D(P) \xrightarrow{t_P} D'(P) \xrightarrow{\iota_P} \mathsf{colim} D'.$$

The naturality property shows that this family is a cocone. ∎

Proposition 26.26. Colimits in the category of simplicial sets exist and can be computed pointwise as described in the proof.

Proof. A diagram

$$D: I \to \mathsf{sSet} = \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{Set})$$

can be rewritten as

$$D: I \times \Delta^{\mathsf{op}} \to \mathsf{Set},$$

which can be rewritten as

$$D: \Delta^{\mathsf{op}} \to \mathsf{Fun}(I, \mathsf{Set}) = \mathsf{Set}^I.$$

That is, there is a tautological functor

$$\mathsf{sSet}^I \to \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{Set}^I).$$

We can now apply the colimit functor

$$\mathsf{colim}_{\mathsf{Set}}\colon \mathsf{Set}^I \to \mathsf{Set}$$

in the category of sets degreewise:

$$\mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{colim}_{\mathsf{Set}})\colon \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{Set}^I) \to \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{Set}),$$

yielding the colimit functor in the category of simplicial sets. ▮

Another important class of colimits is given by filtered colimits, which generalizes sequential colimits by allowing diagrams whose objects are not linearly ordered.

Definition 26.27. A category I is *filtered* if any finite diagram $J \to I$ admits a cocone in I. Used in 26.26*, 26.28, 26.30, 39.6, 39.6*.

Lemma 26.28. A category I is filtered if and only if the following three conditions are met, which are special cases of the above general condition for a specific J:

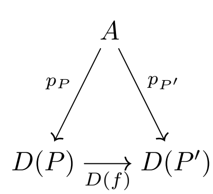
- I has an object ($J = \emptyset$);
- for any two objects $A, B \in I$ there is an object $C \in I$ with maps $A \to C$ and $B \to C$ ($J = \{0, 1\}$);
- for any two morphisms $f, g\colon A \to B$ in I there is a morphism $h\colon B \to C$ such that $hf = hg$ ($J = \{0 \rightrightarrows 1\}$).

Example 26.29. A category induced by a poset P if and only if P is a directed poset: any finite subset of P has an upper boundary.

Example 26.30. Consider the category I with a single object 1 and a single nonidentity morphism $e\colon 1 \to 1$ such that $e \circ e = e$, i.e., an *idempotent morphism*, or simply an *idempotent*. (In linear algebra and related areas, idempotents are known as projections.) A diagram $D\colon I \to \mathsf{C}$ picks a single object $X \in \mathsf{C}$ and a morphism $D(e)\colon X \to X$ such that $D(e) \circ D(e) = D(e)$. If the colimit of D exists, we say that $D(e)$ is a *split idempotent*. In this case, the colimit of D is known as the *retract* of $D(e)$. The category I is filtered but is not induced by any poset. Colimits indexed by this category compute the splitting of a given idempotent, i.e., if we have an I-diagram D in C given by an object $X \in \mathsf{C}$ with an idempotent morphism $e\colon X \to X$, then the colimit of this diagram is an object $R \in \mathsf{C}$ together with an injection map $r\colon X \to R$ such that $r = re$ and the universal property of colimits is satisfied. In particular, applying the universal property to another cocone under D given by the object X itself together with the injection map $e\colon X \to X$, we can construct a morphism $i\colon R \to X$ such that $ir = e$. Applying the universal property to yet another cocone under D given by the object R with the injection map $re\colon X \to R$, we see that both id_R and $ri\colon R \to R$ are morphisms of cocones $(R, r) \to (R, re)$, so by the uniqueness part we have $ri = \mathrm{id}_R$. This, $ri = \mathrm{id}_R$ and $ir = e$ is an idempotent, so $r\colon X \to R$ exhibits R as a retract of X, with the other composition $ir = e$ being the corresponding idempotent. Used in 26.30.

We now define the dual notion of limits. By definition, limits in a category C are colimits in C^{op}.

Definition 26.31. A *cone* over a diagram $D\colon I \to \mathsf{C}$ is an object $A \in \mathsf{C}$ together with a family of *projection morphisms* $p_P\colon A \to D(P)$ for any $P \in I$ such that the following triangle commutes for an arbitrary morphism $f\colon P \to P'$ in I:



A *morphism of cones* $(A, p) \to (A', p')$ over a diagram $D\colon I \to \mathsf{C}$ is a morphism $g\colon A \to A'$ in the category C such that the following triangle commutes for an arbitrary object $P \in I$:

$$\begin{array}{ccc} A & \xrightarrow{g} & A' \\ & {\scriptstyle p_P}\searrow \quad \swarrow{\scriptstyle p'_P} & \\ & D(P) & \end{array}$$

The *category of cones* is defined in the obvious way. Used in 26.33, 43.6.

Definition 26.32. Suppose A is an object of a category C. We say that A is *terminal* if for any object $B \in \mathsf{C}$ there is exactly one morphism $B \to A$. Used in 7.9, 26.20, 26.33, 33.5, 44.8.

Definition 26.33. The *limit* of a diagram $D\colon I \to \mathsf{C}$ is an terminal object in the category of cones under D. If a category C admits limits of all small diagrams, then we say that C is *complete*. Likewise, if C admits limits of all finite diagrams, then we say that C is *finitely complete*. Abusing language, the object A is often referred to as the limit of D. We can say "*limit cone*" if we want to emphasize the projection maps. The *universal property of limits* says that there is a natural bijective correspondence between morphisms

$$A \to \mathtt{lim} D$$

and families of maps

$$p_P\colon A \to D(P)$$

that form a cone over D. Used in 26.30*, 26.33, 26.34, 26.35, 26.38, 26.39, 26.40, 26.43, 29.10, 31.4, 33.5, 35.6*, 39.6, 39.6*, 42.1*, 42.5, 43.7, 44.5, 44.6, 45.8*.

Proposition 26.34. Suppose a category C admits limits of small diagrams. If $D\colon I \to \mathsf{C}$ is a diagram in C, then we can establish the following canonical bijective correspondence between cones over D and equalizer forks of the following pair of morphisms:

$$\prod_{Q \in I} D(Q) \overset{u}{\underset{v}{\rightrightarrows}} \prod_{f\colon P \to P'} D(P'),$$

where the top morphism u has components

$$\prod_{Q \in I} D(Q) \xrightarrow{p_{P'}} D(P')$$

given by the projection morphisms p_Q, whereas the bottom morphism v has components

$$\prod_{Q \in I} D(Q) \xrightarrow{p_P} D(P) \xrightarrow{D(f)} D(P').$$

As a consequence, the limit of D can be computed as the equalizer of u and v.

Corollary 26.35. If a category C admits equalizers and small products, then it is complete.

Remark 26.36. *Cartesian squares*, *base changes*, and the *universal property of pullbacks* are defined by reversing all arrows in the corresponding definitions. If a square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is cartesian, we write $A = B \times_D C$. In other words, $B \to D \leftarrow C$ is the diagram, A is the apex of a limit cone, and $A \to B$ and $A \to C$ are projection maps. (The remaining projection map $A \to D$ can be computed as the composition $A \to C \to D$, equivalently, $A \to B \to D$.) We also say that the left arrow is the base change of the right arrow along the bottom arrow. Used in 32.2, 39.6*, 43.7, 45.7, 45.8*, 46.4*, 59.4*.

Exercise 26.37. Prove that in the case $\mathsf{C} = \mathsf{Set}$ the *invariant object* of a G-set X is naturally isomorphic to the set of fixed points of the action of G on X.

Proposition 26.38. Suppose C is a complete category and I is an indexing category. Then there is a *limit functor*

$$\mathrm{lim}\colon \mathsf{C}^I \to \mathsf{C}$$

whose value on objects of C^I, i.e., I-indexed diagrams, is given by the limit. Used in 26.33, 30.8, 34.6, 42.1*, 42.2, 42.7, 43.6.

Proposition 26.39. Limits in the category of simplicial sets exist and can be computed pointwise.

Definition 26.40. A functor F *preserves colimits* (we also say that F is a *cocontinuous functor*) if the image of a colimit cocone under F is again a colimit cocone. Likewise, F *preserves limits* (we also say that F is a *continuous functor*) if it sends limit cones to limit cones. Used in 26.41, 28.5, 28.5*, 28.6, 28.9, 30.9.

Example 26.41. Many forgetful functors preserve limits. For instance, $\mathsf{U}\colon \mathsf{Ab} \to \mathsf{Set}$ and $\mathsf{U}\colon \mathsf{Ring} \to \mathsf{Ab}$ preserve limits. Likewise, many free functors preserve colimits. For instance, $\mathbf{Z}[-]\colon \mathsf{Set} \to \mathsf{Ab}$ and $\mathbf{Z}[-]\colon \mathsf{Set} \to \mathsf{Ring}$.

Definition 26.42. An *equivalence of categories* is a functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ such that there is an *inverse functor* $\mathsf{G}\colon \mathsf{D} \to \mathsf{C}$ together with natural isomorphisms $\mathrm{id}_\mathsf{C} \to \mathsf{G} \circ \mathsf{F}$ and $\mathsf{F} \circ \mathsf{G} \to \mathrm{id}_\mathsf{D}$. Used in 26.43, 27.4, 28.5, 29.13*, 31.5, 32.8.

Proposition 26.43. An equivalence of categories preserves all limits and colimits.

Proof. It suffices to observe that an equivalence of categories induces an equivalence of the categories of cocones under a diagram D in C and cocones under the diagram $\mathsf{F} \circ D$ in D. ∎

27 Full, faithful, and essentially surjective functors

Definition 27.1. We say that $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ is a *fully faithful functor* if

$$\mathrm{Mor}_\mathsf{C}(X, Y) \to \mathrm{Mor}_\mathsf{D}(\mathsf{F}(X), \mathsf{F}(Y))$$

is an isomorphism of sets for all $X, Y \in \mathsf{C}$. If the above map is always injective, we say that F is *faithful* and if it is surjective, we say that F is *full.* Used in 27.2, 27.4, 27.4*, 28.5*, 30.10.

Definition 27.2. We say that C is a *full subcategory* of D if $\mathrm{Ob}(\mathsf{C}) \subset \mathrm{Ob}(\mathsf{D})$, $\mathrm{Mor}_\mathsf{C}(X, Y) \subset \mathrm{Mor}_\mathsf{D}(X, Y)$, and the inclusion $\mathsf{C} \subset \mathsf{D}$ is a fully faithful functor. Used in 29.2, 31.5, 32.1, 32.10, 33.5, 39.1.

Definition 27.3. A functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ is *essentially surjective* if for any object $Y \in \mathsf{D}$ there is an object $X \in \mathsf{C}$ such that $\mathsf{F}(X)$ is isomorphic to Y. Used in 27.4, 27.4*, 28.5*.

Lemma 27.4. A functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ is an equivalence of categories if and only if it is essentially surjective and fully faithful. Used in 28.5*.

Proof sketch. Essential surjectivity allows us to define an inverse functor $\mathsf{G}\colon \mathsf{D} \to \mathsf{C}$ on objects: we set $\mathsf{G}(Y) = X$, where $X \in \mathsf{C}$ is any objects such that $\mathsf{F}(X)$ is isomorphic to Y. Full faithfulness alllows us to define G on morphisms as the inverse of the map of sets

$$\mathrm{hom}(\mathsf{G}(Y), \mathsf{G}(Y')) \to \mathrm{hom}(\mathsf{F}(\mathsf{G}(Y)), \mathsf{F}(\mathsf{G}(Y'))) \to \mathrm{hom}(Y, Y'),$$

given by the composition of two maps, the first of which is an isomorphism by full faithfulness and the second is an isomorphism of sets induced by isomorphisms $Y \to \mathsf{F}(\mathsf{G}(Y))$ and $\mathsf{F}(\mathsf{G}(Y')) \to Y'$. ∎

28 Nerve-realization adjunction

In this section, we define the nerve functor (Definition 28.1), the realization functor (Definition 28.6), and show that the nerve functor is right adjoint to the realization functor.

Definition 28.1. The *nerve functor* associated to a pair (C, R), where C is a category and $R\colon \Delta \to \mathsf{C}$ is a functor, sends an object $X \in \mathsf{C}$ to the simplicial set $\mathrm{N}_R(X)$ defined as

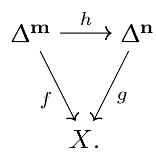
$$\mathrm{N}_R(X)_m = \mathsf{hom}(R(\mathbf{m}), X)$$

for any simplex $\mathbf{m} \in \Delta$ and

$$\mathrm{N}_R(X)_f = \mathsf{hom}(R(f), X)\colon \mathsf{hom}(R(\mathbf{n}), X) \to \mathsf{hom}(R(\mathbf{m}), X)$$

for any map of simplices $f\colon \mathbf{m} \to \mathbf{n}$. Used in 28.0*, 28.7.

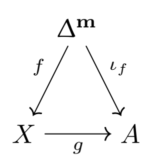
Definition 28.2. The *category of simplices of a simplicial set* X, denoted by Δ/X, has simplicial maps $f\colon \Delta^{\mathbf{m}} \to X$ as objects ($\mathbf{m}$ is an arbitrary simplex) and morphisms $f \to g$ are simplicial maps $h\colon \Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$ that make the following triangle commutative:



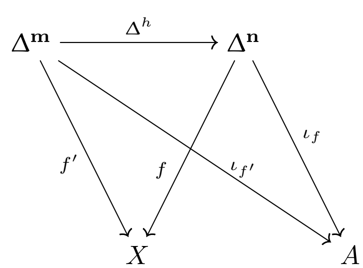
Definition 28.3. The *canonical diagram* of a simplicial set X is a functor $\Delta/X \to \mathsf{sSet}$ that sends an object $f\colon \Delta^{\mathbf{m}} \to X$ to $\Delta^{\mathbf{m}}$ and a morphism $f \to g$ to the underlying simplicial map $h\colon \Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$. The *canonical cocone* of a simplicial set X is a cocone under the canonical diagram of X whose apex is X and the injection morphism for an object $f\colon \Delta^{\mathbf{m}} \to X$ is the morphism f. Used in 28.3, 28.4, 31.1.

Proposition 28.4. The canonical cocone of a simplicial set X is a colimit cocone, i.e., an initial object in the category of cocones under the canonical diagram $\Delta/X \to \mathsf{sSet}$. Used in 31.2.

Proof. Given another cocone $(A, \{\iota_f\})$, we have to show that there is a unique morphism of cocones $g\colon X \to A$. Indeed, the commutativity triangle for $f\colon \Delta^{\mathbf{m}} \to X$, namely,



uniquely determines the value of g on the simplex f. Since f is an arbitrary simplex of X, this show uniqueness. For existence, we have to show that g, as defined by the above relations, is indeed a simplicial map, which follows from the following commutative diagram:



Here $f' = f \circ \Delta^h$ by definition, so the corresponding triangle commutes. Also $\iota_{f'} = \iota_f \circ \Delta^h$ because (A, ι) is a cocone, so the other triangle also commutes. In term of this diagram, evaluating g on a simplex of X, expressed as an arrow with codomain X, simply switches the codomain to A. Applying the simplicial structure map associated to h amount to precomposing with the map Δ^h. We start with the simplex f and observe that applying both operations in either order produces $\iota_{f'}$ in both cases, which proves that g is indeed a simplicial map. ∎

Proposition 28.5. Suppose C is a cocomplete category. The functor

$$\mathsf{CocoFun}(\mathsf{sSet}, \mathsf{C}) \to \mathsf{Fun}(\Delta, \mathsf{C})$$

(where the left side denotes cocontinuous functors) given by the restriction along the Yoneda embedding $\Delta\colon \Delta \to \mathsf{sSet}$ is an equivalence of categories.

Proof. Using Lemma 27.4, it suffices to show that the restriction functor is fully faithful and essentially surjective.

To show essential surjectivity, for a functor $\mathsf{R}\colon \Delta \to \mathsf{C}$ consider the functor

$$|{-}|_{\mathsf{R}}\colon \mathsf{sSet} \to \mathsf{C}, \qquad |X|_{\mathsf{R}} = \mathsf{colim}_{s\in\Delta/X}\mathsf{R}(p(s)),$$

where $p\colon \Delta/X \to \Delta$ is the forgetful functor. Restricting to simplicial sets in the image of the Yoneda embedding yields the original functor R because the indexing category $\Delta/\Delta^{\mathbf{n}}$ has a terminal object given by the identity map on $\Delta^{\mathbf{n}}$ and colimits over indexing categories with terminal objects can be computed by evaluating on the terminal object.

To show full faithfulness, observe that faithfulness follows from the fact that extending two given natural transformations $t, t'\colon R \to R'$ to sSet and then restricting back to Δ gives back t and t'. Thus, if the extensions of t and t' to sSet are equal, then so are t and t' themselves. To show fullness, consider a natural transformation $t\colon |{-}|_{\mathsf{R}} \to |{-}|_{\mathsf{R}'}$. We claim that t coincides with the extension to sSet of its restrction to Δ. Indeed, the latter natural transformation takes the same values on simplices, so it suffices to show that taking the same values on simplices implies taking the same values on all simplicial sets. This follows from the cocontinuity property, which allows us to compute t_X for some $X \in \mathsf{sSet}$ as a colimit (in the category $\mathsf{sSet}^{\to}$ of simplicial maps and commutative squares) of a diagram consisting of maps of the form $t_{\Delta^{\mathbf{m}}}$. ∎

Definition 28.6. Given a functor $R\colon \Delta \to \mathsf{C}$, the associated cocontinuous functor $\mathsf{sSet} \to \mathsf{C}$ is denoted by $|{-}|_R$ and is referred to as the *realization functor* associated to R. If C is a "geometric" category (e.g., some kind of spaces), then $|{-}|_R$ is also known as the *geometric realization* functor. Used in 2.0*, 28.0*, 28.7, 29.16.

Proposition 28.7. Given a functor $R\colon \Delta \to \mathsf{C}$, the nerve functor N_R (Definition 28.1) is right adjoint to the realization functor $|{-}|_R$ (Definition 28.6).

Proof. Given a simplicial set X and an object $Y \in \mathsf{C}$, we have to construct a bijection of sets

$$\mathsf{hom}_{\mathsf{C}}(|X|_R, Y) \to \mathsf{hom}_{\mathsf{sSet}}(X, \mathrm{N}_R(Y)).$$

Both sides are cocontinuous in X, so it suffices to establish a natural bijection for $X \in \Delta$. Expanding both sides yields the same set $\mathsf{hom}_{\mathsf{C}}(R(X), Y)$, as desired. ∎

Example 28.8. Consider the functor $\mathrm{C}\colon \Delta \to \Delta$ that takes a simplex $\mathbf{m} = (V, \leq)$ to the simplex $\mathrm{C}(\mathbf{m}) = (V \sqcup \{*\}, \leq')$, where $v <' *$ for all $v \in V$. Thus, the functor $\Delta \circ \mathrm{C}\colon \Delta \to \mathsf{sSet}$ admits a unique cocontinuous extension to a functor $\mathrm{C}\colon \mathsf{sSet} \to \mathsf{sSet}$ that we also denote by C. The simplicial set $\mathrm{C}X$ is known as the *simplicial cone* of X.

Example 28.9. Define the *barycentric subdivision* functor $\mathrm{sd}\colon \mathsf{sSet} \to \mathsf{sSet}$ as a unique cocontinuous functor such that $\mathrm{sd}\,\Delta^{\mathbf{0}} = \Delta^{\mathbf{0}}$ and $\mathrm{sd}\,\Delta^{\mathbf{n}} = \mathrm{C}(\mathrm{sd}(\partial\Delta^{\mathbf{n}}))$. The definition of $\mathrm{sd}\,\Delta^{\mathbf{n}}$ uses the value of sd on a simplicial set whose nondegenerate simplices have dimension less than n, so this construction is well-defined. Used in 28.9, 36.4, 36.5, 39.2, 39.3, 39.4, 39.6*, 39.7*, 40.12.

29 The fundamental groupoid

29.1. ***Groupoids***

Definition 29.2. A *groupoid* is a category in which all morphisms are isomorphisms. The *category of groupoids* Grpd is the full subcategory of the category of categories (assumed to be small categories, as usual). Used in 11.0*, 29.8, 29.8*, 29.10, 29.15, 29.16, 30.6, 30.7, 31.6, 31.6*.

Example 29.3. The *discrete category* on a set of objects (defined as having only identity morphisms) is a groupoid because all morphisms are identities. Used in 29.18, 42.2.

Example 29.4. The delooping category $\mathsf{B}G$ of a monoid G is a groupoid if and only if G is a group.

Definition 29.5. A *system of generators and relations for a groupoid* is specified as follows. First, one specifies a set of objects O. Next, one specifies a set of generating morphisms $f_i\colon x_i \to y_i$, where $x_i, y_i \in O$. Finally, one specifies a set of relations of the form $f_{i_k}^{\pm 1} \circ \cdots \circ f_{i_0}^{\pm 1} = f_{i'_{k'}}^{\pm 1} \circ \cdots \circ f_{i'_0}^{\pm 1}$, where all generators must be composable in the obvious sense, and the domain and codomain must be the same on both sides. Used in 29.17.

Definition 29.6. The *groupoid generated by a system of generators and relations* has the given set of objects, and morphisms between two objects are equivalence classes of chains of composable generators of length zero or more, with the equivalence relation allowing for zero or more substitutions from the given set of relations. Used in 29.7, 29.17*.

Exercise 29.7. Formulate and prove the universal property of the groupoid generated by a system of generators and relations, and prove that it always exists. Used in 29.8*.

Proposition 29.8. The category of groupoids is cocomplete.

Proof. Given a small diagram $D\colon I \to \mathsf{Grpd}$ of groupoids, we take $A = \mathtt{colim}_{i\in I} \mathsf{U}(D(i))$ as the set of objects. Denote by $\iota_i\colon \mathsf{U}(D(i)) \to A$ the injection morphisms (which need not be injective maps of sets). For each $i \in I$ and morphism $f\colon x \to y$ in $D(i)$, we add a generating morphism $\iota_f\colon \iota_i(x) \to \iota_i(y)$. For each $i \in I$ and composable pair of morphisms $f\colon x \to y$ and $f'\colon y \to z$ in $D(i)$ we add a relation $\iota_i(f') \circ \iota_i(f) = \iota_i(f' \circ f)$. For each $i \in I$ and object x in $D(i)$ we add a relation $\mathrm{id}_{\iota_i(x)} = \iota_i(\mathrm{id}_x)$. Finally, for each morphism $h\colon i \to i'$ in I and morphism $f\colon x \to y$ in $D(i)$ we add a relation $\iota_i(f) = \iota_{i'}(h(f))$. To establish the universal property of colimits, invoke Exercise 29.7. Unfolding the universal property of Exercise 29.7 reproduces precisely the universal property of colimits. ∎

Remark 29.9. An identical proof shows that the category of categories is cocomplete, using generators and relations for categories instead of groupoids, provided that we disallow inverses in relations.

Exercise 29.10. Show that the category of groupoids is complete.

29.11. ***Classification of groupoids***

We start by making a trivial observation.

Proposition 29.12. Every groupoid G decomposes into a coproduct of nonempty groupoids that themselves cannot be further decomposed into such a coproduct with two or more summands, which we refer to as *connected groupoids*. This decomposition is unique up to a unique isomorphism. Its indexing set is denoted by $\pi_0(G)$ and can be computed as the set of objects of G modulo the equivalence relation of isomorphism. Used in 29.12*, 29.13.

Next, we classify connected groupoids.

Proposition 29.13. Suppose G is a connected groupoid and $x \in G$. (Such x exists because connected groupoids are by definition nonempty.) The canonical inclusion $\mathsf{B}\mathtt{Aut}_G(x) \to G$ is an equivalence of groupoids, where $\mathtt{Aut}_G(x)$ denotes the group of automorphisms of the object x in the groupoid G. Used in 29.13, 29.13*, 29.16.

Proof. We choose for any object $y \in G$ an isomorphism $p_y\colon x \to y$. (Such an isomorphism exists because G is connected, so $\pi_0(G) = \{*\}$ and all objects belong to the same isomorphism class.) We set $p_x = \mathrm{id}_x$.

First, we construct an inverse functor $G \to \mathsf{BAut}_G(x)$. On the level of objects, there is nothing to specify. We send a morphism $f\colon y \to y'$ to the morphism $p_{y'}^{-1} f p_y \in \mathsf{Aut}_G(x)$. This indeed defines a functor: id_y is sent to $p_y^{-1}\,\mathrm{id}_y\,p_y = \mathrm{id}_x$ and $f' \circ f$ (where $f'\colon y' \to y''$ is sent to $p_{y''}^{-1} f' f p_y = p_{y''}^{-1} f' p_{y'} p_{y'}^{-1} f p_y$.

By construction, the composition $\mathsf{BAut}_G(x) \to G \to \mathsf{BAut}_G(x)$ equals the identity functor because $p_x = \mathrm{id}_x$. It remains to show that the other composition $G \to \mathsf{BAut}_G(x) \to G$ is isomorphic to the identity functor on G. Indeed, the collection of morphisms $p_x\colon x \to y$ defines such a natural isomorphism. ∎

29.14. ***Construction of the functor***

Definition 29.15. Consider the functor $\Delta \to \mathsf{Grpd}$ that sends a simplex $\mathbf{m}$ to the groupoid $[m]^{\leftrightarrow}$ with $\mathsf{U}(\mathbf{m})$ as its set of objects and exactly one morphism between any pair of objects. A map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ is sent to the unique functor that has $\mathsf{U}(f)$ as its underlying map on objects.

Definition 29.16. The *fundamental groupoid* functor $\pi_{\le 1}\colon \mathsf{sSet} \to \mathsf{Grpd}$ is the realization functor associated to the functor $\Delta \to \mathsf{Grpd}$ constructed above. Given a vertex v in a simplicial set X, the *fundamental group* of v in X is the group $\mathsf{Aut}_{\pi_{\le 1}(X)}(v)$. Used in 1.0*, 29.17*, 29.20, 29.21, 29.22, 33.3.

Proposition 29.17. A system of generators and relations for the fundamental groupoid of a simplicial set X can be constructed as follows. The set of objects is X_0, the set of vertices of X. There is a generator for every nondegenerate 1-simplex of X, which is a morphism from the 0th to the 1st vertex. Finally, there is a relation for every nondegenerate 2-simplex σ of X:

$$\mathrm{d}_1\sigma = \mathrm{d}_0\sigma \circ \mathrm{d}_2\sigma.$$

Proof. We have to show that the fundamental groupoid $\pi_{\le 1}X$ of X is equivalent to the groupoid G specified by the above system of generators and relations. We construct maps both ways and show they are equivalences. The map $\pi_{\le 1}X \to G$ is constructed using the universal property of colimits: for any simplex $\sigma\colon \Delta^{\mathbf{m}} \to X$ we have to construct a functor $[m]^{\leftrightarrow} \to G$ and show these functors are compatible for all σ. Such a functor simply sends objects of $[m]^{\leftrightarrow}$ to the corresponding objects of G (given by the vertex map of $\Delta^{\mathbf{m}} \to X$) and morphisms likewise. Composition is clearly preserved and different σ give compatible choices.

A functor $G \to \pi_{\le 1}X$ is constructed using the universal property of a groupoid generated by a system of generators and relations. Each object of G is a vertex of X, i.e., a map $\Delta^{\mathbf{0}} \to X$ and we map to itself in $\pi_{\le 1}X$. Each generator of G is a map $\sigma\colon \Delta^{\mathbf{1}} \to X$ and we map it to the morphism $0 \to 1$ in the groupoid $[1]^{\leftrightarrow}$ with index σ. Finally, each relation of G comes from a map $\tau\colon \Delta^{\mathbf{2}}$ and we see that once we map to the groupoid $[2]^{\leftrightarrow}$ with index τ, it is satisfied, and therefore it is satisfied in $\pi_{\le 1}$.

Both compositions $G \to \pi_{\le 1}X \to G$ and $\pi_{\le 1}X \to G \to \pi_{\le 1}X$ are equal to identities by construction. ∎

Definition 29.18. A *spanning tree* for a connected simplicial set X is a simplicial subset $T \subset X$ such that $T_{\mathbf{0}} = X_{\mathbf{0}}$, all nondegenerate simplices of T have dimension 0 or 1, the map $\pi_0(T) \to \pi_0(X)$ is an isomorphism and the groupoid $\pi_{\le 1}(T)$ is a discrete category.

Proposition 29.19. Given a connected simplicial set X and a vertex $x \in X_{\mathbf{0}}$, a system of generators and relations for $\pi_1(X, x)$ can be constructed as follows. Choose a spanning tree T for X. The set of generators is $X_{\mathbf{1}}$. For each 1-simplex $\sigma \in T_{\mathbf{1}} \subset X_{\mathbf{1}}$ introduce a relation $\sigma = 1$. For each nondegenerate 2-simplex $\tau \in X_{\mathbf{2}}$ introduce a relation $\mathrm{d}_1\tau = \mathrm{d}_0\tau\mathrm{d}_2\tau$.

Proof. We construct isomorphisms $\pi_1(X, x) \to G$ and $G \to \pi_1(X, x)$, where G is the group generated by the above system of generators and relations. For the map $\pi_1(X, x) \to G$, observe that any element of $\pi_1(X, x)$ is a loop of 1-simplices in X, traversed in either direction (with inverses added for traversing in the wrong direction). We send such a loop to the product of the corresponding generators in G or their inverses, accordingly. This map preserves composition and identity by construction.

The map $G \to \pi_1(X, x)$ is specified using the universal property of groups generated by a system of generators and relations. A generator $\sigma \in X_{\mathbf{1}}$ is sent to the element $t_{\mathrm{d}_0\sigma}^{-1}\sigma t_{\mathrm{d}_1\sigma}$ of $\pi_1(X, x)$ that goes from x to $\mathrm{d}_1\sigma$ using the edges in the spanning tree T, then traverses σ, and then goes from $\mathrm{d}_0\sigma$ to x also using the edges in T. By definition of a spanning tree, there is a unique path in the tree between any pair of vertices. In particular, relations are preserved because $\mathrm{d}_0\tau\mathrm{d}_2\tau$ is sent to

$$t_{\mathrm{d}_0\mathrm{d}_0\tau}^{-1}\mathrm{d}_0\tau t_{\mathrm{d}_1\mathrm{d}_0\tau} t_{\mathrm{d}_0\mathrm{d}_2\tau}^{-1}\mathrm{d}_2\tau t_{\mathrm{d}_1\mathrm{d}_2\tau},$$

where $d_1 d_0 \tau = d_0 d_2 \tau$, so the middle terms disappear and we get

$$t^{-1}_{d_0 d_0 \tau} d_0 \tau d_2 \tau t_{d_1 d_2 \tau} = t^{-1}_{d_0 d_0 \tau} d_1 \tau t_{d_1 d_2 \tau} = t^{-1}_{d_0 d_1 \tau} d_1 \tau t_{d_1 d_1 \tau},$$

as desired. The other relation $\sigma = 1$ for $\sigma \in T_{\mathbf{1}}$ is preserved because σ maps to $t^{-1}_{d_0 \sigma)} \sigma t_{d_1 \sigma}$, and this loop from x to x is contained in the spanning tree T, and since $\pi_{\leq 1}(T)$ is a discrete category, we know that the loop composes to id_x also in $\pi_{\leq 1}(X)$.

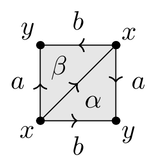
The composition $\pi_1(X, x) \to G \to \pi_1(X, x)$ is identity because a loop $\sigma_k^{\pm 1} \circ \cdots \circ \sigma_1^{\pm 1}$ of 1-simplices in X from x to x is sent to the same expression $\sigma_k^{\pm 1} \circ \cdots \circ \sigma_1^{\pm 1}$ in G, which is then mapped to a loop of 1-simplices in X given by the original loop in which we insert $t_v t_v^{-1} = \mathrm{id}_v$ for each intermediate position corresponding to a vertex v, whereas at the beginning or end we insert $t_x^{\pm 1} = \mathrm{id}_x$. Thus, nothing changes and the original loop maps to itself.

The composition $G \to \pi_{\leq 1}(X, x) \to G$ sends $\sigma \in X_{\mathbf{1}}$ to $t^{-1}_{d_0 \sigma} \sigma t_{d_1 \sigma}$, which is then sent to σ because each t is a composition of edges in the spanning tree, which map to 1 by definition of the map $\pi_{\leq 1}(X, x) \to G$. ▮

Example 29.20. We compute the fundamental groupoid of the circle $S^{\mathbf{1}}$. We have a single object x and a single generator $u: x \to x$, with no relations. Thus, we have arbitrary integer powers of u as morphisms, and there are no other morphisms. Thus, the fundamental groupoid is $\mathrm{B}\mathbf{Z}$ and the fundamental group is $\mathbf{Z}$.

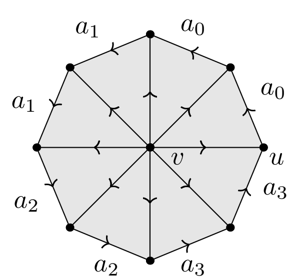
Example 29.21. We compute the fundamental groupoid of the sphere $S^{\mathbf{m}}$, where $m > 1$. We have a single object x. There are no generators or relations since there is only a single 1-simplex, which is degenerate. Thus, the fundamental groupoid is $\{x\}$ and the fundamental group is trivial.

Example 29.22. We compute the fundamental groupoid of the real projective plane X:



There are two objects, x and y. There are three generating isomorphisms: $a: x \to y$, $b: x \to y$, and $d: x \to x$. There are two relations: $a = b \circ d$ and $b = a \circ d$. The first relation allows us to eliminate a from the list of generators. Substituting the expression for a into the second relation, we get $b = b \circ d \circ d$. Since b is invertible, the latter relation is equivalent to $\mathrm{id}_x = d \circ d$. To summarize, the fundamental groupoid of the real projective plane is freely generated by two objects, a morphism $b: x \to y$, and a morphism $d: x \to x$ such that $d \circ d = \mathrm{id}_x$. In particular, we can extract the fundamental group $\pi_1(X, x)$ as the group generated by a single generator d with a relation $d^2 = 1$. This is the group $\mathbf{Z}/2$.

Example 29.23. We compute the fundamental group of the nonorientable surface with g crosscaps, with the basepoint v:



The spanning tree has a single edge that connects u and v. The $2g$ edges connecting v to u are marked b_i and c_i. According to the recipe, we have relations

$$a_i b_i = c_i,$$

$$a_i c_i = b_{i+1},$$

and

$$b_0 = 1.$$

By a telescoping argument, we get

$$c_0 = a_0, \qquad b_1 = a_0 a_0, \qquad c_1 = a_1 a_0 a_0, \qquad b_2 = a_1 a_1 a_0 a_0,$$

and in general,

$$c_i = a_i a_{i-1}^2 \cdots a_0^2,$$

$$b_i = a_{i-1}^2 \cdots a_0^2,$$

with the final relation being

$$b_0 = a_{g-1}^2 \cdots a_0^2.$$

Thus, b_i and c_i can be expressed via a_i, so the only relation that remains is

$$a_{g-1}^2 \cdots a_0^2 = 1.$$

Thus, the fundamental group with respect to the basepoint v is

$$\langle a_0, \ldots, a_{g-1} \mid a_{g-1}^2 \cdots a_0^2 = 1 \rangle.$$

Exercise 29.24. For each of the simplicial sets listed in Exercise 15.13 as well as for the lens spaces, compute its fundamental group and fundamental groupoid with respect to the basepoint.

We now explain how the fundamental group depends on the basepoint.

Proposition 29.25. (*Fundamental group as a functor of the basepoint.*) For any simplicial set X there is a functor

$$\pi_{\leq 1}(X) \to \mathsf{Group}$$

that sends an vertex $x \in X_0$ to the group $\pi_1(X, x)$ and a path $p\colon x \to y$ to the homomorphism of groups

$$\pi_1(X, x) \to \pi_1(X, y)$$

given by the formula

$$g \mapsto pgp^{-1}.$$

Used in 32.13*.

Proof. The given map is indeed a homomorphism:

$$gg' \mapsto pgg'p^{-1} = pgp^{-1}pg'p^{-1}.$$

It is also functorial:

$$(p'p)g(p'p)^{-1} = p'(pgp^{-1})(p')^{-1}.$$

Identities are preserved, so it a functor. ▮

Corollary 29.26. If $X \in \mathsf{sSet}$ is connected, all fundamental groups are isomorphic. These isomorphisms are noncanonical and depend on the choice of a connecting path.

30 Adjoint functors

Adjoint functors are omnipresent in mathematics.

Definition 30.1. Suppose $\mathsf{L}\colon \mathsf{C} \to \mathsf{D}$ and $\mathsf{R}\colon \mathsf{D} \to \mathsf{C}$ are functors. We say that L is *left adjoint* to R (alias R is *right adjoint* to L, or simply $\mathsf{L} \dashv \mathsf{R}$) if there is a natural isomorphism $u \to v$, where both u and v are functors of the form

$$\mathsf{C}^{\mathsf{op}} \times \mathsf{D} \to \mathsf{Set}$$

and

$$u(c,d) = \mathrm{hom}_{\mathsf{D}}(\mathsf{L}(c), d), \qquad v(c,d) = \mathrm{hom}_{\mathsf{C}}(c, \mathsf{R}(d)).$$

Slightly more informally, we say that there is a natural isomorphism

$$\mathrm{hom}_{\mathsf{D}}(\mathsf{L}(c), d) \to \mathrm{hom}_{\mathsf{C}}(c, \mathsf{R}(d)),$$

and even more informally, we say that morphisms

$$f\colon \mathsf{L}(c) \to d$$

in the category D can be identified with morphisms

$$g\colon c \to \mathsf{R}(d)$$

in the category C. This is known as the *universal property of adjoint functors*. In this case we say that f is a *left adjunct* of g and g is a *right adjunct* of f. Used in 39.2, 39.6*, 39.7*, 42.1*, 48.1, 48.2, 48.5.

Example 30.2. Consider $\mathsf{C} = \mathsf{Set}$, $\mathsf{D} = \mathsf{Ab}$, $\mathsf{L}\colon \mathsf{Set} \to \mathsf{Ab}$ the free abelian group functor, $\mathsf{R}\colon \mathsf{Ab} \to \mathsf{Set}$ the forgetful functor. The L is left adjoint to R. Indeed, homomorphisms of abelian groups $\mathsf{L}(S) \to A$ are uniquely determined by their values on the basis elements in S, i.e., by the map of sets $S \to \mathsf{R}(A)$.

Example 30.3. The functor $\pi_0\colon \mathsf{sSet} \to \mathsf{Set}$ is left adjoint to the functor $\mathrm{dis}\colon \mathsf{Set} \to \mathsf{sSet}$. Indeed, a map $X \to \mathrm{dis}(S)$ must be constant on each connected component of X, where it maps to some point of S. This is nothing else than a map $\pi_0(X) \to S$.

Exercise 30.4. What is the right adjoint functor of $\pi_{\le 1}$? Hint: it is closely related to the nerve construction.

Example 30.5. The functor $\mathsf{sSet} \to \mathsf{Cat}$ $(X \mapsto \Delta/X)$ is left adjoint to the functor $\mathsf{Cat} \to \Delta/X$ that sends a small category C to the simplicial set whose set of $\mathbf{m}$-simplices is $\mathrm{Fun}(\Delta/\Delta^{\mathbf{m}}, C)$.

Example 30.6. The functor $\mathsf{Cat} \to \mathsf{Grpd}$ $(C \to C[C^{-1}])$ that inverts all morphisms is left adjoint to the inclusion functor $\mathsf{Grpd} \to \mathsf{Cat}$. Indeed, a functor $C[C^{-1}] \to G$ is the same data as a functor $C \to G$ such that the image of any morphism in C is invertible in G. If G is a groupoid, the latter condition is trivial.

Example 30.7. The inclusion functor $\mathsf{Grpd} \to \mathsf{Cat}$ is left adjoint to the functor $\mathsf{Cat} \to \mathsf{Grpd}$ $(C \mapsto C^{\times})$ that removes all noninvertible morphisms from a category. Indeed, a functor $G \to C$ must land in invertible morphisms of C because functors preserves isomorphisms, and in a groupoid all morphisms are isomorphisms. Thus, the set of functors $G \to C$ and $G \to C^{\times}$ coincides.

Example 30.8. The colimit functor

$$\mathrm{colim}\colon \mathrm{Fun}(I, \mathsf{C}) \to \mathsf{C}$$

is left adjoint to the *constant diagram functor*

$$\mathrm{const}\colon \mathsf{C} \to \mathrm{Fun}(I, \mathsf{C})$$

that sends all objects to a given object of C and all morphisms to identities. The limit functor

$$\lim\colon \mathrm{Fun}(I, \mathsf{C}) \to \mathsf{C}$$

is right adjoint to the constant diagram functor $\mathsf{const}\colon \mathsf{C} \to \mathsf{Fun}(I, \mathsf{C})$. Used in 30.8, 30.9*, 42.1*.

Proposition 30.9. Suppose C and D are cocomplete categories, and $\mathsf{L}\colon \mathsf{C} \to \mathsf{D}$ is a functor that is left adjoint to a functor $\mathsf{R}\colon \mathsf{D} \to \mathsf{C}$. Then L is a cocontinuous functor.

First proof. Suppose $D\colon I \to \mathsf{C}$ is a diagram in C. We want to show that the canonical morphism

$$\mathsf{colim}(\mathsf{L} \circ D) \to \mathsf{L}(\mathsf{colim} D)$$

is an isomorphism.

We construct a map

$$\mathsf{L}(\mathsf{colim} D) \to \mathsf{colim}(\mathsf{L} \circ D)$$

by constructing a map

$$\mathsf{colim} D \to \mathsf{R}(\mathsf{colim}(\mathsf{L} \circ D)),$$

which itself can be constructed as a compatible family of maps

$$D(i) \to \mathsf{R}(\mathsf{colim}(\mathsf{L} \circ D))$$

that we can construct from maps

$$\iota_i^{\mathsf{L}\circ D}\colon \mathsf{L}(D(i)) \to \mathsf{colim}(\mathsf{L} \circ D)$$

given by the injection morphisms.

We verify that the composition

$$\mathsf{colim}(\mathsf{L} \circ D) \to \mathsf{L}(\mathsf{colim} D) \to \mathsf{colim}(\mathsf{L} \circ D)$$

equals the identity map. Indeed, pick an object $i \in I$ and consider the component map

$$\mathsf{L}(D(i)) \xrightarrow{\mathsf{L}(\iota_i^D)} \mathsf{L}(\mathsf{colim} D) \to \mathsf{colim}(\mathsf{L} \circ D).$$

Then the composition equals $\iota_i^{\mathsf{L}\circ D}$, as required.

We verify that the composition

$$\mathsf{L}(\mathsf{colim} D) \to \mathsf{colim}(\mathsf{L} \circ D) \to \mathsf{L}(\mathsf{colim} D)$$

equals the identity map. Indeed, passing to adjuncts, we get

$$\mathsf{colim} D \to \mathsf{R}(\mathsf{colim}(\mathsf{L} \circ D)) \to \mathsf{R}(\mathsf{L}(\mathsf{colim} D)).$$

Restricting to a single component, we get

$$D(i) \to \mathsf{R}(\mathsf{colim}(\mathsf{L} \circ D)) \to \mathsf{R}(\mathsf{L}(\mathsf{colim} D))$$

and passing to adjuncts again, we get

$$\mathsf{L}(D(i)) \to \mathsf{colim}(\mathsf{L} \circ D) \to \mathsf{L}(\mathsf{colim} D),$$

which we know to be precisely the injection morphism $\iota_i^{\mathsf{L}\circ D}$. ▮

Second proof. We want to show that the canonical morphism

$$\mathsf{colim}(\mathsf{L} \circ D) \to \mathsf{L}(\mathsf{colim} D)$$

is an isomorphism. By the weak Yoneda lemma applied to C^{op}, it suffices to show that for any object $Y \in \mathsf{D}$, the induced map of sets

$$\mathsf{hom}(\mathsf{colim}(\mathsf{L} \circ D), Y) \to \mathsf{hom}(\mathsf{L}(\mathsf{colim} D), Y)$$

is an isomorphism. Using the adjunction property, we may instead verify that the map of sets

$$\mathsf{hom}(\mathsf{L} \circ D, \mathsf{const}(Y)) \to \mathsf{hom}(\mathsf{colim} D, \mathsf{R}(Y))$$

is an isomorphism. Using the adjunction property one more time, we may instead verify that the map of sets

$$\mathsf{hom}(D, \mathsf{R} \circ \mathsf{const}(Y))) \to \mathsf{hom}(D, \mathsf{const}(\mathsf{R}(Y)))$$

is an isomorphism. But $\mathsf{R} \circ \mathsf{const}(Y) = \mathsf{const}(\mathsf{R}(Y))$, which completes the proof. ▮

Theorem 30.10. (The *weak Yoneda lemma.*) Given a small category C, the functor

$$\mathsf{Y}: C \to \mathsf{Fun}(C^{\mathsf{op}}, \mathsf{Set})$$

that sends $c \in C$ to the functor $d \mapsto \mathsf{hom}_{\mathsf{C}}(d, c)$ is a fully faithful functor. Likewise, substituting C^{op} for C, the functor

$$\mathsf{Y}: C^{\mathsf{op}} \to \mathsf{Fun}(C, \mathsf{Set})$$

that sends $c \in C$ to the functor $d \mapsto \mathsf{hom}_{\mathsf{C}}(c, d)$ is a fully faithful functor. Used in 30.9*.

Proof. This follows from the (strong) Yoneda lemma:

$$\mathsf{hom}(\mathsf{Y}(c), \mathsf{Y}(c')) = \mathsf{Y}(c')(c) = \mathsf{hom}(c, c'). \blacksquare$$

31 Fiber functors

Supplementary sources: §2.4 and §2.5 in Joyal and Tierney [NSHT].

First, we examine how simplicial maps can be analyzed through their fibers.

Definition 31.1. The functor

$$\mathsf{Fiber}: \mathsf{sSet}/Y \to \mathsf{Fun}(\Delta/Y, \mathsf{sSet})/c_Y$$

sends an object of sSet/Y, i.e., a simplicial map $f: X \to Y$, to the object of $\mathsf{Fun}(\Delta/Y, \mathsf{sSet})/c_Y$, i.e., a natural transformation $\mathsf{f}: \mathsf{F} \to c_Y$ of two functors $\Delta/Y \to \mathsf{sSet}$, where c_Y is the canonical diagram of Y and F is a functor that sends a simplex $s: \Delta^{\mathbf{m}} \to Y$ to the pullback $\Delta^{\mathbf{m}} \times_Y X$ and a morphism of simplices $h: \Delta^{\mathbf{m}} \to \Delta^{\mathbf{m}'}$ to the induced map $h \times_Y X: \Delta^{\mathbf{m}} \times_Y X \to \Delta^{\mathbf{m}'} \times_Y X$. The natural transformation f on a simplex $s: \Delta^{\mathbf{m}} \to Y$ is given by the projection $p_s: \Delta^{\mathbf{m}} \times_Y X \to \Delta^{\mathbf{m}}$. Used in 31.3, 31.5.

Definition 31.2. The functor

$$\mathsf{Assemble}: \mathsf{Fun}(\Delta/Y, \mathsf{sSet})/c_Y \to \mathsf{sSet}/Y$$

sends a natural transformation $\mathsf{f}: \mathsf{F} \to c_Y$ to its colimit, i.e., the map

$$\mathsf{colim}(\mathsf{f}): \mathsf{colim}(\mathsf{F}) \to \mathsf{colim}(c_Y) \cong Y,$$

where the last isomorphism is supplied by Proposition 28.4. For a morphism $\mathsf{F} \to \mathsf{F}'$ it yields $\mathsf{colim}(\mathsf{F}) \to \mathsf{colim}(\mathsf{F}')$. Used in 31.3, 31.5.

Proposition 31.3. The functor

$$\mathsf{Assemble}: \mathsf{Fun}(\Delta/Y, \mathsf{sSet})/c_Y \to \mathsf{sSet}/Y$$

is left adjoint to the functor

$$\mathsf{Fiber}: \mathsf{sSet}/Y \to \mathsf{Fun}(\Delta/Y, \mathsf{sSet})/c_Y.$$

Proof. This is nothing else than the universal property of colimits indexed by the category Δ/Y. ∎

Definition 31.4. A natural transformation $t: \mathsf{F} \to \mathsf{G}$ of functors $\mathsf{F}, \mathsf{G}: \mathsf{C} \to \mathsf{D}$ between categories that admits finite limits is *equifibered* if the commutative square

$$\begin{array}{ccc} \mathsf{F}(s) & \xrightarrow{\mathsf{F}(h)} & \mathsf{F}(s') \\ \downarrow t_s & & \downarrow t_{s'} \\ \mathsf{G}(s) & \xrightarrow{\mathsf{G}(h)} & \mathsf{G}(s') \end{array}$$

is cartesian for any morphism $h: s \to s'$ in C. Used in 31.5.

Proposition 31.5. The functor Fiber lands in the full subcategory of $\mathsf{Fun}(\Delta/Y, \mathsf{sSet})/c_Y$ consisting of equifibered natural transformations, denoted by $\mathsf{Equi}(Y)$. If we restrict the domain of $\mathsf{Assemble}$ to $\mathsf{Equi}(Y)$ and corestrict the codomain of Fiber to $\mathsf{Equi}(Y)$, the resulting adjunction is an equivalence of categories between $\mathsf{Equi}(Y)$ and sSet/Y. Used in 31.5.

Proof. The equifibration condition is necessary because the fiber of a fiber is again a fiber. ∎

Next, we consider yet another model for the fundamental groupoid.

Proposition 31.6. (*Category of simplices model for fundamental groupoids.*) We have a natural equivalence

$$\Delta/X[\Delta/X^{-1}] \to \pi_{\leq 1}(X),$$

where the left side denotes the image of the small category Δ/X under the functor

$$\mathsf{Cat} \to \mathsf{Grpd}$$

that inverts all morphisms. Used in 33.10.

Proof. The functor $\pi_{\leq 1}$ is cocontinuous by definition. Below we establish that the functor $\mathsf{sSet} \to \mathsf{Cat}$ ($X \mapsto \Delta/X$) is also cocontinuous, as well as the functor $\mathsf{Cat} \to \mathsf{Grpd}$. Thus their composition is cocontinuous, so $X \mapsto \Delta/X[\Delta/X^{-1}]$ is a cocontinuous functor $\mathsf{sSet} \to \mathsf{Grpd}$. Thus, to construct a natural equivalence of functors $\mathsf{sSet} \to \mathsf{Grpd}$,

$$\Delta/X[\Delta/X^{-1}] \to \pi_{\leq 1}(X),$$

it suffices to construct a natural equivalence of functors $\Delta \to \mathsf{Grpd}$,

$$\Delta/\Delta^{\mathbf{m}}[\Delta/\Delta^{\mathbf{m}^{-1}}] \to [m]^{\to}.$$

Given an object of the left side, i.e., a simple $\Delta^{\mathbf{k}} \to \Delta^{\mathbf{m}}$ of $\Delta^{\mathbf{m}}$, we send it to its last vertex, which is an object of the right side. On morphisms, the map is trivial. In the opposite direction, objects of the right side map to the corresponding vertices on the left side. A morphism $v \to v'$ maps to the composition of $v \to e$ and the inverse of $v' \to e$, where e is the edge that contains v and v'. ▮

32 Coverings

Supplementary sources: §2.4 and §2.5 in Joyal and Tierney [NSHT].

Definition 32.1. The *category of coverings* Cov/Y of a simplicial set Y is the full subcategory of sSet/Y consisting of *coverings* of Y, defined as simplicial maps $f\colon X \to Y$ such that any commutative square

$$\begin{array}{ccc} \Delta^{\mathbf{0}} & \longrightarrow & X \\ {\scriptstyle k}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \Delta^{\mathbf{m}} & \longrightarrow & Y \end{array}$$

has a unique diagonal filler that makes both triangles commute. Used in 32.2, 32.3, 32.4, 32.6, 32.7*, 32.8, 32.9, 32.10, 32.12, 32.13, 32.14, 32.15, 33.4, 33.6.

Proposition 32.2. A simplicial map $f\colon X \to Y$ is a covering if and only if for any $s\colon \Delta^{\mathbf{m}} \to Y$ we have a commutative triangle

$$\begin{array}{ccc} \Delta^{\mathbf{m}} \times_Y X & \xrightarrow{\cong} & \Delta^{\mathbf{m}} \times \operatorname{dis}(S) \\ & {\scriptstyle p_Y}\searrow \quad \swarrow{\scriptstyle s \circ p_{\Delta^{\mathbf{m}}}} & \\ & Y, & \end{array}$$

where the top map is an isomorphism and S is a set that will turn out to be canonically isomorphic to $\pi_0(\Delta^{\mathbf{m}} \times_Y X)$. The map p_Y is supplied by the definition of cartesian squares. It can be obtained by projecting onto either $\Delta^{\mathbf{m}}$ or X and then mapping to Y. Used in 32.7*.

Lemma 32.3. If $f\colon X \to Y$ is a covering, then for any $\Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}} \to Y$ the map of sets $\pi_0(\Delta^{\mathbf{m}} \times_Y X) \to \pi_0(\Delta^{\mathbf{n}} \times_Y X)$ is an isomorphism. Used in 32.5*.

Definition 32.4. The *monodromy functor*

$$\mathsf{Mono}\colon \mathsf{Cov}/Y \to \mathsf{Fun}(\pi_{\leq 1}(Y), \mathsf{Set})$$

sends a covering $f\colon X \to Y$ to the functor $\pi_{\leq 1}(Y) \to \mathsf{Set}$ that sends a vertex $v\colon \Delta^{\mathbf{0}} \to Y$ to the set $v \times_Y X$ and an edge $e\colon \Delta^{\mathbf{1}} \to Y$ from v_0 to v_1 to the composition below (where the map going from right to left is replaced by its inverse):

$$v_0 \times_Y X \cong \pi_0(v_0 \times_Y X) \to \pi_0(e \times_Y X) \leftarrow \pi_0(v_1 \times_Y X) \cong v_1 \times_Y X.$$

The monodromy functor sends a morphism of coverings $f_1 \to f_2$ over Y to the natural transformation with components $v \times_Y X_1 \to v \times_Y X_2$. Used in 32.8, 32.8*, 32.9, 32.10.

Lemma 32.5. The monodromy functor is well-defined.

Proof. Given a triangle $t\colon \Delta^{\mathbf{2}} \to Y$, we have to verify that the corresponding triangular diagram commutes. This is obvious from Lemma 32.3. ▮

Definition 32.6. The *reconstruction functor*

$$\mathsf{Recon}\colon \mathsf{Fun}(\pi_{\leq 1}(Y), \mathsf{Set}) \to \mathsf{Cov}/Y$$

is defined as follows. First, given a functor

$$\mathsf{M}\colon \pi_{\leq 1}(Y) \to \mathsf{Set},$$

construct a functor

$$\mathsf{F}\colon \Delta/Y \to \mathsf{sSet}$$

such that $\mathsf{F}(s) = \mathsf{M}(s_L) \times \Delta^{\mathbf{m}}$ for any simplex $s\colon \Delta^{\mathbf{m}} \to Y$, where s_L denotes the last vertex of s. Given a morphism $s \to s'$ of Δ/Y (with a map of simplices $\sigma\colon \Delta^{\mathbf{m}} \to \Delta^{\mathbf{m}'}$), the functor F sends it to the induced map

$$\mathsf{M}(s_L) \times \Delta^{\mathbf{m}} \xrightarrow{\mathsf{M}(e_L)\times\sigma} \mathsf{M}(s'_L) \times \Delta^{\mathbf{m}'}.$$

There is a natural transformation

$$p\colon \mathsf{F} \to c_Y,$$

where

$$c_Y\colon \Delta/Y \to \mathsf{sSet}$$

is the canonical diagram of Y. We set

$$\mathsf{Recon}(\mathsf{M}) = \mathsf{colim}(p)\colon \mathsf{colim}(\mathsf{F}) \to \mathsf{colim}(c_Y) \cong Y.$$

Used in 32.6, 32.7*, 32.8, 32.8*, 33.4.

Lemma 32.7. The reconstruction functor is well-defined.

Proof. By Proposition 32.2, given a functor

$$\mathsf{M}\colon \pi_{\leq 1}(Y) \to \mathsf{Set},$$

it suffices to show that the fiber of

$$\mathsf{Recon}(\mathsf{M}) \in \mathsf{Cov}/Y$$

over some simplex $s\colon \Delta^{\mathbf{m}} \to Y$ (i.e., $s \in \Delta/Y$) is isomorphic to the simplicial set

$$\mathsf{F}(s) = \mathsf{M}(s_L) \times \Delta^{\mathbf{m}},$$

where s_L denotes the last vertex of s. Indeed, by construction we have $\mathsf{F}(s) = \mathsf{M}(s_L) \times \Delta^{\mathbf{m}}$. When passing to $\mathsf{colim}(\mathsf{F})$, the fiber over s remains the same because all structure maps in the diagram

$$\mathsf{F}\colon \Delta/Y \to \mathsf{sSet}$$

are monomorphisms. ▮

Proposition 32.8. The functors

$$\mathsf{Mono}\colon \mathsf{Cov}/Y \to \mathsf{Fun}(\pi_{\le 1}(Y), \mathsf{Set})$$

and

$$\mathsf{Recon}\colon \mathsf{Fun}(\pi_{\le 1}(Y), \mathsf{Set}) \to \mathsf{Cov}/Y$$

form an equivalence of categories.

Proof. Given a covering $f\colon X \to Y$, we construct a natural isomorphism

$$\mathsf{Recon}(\mathsf{Mono}(f)) \to f$$

using the universal property of colimits.

Given a functor $\mathsf{M}\colon \pi_{\le 1}(Y) \to \mathsf{Set}$, we construct a natural isomorphism

$$\mathsf{M} \to \mathsf{Mono}(\mathsf{Recon}(\mathsf{M}))$$

by defining its component indexed by an object $v \in \pi_{\le 1}(Y)$ as the isomorphism

$$\mathsf{M}(v) \to v \times_Y \mathsf{Recon}(\mathsf{M}) \cong \mathsf{M}(v) \times \Delta^0 \cong \mathsf{M}(v). \blacksquare$$

Corollary 32.9. (Classification of coverings of connected simplicial set.) Suppose $Y \in \mathsf{sSet}$ is a connected simplicial set and $y \in Y_0$ is a vertex of Y. Then the monodromy functor induces an equivalence of categories

$$\mathsf{Mono}\colon \mathsf{Cov}/Y \to \mathsf{Fun}(\pi_{\le 1}(Y), \mathsf{Set}) \to \mathsf{Fun}(\mathrm{B}\pi_1(Y, y), \mathsf{Set}) = \mathsf{Set}^{\pi_1(Y,y)},$$

where $\mathsf{Set}^{\pi_1(Y,y)}$ denotes the category of sets equipped with an action of the group $\pi_1(Y, y)$.

Corollary 32.10. (Classification of connected coverings of a connected simplicial set.) Suppose $Y \in \mathsf{sSet}$ is a connected simplicial set and $y \in Y_0$ is a vertex of Y. Then the monodromy functor induces an equivalence of categories

$$\mathsf{CCov}/Y \to \mathsf{Orb}_{\pi_1(Y,y)},$$

where CCov/Y denotes the full subcategory of Cov/Y consisting of coverings with connected total space (alias *connected coverings*) and Orb_G denotes the category of *orbits* of a group G, i.e., nonempty sets equipped with a transitive action of G. Orbits are uniquely determined by their stabilizer groups, so connected coverings correspond to subgroups of $\pi_1(Y, y)$. Used in 32.10, 32.11, 32.15, 32.18, 32.19.

Remark 32.11. (*Orbits via stabilizers.*) Recall the following alternative description of the category Orb_G:
- objects are subgroups $H \subset G$;
- morphisms $H_1 \to H_2$ are elements $[g] \in G/H_2$ such that $H_1 \subset gH_2g^{-1}$.

In particular, the group of automorphisms of H is precisely the group $N_G(H)/H$.

An equivalence to the category Orb_G is supplied by the functor that sends $H \subset G$ to the G-orbit G/H and a morphism $[g]\colon H_1 \to H_2$ to the map $G/H_1 \to G/H_2$ that sends $[x] \mapsto [gx]$. The latter formula descends to equivalence classes and does not depend on the choice of g because $H_1 \subset gH_2g^{-1}$. Used in 32.19*.

Lemma 32.12. If $f\colon X \to Y$ is a covering and both X and Y are connected, then for any $x \in X_0$ the homomorphism of groups

$$\pi_1(f, x)\colon \pi_1(X, x) \to \pi_1(Y, f(x))$$

is injective.

Proof. If two elements of $\pi_1(X, x)$ map to the same elements of $\pi_1(Y, f(x))$, then the representing loops in X map to homotopic loops in Y. This homotopy can be lifted to X using the unique lifting property. $\blacksquare$

Lemma 32.13. If $f\colon X \to Y$ is a covering and both X and Y are connected, then for any $x, x' \in X_0$ such that $f(x) = f(x')$ the homomorphisms of groups

$$\pi_1(f,x)\colon \pi_1(X,x) \to \pi_1(Y, f(x))$$

and

$$\pi_1(f,x')\colon \pi_1(X,x') \to \pi_1(Y, f(x))$$

yield conjugate subgroups of $\pi_1(Y, f(x))$ as images. Vice versa, conjugating the subgroup $\pi_1(X,x) \subset \pi_1(Y, f(x))$ by any element of $\pi_1(Y, f(x))$ produces a subgroup of the form $\pi_1(X, x')$ for some $x' \in X_0$ such that $f(x') = f(x)$.

Proof. Since X is connected, there is a morphism $p\colon x \to x'$ in $\pi_{\le 1}(X)$. According to Proposition 29.25, p induces an isomorphism $\pi_1(p)\colon \pi_1(X,x) \to \pi_1(X,x')$ given by $g \mapsto pgp^{-1}$. The image of p in $\pi_{\le 1}(Y)$ is an automorphism of $f(x) = f(x')$, i.e., an element of $\pi_1(Y, f(x))$. Thus, taking the images under $\pi_{\le 1}(f)$, the isomorphism $\pi_1(p)$ becomes the conjugation by the element $\pi_{\le 1}(p) \in \pi_1(Y, f(x))$.

Vice versa, given an element $q \in \pi_1(Y, f(x))$, we can lift $q\colon f(x) \to f(x)$ to an isomorphism $p\colon x \to x'$ in $\pi_{\le 1}(X)$, which induces an isomorphism $\pi_1(X,x) \to \pi_1(X,x')$ whose image under $\pi_{\le 1}(f)$ is the conjugation by q. ∎

Proposition 32.14. Suppose Y is a connected simplicial set with a vertex $y \in Y_0$. Given a connected covering $X \to Y$ corresponding to a G-orbit O (which is isomorphic to the fiber over y), the fundamental group of X with respect to a vertex $x \in X_0$ over y is isomorphic to the stabilizer group of x in O.

Proof. Any loop in Y from y to itself lifts to a path in X from x to some point in the fiber over y. This points equals x if and only if the corresponding element of $\pi_1(Y, y)$ acts trivially on x, i.e., it belongs to the stabilizer group. ∎

Corollary 32.15. Suppose Y is a connected simplicial set with a vertex $y \in Y_0$. The covering $X \to Y$ corresponding to the left regular action of $\pi_1(Y, y)$ on itself has a simply connected total space, i.e., $\pi_{\le 1}(X)$ is trivial. The covering $X \to Y$ is known as the *universal covering* of X. The choice of terminology is justified by the fact that any other connected covering is a quotient of the universal covering because any G-orbit is a quotient of the universal orbit G. If we make the base space and total space pointed, then the requisite maps become unique. Used in 32.15, 32.16.

Example 32.16. Consider the real projective plane. Its fundamental group has been computed as $\mathbf{Z}/2$. Thus, apart from the trivial connected covering, which corresponds to a singleton orbit, it only has a universal covering, corresponding to the left action of $\mathbf{Z}/2$ on itself.

Definition 32.17. The *deck transformation group* of a (typically connected) covering $f\colon X \to Y$ is the automorphism group of f in the category of coverings of Y. Used in 32.18.

Definition 32.18. A *Galois covering* (alias *normal covering*) is a connected covering $f\colon X \to Y$ such that the action of the deck transformation group on the fiber of one (hence any) $y \in Y_0$ is transitive.

Proposition 32.19. A connected covering $f\colon X \to Y$ is Galois if and only if the corresponding subgroup $\pi_1(X,x) \subset \pi_1(Y, f(x))$ is normal.

Proof. If the subgroup is normal, then the quotient orbit $\pi_1(Y, f(x))/\pi_1(X,x)$ is itself a group, and its automorphism group is the normalizer of $\pi_1(X,x)$, which coincides with $\pi_1(Y, f(x))$ and hence acts transitively.

Vice versa, if the deck transformation group acts transitively, by Remark 32.11 the normalizer of $\pi_1(X,x)$ must coincide with $\pi_1(Y, f(x))$, i.e., $\pi_1(X,x)$ must be a normal subgroup. ∎

33 Local systems

Definition 33.1. A *local system* (of abelian groups) over a simplicial set Y is a functor $\pi_{\leq 1}(Y) \to \mathsf{Ab}$. The *category of local systems* Loc/Y is the category of such functors. Used in 33.3, 33.4, 33.6, 33.7, 33.8, 33.10, 33.11, 33.12, 36.14, 37.3, 37.4.

Example 33.2. The *trivial local system* on Y with fiber $A \in \mathsf{Ab}$ is given by the constant functor $\pi_{\leq 1}(Y) \to \mathsf{Ab}$ that sends all objects to A and all morphisms to identities.

Example 33.3. The local system of twisted integers on the real projective plane sends all objects to $\mathbf{Z}$, and generating morphisms a, b, d are sent to the multiplication by -1, 1, and -1 respectively. The relations for the fundamental groupoid are satisfied, so this indeed defines a local system. Used in 33.12.

Definition 33.4. The forgetful functor

$$\mathsf{Loc}/Y \to \mathsf{Cov}/Y$$

sends a local system, i.e., a functor

$$\pi_{\leq 1}(Y) \to \mathsf{Ab}$$

to the composition

$$\pi_{\leq 1}(Y) \to \mathsf{Ab} \to \mathsf{Set}$$

and then applies the reconstruction functor

$$\mathsf{Recon}\colon \mathsf{Fun}(\pi_{\leq 1}(Y), \mathsf{Set}) \to \mathsf{Cov}/Y.$$

Proposition 33.5. Given a complete category C, the *category of group objects* in C is defined as the category of quadruples (G, m, i, u), where $G \in \mathsf{C}$ is an object of C,

$$m\colon G \times G \to G, \qquad i\colon G \to G \qquad u\colon 1 \to G$$

are morphisms in C (where 1 denotes the terminal object of C) such that the following diagrams commute:

$$\begin{array}{ccc} G\times G\times G & \xrightarrow{m\times G} & G\times G \\ {\scriptstyle G\times m}\downarrow & & \downarrow{\scriptstyle m} \\ G\times G & \xrightarrow[m]{} & G, \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\mathrm{id}} & G \\ {\scriptstyle d}\downarrow & & \uparrow{\scriptstyle m} \\ G\times G & \xrightarrow[\mathrm{id}\times i]{} & G\times G, \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\mathrm{id}} & G \\ {\scriptstyle d}\downarrow & & \uparrow{\scriptstyle m} \\ G\times G & \xrightarrow[i\times \mathrm{id}]{} & G\times G \end{array}$$

(associativity and left and right inverses) and

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{id}} & G \\ {\scriptstyle \cong}\downarrow & & \uparrow{\scriptstyle m} \\ G\times 1 & \xrightarrow[\mathrm{id}\times u]{} & G\times G, \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\mathrm{id}} & G \\ {\scriptstyle \cong}\downarrow & & \uparrow{\scriptstyle m} \\ 1\times G & \xrightarrow[u\times \mathrm{id}]{} & G\times G \end{array}$$

(left and right unitality). Furthermore, *abelian group objects* are distinguished by the following additional diagram that must be commutative:

$$\begin{array}{ccc} G\times G & & \\ & \searrow^{m} & \\ {\scriptstyle \gamma}\downarrow & & G, \\ & \nearrow_{m} & \\ G\times G & & \end{array}$$

where $\gamma\colon G\times G\to G\times G$ permutes the two factors. The *category of abelian group objects* is defined as the full subcategory of the category of group objects. Used in 33.5, 33.6.

Proposition 33.6. The category of local systems over $Y\in\mathsf{sSet}$ is equivalent to the category of abelian group objects in the category of coverings of Y.

Lemma 33.7. If L is a local system over a connected simplicial set Y, then all values of L are (noncanonically) isomorphic to each other. If all values of L are isomorphic to some abelian group A, we say that L has a (typical) fiber A.

Proposition 33.8. The category of local systems over a connected simplicial set Y with typical fiber A is equivalent to the category of representations of $\pi_1(Y,y)$ on A, where $y\in Y_0$.

Proof. We have an equivalence

$$\mathsf{B}\pi_1(Y,y)\to\pi_{\le 1}(Y).$$

Restriction along this inclusion produces an equivalence

$$\mathsf{Fun}(\pi_{\le 1}(Y),\mathsf{Ab})\to\mathsf{Fun}(\mathsf{B}\pi_1(Y,y),\mathsf{Ab})=\mathsf{Ab}^{\pi_1(Y,y)},$$

where the latter category is the category of abelian groups equipped with an action of $\pi_1(Y,y)$. ∎

Example 33.9. The fundamental group of the real projective plane is $\mathbf{Z}/2$. There are exactly two representations of $\mathbf{Z}/2$ on $\mathbf{Z}$: the trivial representation and the sign representation. The latter corresponds to a nontrivial covering with fiber $\mathbf{Z}$: as we go around the diagonal d, we get multiplied by -1. More generally, this construction works for an arbitrary abelian group A instead of $\mathbf{Z}$.

Definition 33.10. Suppose $L\colon\pi_{\le 1}(Y)\to\mathsf{Ab}$ is a local system on a simplicial set Y. According to Proposition 31.6, the functor

$$\Delta/Y[\Delta/Y^{-}1]\to\pi_{\le 1}(Y)$$

is an equivalence of categories. The composition

$$\Delta/Y\to\Delta/Y[\Delta/Y^{-1}]\to\pi_{\le 1}(Y)\to\mathsf{Ab}$$

is denoted by A. The *twisted simplicial chains* on Y with coefficients in L form a chain complex defined as follows. In degree n we place the direct sum (primes denote nondegenerate simplices)

$$\bigoplus_{\sigma\in Y'_n}A(\sigma).$$

The differentials are given by the alternating sum of face maps, as usual. The ith face map applied to σ is a homomorphism

$$A(p^{-1})\colon A(\sigma)\to A(\mathrm{d}_i\sigma),$$

where p is the unique morphism in $\pi_{\le 1}(Y)$ from the last vertex of $\mathrm{d}_i\sigma$ to the last vertex of σ. *Twisted simplicial cochains* are defined in a similar way, replacing direct sums with products, and defining the coboundary map as an alternating sum of coface maps. Used in 33.11, 33.12.

An analogous argument to Lemma 15.8 shows that we indeed have a chain complex.

Definition 33.11. The *twisted homology* of a simplicial set Y with coefficients in a local system L of abelian groups on Y is the homology of the twisted simplicial chains on Y with coefficients in L. Likewise for the *twisted cohomology.* Used in 33.12, 36.16.

Example 33.12. We compute the twisted homology of the real projective plane with coefficients in the local system constructed in Example 33.3. The twisted simplicial chains are

$$A\xleftarrow{\;2\;}A\xleftarrow{\;0\;}A.$$

Its homology is

$$A/2A,\qquad \mathsf{Tor}(\mathbf{Z}/2,A),\qquad A.$$

This is different from the untwisted homology, which is

$$A,\qquad A/2A,\qquad \mathsf{Tor}(\mathbf{Z}/2,A).$$

34 Function complexes

Given two sets X and Y, we can construct another set $\mathsf{hom}(X,Y)$, whose elements are maps $X \to Y$. There is a natural isomorphism between the set of maps of the form $W \to \mathsf{hom}(X,Y)$ and the set of maps of the form $W \times X \to Y$, i.e., an isomorphism

$$\mathsf{hom}(W, \mathsf{hom}(X,Y)) \to \mathsf{hom}(W \times X, Y).$$

Given a map $f\colon W \to \mathsf{hom}(X,Y)$, we send it to the map $g\colon W \times X \to Y$ such that $g(w,x) = f(w)(x)$. Given a map $g\colon W \times X \to Y$, we send it to the map $f\colon W \to \mathsf{hom}(X,Y)$ such that $f(w)(x) = g(w,x)$. These maps are manifestly mutually inverse to each other.

We would like to extend such a construction to simplicial sets, i.e., given $X, Y \in \mathsf{sSet}$, we would like to construct $\mathsf{Hom}(X,Y) \in \mathsf{sSet}$ such that there is a natural isomorphism

$$\mathsf{hom}(W, \mathsf{Hom}(X,Y)) \to \mathsf{hom}(W \times X, Y).$$

Recall that $\mathsf{hom}(X,Y)$ denotes the set of simplicial maps $X \to Y$.

We substitute $W = \Delta^{\mathbf{n}}$ in the above isomorphism, obtaining

$$\mathsf{hom}(\Delta^{\mathbf{n}}, \mathsf{Hom}(X,Y)) \to \mathsf{hom}(\Delta^{\mathbf{n}} \times X, Y).$$

By the Yoneda lemma, the left side is isomorphic to $\mathsf{Hom}(X,Y)_n$, the set of n-simplices of the simplicial set $\mathsf{Hom}(X,Y)$. The right side only uses hom, whose definition is known to us. Thus, we can define the left side as the right side.

Definition 34.1. Given simplicial sets $X, Y \in \mathsf{sSet}$, the *internal hom* (alias *function complex* or *mapping simplicial set*) $\mathsf{Hom}(X,Y)$ is a simplicial set such that

$$\mathsf{Hom}(X,Y)_n = \mathsf{hom}(\Delta^{\mathbf{n}} \times X, Y)$$

and for a map of simplices $f\colon \mathbf{m} \to \mathbf{n}$ the simplicial structure map

$$\mathsf{Hom}(X,Y)_f\colon \mathsf{hom}(\Delta^{\mathbf{n}} \times X, Y) \to \mathsf{hom}(\Delta^{\mathbf{m}} \times X, Y)$$

is the map

$$\mathsf{hom}(\Delta^f \times X, Y)\colon \mathsf{hom}(\Delta^{\mathbf{n}} \times X, Y) \to \mathsf{hom}(\Delta^{\mathbf{m}} \times X, Y).$$

Used in 7.5, 35.2*, 40.12.

Remark 34.2. As an important special case of the above definition, we obtain a natural isomorphism

$$\mathsf{Hom}(X,Y)_{\mathbf{0}} = \mathsf{hom}(\Delta^{\mathbf{0}} \times X, Y) \cong \mathsf{hom}(X,Y),$$

i.e., 0-simplices of $\mathsf{Hom}(X,Y)$ can be naturally identified with simplicial maps $X \to Y$.

Proposition 34.3. (The universal property of internal homs.) For any simplicial sets X, Y, Z there is a natural bijection between the set of maps of the form

$$X \to \mathsf{Hom}(Y,Z)$$

and the set of maps of the form

$$X \times Y \to Z.$$

In other words, the functor

$$\mathsf{sSet} \to \mathsf{sSet}, \qquad X \mapsto X \times Y$$

is left adjoint to the functor

$$\mathsf{sSet} \to \mathsf{sSet}, \qquad Z \mapsto \mathsf{Hom}(Y,Z).$$

Definition 34.4. The functor
$$\mathsf{Hom}\colon \mathsf{sSet}^{\mathsf{op}} \times \mathsf{sSet} \to \mathsf{sSet}$$
sends a pair of simplicial sets $X, Y \in \mathsf{sSet}$ to $\mathsf{Hom}(X, Y)$ and a pair of simplicial maps $f\colon X \leftarrow X'$, $g\colon Y \to Y'$ to a simplicial map
$$\mathsf{Hom}(f, g)\colon \mathsf{Hom}(X', Y) \to \mathsf{Hom}(X, Y')$$
whose component in degree n
$$\mathsf{Hom}(f, g)_n\colon \mathsf{Hom}(X', Y)_n \to \mathsf{Hom}(X, Y')_n$$
is the map
$$\mathsf{hom}(\Delta^{\mathbf{n}} \times f, g)_n\colon \mathsf{hom}(\Delta^{\mathbf{n}} \times X', Y) \to \mathsf{hom}(\Delta^{\mathbf{n}} \times X, Y').$$

Proposition 34.5. We have natural isomorphisms
$$X \to \mathsf{Hom}(\Delta^{\mathbf{0}}, X)$$
and
$$\Delta^{\mathbf{0}} \to \mathsf{Hom}(X, \Delta^{\mathbf{0}}).$$

Proof. The codomain of the first map is the value of the right adjoint functor $Z \mapsto \mathsf{Hom}(\Delta^{\mathbf{0}}, Z)$ on $Z = X$, so the first map is adjoint to the isomorphism
$$\Delta^{\mathbf{0}} \times X \to X.$$
The natural map
$$X \to \mathsf{Hom}(\Delta^{\mathbf{0}}, X)$$
has as its nth component an isomorphism of sets
$$X_n \to \mathsf{hom}(\Delta^{\mathbf{n}} \times \Delta^{\mathbf{0}}, X) \cong \mathsf{hom}(\Delta^{\mathbf{n}}, X) \cong X_n,$$
hence the simplicial map is itself an isomorphism.

Likewise, the second map is adjoint to the isomorphism
$$X \times \Delta^{\mathbf{0}} \to X.$$
The simplicial map
$$\Delta^{\mathbf{0}} \to \mathsf{Hom}(X, \Delta^{\mathbf{0}})$$
has as its nth component an isomorphism of sets
$$1 \to \mathsf{hom}(\Delta^{\mathbf{n}} \times X, \Delta^{\mathbf{0}}) \cong 1,$$
hence the simplicial map is itself an isomorphism. ∎

Proposition 34.6. The functor $\mathsf{Hom}\colon \mathsf{sSet}^{\mathsf{op}} \times \mathsf{sSet} \to \mathsf{sSet}$ preserves limits separately in each argument. This means that the canonical maps
$$\mathsf{Hom}(X, \mathsf{lim} D) \to \mathsf{lim}_{i \in I} \mathsf{Hom}(X, D_i)$$
and
$$\mathsf{Hom}(\mathsf{colim} D, X) \to \mathsf{lim}_{i \in I} \mathsf{Hom}(D_i, X)$$
are isomorphisms. (For the second map, recall that limits in $\mathsf{sSet}^{\mathsf{op}}$ are precisely colimits in sSet, which is what we used for the left side.)

35 Homotopies, homotopy equivalences, and invariance of homology

Definition 35.1. A *simplicial homotopy* between simplicial maps $f, g\colon X \to Y$ is a simplicial map

$$h\colon \Delta^{\mathbf{1}} \times X \to Y$$

such that

$$f = h \circ (\mathrm{d}^1 \times X)$$

and

$$g = h \circ (\mathrm{d}^0 \times X).$$

Used in 35.2, 35.3, 35.6, 35.7*.

Proposition 35.2. Equivalently, a simplicial homotopy between $f, g\colon X \to Y$ is a simplicial map $h\colon X \to Y^{\Delta^{\mathbf{1}}}$ such that

$$f = Y^{\mathrm{d}^1} \circ h$$

and

$$g = Y^{\mathrm{d}^0} \circ h.$$

Another alternative: a simplicial homotopy between $f, g\colon X \to Y$ is a 1-simplex $h \in Y^X_{\mathbf{1}}$ whose two endpoints are f and g respectively.

Proof. This follows immediately from the universal property of internal hom: there is a natural bijective correspondence between simplicial maps of the form

$$\Delta^{\mathbf{1}} \times X \to Y,$$

$$\Delta^{\mathbf{1}} \to Y^X,$$

and

$$X \to Y^{\Delta^{\mathbf{1}}}. \blacksquare$$

Definition 35.3. A *simplicial homotopy equivalence* is a simplicial map $f\colon X \to Y$ such that there is a simplicial map $g\colon Y \to X$ so that there is a simplicial homotopy from id_X to $g \circ f$ and a simplicial homotopy from $f \circ g$ to id_Y. Used in 35.7, 35.12, 35.14, 39.10, 39.13, 39.18, 39.20, 39.21, 40.3, 41.2, 42.4, 45.2*, 45.5*, 45.8*, 46.2*, 46.4*.

Definition 35.4. A *continuous homotopy* between continuous maps $f, g\colon X \to Y$ of metric or topological spaces is a continuus map

$$h\colon [0,1] \times X \to Y$$

such that $h|_{0\times X} = f$ and $h|_{1\times X} = g$. Used in 35.5, 35.6.

Definition 35.5. A *continuous homotopy equivalence* is a continuous map $f\colon X \to Y$ such that there is a continuous map $g\colon Y \to X$ so that there is a continuous homotopy from id_X to $g \circ f$ and a continuous homotopy from $f \circ g$ to id_Y. Used in 35.7, 35.14.

Proposition 35.6. The singular simplicial set functor sends continuous homotopies to simplicial homotopies.

Proof. The singular simplicial set functor preserves limits, so we have an isomorphism

$$\mathsf{Sing}([0,1] \times X) \cong \mathsf{Sing}([0,1]) \times \mathsf{Sing}(X).$$

Thus we can write

$$\mathsf{Sing}(h)\colon \mathsf{Sing}([0,1]) \times \mathsf{Sing}(X) \to \mathsf{Sing}(Y).$$

Consider the simplicial map

$$\alpha\colon \Delta^{\mathbf{1}} \to \mathsf{Sing}([0,1])$$

that picks the singular simplex $|\mathbf{1}| \to [0,1]$ given by the obvious homeomorphism. Precomposing with the simplicial map $\alpha \times \mathsf{Sing}(X)$ yields a simplicial map

$$\Delta^{\mathbf{1}} \times \mathsf{Sing}(X) \to \mathsf{Sing}(Y),$$

which completes the proof. $\blacksquare$

Corollary 35.7. The singular simplicial set functor sends continuous homotopy equivalences to simplicial homotopy equivalences.

We now examine the behavior of simplicial homotopies unde the simplicial chain functor $\mathsf{C}\colon \mathsf{sSet} \to \mathsf{Ch}$. If $h\colon \Delta^{\mathbf{1}} \times X \to Y$ is such a homotopy, then after passing to simplicial chains, we can precompose with the Eilenberg–Zilber map for simplicial chains:

$$\mathsf{C}(\Delta^{\mathbf{1}}) \otimes \mathsf{C}(X) \xrightarrow{\nabla_{\Delta^{\mathbf{1}},X}} \mathsf{C}(\Delta^{\mathbf{1}} \times X) \xrightarrow{\mathsf{C}(h)} \mathsf{C}(Y).$$

Recall that $\mathsf{C}(\Delta^{\mathbf{1}})$ is the chain complex

$$\mathbf{Z}[0] \oplus \mathbf{Z}[0] \xleftarrow{-1,1} \mathbf{Z}[1].$$

Furthermore, the requirement that h is a simplicial homotopy from $f\colon X \to Y$ to $g\colon X \to Y$ now translates into the requirement that precomposing the above composition with the maps $\mathsf{C}(\mathrm{d}^1)\otimes\mathsf{C}(X)$ and $\mathsf{C}(\mathrm{d}^0)\otimes\mathsf{C}(X)$ yields $\mathsf{C}(f)$ respectively $\mathsf{C}(g)$. Observe that the above definition makes no use of the nature of the chain complexes $\mathsf{C}(X)$, $\mathsf{C}(Y)$ or the chain maps $\mathsf{C}(f)$, $\mathsf{C}(g)$. This motivates the following definition.

Definition 35.8. A chain homotopy between chain maps $f, g\colon C \to D$ is a chain map $h\colon \mathsf{C}(\Delta^{\mathbf{1}}) \otimes C \to D$ such that

$$f = h \circ (\mathsf{C}(\mathrm{d}^1) \otimes C)$$

and

$$g = h \circ (\mathsf{C}(\mathrm{d}^0) \otimes C).$$

The chain complex $\mathsf{C}(\Delta^{\mathbf{1}})$ has the abelian group $\mathbf{Z} \oplus \mathbf{Z}$ in degree 0, and the value of h on $\mathsf{C}(\Delta^{\mathbf{1}})_0 \otimes C$ is prescribed by the two conditions on f and g. Thus, the only remaining piece of data is the value of h on $\mathsf{C}(\Delta^{\mathbf{1}})_1 \otimes C \cong \mathbf{Z}[1] \otimes C$. This motivates the following alternative characterization of chain homotopies.

Definition 35.9. A *chain homotopy* between chain maps $f, g\colon C \to D$ is a family of homomorphisms of abelian groups $h_n\colon C_n \to D_{n+1}$ such that $\mathrm{d}_{n+1} \circ h_n + h_{n-1} \circ \mathrm{d}_n = g - f$. Used in 22.18*, 35.8, 35.8*, 35.10, 35.11, 35.13.

Lemma 35.10. There is a natural bijective correspondence between the two variants of chain homotopies.

Proof. Given a chain map

$$h\colon \mathsf{C}(\Delta^{\mathbf{1}}) \otimes C \to D$$

we evaluate it on the degree 1 part of $\mathsf{C}(\Delta^{\mathbf{1}})$, which is isomorphic to $\mathbf{Z}$, obtaining a family of homomorphisms of abelian groups

$$h_n\colon C_n \to D_{n+1}.$$

This, $h_n(c) = h(1_1 \otimes c)$, where $1_1 \in \mathsf{C}(\Delta^{\mathbf{1}})_1$ is a generator. Since h is a chain map, for any $c \in C_n$ we have

$$\mathrm{d}(h(1_1 \otimes c)) = h(\mathrm{d}(1_1 \otimes c)).$$

Expanding both sides, we get

$$\mathrm{d}_{n+1} h_n(c) = h((-1 \oplus 1) \otimes c - 1_1 \otimes \mathrm{d}c) = -f(c) + g(c) - h_{n-1}\mathrm{d}_n c,$$

which completes the construction.

In the opposite direction, given $h_n\colon C_n \to D_{n+1}$ such that $\mathrm{d}_{n+1} \circ h_n + h_{n-1} \circ \mathrm{d}_n = g - f$, we construct a chain map

$$h\colon \mathsf{C}(\Delta^{\mathbf{1}}) \otimes C \to D$$

by setting it to $f \oplus g$ on $\mathsf{C}(\Delta^{\mathbf{1}})_0 \otimes C$ and h_n on $\mathsf{C}(\Delta^{\mathbf{1}})_1 \otimes C_n$. It remains to see that this is a chain map, which is done by reversing the computation in the previous paragraph. ∎

Definition 35.11. A *chain homotopy equivalence* is a chain map $f: C \to D$ such that there is a chain map $g: D \to C$ so that there is a chain homotopy from id_C to $g \circ f$ and a chain homotopy from $f \circ g$ to id_D. Used in 15.7, 35.12, 35.14, 40.7.

Corollary 35.12. The simplicial chain functor sends simplicial homotopy equivalences to chain homotopy equivalences.

Proposition 35.13. If $f, g: C \to D$ are chain homotopic, then $\mathsf{H}(f) = \mathsf{H}(g)$.

Proof. We have to show that for any $c \in \mathsf{Z}_n(C)$, the chain $g(c) - f(c)$ is a boundary, so that $\mathsf{H}(f)([c]) = \mathsf{H}(g)([c])$, i.e., $\mathsf{H}(f) = \mathsf{H}(g)$. Indeed,

$$g(c) - f(c) = \mathrm{d}_{n+1}(h_n(c)) + h_{n-1}(\mathrm{d}_n(c)) = \mathrm{d}_{n+1}(h_n(c)) + h_{n-1}(0) = \mathrm{d}_{n+1}(h_n(c)),$$

which completes the proof. ∎

Corollary 35.14. The homology functor sends chain homotopy equivalences to isomorphisms. The simplicial homology functor sends simplicial homotopy equivalences to isomorphisms. The singular homology functor sends continuous homotopy equivalences to isomorphisms.

Manifolds

36 Combinatorial manifolds

Definition 36.1. The *star* of an n-simplex $x \in X_{\mathbf{n}}$ in a simplicial set X is the simplicial subset $\mathrm{star}(x)$ of X generated by all simplices of X that contain x. The *link* of x is the simplicial subset $\mathrm{link}(x)$ of the star of x generated by all simplices of the star of X that do not contain any vertex of x. Used in 36.1, 36.2, 36.4, 36.5, 36.13, 36.15, 36.16*.

Definition 36.2. A *bistellar move* in a simplicial set X replaces $a \star \partial b$ with $\partial a \star b$, where $a: \Delta^r \to X$ and $\partial b: \partial\Delta^{n-r} \to X$ are injective simplicial maps such that ∂b is the link of a and $a \star \partial b$ is the star of a. Used in 36.2*, 36.3.

In 1991 Udo Pachner established that bistellar moves are sufficient to relate any two piecewise linearly homeomorphic manifolds. We take his result as a definition.

Definition 36.3. Two simplicial sets are *combinatorially equivalent* if they can be related by a finite sequence of bistellar moves and reorientations of simplices. Used in 36.4, 36.5, 36.9*.

Definition 36.4. A *combinatorial manifold* is a simplicial set X such that the link of any r-simplex in $\mathrm{sd}(\mathrm{sd}(X))$ is combinatorially equivalent to $\partial\Delta^{n-r}$ for some $n \geq 0$. Used in 36.9*, 36.10, 36.11, 36.13, 36.16, 37.0*, 37.4.

Definition 36.5. A *combinatorial manifold with boundary* is a simplicial set X such that the star of any r-simplex in $\mathrm{sd}(\mathrm{sd}(X))$ is combinatorially equivalent to Δ^{n-r} for some $n \geq 0$. The *boundary of a combinatorial manifold* X is the simplicial subset of X consisting of all simplices $x: \Delta^{\mathbf{r}} \to X$ of X such that the link of any simplex in the image of $\mathrm{sd}(\mathrm{sd}(x)): \mathrm{sd}(\mathrm{sd}(\Delta^{\mathbf{n}})) \to X$ is combinatorially equivalent to Δ^{n-r-1}.

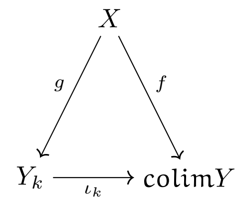
Remark 36.6. The number n that appears in the definition is constant on every connected component of X and is known as the *dimension* of that connected component. In particular, all nondegenerate simplices have dimension at most n and are faces of some nondegenerate n-simplex.

Definition 36.7. A simplicial set is *compact* if it has finitely many nondegenerate simplices. A simplicial set is *locally compact* if any simplex is contained in finitely many nondegenerate simplices. Used in 36.8, 36.9, 36.11, 36.12, 37.4, 39.6*, 39.7*.

Proposition 36.8. A simplicial set X is compact if and only if for any infinite chain

$$Y_0 \to Y_1 \to Y_2 \to \cdots$$

and any simplicial map $f\colon X \to \mathsf{colim} Y$ there is $k \geq 0$ and a map $g\colon X \to Y_k$ such that the following diagram commutes:



Used in 39.7*.

Proposition 36.9. Suppose X is a connected compact n-manifold. Then $\mathsf{H}_n(X, A)$ is isomorphic to either A or $\mathsf{Tor}(\mathbf{Z}/2, A)$. In the former case the isomorphism is canonical up to a sign, and in the latter case it is canonical. We say that X is *orientable* if the former holds and *nonorientable* if the latter holds. In the orientable case, the choice of one of two canonical isomorphisms $\mathsf{H}_n(X, A) \cong A$ is known as an *orientation* of X. Additionally, if A is a ring, the element of $\mathsf{H}_n(X, A) = \mathsf{Z}_n(X, A)$ corresponding to $1 \in A$ under the isomorphism $\mathsf{H}_n(X, A) \cong A$ is known as the *fundamental class* or the *fundamental cycle* of X. Used in 24.10, 24.11, 36.9.

Proof. Replace X with its second barycentric subdivision, which induces an isomorphism on homology. By definition of a combinatorial manifold, the link of any $(n-1)$-simplex σ in X is combinatorially equivalent to $\partial\Delta^1$, i.e., two points. Combinatorial equivalences do nothing to 0-simplices, so the link of σ also consists of two points. This means that the star of σ consists of two n-simplices glued along their common $(n-1)$-dimensional face. The two vertices not in this face form the link of σ.

Since X has no nondegenerate simplices of dimension higher than n, we have $\mathsf{H}_n(X, A) = \mathsf{Z}_n(X, A)$. The boundary of an n-chain s vanishes if and only if its coefficient before any $(n-1)$-simplex σ vanishes. The latter is

$$(-1)^i s_\alpha + (-1)^j s_\beta,$$

where $\sigma = \mathrm{d}_i\alpha = \mathrm{d}_j\beta$. Thus we get a system of equations

$$s_\alpha = s_\beta$$

or

$$s_\alpha = -s_\beta,$$

where α and β are neighboring n-simplices in the above sense.

Since X is connected, any two nondegenerate n-simplices α and β can be connected by a sequence of jumps between neighboring simplices. According to the above equations, this implies that $s_\alpha = \pm s_\beta$.

The answer now depends whethere there is a loop of neighboring n-simplices that starts and ends at some n-simplex α and such that the total change of parity in the loop is odd. If there is such a loop, it yields an equation $s_\alpha = -s_\alpha$, which forces s_α, hence all of s_β, to be 2-torsion, and also $s_\alpha = s_\beta$ for all α and β. The above equations are then satisfied. The common value of s_α yields an isomorphism $\mathsf{Z}_n(X, A) \cong \mathsf{Tor}(\mathbf{Z}/2, A)$. If there is no such loop, then assigning an arbitrary value to s_α yields unique values for all s_β, so $\mathsf{Z}_n(X, A) = A$. A different choice of α will either yield the same isomorphism, or its additive inverse. ▮

Exercise 36.10. Consider $n \geq 0$ simplices of dimension $d \geq 0$ glues along their common boundary $\partial\Delta^d$. For which pairs (n, d) is the resulting simplicial set a combinatorial manifold?

Exercise 36.11. For a compact connected n-dimensional combinatorial manifold X compute $\mathsf{H}^n(X, A)$. Hint: both the answer and the proof will be very similar to the above proof.

Remark 36.12. The above proof does not work for noncompact manifolds. However, we can obtain an analog of the above proposition by replacing homology with *Borel–Moore homology.* The latter is defined using *Borel–Moore simplicial chains* C^{BM}, which are defined like ordinary simplicial chains, but using products of abelian groups instead of direct sums. In order for this construction to make sense, we must make sure that no infinite sums occur in the definition of a boundary map. This is the case precisely for locally compact simplicial sets. Likewise, one can define *cohomology with compact support* using *simplicial cochains with compact support* $\mathsf{C}^*_{\mathsf{cs}}$, which are define like simplicial cochains, but using direct sums of abelian groups instead of products. Once again, the construction makes sense precisely when the simplicial set is locally compact. With these modifications, the above computations extend to noncompact manifolds. Both of these construction are not functorial with respect to all simplicial maps, but only with respect to *proper simplicial maps*. These are defined as simplicial maps $f\colon X \to Y$ such that for any $\Delta^{\mathbf{m}} \to Y$ the fiber $\Delta^{\mathbf{m}} \times_Y X$ is compact. This guarantees that when we write down formulas the homological pushforward and cohomological pullback, the relevant sums will remain finite. Used in 37.5.

Definition 36.13. The *orientation bundle* (alias *orientation covering*, *orientation local system*) with a typical fiber A (the default choice is $\mathbf{Z}$) of an n-dimensional combinatorial manifold M is denoted by $\tilde{A}$ and is constructed as follows. We pass to the second barycentric subdivision. We send a vertex v of M to the group $\mathsf{H}_n(\operatorname{star}(v)/\operatorname{link}(v), A)$, which is isomorpic to A, canonically up to a noncanonical sign. We send an edge $e\colon u \to v$ of M to an isomorphism $\mathsf{H}_n(\operatorname{star}(u)/\operatorname{link}(u), A) \to \mathsf{H}_n(\operatorname{star}(v)/\operatorname{link}(v), A)$ constructed as follows. Denote by W the simplicial subset of $\operatorname{star}(u) \cup \operatorname{star}(v)$ consisting of those simplices that do not intersect with e. Observe that $W \subset \operatorname{link}(u) \cup \operatorname{link}(v)$. Furthermore, W is a combinatorial n-disk because $\operatorname{link}(u)$, $\operatorname{link}(v)$, and $\operatorname{link}(u) \cap \operatorname{link}(v)$ are combinatorial n-disks. Thus, we have isomorphisms

$$\mathsf{H}_n((\operatorname{star}(u) \cup \operatorname{star}(v))/W, A) \to \mathsf{H}_n(\operatorname{star}(u)/\operatorname{link}(u), A)$$

and

$$\mathsf{H}_n((\operatorname{star}(u) \cup \operatorname{star}(v))/W, A) \to \mathsf{H}_n(\operatorname{star}(v)/\operatorname{link}(v), A).$$

Composing the inverse of the former with the latter yields the desired isomorphism. Geometrically, we "transport" the given cycle of the link of u along the edge e, obtaining a cycle of the link of v. Used in 36.14, 36.15, 36.16.

Definition 36.14. The local system of *twisted integers* is the orientation local system for $A = \mathbf{Z}$, i.e., $\tilde{\mathbf{Z}}$.

Remark 36.15. A manifold M is orientable if and only if the orientation local system is trivial. Indeed, in this case we have isomorphisms

$$\mathsf{H}_n(M, A) \to \mathsf{H}_n(\operatorname{star}(u)/\operatorname{link}(u), A).$$

Proposition 36.16. The twisted homology of a combinatorial manifold M with coefficients in the orientation local system of M with typical fiber A is canonically isomorphic to A:

$$\mathsf{H}_n(M, \tilde{A}) \xrightarrow{\cong} A.$$

Proof. The proof proceeds along similar lines as above. The crucial difference now is that an n-cycle assigns elements of $\mathsf{H}_n(\operatorname{link}(v)/\operatorname{star}(v), A)$ to n-simplices, instead 1 or -1, and such elements can be chosen canonically, since the sign problem disappears. ∎

Definition 36.17. The *twisted fundamental class* is the element of $\mathsf{H}_n(M, \tilde{A})$ corresponding to $1 \in A$ under the above isomorphism. Here A is an arbitrary commutative ring. Used in 37.3, 37.5.

37 Poincaré duality

Supplementary sources: Clavier [VPD].

In this section M denotes an n-dimensional combinatorial manifold.

Lemma 37.1. Suppose X is a simplicial set and $u \in \mathsf{C}^m(X, A)$ and $v \in \mathsf{C}_n(X, \tilde{A})$ are a simplicial cochain and simplicial chain on X, where (abusing notation) $v \in X_n$ is a single simplex in X. The *twisted cap product* $u \cap v \in \mathsf{C}_{n-m}(X, \tilde{A})$ is the simplicial chain

$$u(v_{n-m,\ldots,n})v_{0,\ldots,n-m},$$

where $v_{0,\ldots,n-m}$ and $v_{n-m,\ldots,n}$ denote the $n-m$- and m-simplices of X given by the first $n-m$ and the last m vertices of v. Used in 37.2, 37.3.

Proposition 37.2. For any simplicial set X and ring A, the twisted cap product turns $\mathsf{C}(X, \tilde{A})$ into a differential graded module over the differential graded ring $\mathsf{C}^*(X, A)$.

Definition 37.3. The *Poincaré duality* morphism

$$\mathsf{H}^k(M, A) \to \mathsf{H}_{n-k}(M, \tilde{A})$$

is given by the twisted cap product with the twisted fundamental class f (with integer coefficients):

$$c \mapsto c \cap f.$$

More generally, for any local system L we have a morphism

$$\mathsf{H}^k(M, L) \to \mathsf{H}_{n-k}(M, \tilde{\mathbf{Z}} \otimes L)$$

Used in 24.11, 37.4.

Theorem 37.4. (Poincaré, 1893, 1895, 1899, 1900.) The Poincaré duality morphism is an isomorphism for any compact combinatorial manifold M and any local system L.

Remark 37.5. For noncompact manifolds one must use either Borel–Moore homology or cohomology with compact support. Thus, the following two morphisms, given by the cap product with the twisted fundamental class (with integer coefficients) in the Borel–Moore homology are isomorphisms:

$$\mathsf{H}^k(M, L) \to \mathsf{H}^{\mathsf{BM}}_{n-k}(M, \tilde{\mathbf{Z}} \otimes L),$$

$$\mathsf{H}^k_{\mathsf{cs}}(M, L) \to \mathsf{H}_{n-k}(M, \tilde{\mathbf{Z}} \otimes L).$$

38 Cellular homology

Definition 38.1. A *cellular structure* on a simplicial set X is a collection $\{X_{<i}\}_{i\geq 0}$ of simplicial subsets of X (we also write $X_i = X_{<i+1}$), $X_{<i}$ is a simplicial subset of $X_{<j}$ whenever $i \leq j$, and the following square is a pushout square

$$\begin{array}{ccc} \coprod_{i\in C_n} \partial D_i^n & \longrightarrow & X_{<n} \\ \Big\downarrow \coprod_{i\in C_n} \iota_i^n & & \Big\downarrow \kappa_n \\ \coprod_{i\in C_n} D_i^n & \longrightarrow & X_{<n+1}, \end{array}$$

where D_i^n denotes an arbitrary combinatorial oriented n-disk (alias cell) (we can take different disks for different i) and C_n is known as the indexing set of n-dimensional cells. The map $\partial D_i^n \to X_{<n}$ is known as the attaching map of the cell D_i^n. Used in 38.3, 38.4.

Definition 38.2. A CW-complex is a topological space equipped with a cellular structure, defined in the same way as for simplicial sets, but in the category of topological spaces and D^n being the topological n-disk.

Definition 38.3. The *cellular chain complex* associated to a simplicial set with a cellular structure C or to a CW-complex is defined as follows. In degree n we place the abelian group $\mathbf{Z}^{C_n}$. Given $c \in C_n$, its boundary is an $(n-1)$-chain computed as follows. We take the attaching map of c, namely, $a\colon \partial D_c^n \to X_{<n}$, and compose it with the quotient map $q\colon X_{<n} \to X_{<n}/X_{<n-1}$. We compute its homology, which is a map

$$\mathsf{H}_{n-1}(qa)\colon \mathsf{H}_{n-1}(\partial D_i^n) \to \mathsf{H}_{n-1}(X_{<n}/X_{<n-1}).$$

We have an isomorphism $\bigvee_{i\in C_{n-1}} D_i^{n-1}/\partial D_i^{n-1} \to X_{<n}/X_{<n-1}$. The homology of a wedge can be computed using direct sums, with a proviso that in degree 0 we must take reduced homology (i.e., the cokernel of the map induced by the inclusion of the basepoint). Furthermore, both ∂D_c^n and $D_i^{n-1}/\partial D_i^{n-1}$ are oriented spheres of dimension $n-1$, so applying H_{n-1} yields a group canonically isomorphic to $\mathbf{Z}$. Thus $\mathsf{H}(a)$ can be identified with a map $\mathbf{Z} \to \mathbf{Z}^{C_{n-1}}$, whose codomain is precisely the group of cellular $(n-1)$-chains. We now define ∂c as the image of 1. Used in 38.4, 38.6, 38.7.

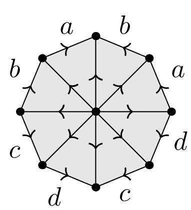
Definition 38.4. The canonical map from the cellular chain complex associated to a simplicial set with a cellular structure C (respectively CW-complex) to the corresponding simplicial chains (respectively singular chains) is defined as follows. We send a generator $c \in C_n$ in degree n to a simplicial chain of X in degree n given by the image of the fundamental chain of D_c^n under the induced map

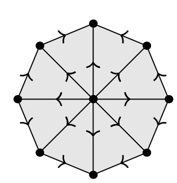
$$\mathsf{C}(\iota)\colon \mathsf{C}(D_c^n) \to \mathsf{C}(X),$$

where $\iota\colon D_c^n \to X$ is the canonical map induced by the cellular structure.

Proposition 38.5. The above map is indeed a chain map. Furthermore, it is a quasi-isomorphism, i.e., its homology is an isomorphism of graded abelian groups.

Example 38.6. Consider the orientable surface of genus $g \geq 1$, depicted by the left figure below:





The right figure denotes a combinatorial 2-disk, given by a similar figure, but without any identifications between edges. We construct a cellular structure as follows. The simplicial subset X_0 consists of the only vertex located on the exterior polygon, so C_0 has a single element. Its orientation is chosen to be the canonical element $1 \in \mathbf{Z} \cong \mathsf{H}_0(S^0)/\mathsf{H}_0(\Delta^0)$. The simplicial subset X_1 is the exterior polygon, and the elements of C_1 are given by the $2g$ exterior edges (after identification), e.g., $C_1 = \{a, b, c, d\}$ in the above picture. We choose their orientations to coincide with the directions of arrows. The simplicial subset X_2 coincides with X and C_2 consists of a single element, corresponding to the disk on the right. Its orientation is chosen so that its boundary is oriented counterclockwise.

We now compute the cellular chain complex:

$$A \leftarrow A^{2g} \leftarrow A.$$

The differential ∂_1 sends a 1-cell to its boundary, i.e., the difference of terminal and initial 0-cells, which in our case coincide, so $\partial_1 = 0$. The differential ∂_2 sends the only 2-cell to the homology class of the image fundamental cycle of its boundary (which is simply the boundary of the fundamental chain of the disk itself) unded the map that lands in X_1/X_0. The boundary chain, being oriented counterclockwise, has coefficients in a repeating patter 1, 1, −, −1. Once we map it to the surface, edges with indices differing by 2 will be identified, so the corresponding coefficients will annihilate each other. Thus $\partial_2 = 0$. Hence $\mathsf{H}_0 \cong \mathsf{H}_2 \cong A$ and $\mathsf{H}_1 \cong A^{2g}$.

Example 38.7. The nonorientable surface with g crosscaps is treated very similarly: there is a single 0-cell, g 1-cells, and a single 2-cell. The cellular chain complex is

$$A \leftarrow A^{g} \leftarrow A,$$

with $\partial_1 = 0$ for the same reason as before. When we compute ∂_2, we no longer have the same cancellation effect, but rather both coefficients will be 1, for the total coefficient of 2. Thus $\partial_2(a) = 2a\sum_i e_i$, where the sum is taken over all 1-cells. We immediately deduce that $\mathsf{H}_0 \cong A$, $\mathsf{H}_1 \cong A^{g-1} \oplus A/2A$, and $\mathsf{H}_2 \cong \mathsf{Tor}(\mathbf{Z}/2, A)$.

Homotopy theory of simplicial sets and homotopical algebra

39 Kan complexes

Supplementary sources: Kerodon §3

Definition 39.1. A *Kan complex* is a simplicial set X that has a (nonunique) lifting property with respect to horn inclusions: for any map $\Lambda^n_k \to X$ there is a (noncanical and nonunique) map $\Delta^n \to X$ so that the following diagram commutes:

$$\begin{array}{ccc} \Lambda^n_k & \searrow & \\ \downarrow{\scriptstyle \iota} & & X. \\ \Delta^n & \nearrow & \end{array}$$

The full subcategory of all Kan complexes is denoted by $\mathsf{sSet}_{\mathsf{Kan}}$. Used in 2.0*, 7.5, 39.1, 39.7, 39.10, 39.10*, 39.20, 41.2, 42.4, 45.1, 45.5*, 46.3, 46.4.

Definition 39.2. The functor

$$\mathsf{Ex}\colon \mathsf{sSet} \to \mathsf{sSet}$$

is defined as the right adjoint of the functor sd. Thus,

$$\mathsf{Ex}(X)_n = \mathsf{hom}(\mathrm{sd}(\Delta^n), X)$$

and likewise for simplicial structure maps. Used in 39.2, 39.3, 39.4, 39.5, 39.6, 39.6*, 39.7, 39.7*, 39.13, 39.14, 39.15, 39.16, 39.17, 41.2, 42.3, 42.4, 42.7, 45.4, 45.5*, 45.8*, 46.5*, 46.8.

Definition 39.3. We have a natural transformation $\mathrm{id}_{\mathsf{sSet}} \to \mathsf{Ex}$ with components

$$X \to \mathsf{Ex}(X)$$

that are adjoints of the *last vertex maps*

$$\mathrm{sd}(X) \to X.$$

The latter are induced by the natural transformation $\mathrm{sd} \to \mathrm{id}_{\mathbf{\Delta}}$ with components

$$\mathrm{sd}\,\Delta^{\mathbf{0}} = \Delta^{\mathbf{0}} \xrightarrow{\mathrm{id}} \Delta^{\mathbf{0}}$$

and

$$\mathrm{sd}\,\Delta^{\mathbf{n}} = \mathrm{C}(\mathrm{sd}(\partial\Delta^{\mathbf{n}})) \longrightarrow \Delta^{\mathbf{n}}$$

induced by sending the apex to the last vertex of $\Delta^{\mathbf{n}}$ and using the inductively defined map

$$\mathrm{sd}(\partial\Delta^{\mathbf{n}}) \to \partial\Delta^{\mathbf{n}}.$$

Remark 39.4. Concretely, an n-simplex of $\mathsf{Ex}X$ is a simplicial map $\mathrm{sd}\,\Delta^{\mathbf{n}} \to X$. The inclusion $X \to \mathsf{Ex}X$ sends an n-simplex of X, i.e., a simplicial map $\Delta^{\mathbf{n}} \to X$, to the composition $\mathrm{sd}\,\Delta^{\mathbf{n}} \to \Delta^{\mathbf{n}} \to X$, i.e., an n-simplex in $\mathsf{Ex}X$.

Definition 39.5. The functor

$$\mathsf{Ex}^{\infty}\colon \mathsf{sSet} \to \mathsf{sSet}$$

sends a simplicial set X to the colimit of the diagram

$$X \to \mathsf{Ex}(X) \to \mathsf{Ex}(\mathsf{Ex}(X)) \to \mathsf{Ex}(\mathsf{Ex}(\mathsf{Ex}(X))) \to \cdots$$

Likewise for morphisms. We have a natural transformation $\mathrm{id}_{\mathsf{sSet}} \to \mathsf{Ex}^{\infty}$ whose components

$$X \to \mathsf{Ex}^{\infty}(X)$$

are given by the injection map of the first term in the colimit.

Proposition 39.6. The functor Ex^{∞} preserves finite limits, filtered colimits, monomorphisms.

Proof. The functors Ex^{n} (for any $n \geq 0$) are right adjoint functors, so preserve small limits, in particular, finite limits. They also preserve filtered colimits because $\mathrm{sd}^{k}\,\Delta^{\mathbf{n}}$ is a compact simplicial set for any $k \geq 0$. Finally, filtered colimits of simplicial sets commute with filtered colimits and finite limits. Monomorphisms are preserved because cartesian squares are preserved. ∎

Proposition 39.7. For any simplicial set X the simplicial set $\mathsf{Ex}^{\infty}X$ is a Kan complex.

Proof. We have to show that any diagram

$$\begin{array}{ccc} \Lambda^{\mathbf{n}}_{k} & \longrightarrow & \mathsf{Ex}^{\infty}X \\ \downarrow & \overset{d}{\nearrow} & \\ \Delta^{\mathbf{n}} & & \end{array}$$

there is a lift d as depicted. Recall that $\Lambda^{\mathbf{n}}_{k}$ is a compact simplicial set, so by Proposition 36.8, the map $\Lambda^{\mathbf{n}}_{k} \to \mathsf{Ex}^{\infty}X$ factors through some inclusion $\mathsf{Ex}^{m}X \to \mathsf{Ex}^{\infty}X$, as depicted by the top maps in the diagram below.

We are going to construct a map $e\colon \Delta^{\mathbf{n}} \to \mathsf{Ex}^{m+1}X$ so that the left square in the diagram

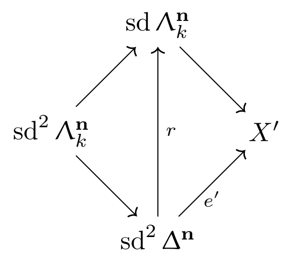
$$\begin{array}{ccccc} \Lambda^{\mathbf{n}}_{k} & \longrightarrow & \mathsf{Ex}^{m}X & & \\ & & & \searrow & \\ \downarrow & & \downarrow & & \mathsf{Ex}^{\infty}X \\ & & & \nearrow & \\ \Delta^{\mathbf{n}} & \xrightarrow[e]{} & \mathsf{Ex}^{m+1}X & & \end{array}$$

commutes. If we define d as the composition of the bottom two maps, then the original triangle with d commutes by definition of the maps involved.

Replacing $\mathsf{Ex}^{m-1}X$ with X', we simplify the lifting problem to

$$\begin{array}{ccc} \Lambda^{\mathbf{n}}_k & \longrightarrow & \mathsf{Ex}X' \\ \downarrow & & \downarrow \\ \Delta^{\mathbf{n}} & \xrightarrow[e]{} & \mathsf{Ex}^2 X' \end{array}$$

Using the adjunction $\mathrm{sd} \dashv \mathsf{Ex}$, we rewrite the problem as



,

where the map r is constructed below and e' is the left adjunct of e. We construct e' by declaring the right triangle to be commutative.

It remains to construct r so that the left triangle is commutative. Simplices of $\mathrm{sd}\,\Lambda^{\mathbf{n}_k}$ are determined by their vertices, so it suffices to construct r_0 and verify that r maps simplices to simplices. ∎

Definition 39.8. The *skeletal filtration* of a simplicial set X is the diagram

$$\mathsf{sk}_0 X \to \mathsf{sk}_1 X \to \mathsf{sk}_2 X \to \cdots,$$

where $\mathsf{sk}_n X$ is the simplicial subset of X generated by simplices of dimension at most n. The colimit of this diagram is canonically isomorphic to X. Used in 39.8, 39.9, 39.9*, 39.10*, 45.3*.

Proposition 39.9. For any simplicial set X and any $n \geq 0$ the inclusion map $\mathsf{sk}_{n-1}X \to \mathsf{sk}_n X$ in the skeletal filtration of X fits in the pushout square

$$\begin{array}{ccc} \coprod_{\sigma\in X'_n} \partial\Delta^{\mathbf{n}} & \longrightarrow & \mathsf{sk}_{n-1}X \\ \downarrow & & \downarrow \\ \coprod_{\sigma\in X'_n} \Delta^{\mathbf{n}} & \longrightarrow & \mathsf{sk}_n X, \end{array}$$

where X'_n is an ad hoc notation for the set of nondegenerate n-simplices of X. The bottom map is induced by the universal property of coproducts and the Yoneda lemma. The top map is obtained by factoring the left-bottom composition through $\mathsf{sk}_{n-1}X$, which is possible because the nondegenerate simplices of $\partial\Delta^{\mathbf{n}}$ have dimension less than n. Used in 45.3*.

Proof. The square is commutative by construction. Pushouts of simplicial sets are computed degreewise. Thus, we have to show that for any simplex $\mathbf{k}$ the induced commutatative square of sets of $\mathbf{k}$-simplices is a pushout square of sets. Since the left map is an injection of sets, so is the right map and it suffices to show that the bottom map induces a bijection of sets

$$\coprod_{\sigma\in X'_n} (\Delta^{\mathbf{n}_\mathbf{k}} \setminus \partial\Delta^{\mathbf{n}_\mathbf{k}}) = \coprod_{\sigma\in X'_n} \Delta^{\mathbf{n}_\mathbf{k}} \setminus \coprod_{\sigma\in X'_n} \partial\Delta^{\mathbf{n}_\mathbf{k}} \to (\mathsf{sk}_n X)_\mathbf{k} \setminus (\mathsf{sk}_{n-1}X)_\mathbf{k}.$$

By the Eilenberg–Zilber lemma an element of the right side is a pair $(\sigma\colon \Delta^{\mathbf{n}} \to X, \alpha\colon \mathbf{k} \to \mathbf{n})$, where σ is a nondegenerate n-simplex of X and α is a surjective map of simplices. An element of the left side indexed by some $\sigma \in X'_n$ is a map $\alpha\colon \Delta^{\mathbf{k}} \to \Delta^{\mathbf{n}}$ that does not factor through $\partial\Delta^{\mathbf{n}}$, i.e., is surjective. Thus both sides are isomorphic. ∎

Proposition 39.10. (*Simplicial Whitehead theorem*; J. H. C. Whitehead, 1949.) Suppose $f\colon X \to Y$ is a simplicial map between Kan complexes. The map f is a simplicial homotopy equivalence if and only if for any commutative square

$$\begin{array}{ccc} \partial\Delta^{\mathbf{n}} & \longrightarrow & X \\ \downarrow & \overset{d}{\nearrow} & \downarrow f \\ \Delta^{\mathbf{n}} & \longrightarrow & Y \end{array}$$

there is a diagonal arrow d such that the upper triangle commutes and the lower triangle commutes up to a homotopy relative boundary, meaning there is a homotopy $\Delta^{\mathbf{1}} \times \Delta^{\mathbf{n}} \to Y$ from $f \circ d$ to the bottom map whose restriction to $\Delta^{\mathbf{1}} \times \partial\Delta^{\mathbf{n}}$ is a constant homotopy, i.e., it factors as $\Delta^{\mathbf{1}} \times \partial\Delta^{\mathbf{n}} \to \partial\Delta^{\mathbf{n}} \to Y$, where the first map is a projection. Used in 39.11, 39.12, 39.14, 39.15*, 42.4, 45.5*.

Proof. Consider a simplicial map $f\colon X \to Y$ of Kan complexes with a lifting property as in the statement. We are going to construct an inverse map $g\colon Y \to X$ together with homotopies $h\colon \Delta^{\mathbf{1}} \times X \to X$ and $i\colon \Delta^{\mathbf{1}} \times Y \to Y$ by induction on the skeletal filtration. Specifically, the inductive assumption is that we have already constructed the map $\mathsf{sk}_{n-1}g\colon \mathsf{sk}_{n-1}Y \to \mathsf{sk}_{n-1}X$ together with homotopies $h_n\colon \Delta^{\mathbf{1}} \times \mathsf{sk}_{n-1}X \to \mathsf{sk}_{n-1}X$ and $i_n\colon \Delta^{\mathbf{1}} \times \mathsf{sk}_{n-1}Y \to \mathsf{sk}_{n-1}Y$, and we now want to construct the same data for n so that the resulting maps $\mathsf{sk}_n g$, h_n, and i_n extend the ones we already constructed. The base of the induction ($n = -1$) is trivial, since all simplicial sets involved are empty.

Recall now that the inclusion map $\mathsf{sk}_{n-1}Y \to \mathsf{sk}_n Y$ in the skeletal filtration of the simplicial set Y fits in the pushout square

$$\begin{array}{ccc} \coprod_{\sigma \in Y'_n} \partial\Delta^{\mathbf{n}} & \longrightarrow & \mathsf{sk}_{n-1}Y \\ \downarrow & & \downarrow \\ \coprod_{\sigma \in Y'_n} \Delta^{\mathbf{n}} & \longrightarrow & \mathsf{sk}_n Y, \end{array}$$

where Y'_n is an ad hoc notation for the set of nondegenerate n-simplices of Y.

The universal property of pushouts allows us to construct the relevant maps on the coproducts of simplices instead, while verifying that they are compatible with the existing maps on the boundary. The universal property of coproducts allows us to construct the relevant maps individually for some fixed nondegenerate n-simplex $\sigma\colon \Delta^{\mathbf{n}} \to Y$ and its boundary $\partial\sigma\colon \partial\Delta^{\mathbf{n}} \to Y$, which factors through $\mathsf{sk}_{n-1}Y$. Consider the composition $\mathsf{sk}_{n-1}g \circ \partial\sigma\colon \partial\Delta^{\mathbf{n}} \to X$. The simplicial map $f \circ g \circ \partial\sigma\colon \partial\Delta^{\mathbf{n}} \to Y$ is homotopic to $\partial\sigma$ via the simplicial homotopy $\Delta^{\mathbf{1}} \times \partial\Delta^{\mathbf{n}} \to Y$ given by the restriction of i_{n-1}. This homotopy, combined with the map σ, yields a map

$$A = \Delta^{\mathbf{1}} \times \partial\Delta^{\mathbf{n}} \sqcup_{1 \times \partial\Delta^{\mathbf{n}}} \Delta^{\mathbf{n}} \to Y.$$

Using the Kan condition on Y, we can extend the map $A \to Y$ to a map

$$\pi\colon B = \Delta^{\mathbf{1}} \times \Delta^{\mathbf{n}} \to Y.$$

Define the map

$$\tau\colon \Delta^{\mathbf{n}} \to Y$$

to be the restriction of the map $B \to Y$ to the simplicial subset $0 \times \Delta^{\mathbf{1}}$. We have $\partial\tau = f \circ g \circ \partial\sigma$.

We have constructed a pair $g \circ \partial\sigma\colon \partial\Delta^{\mathbf{n}} \to X$ and $\tau\colon \Delta^{\mathbf{n}} \to Y$, which together form an input data for the lifting property in the statement. Thus we get a diagonal arrow $d\colon \Delta^{\mathbf{n}} \to X$ such that $\partial d = g \circ \partial\sigma$ and $f \circ d\colon \Delta^{\mathbf{n}} \to Y$ is homotopic to τ relative boundary via a homotopy $\rho\colon \Delta^{\mathbf{1}} \times \Delta^{\mathbf{n}} \to Y$. We define the value g on σ (i.e., the map $g \circ \sigma\colon \Delta^{\mathbf{n}} \to X$) to be equal to d. The above condition on ∂d guarantees that g respects the previously defined values on $\partial\sigma$. We define the value of the homotopy $i\colon \Delta^{\mathbf{1}} \times Y \to Y$ on σ to be the gluing of the homotopies π and ρ constructed above.

Finally, we define the value of the homotopy $h\colon \Delta^{\mathbf{1}} \times X \to X$ on some arbitrary simplex $\kappa\colon \Delta^{\mathbf{n}} \to X$ as follows. First, for the simplex $\sigma = f(\kappa)\colon \Delta^{\mathbf{n}} \to Y$ we have the entire collection of maps constructed above. In particular, we have the simplex $d\colon \Delta^{\mathbf{n}} \to X$ such that $f(d)$ is homotopic to $\sigma = f(\kappa)$. Furthermore, the inductively constructed homotopy h yields a homotopy $\varepsilon\colon \Delta^{\mathbf{1}} \times \partial\Delta^{\mathbf{n}} \to X$ between κ and d. The maps κ, d, and ε combine together into a map $C = \partial(\Delta^{\mathbf{1}} \times \Delta^{\mathbf{n}}) \to X$, whose domain is a subdivided sphere. The composition $C \to X \to Y$ has a filling by a disk constructed in the previous paragraph. Using the lifting property, we lift this filling to a map $\Delta^{\mathbf{1}} \times \Delta^{\mathbf{n}} \to X$, which is the value of h on κ. ∎

Exercise 39.11. Show that the lifting condition in the simplicial Whitehead theorem is equivalent to the following two conditions:

- the map $\pi_0 f\colon \pi_0 X \to \pi_0 Y$ is an isomorphism of sets;
- the map $\pi_n(f,x)\colon \pi_n(X,x) \to \pi_n(Y,f(x))$ is an isomorphism of sets (or groups) for any $n \geq 1$ and any vertex $x \in X_{\mathbf{0}}$.

Example 39.12. Given a smooth manifold X (such as an open subset of $\mathbf{R}^n$), we can define two different version of the singular simplicial set: $\mathsf{Sing}_{\mathsf{cont}}(X)$ using continuous maps $\Delta^n \to X$ and $\mathsf{Sing}_{\mathsf{smooth}}(X)$ using smooth (i.e., infinitely differentiable maps) $\Delta^n \to X$. The inclusion $\mathsf{Sing}_{\mathsf{smooth}}(X) \to \mathsf{Sing}_{\mathsf{cont}}(X)$ is a simplicial weak equivalence. Indeed, the simplicial Whitehead theorem requires us to show that any continuous map $\Delta^n \to X$ whose faces are smooth maps $\Delta^{n-1} \to X$ can be continuously deformed to a smooth map $\Delta^n \to X$ with its boundary not moving. This follows, for example, from the Weierstrass approximation theorem or more directly from the Whitney approximation theorem. Used in 39.12.

Definition 39.13. A simplicial map $f\colon X \to Y$ is a simplicial weak equivalence (alias *simplicial weak homotopy equivalence* if $\mathsf{Ex}^\infty(f)$ is a simplicial homotopy equivalence.

Corollary 39.14. If $f\colon X \to Y$ is a simplicial map of arbitrary simplicial sets (not necessarily Kan), we can expand the meaning of the simplicial Whitehead theorem for $\mathsf{Ex}^\infty(f)$ as follows: f is a simplicial weak equivalence if and only if for any commutative square

$$\begin{array}{ccc} \mathrm{sd}^k\,\partial\Delta^{\mathbf{n}} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{sd}^k\,\Delta^{\mathbf{n}} & \longrightarrow & Y \end{array}$$

Proposition 39.15. For any simplicial set X the map $X \to \mathsf{Ex}^\infty(X)$ is a simplicial weak equivalence.

Proof. We use the simplicial Whitehead theorem. ▮

Definition 39.16. The *derived internal hom* functor

$$\mathsf{RHom}\colon \mathsf{sSet}^{\mathsf{op}} \times \mathsf{sSet} \to \mathsf{sSet}$$

is defined by

$$\mathsf{RHom}(X,Y) = \mathsf{Hom}(\mathsf{Ex}^\infty(X), \mathsf{Ex}^\infty(Y)).$$

Used in 39.16, 39.17, 39.18, 40.13, 40.13*, 40.14, 42.7.

Remark/Exercise 39.17. Sometimes

$$\mathsf{RHom}(X,Y) = \mathsf{Hom}(X, \mathsf{Ex}^\infty(Y))$$

is used as the definition of RHom. This definition produces weakly equivalent answers, but there is no canonical way to define composition for it.

Proposition 39.18. The derived internal hom sends pairs of simplicial weak equivalences to simplicial weak equivalences (and even simplicial homotopy equivalences).

Remark 39.19. In modern homotopy theory, a *space* is simplicial set considered up to a simplicial weak equivalence. Used in 1.0*, 40.13*.

Proposition 39.20. A simplicial map $f\colon X \to Y$ is a *simplicial weak equivalence* or simply *weak equivalence*) such that for any Kan complex Z the induced map

$$\mathsf{Hom}(f,Z)\colon \mathsf{Hom}(Y,Z) \to \mathsf{Hom}(X,Z)$$

is a simplicial homotopy equivalence. Used in 2.0*, 9.10, 16.0*, 25.2, 39.12, 39.13, 39.14, 39.15, 39.18, 39.19, 39.21, 39.22, 39.23, 40.2, 40.5, 40.8, 40.10*, 40.11, 40.12, 41.1, 41.2, 42.3, 42.4, 45.1*, 45.2, 45.2*, 45.5, 45.5*, 45.6, 45.6*, 45.7, 45.8*, 46.2, 46.2*, 46.4, 46.4*, 46.5*, 46.6, 46.6*, 48.3*, 50.3, 52.1, 52.2, 52.4.

Proposition 39.21. Any simplicial homotopy equivalence is a simplicial weak equivalence.

Definition 39.22. Two simplicial sets X and Y are *weakly equivalent* if they can be connected by a zigzag of simplicial weak equivalences (going in either direction). A simplicial set X is *weakly contractible* if it is weakly equivalent to $\Delta^{\mathbf{0}}$. Used in 6.9, 17.17*, 39.22.

Proposition 39.23. If $f\colon X \to Y$ is a simplicial weak equivalence, then the maps

$$\mathsf{H}(f)\colon \mathsf{H}(X) \to \mathsf{H}(Y),$$

$$\mathsf{H}^*(f)\colon \mathsf{H}^*(X) \to \mathsf{H}^*(Y),$$

are isomorphisms (of graded abelian groups) and

$$\pi_{\leq 1}(f)\colon \pi_{\leq 1}(X) \to \pi_{\leq 1}(Y)$$

is an equivalence of groupoids.

40 Relative categories

Definition 40.1. A *relative category* is a category C together with a subcategory $\mathsf{W} \subset \mathsf{C}$ with the same objects as C. Morphisms in W are known as weak equivalences. A *relative functor* $(\mathsf{C}, \mathsf{W}) \to (\mathsf{C}', \mathsf{W}')$ is a functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{C}'$ that maps W to W'. Small relative categories and relative functors form a category RelCat. Used in 40.1, 40.3, 40.6, 40.7, 40.10*, 40.13*, 40.14, 41.1, 41.2, 41.3, 41.5, 42.1, 42.3, 42.5, 44.4, 44.5, 52.0*, 52.1, 52.3, 52.4.

Example 40.2. The *relative category of simplicial sets* is formed by simplicial sets and simplicial weak equivalences. Used in 40.3, 45.1.

Example 40.3. Simplicial sets and simplicial homotopy equivalences form a very different relative category from the relative category of simplicial sets, even though their underlying categories are the same.

Definition 40.4. A chain map $f\colon C \to D$ of chain complexes is a *quasi-isomorphism* if the induced homology map $\mathsf{H}(f)\colon \mathsf{H}(C) \to \mathsf{H}(D)$ is an isomorphism of graded abelian groups. Used in 40.5, 40.6.

Example 40.5. If $f\colon X \to Y$ is a simplicial weak equivalence, then $\mathsf{C}(f)\colon \mathsf{C}(X) \to \mathsf{C}(Y)$ is a quasi-isomorphism.

Example 40.6. Chain complexes and quasi-isomorphisms form a relative category, which we will refer to as the *relative category of chain complexes*. Used in 40.7.

Example 40.7. Chain complexes and chain homotopy equivalences form a relative category, different from the relative category of chain complexes.

Example 40.8. Topological spaces and weak homotopy equivalences of topological spaces (defined as the preimage of simplicial weak equivalences under the singular simplicial set functor).

Definition 40.9. Two object $A, B \in \mathsf{C}$ in a relative category (C, W) are weakly equivalent if there is a finite zigzag of weak equivalences that connects A and B:

$$A = X_0 \leftarrow X_1 \to X_2 \leftarrow X_3 \to X_4 \leftarrow \cdots X_n = B.$$

Example 40.10. Denote by $S^{\mathbf{1}}_1$ and $S^{\mathbf{1}}_2$ simplicial circles comprising one respectively two nondegenerate 1-simplices. Denote by $S^{\mathbf{2}}_1$ and $S^{\mathbf{2}}_2$ simplicial spheres comprising one respectively two nondegenerate 2-simplices. We have weak equivalences $S^{\mathbf{1}}_2 \to S^{\mathbf{1}}_1$ and $S^{\mathbf{2}}_2 \to S^{\mathbf{2}}_1$. No maps $S^{\mathbf{1}}_1 \to S^{\mathbf{1}}_2$ or $S^{\mathbf{2}}_1 \to S^{\mathbf{2}}_2$ could be weak equivalences since a single nondegenerate simplex cannot be "stretched" to span two nondegenerate ones. Thus, the simplicial sets $S^{\mathbf{1}}_1 \sqcup S^{\mathbf{2}}_2$ and $S^{\mathbf{1}}_2 \sqcup S^{\mathbf{2}}_1$ are weakly equivalent, but only through a zigzag of length 2, with an intermediate simplicial set $S^{\mathbf{1}}_2 \sqcup S^{\mathbf{2}}_2$ that maps to both of them via weak equivalences.

When working in a relative category, we want all constructions to respect weak equivalences. This means that replacing some data used in a construction with a weakly equivalent data produces a weakly equivalent answer.

Example 40.11. The hom-set functor $\mathsf{hom}(X,Y)$ between simplicial sets X and Y does not respect weak equivalences. Indeed, the simplicial sets $\Delta^{\mathbf{m}}$ are all weakly equivalent to each other. However, the mapping sets $\mathsf{hom}(\Delta^{\mathbf{0}},\Delta^{\mathbf{m}}) = \mathsf{U}(\mathbf{m})$ are all nonisomorphic.

Example 40.12. The mapping simplicial set functor $\mathsf{Hom}(X,Y)$ between simplicial sets X and Y does not respect weak equivalences. Take $X = S^{\mathbf{1}}$, the simplicial circle, $Y = S^{\mathbf{1}}$, and $Y' = \operatorname{sd} Y$. Then Y and Y' are weakly equivalent via a weak equivalence $g\colon Y' \to Y$. However, $\mathsf{Hom}(X,Y)$ has two vertices, which lie in different connected component (so $\pi_0\mathsf{Hom}(X,Y)$ has cardinality 2), whereas $\mathsf{Hom}(X,Y')$ has a single vertex (so $\pi_0\mathsf{Hom}(X,Y')$ has cardinality 1). If $\mathsf{Hom}(X,g)\colon \mathsf{Hom}(X,Y') \to \mathsf{Hom}(X,Y)$ was a simplicial weak equivalence, then $\pi_0\mathsf{Hom}(X,g)$ would be an isomorphism of sets, since π_0 sends weak equivalences to isomorphism. But in our case, both sides have different cardinality.

Example 40.13. The derived mapping space functor $\mathsf{RHom}(X,Y)$ preserves weak equivalences. Used in 40.13*.

These examples exhibit an important feature of relative categories: the set $\mathsf{hom}(X,Y)$ of morphisms $X \to Y$ and the internal hom $\mathsf{Hom}(X,Y)$ can play at best a technical auxiliary role, since these functors do not respect weak equivalences. What really matters is the derived mapping space.

Thus, we naturally are interested in category-like structures that have a space of morphisms between any two objects, as opposed to a set of morphisms. Such a structure is known as an (∞,1)-category, or previously also as an ∞-category, though the latter term is often used interchangeably with quasicategories, an important model for (∞,1)-categories.

The term "(∞,1)-category" is not rigorously defined and what is actually studied in mathematics are *models* of (∞,1)-categories, such as relative categories or quasicategories.

As it turns out, for any relative category (C,W) one can define a functor $\mathsf{RMap}\colon \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{sSet}$. For the relative category of simplicial sets this functor is guaranteed to be weakly equivalent to the derived mapping space RHom functor defined above.

Even more generally, we will describe a procedure that converts a given functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ (often a right adjoint) between relative categories (C,W) and (D,Y) that does not send W to Y (i.e., does not respect weak equivalences) and satisfies some mild conditions into a new functor $\mathsf{RF}\colon \mathsf{C} \to \mathsf{D}$ that does respect weak equivalence. The functor RF is known as the right derived functor of F. Often, we have $\mathsf{RF} = \mathsf{F} \circ \mathsf{Q}$, where $\mathsf{Q}\colon \mathsf{C} \to \mathsf{C}$ is a functor that respects weak equivalences, sends any $X \in \mathsf{C}$ to a weakly equivalent object $\mathsf{Q}X$ and induces weak equivalences $\mathsf{RMap}(X,Y) \to \mathsf{RMap}(\mathsf{Q}X,\mathsf{Q}Y)$ for all object $X, Y \in \mathsf{C}$. Thus, from the above point of view, Q "does nothing". Yet, the composition $\mathsf{F}\circ\mathsf{Q}$ preserves weak equivalences, unlike F. Applying this construction to the functor $\mathsf{F} = \mathsf{Map}\colon \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{sSet}$ reconstructs the functor $\mathsf{RMap}\colon \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{sSet}$ mentioned above. Likewise, a functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ (where F is often a left adjoint) can often be converted to a left derived functor $\mathsf{LF}\colon \mathsf{C} \to \mathsf{D}$ that respects weak equivalences.

Example 40.14. We work in the relative category of chain complexes and consider the internal functor

$$\mathsf{Hom}\colon \mathsf{Ch}^{\mathsf{op}} \times \mathsf{Ch} \to \mathsf{Ch}$$

and the tensor product functor

$$\otimes\colon \mathsf{Ch} \times \mathsf{Ch} \to \mathsf{Ch}.$$

The former functor is left derivable and the latter functor is right derivable. For abelian groups A and B we set

$$\mathsf{Ext}^n(A,B) = \mathsf{H}_n\mathsf{RHom}(A,B[n])$$

and

$$\mathsf{Tor}^n(A,B) = \mathsf{H}_n(A \otimes^{\mathsf{L}} B).$$

Used in 18.3, 20.8, 23.10, 33.12, 36.9, 36.9*, 38.7.

41 Derived functors

Our first (preliminary) definition of a derived functor is based on what we have learned about deriving the internal hom on simplicial sets. It suffers from theoretical defects that will be explained later, but it is also a very practical way to computer derived functors.

Definition 41.1. Suppose $(\mathsf{C}, \mathsf{W}_\mathsf{C})$ and $(\mathsf{D}, \mathsf{W}_\mathsf{D})$ are relative categories and $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ is a functor that need not preserve weak equivalences. We say that F is *right derivable* if there is a full subcategory $\mathsf{C}' \subset \mathsf{C}$ (with the inclusion functor denoted by ι) and a *resolution functor* $R\colon \mathsf{C} \to \mathsf{C}'$ that preserves weak equivalences together with a natural weak equivalence $r\colon \mathrm{id}_\mathsf{C} \to \iota \circ R$ (i.e., a natural transformation whose components $X \to RX$ are weak equivalences for any $X \in \mathsf{C}$) such that the restriction of F to C' preserves weak equivalences. In this case, the composition $\mathsf{F} \circ R$ will be referred to as a *right derived functor* of F and will be denoted by $\mathrm{R}\, F$. The notions of *left derivable functors* and *left derived functors* are defined analogously using $\iota \circ L \to \mathrm{id}_\mathsf{C}$ as a natural weak equivalence. Used in 40.13*, 40.14, 41.5, 42.2, 48.3, 48.4.

Example 41.2. For the relative category $(\mathsf{sSet}, \mathsf{W}_\mathsf{sSet})$ of simplicial sets and simplicial weak equivalences we typically will take $\mathsf{C}' = \mathsf{sSet}_\mathsf{Kan}$, $R = \mathsf{Ex}^\infty$ and $r\colon \mathrm{id}_\mathsf{sSet} \to \iota \circ \mathsf{Ex}^\infty$ will have as its components the canonical weak equivalences $X \to \mathsf{Ex}^\infty X$ that we constructed previously. Simplicial weak equivalences between Kan complexes are automatically simplicial homotopy equivalences, and most functors that we deal with automatically preserve simplicial homotopy equivalences because they are enriched over the category of simplicial sets.

Definition 41.3. The *homotopy category* of a relative category (C, W) is defined as follows. It is an ordinary category D equipped with a relative functor $\mathsf{F}\colon (\mathsf{C}, \mathsf{W}) \to (\mathsf{D}, \mathrm{Iso}_\mathsf{D})$ such that for any other such pair $(\mathsf{D}', \mathsf{F}')$ the category of functors $\mathsf{D} \to \mathsf{D}'$ that make the diagram

$$\begin{array}{ccc} & \mathsf{C} & \\ {}^{\mathsf{F}}\swarrow & & \searrow^{\mathsf{F}'} \\ \mathsf{D} & \longrightarrow & \mathsf{D}' \end{array}$$

commutative is contractible.

Definition 41.4. A *contractible category* is a category C that is equivalent to the category with one object and a single identity morphism.

Another way to phrase this is to say that a contractible category admits a terminal object, and the unique map to the terminal object from any other object is an isomorphism.

Proposition 41.5. A relative functor induces a functor on homotopy categories. Moreover, we have a functor

$$\mathsf{Ho}\colon \mathsf{RelCat} \to \mathsf{Cat}.$$

Thus, a right derivable or left derivable functor induces a functor on homotopy categories by virtue of its right respectively left derived functor.

42 Homotopy limits and colimits

Definition 42.1. Suppose I is a small category and (C, W) is a relative category. The relative category $(\mathsf{C}, \mathsf{W})^I$ is defined as follows. Its underlying category is the category C^I of functors $I \to \mathsf{C}$, i.e., I-indexed diagrams in C. Its class of weak equivalences consists precisely of those natural transformations $t\colon \mathsf{F} \to \mathsf{G}$ of functors $\mathsf{F}, \mathsf{G}\colon I \to \mathsf{C}$ for which the morphism $t(i)\colon \mathsf{F}(i) \to \mathsf{G}(i)$ belongs to W for any object $i \in I$.

Recall that for any small category I and a category C the unique functor $\pi\colon I \to 1$ induces the constant diagram functor

$$\mathrm{const}\colon \mathrm{Fun}(\pi, \mathsf{C})\colon \mathrm{Fun}(1, \mathsf{C}) = \mathsf{C} \to \mathrm{Fun}(I, \mathsf{C}) = \mathsf{C}^I.$$

Its left adjoint functor exists whenever C is cocomplete, in which case it is the colimit functor

$$\mathrm{colim}\colon \mathsf{C}^I \to \mathsf{C}$$

and its right adjoint functor exists whenever C is complete, in which case it is the limit functor

$$\mathrm{lim}\colon \mathsf{C}^I \to \mathsf{C}.$$

Altogether, we have an adjoint triple of functors

$$\mathrm{colim} \dashv \mathrm{const} \dashv \mathrm{lim}.$$

Definition 42.2. The *homotopy limit functor* (alias *derived limit functor*) is the right derived functor of lim. *homotopy limit* The *homotopy colimit functor* (alias *derived colimit functor*) is the left derived functor of colim. *homotopy colimit* If I is a discrete category, i.e., all morphisms in I are identities, then $\mathrm{colim}_I = \coprod_I$ and $\mathrm{lim}_I = \prod_I$. These special cases are known as *homotopy coproducts* and *homotopy products*. Used in 1.0*, 42.3, 42.5, 42.8, 42.9.

Example 42.3. Consider the relative category sSet of simplicial sets and simplicial weak equivalences. If I is discrete, the coproduct functor $\coprod_I$ preserves weak equivalences. (For instance, observe that Ex^∞ preserves coproducts of simplicial sets.) Thus homotopy coproducts of simplicial sets can be computed as ordinary coproducts. If, in addition, I is finite, then the product functor $\prod_I$ also preserves weak equivalences. (For instance, observe that Ex^∞ preserves finite products of simplicial sets.) However, for infinite I the product functor $\prod_I$ does not preserve weak equivalences.

Example 42.4. (Infinite products of simplicial sets do not preserve weak equivalences.) Continuing the previous example, consider the case of infinite products of simplicial sets. We exhibit a weak equivalence $f\colon A \to B$ such that $\prod_I f\colon \prod_I A \to \prod_I B$ is not a weak equivalence for any infinite set I. Take $B = \Delta^{\mathbf{0}}$ and A to be the simplicial set generated by vertices n ($n \in \mathbf{Z}$) and edges $n \to n+1$ ($n \in \mathbf{Z}$). Using the simplicial Whitehead theorem, one can immediately see that f is a simplicial weak equivalence. Computing $\pi_0 \left(\prod_I f\right)$ shows that it is not an isomorphism. Thus, $\prod_I f$ is not a weak equivalence. Thus, infinite products of simplicial sets must be derived. In fact, we can take $\mathsf{C}' = \mathsf{sSet}^I_{\mathsf{Kan}} \subset \mathsf{sSet}^I = \mathsf{C}$, $R = (\mathsf{Ex}^\infty)^I$, and r the indexwise inclusion. When restricted to Kan complexes, simplicial weak equivalences are precisely simplicial homotopy equivalences, and $\prod_I$ preserves simplicial homotopy equivalences. Thus, $\mathsf{R}\prod_{i \in I} A_i$ can be computed as $\prod_{i \in I} \mathsf{Ex}^\infty A_i$. Used in 42.7.

Example 42.5. Any category can be turned into a relative categories by postulating that weak equivalence coincide with isomorphisms. In this case, homotopy limits are precisely ordinary limits, and likewise for colimits.

In order to derive more complicated shapes of limits than those given by discrete I, we need a new idea.

Philosophy 42.6. When promoting ordinary categorical constructions to ∞-categorical constructions, equalities should be replaced by homotopies and be made part of the data of the construction under consideration.

We illustrate this idea with the case of pullbacks.

Example 42.7. Consider the case $I = \{0 \to 1 \leftarrow 2\}$, so that I-limits are pullbacks (alias fibered products). The functor $\mathtt{lim}_I$ does not preserve limits. For instance, consider the following weak equivalence (depicted vertically) of I-diagrams (depicted horizontally)

$$\begin{array}{ccccc} \Delta^0 & \longrightarrow & \mathrm{S}^1 & \longleftarrow & \Delta^0 \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \Delta^0 & \longrightarrow & \mathrm{S}^1 & \longleftarrow & A \end{array}$$

where A is the universal cover of S^1 as constructed in Example 42.4. Applying the functor $\mathtt{lim}_I$ yields a morphism

$$\begin{array}{c} \Delta^0 \\ \big\downarrow \\ \mathbf{Z} \end{array}$$

that is not a weak equivalence. Thus, pullbacks of simplicial sets must be derived.

To apply the above philosophy to this case, recall that the ordinary pullback is

$$A \times_B C = \{(a, c) \mid a \in A, c \in C, f(a) = g(c)\}.$$

Replacing equalities with homotopies and making them part of the data yields the following informal formula

$$A \times^h_B C = \{(a, c, h) \mid a \in A, c \in C, h\colon f(a) \to g(c)\},$$

where the notation for h means that h is a path in B from $f(a)$ to $g(c)$. To make this precise, we formalize the space of paths as

$$\mathsf{RHom}(\Delta^1, B) = (\mathsf{Ex}^\infty B)^{\Delta^1}.$$

The formula then becomes

$$A \times^h_B C = A \times_{\mathsf{Ex}^\infty B} \times (\mathsf{Ex}^\infty B)^{\Delta^1} \times_{\mathsf{Ex}^\infty B} C.$$

As we will show later using more powerful abstract tools, this formula does indeed preserve weak equivalences.

Exercise 42.8. Explain how to compute the homotopy limit of an infinite tower of simplicial sets:

$$\cdots \to X_3 \to X_2 \to X_1 \to X_0.$$

Show that the limit functor does not preserve weak equivalences of such towers.

Exercise 42.9. Show that the homotopy colimit of an infinite cotower of simplicial sets

$$X_0 \to X_1 \to X_2 \to X_3 \to \cdots$$

can be computed as its colimit. Bonus points: show that the homotopy colimit of an infinite cotower of topological spaces cannot be computed as its colimit and explain how to resolve this problem. Used in 1.0*.

43 Lifting properties and Kan fibrations

Definition 43.1. In a category C, we write $f \pitchfork g$ for morphisms $f\colon A \to B$ and $g\colon C \to D$ and say that f has a *left lifting property* with respect to g and g has a *right lifting property* with respect to f if any commutative square

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & D \end{array}$$

can be extended to a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle g} \\ B & \longrightarrow & D. \end{array}$$

The diagonal map d is sometimes referred to as the *lift* of f with respect to g. Used in 45.5*.

Definition 43.2. For a class S of morphisms in a category C we set

$$S^{\pitchfork} = \{g \mid f \pitchfork g \text{ for all } f \in S\}$$

(the right complement of S) and

$${}^{\pitchfork}S = \{g \mid g \pitchfork f \text{ for all } f \in S\}$$

(the left complement of S).

Definition 43.3. The class of *Kan fibrations* (of simplicial sets) is defined as $S^{\pitchfork}$, where $S = \{\Lambda^n_k \to \Delta^n\}$ is the set of inclusions of horns into simplices. The class of *trivial Kan fibrations* (alias *acyclic Kan fibrations*) is defined as $S^{\pitchfork}$, where $S = \{\partial\Delta^n \to \Delta^n\}$ is the set of inclusions of boundaries of simplices into simplices. Used in 45.1.

Definition 43.4. Suppose S is a well-founded totally ordered set and $I \subset \mathsf{Mor}(\mathsf{C})$ is a class of morphisms in a category C. An S-indexed *transfinite sequence* of morphisms in I is a functor $X\colon S \to \mathsf{C}$ such that for any $s \in S$ the morphism $X(s) \to X(s+1)$ belongs to I (where $s+1$ denotes the successor of s, i.e., the smallest element of S greater than s) and for any $t \in S$ that is not a successor and not the smallest element of S, the object $X(t)$ together with injection maps $X(s) \to X(t)$ for all $s < t$ is a colimit cocone for the diagram obtained by restricting X to elements smaller than t. Used in 43.6.

Remark 43.5. The most commonly used transfinite sequences of morphisms are indexed by the first infinite ordinal and are simply functors $\{0 < 1 < 2 < \cdots\} \to \mathsf{C}$ valued in I on morphisms.

Definition 43.6. The *transfinite composition* of a transfinite sequence $X\colon S \to \mathsf{C}$ is the injection map

$$X(0) \to \mathtt{colim} X,$$

where $0 \in S$ is the smallest element. Likewise, the *cotransfinite composition* of a *cotransfinite sequence* $X\colon S^{\mathsf{op}} \to \mathsf{C}$ (where S is a well-founded totally ordered set) is the projection map

$$\mathtt{lim} X \to X(0).$$

Used in 43.7.

Lemma 43.7. For any class S of morphisms in a category C, the classes $S^{\pitchfork}$ and ${}^{\pitchfork}S$ contain all isomorphisms and are closed under compositions. Additionally, $S^{\pitchfork}$ is closed under the following types of limits:

- products;
- base changes;
- cotransfinite compositions.

Likewise, ${}^{\pitchfork}S$ is closed under the following types of colimits:

- coproducts;
- cobase changes;
- transfinite compositions.

44 Weak factorization systems and model categories

Supplementary sources: Joyal [WFS].

Given a category C, we can construct the functor category $\mathsf{C}^{0\to 1}$ of morphisms in C (objects are morphisms in C and morphisms are commutative squares) as well as the functor category $\mathsf{C}^{0\to 1\to 2}$ of composable pairs of morphisms in C (objects are pairs (f,g) such that $g\circ f$ is defined and morphisms are commutative diagrams made of two squares). The functor $\{0\to 1\}\cong\{0\to 2\}\to\{0\to 1\to 2\}$ induces the composition functor $\circ\colon \mathsf{C}^{0\to 1\to 2}\to \mathsf{C}^{0\to 1}$.

Definition 44.1. A *functorial factorization* on a category C is a section of the composition functor

$$\circ\colon \mathsf{C}^{0\to 1\to 2}\to \mathsf{C}^{0\to 1},$$

i.e., a functor

$$\mathsf{F}\colon \mathsf{C}^{0\to 1}\to \mathsf{C}^{0\to 1\to 2}$$

such that $\circ\mathsf{F}=\mathrm{id}_{\mathsf{C}^{0\to 1}}$. Used in 44.2, 44.6, 45.1*, 46.2, 46.2*, 46.3, 46.4*, 46.5, 46.6, 46.7, 46.8, 48.3, 48.4.

Concretely, F sends a morphism $f\colon X\to Y$ in C to a pair of morphisms $\mathsf{F}_1(f)\colon X\to Z$ and $\mathsf{F}_2(f)\colon Z\to Y$ such that $\mathsf{F}_2(f)\circ\mathsf{F}_1(f)=f$, in particular, the left side is always defined. Furthermore, both F_1 and F_2 are functors.

Definition 44.2. A (functorial) *weak factorization system* on a category C is a functorial factorization $\mathsf{F}=(\mathsf{F}_1,\mathsf{F}_2)$ on C such that the left class $L=\mathsf{F}_1(\mathsf{C}^{0\to 1})$ and the right class $R=\mathsf{F}_2(\mathsf{C}^{0\to 1})$ (here in both cases we take essential images, i.e., the closure of image under isomorphisms) satisfy the additional properties $L={}^{\pitchfork}R$ and $R=L^{\pitchfork}$. Used in 44.3, 44.4, 45.1, 45.1*.

Definition 44.3. A *weak factorization system generated by a set of morphisms* S in a category C is a weak factorization system $\mathsf{F}=(\mathsf{F}_1,\mathsf{F}_2)$ on C such the right class $R=\mathsf{F}_2(\mathsf{C}^{0\to 1})=S^{\pitchfork}$ and the left class $L=\mathsf{F}_1(\mathsf{C}^{0\to 1})={}^{\pitchfork}R={}^{\pitchfork}(S^{\pitchfork})$.

Definition 44.4. A *model structure* on a relative category (D,W) is a pair of weak factorization systems (C,AF) (*cofibrations* and *acyclic fibrations*) and (AC,F) (*acyclic cofibrations* and *fibrations*) on D such that $\mathsf{AC}=\mathsf{C}\cap\mathsf{W}$ and $\mathsf{AF}=\mathsf{F}\cap\mathsf{W}$. Used in 2.0*, 6.9, 44.4, 44.5, 44.7, 44.8, 45.1, 45.1*, 45.3, 45.3*, 45.4, 45.5, 45.5*, 45.6, 45.6*, 45.7, 45.8*, 46.2, 46.2*, 46.3, 46.4, 46.4*, 46.5, 46.6, 46.6*, 46.7, 47.1, 48.1, 48.3, 48.3*, 48.4, 48.5, 50.3.

Definition 44.5. A *model category* is a relative category (C,W) equipped with a model structure such that W satisfies the *2-out-of-3 property*: if $g\circ f\in\mathsf{W}$ and one of f or g also belongs to W, then both f and g belong to W. Additionally, C must be complete and cocomplete. Used in 1.0*, 1.2*, 17.6*, 44.6, 44.8, 45.0*, 45.1, 45.5*, 45.7, 47.1, 48.1, 48.5, 50.3, 50.4, 51.1, 51.2.

Remark 44.6. As stated, the above definition is due to Hovey (2001). An older version of the definition of a model category due to Kan (1997) does not include the data of functorial factorizations, but merely requires that they exist. An even older version due to Quillen (1967), where it is referred to as a *closed model category*, does not require functoriality and C is required to be only finitely complete and finitely cocomplete.

Remark 44.7. Knowing just one of the four classes C, AC, F, AF allows one to recover the other three. First, the properties $\mathsf{C}={}^{\pitchfork}\mathsf{AF}$, $\mathsf{AC}={}^{\pitchfork}\mathsf{F}$, $\mathsf{F}=\mathsf{AC}^{\pitchfork}$, $\mathsf{AF}=\mathsf{C}^{\pitchfork}$ allow us to recover the complementary class, so that we know either AC and F or C and AF. Next, the properties $\mathsf{AF}=\mathsf{F}\cap\mathsf{W}$ respectively $\mathsf{AC}=\mathsf{C}\cap\mathsf{W}$ allow us to recover AF respectively AC. Finally, $\mathsf{C}={}^{\pitchfork}\mathsf{AF}$ respectively $\mathsf{F}=\mathsf{AC}^{\pitchfork}$ allows us to recover the remaining fourth class. Used in 45.1*.

Definition 44.8. An object X of a model category is *cofibrant* if the unique map $0\to X$ from the initial object is a cofibration. Likewise, X is *fibrant* if the unique map $X\to 1$ to the terminal object is a fibration. Used in 45.1, 46.4*, 48.3, 48.3*, 48.4.

45 Model structure on simplicial sets

In this section we construct our first example of a model category: the model category of simplicial sets.

Proposition 45.1. (Quillen, 1967.) There is a unique model structure on the relative category of simplicial sets that turns it into a model category such that the class of cofibrations coincides with the class of monomorphisms. The weak factorization system $(\mathsf{C}, \mathsf{AF})$ is generated by boundary inclusions $\partial\Delta^{\mathbf{n}} \to \Delta^{\mathbf{n}}$ for all $n \geq 0$. The weak factorization system $(\mathsf{AC}, \mathsf{F})$ is generated by horn inclusions $\Lambda^{\mathbf{n}}_k \to \Delta^{\mathbf{n}}$ for all $n \geq 1$ and $0 \leq k \leq n$. All objects are cofibrant. Fibrant objects are precisely Kan complexes. Fibrations are precisely Kan fibrations. Used in 45.0*, 45.8.

Proof. Uniqueness follows immediately from 44.7, since we specified the class of cofibrations. Lemma 45.2 shows that weak equivalences satisfy the 2-out-of-3 property. In the next section, we establish functorial factorizations for both weak factorization systems. Since both weak factorization systems are generated by a set of morphisms, the lifting properties hold by definition. Lemma 45.3 shows that the class of cofibrations coincides with monomorphisms. ▮

Lemma 45.2. The class of simplicial weak equivalences satisfies the 2-out-of-3 property. Used in 45.1*.

Proof. By definition of simplicial weak equivalences, the problem is immediately reduced to showing that simplicial homotopy equivalences satisfy the 2-out-of-3 property. The latter is accomplished by constructing the necessary homotopy inverse by composing the given maps and their homotopy inverses. ▮

Lemma 45.3. The class of cofibrations coincides with monomorphisms. Used in 45.1*.

Proof. The skeletal filtration and Proposition 39.9 imply that $\emptyset \to B$ is a cofibration for any simplicial set B. More generally, given an arbitrary monomorphism $A \to B$, one can construct in an analogous way the relative skeletal filtration

$$A = B_0 \to B_1 \to B_2 \to \cdots$$

whose colimit is canonically isomorphic to B and show that individual transition maps $B_{n-1} \to B_n$ satisfy an analog of Proposition 39.9. ▮

Lemma 45.4. The functor Ex^∞ preserves fibrations and acyclic fibrations.

Lemma 45.5. The class of acyclic fibrations of simplicial sets coincides with the intersection of the class of fibrations and simplicial weak equivalences:

$$\mathsf{AF} = \mathsf{F} \cap \mathsf{W}.$$

Proof. Suppose $f\colon X \to Y$ is a simplicial map. If f is an acyclic fibration, then f is also a fibration. Indeed, the right lifting property of f with respect to $\Lambda^{\mathbf{n}_k} \to \Delta^{\mathbf{n}}$ follow from the right lifting property of f with respect to $\Lambda^{\mathbf{n}_k} \to \partial\Delta^{\mathbf{n}}$ and $\partial\Delta^{\mathbf{n}} \to \Delta^{\mathbf{n}}$. This is true by assumpotion for the latter map, whereas the former map is a cobase change of $\partial\Delta^{n-1} \to \Delta^{n-1}$ along the inclusion $\partial\Delta^{n-1} \to \Lambda^{\mathbf{n}_k}$ whose image is the boundary of $\Lambda^{\mathbf{n}_k}$.

Next, if f is an acyclic fibration, then f is also a simplicial weak equivalence. It suffices to show that $\mathsf{Ex}^\infty f$, which is an acyclic fibration between Kan complexes, is a simplicial homotopy equivalence using the simplicial Whitehead theorem. Indeed, the corresponding square in the statement of the simplicial Whitehead theorem

$$\begin{array}{ccc} \partial\Delta^{\mathbf{n}} & \to & \mathsf{Ex}^\infty X \\ \downarrow & \nearrow & \downarrow{\scriptstyle \mathsf{Ex}^\infty f} \\ \Delta^{\mathbf{n}} & \longrightarrow & \mathsf{Ex}^\infty Y, \end{array}$$

admits a lifting where both triangles commute strictly, by definition of an acyclic fibration.

Finally, if f is a fibration and a simplicial weak equivalence, we have to show that f is an acyclic fibration. Using Proposition 46.7, factor $f = gc$, where c is a cofibration and g is an acyclic fibration. The map c is also a simplicial weak equivalence by the 2-out-of-3 property since we already know that g is a simplicial weak equivalence. ▮

Lemma 45.6. The class of acyclic cofibrations of simplicial sets coincides with the intersection of the class of cofibrations and simplicial weak equivalences:

$$\mathsf{AC} = \mathsf{C} \cap \mathsf{W}.$$

Proof. Acyclic cofibrations are cofibrations because acyclic fibrations are fibrations.

Acyclic cofibrations are simplicial weak equivalences because

Finally, if f is a cofibration and a simplicial weak equivalence, we have to show that f is an acyclic cofibration. Using Corollary 46.6, factor $f = gc$, where c is an acyclic cofibration and g is an acyclic fibration. ▮

Definition 45.7. A model category is *right proper* if base changes along fibrations preserve weak equivalences. Used in 6.9, 45.8.

Lemma 45.8. The model category of simplicial sets is right proper. Used in 46.5*.

Proof. Given a cartesian square of simplicial sets

$$\begin{array}{ccc} A' & \longrightarrow & A \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ B' & \xrightarrow[f]{} & B \end{array}$$

we have to show that if f is a fibrations and g is a simplicial weak equivalence, then g' is also a simplicial weak equivalence. We apply the functor Ex^∞ to the entire diagram. Using the fact that Ex^∞ preserves finite limits, fibrations, and creates simplicial weak equivalences from simplicial homotopy equivalences, the above problem is immediately reduce to the special case when all four simplicial sets are Kan and g is a simplicial homotopy equivalence. ▮

46 Functorial factorizations of simplicial maps

Notation 46.1. The maps $\iota_0\colon \Delta^0 \to \Delta^1$ and $\iota_1\colon \Delta^0 \to \Delta^1$ pick the two vertices of Δ^1 in increasing order. The map $\iota\colon \partial\Delta^1 \to \Delta^1$ is the boundary inclusion of Δ^1. The map $p\colon \Delta^1 \to \Delta^0$ is the unique simplicial map from Δ^1 to Δ^0.

Lemma 46.2. (The *mapping cylinder* construction.) If $X, Y \in \mathsf{sSet}$ and $f\colon X \to Y$ is a simplicial map, then the maps $X \to \operatorname{cyl}(f) \to Y$ constructed in the proof form a functorial factorization of f into a cofibration followed by a weak equivalence. Used in 1.0*, 46.2*, 46.6*, 46.7*.

Proof. Denote $\operatorname{cyl}(f) = \Delta^1 \times X \sqcup_X Y$ and $\operatorname{cyl}(X) = \operatorname{cyl}(\mathrm{id}_X) = \Delta^1 \times X$. Consider the following commutative diagram, where the square is cocartesian:

$$\begin{array}{ccccc} & & X & \xrightarrow{f} & Y \\ & & {\scriptstyle \iota_1 \times X}\downarrow & & \downarrow \\ X & \xrightarrow{\iota_0 \times X} & \operatorname{cyl}(X) & \longrightarrow & \operatorname{cyl}(f). \end{array}$$

This diagram yields the map $X \to \operatorname{cyl}(f)$, whereas the map $\operatorname{cyl}(f) \to Y$ is induced by the universal property of pushouts from the maps $f \circ (p \times X)\colon \Delta^1 \times X \to X \to Y$ and $\mathrm{id}_Y\colon Y \to Y$. These definitions imply that the composition $X \to \operatorname{cyl}(f) \to Y$ equals f, so we indeed have a functorial factorization.

Consider the cocartesian square

$$\begin{array}{ccc} Y \sqcup Y & \xrightarrow{f \sqcup \mathrm{id}_Y} & X \sqcup Y \\ {\scriptstyle \iota \times Y}\downarrow & & \downarrow \\ \Delta^1 \times Y & \longrightarrow & X \sqcup_Y \Delta^1 \times Y \end{array}$$

The left map is a cofibration because $\iota\colon \partial\Delta^1 \to \Delta^1$ is a cofibration. Thus, the right map is a cofibration. The map $X \cong X \sqcup \emptyset \to X \sqcup Y$ is a cofibration because $\emptyset \to X$ is a cofibration. Thus, the composition of the map $X \to X \sqcup Y$ and the right map is a cofibration. The resulting map $X \to X \sqcup_Y \Delta^1 \times Y$ is the first map in the factorization.

The map $\mathrm{cyl}(f) \to Y$ is a simplicial homotopy equivalence (hence, a simplicial weak equivalence) because the composition $Y \to \mathrm{cyl}(f) \to Y$ equals id_Y and the map $Y \to \mathrm{cyl}(f)$ is a cobase change of the map $\iota_1 \times X$, which is a simplicial homotopy equivalence because ι_1 is one. ▮

Lemma 46.3. (The *mapping path space* (alias *mapping cocylinder*) construction.) If $X, Y \in \mathsf{sSet}_{\mathsf{Kan}}$ and $f\colon X \to Y$ is a simplicial map, then the maps $X \to \mathrm{cocyl}(f) \to Y$ constructed in the proof form a functorial factorization of the map f into an acyclic cofibration followed by a fibration. Used in 1.0*, 46.4*, 46.5*.

Remark 46.4. This lemma is formally dual to the previous lemma except for two differences. In this lemma we must assume the source and target to be Kan complexes, whereas the previous lemma imposes no such restrictions. Additionally, the previous lemma can produce weak equivalences that are not fibrations, whereas this lemma produces weak equivalences that are also cofibrations.

Proof. Denote $\mathrm{cocyl}(f) = X \times_Y Y^{\Delta^1}$ and $\mathrm{cocyl}(Y) = \mathrm{cocyl}(\mathrm{id}_Y) = Y^{\Delta^1}$. Consider the following commutative diagram, where the square is cartesian:

$$\begin{array}{ccccc} \mathrm{cocyl}(f) & \longrightarrow & \mathrm{cocyl}(Y) & \xrightarrow{\mathsf{Hom}(\iota_0, Y)} & Y \\ \downarrow & & \downarrow{\scriptstyle \mathsf{Hom}(\iota_1, Y)} & & \\ X & \xrightarrow{f} & Y. & & \end{array}$$

This diagram yields the map $\mathrm{cocyl}(f) \to Y$, whereas the map $X \to \mathrm{cocyl}(f)$ is induced by the universal property of pullbacks from the maps $\mathrm{id}_X\colon X \to X$ and $\mathsf{Hom}(p, Y) \circ f\colon X \to Y \to Y^{\Delta^1}$. These definitions imply that the composition $X \to \mathrm{cocyl}(f) \to Y$ equals f, so we indeed have a functorial factorization.

Consider the cartesian square

$$\begin{array}{ccc} X \times_Y Y^{\Delta^1} & \longrightarrow & Y^{\Delta^1} \\ \downarrow & & \downarrow{\scriptstyle \mathsf{Hom}(\iota, Y)} \\ X \times Y & \xrightarrow{f \times \mathrm{id}_Y} & Y \times Y \end{array}$$

The right map is a fibration because $\iota\colon \partial\Delta^1 \to \Delta^1$ is a cofibration and Y is fibrant. Thus, the left map is a fibration. The map $X \times Y \to 1 \times Y \cong Y$ is a fibration because $X \to 1$ is a fibration. Thus, the composition $\mathrm{cocyl}(f) = X \times_Y Y^{\Delta^1} \to X \times Y \to Y$ is also a fibration and it is the second map in the factorization.

The map $X \to \mathrm{cocyl}(f)$ is a simplicial homotopy equivalence (hence, a simplicial weak equivalence) because the composition $X \to \mathrm{cocyl}(f) \to X$ equals id_X and the map $\mathrm{cocyl}(f) \to X$ is a base change of the map $\mathsf{Hom}(\iota_1, X)$, which is an acyclic fibration because ι_1 is an acyclic cofibration.

Finally, the map $X \to \mathrm{cocyl}(f) = X \times_Y Y^{\Delta^1}$ is a cofibration because its composition with the projection $X \times_Y Y^{\Delta^1} \to X$ equals id_X. ▮

We now remove the requirement for the source and target to be Kan complexes.

Proposition 46.5. (The *derived mapping path space* (alias *derived mapping cocylinder*) construction.) If $f\colon X \to Y$ is a simplicial map, then the maps $X \to \mathrm{Rcocyl}(f) \to Y$ constructed in the proof form a functorial factorization of the map f into an acyclic cofibration followed by a fibration. Used in 46.5*, 46.6, 46.7*.

Proof. Consider the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \mathsf{Ex}^\infty X & \xrightarrow{\mathsf{Ex}^\infty f} & \mathsf{Ex}^\infty Y. \end{array}$$

We apply the mapping path space construction of Lemma 46.3 to the bottom map (its source and target are fibrant) and complete the resulting diagram as depicted below, with the right square being cartesian and the top map in the left square induced by the universal property of pullbacks.

$$\begin{array}{ccccc} X & \longrightarrow & \mathrm{Rcocyl}(f) & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{Ex}^\infty X & \longrightarrow & \mathrm{cocyl}(\mathsf{Ex}^\infty f) & \longrightarrow & \mathsf{Ex}^\infty Y. \end{array}$$

By Lemma 46.3 the bottom left map is an acyclic cofibration and the bottom right map is a fibration. Accordingly, the top right map is a fibration because it is a base change of a fibration.

The middle map is a weak equivalence because the right map is one and weak equivalences are stable under base changes along fibrations by Lemma 45.8. Thus, in the left square all maps except for the top one are weak equivalences, hence the top map is also a weak equivalence.

Both maps $X \to \mathsf{Ex}^\infty X \to \mathrm{cocyl}(\mathsf{Ex}^\infty f)$ are cofibrations. Hence, their composition is also a cofibration. Therefore, the top left map $X \to \mathrm{Rcocyl}(f)$ is also a cofibration. ∎

The 2-out-of-3 property of simplicial weak equivalences immediately implies the following claim.

Corollary 46.6. If $f\colon X \to Y$ is a simplicial weak equivalence, then the maps $X \to \mathrm{Rcocyl}(f) \to Y$ constructed in Proposition 46.5 form a functorial factorization of the map f into an acyclic cofibration followed by an acyclic fibration. Used in 45.6*, 46.7*.

We now improve Lemma 46.2, allowing the second map to be an acyclic fibration and not just a simplicial weak equivalence.

Proposition 46.7. (The *derived mapping cylinder* construction.) If $X, Y \in \mathsf{sSet}$ and $f\colon X \to Y$ is a simplicial map, then the maps $X \to \mathrm{Rcyl}(f) \to Y$ constructed in the proof form a functorial factorization of f into a cofibration followed by an acyclic fibration. Used in 45.5*.

Proof. Use Lemma 46.2 to factor the map f as $X \to \mathrm{cyl}(f) \to Y$, where the first map is a cofibration and the second map is a weak equivalence. Use Corollary 46.6 to factor the map $g\colon \mathrm{cyl}(f) \to Y$ as $\mathrm{cyl}(f) \to \mathrm{Rcocyl}(g) \to Y$, where the first map is an acyclic cofibration and the second map is an acyclic fibration. Composing $X \to \mathrm{cyl}(f)$ and $\mathrm{cyl}(f) \to \mathrm{Rcocyl}(g)$, we get a functorial factorization of the map f as $X \to \mathrm{Rcocyl}(g) \to Y$, where the first map is a cofibration and the second map is an acyclic fibration. ∎

Summary 46.8. Suppose $f\colon X \to Y$ is a simplicial map. Expanding the above constructions, we obtain the following formulas for the functorial factorizations of f. Acyclic cofibration followed by a fibration:

$$X \to \mathsf{Ex}^\infty X \times_{\mathsf{Ex}^\infty Y} (\mathsf{Ex}^\infty Y)^{\Delta^1} \times_{\mathsf{Ex}^\infty Y} Y \to Y.$$

Cofibration followed by an acyclic fibration:

$$X \to \mathsf{Ex}^\infty((\Delta^1 \times X) \sqcup_X Y) \times_{\mathsf{Ex}^\infty Y} (\mathsf{Ex}^\infty Y)^{\Delta^1} \times_{\mathsf{Ex}^\infty Y} Y \to Y.$$

47 Cofibrantly generated and combinatorial model categories

Definition 47.1. A model category C is *cofibrantly generated* if there a set GC_C of *generating cofibrations* and a set GAC_C of *generating acyclic cofibrations* such that $\mathsf{F}_\mathsf{C} = (\mathsf{GAC}_\mathsf{C})^{\pitchfork}$ and $\mathsf{AF}_\mathsf{C} = (\mathsf{GC}_\mathsf{C})^{\pitchfork}$. Used in 47.1, 48.5, 50.4.

48 Quillen adjunctions

Definition 48.1. A *left Quillen functor* is a left adjoint functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$, where C and D are model categories and F preserves cofibrations and acyclic cofibrations. A *right Quillen functor* is a right adjoint functor $\mathsf{G}\colon \mathsf{D} \to \mathsf{C}$ such that G preserves fibrations and acyclic fibrations. Used in 48.2, 48.3, 48.4.

Lemma 48.2. If $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ is a left Quillen functor then its right adjoint $\mathsf{G}\colon \mathsf{D} \to \mathsf{C}$ is a right Quillen functor and vice versa. Thus, one also talks about *Quillen adjunction* or *Quillen pairs*.

Proposition 48.3. (Ken Brown's lemma.) Left Quillen functors $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ admit left derived functors that can be derived as follows. The subcategory $\mathsf{C}' \subset \mathsf{C}$ consists of all cofibrant objects. The resolution functor $R\colon \mathsf{C} \to \mathsf{C}'$ sends an object $X \in \mathsf{C}$ to the middle object X' in the functorial factorization $\emptyset \to X' \to X$ of the unique map $\emptyset \to X$ as a cofibration followed by an acyclic fibration. The natural transformation $r\colon R \to \mathrm{id}$ is given by the second map: $r_X\colon X' \to X$.

Proof. We have to show that F preserves weak equivalences between cofibrant objects in C. Suppose $f\colon A \to B$ is such a weak equivalence. Factor $g\colon A \sqcup B \to B$ as a cofibration followed by an acyclic fibration. ∎

We state the dual proposition, for completeness.

Proposition 48.4. (Ken Brown's lemma.) Right Quillen functors $\mathsf{G}\colon \mathsf{D} \to \mathsf{C}$ admit right derived functors that can be derived as follows. The subcategory $\mathsf{C}' \subset \mathsf{C}$ consists of all fibrant objects. The resolution functor $R\colon \mathsf{C} \to \mathsf{C}'$ sends an object $X \in \mathsf{C}$ to the middle object X' in the functorial factorization $X \to X' \to 1$ of the unique map $X \to 1$ as an acyclic cofibration followed by a fibration. The natural transformation $r\colon \mathrm{id} \to R$ is given by the first map: $r_X\colon X \to X'$.

Lemma 48.5. Given a left adjoint functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ between model categories, it suffices to verify that F sends generating cofibrations in C to cofibrations in D and likewise for acyclic cofibrations.

Proof. ∎

49 The Dold–Kan correspondence and the Eilenberg–Zilber theorem

Local homotopy theory

50 Projective model structure on presheaves

Definition 50.1. Suppose $I \in \mathsf{Cat}$ is a small category and C is a category. The category of I-indexed diagrams is defined as the category $\mathsf{Fun}(I, \mathsf{C})$. The category of C-valued presheaves on I is defined as the category $\mathsf{Fun}(I^{\mathsf{op}}, \mathsf{C})$.

Definition 50.2. Suppose $I \in \mathsf{Cat}$ is a small category and C is a category tensored over Set (which simply means that C admits small coproducts). Furthermore, suppose $i \in I$ and $c \in \mathsf{C}$ are objects in these categories. The *hom-presheaf* (alias *represented presheaf*) of i with value c is the C-valued presheaf $\mathsf{hom}(-, i) \otimes c$ on I that sends

$$j \mapsto \mathsf{hom}(j, i) \otimes c$$

and

$$(f\colon j \to j') \mapsto (\mathsf{hom}(f, i) \otimes c\colon \mathsf{hom}(j', i) \otimes c \to \mathsf{hom}(j, i) \otimes c).$$

Definition 50.3. Suppose $I \in \mathsf{Cat}$ is a small category and C is a model category. The *projective model structure* (if it exists) on C-valued presheaves is uniquely defined by the requirement that its weak equivalences, fibrations, and acyclic fibrations are natural transformations whose individual components belong to the corresponding class of maps in the model category C. Used in 50.4, 51.1, 51.2.

Proposition 50.4. If C is a cofibrantly generated model category and I is a small category, then the projective model structure on $\mathsf{Fun}(I^{\mathsf{op}}, \mathsf{C})$ exists. Furthermore, if GC_{C} and $\mathsf{GAC}_{\mathsf{C}}$ are sets of generating cofibrations and generating acyclic cofibrations respectively, then

$$\{\mathfrak{hom}(-, i) \otimes f \mid i \in I, f \in \mathsf{GC}_{\mathsf{C}}\}$$

is a set of generating cofibrations for $\mathfrak{Fun}(I^{\mathsf{op}}, \mathsf{C})$ and likewise for generating acyclic cofibrations.

51 Simplicial presheaves

Definition 51.1. The category of *simplicial presheaves* on a small category S is the model category of functors $\mathsf{S}^{\mathsf{op}} \to \mathsf{sSet}$ equipped with the projective model structure.

Definition 51.2. The category of *presheaves of chain complexes* on a small category S is the model category of functors $\mathsf{S}^{\mathsf{op}} \to \mathsf{Ch}$ equipped with the projective model structure.

However, we really interested in sheaves, not presheaves. The sheaf condition can be encoded as follows.

52 Left Bousfield localization

In the following three definitions, (C, W) is a relative category and $\mathsf{S} \subset \mathfrak{Mor}(\mathsf{C})$ is a class of morphisms.

Definition 52.1. An *S-local object* is an object $X \in \mathsf{C}$ such that for any $s \in S$ the induced morphism $\mathfrak{map}(s, X)$ is a simplicial weak equivalence. (Recall that $\mathfrak{map}$ denotes the simplicial set of maps in a relative category, defined via the Dwyer–Kan simplicial localization or the Dwyer–Kan hammock localization.) Used in 52.1, 52.2, 52.4.

Definition 52.2. An *S-local weak equivalence* is a morphism f such that $\mathfrak{map}(f, W)$ is a simplicial weak equivalence for any S-local object W. Used in 52.3.

Definition 52.3. A *left Bousfield localization* (alias *homotopy reflective localization*) of a relative category (C, W) with respect to a class of morphisms $\mathsf{S} \subset \mathfrak{Mor}(\mathsf{C})$ is the relative category $(\mathsf{C}, \mathsf{W}_S)$, where W_S denotes S-local weak equivalences. Used in 52.4.

Proposition 52.4. The inclusion $(\mathsf{C}_S, \mathsf{W}) \to (\mathsf{C}, \mathsf{W}_S)$ of the relative category of S-local objects and weak equivalences induced from C into the left Bousfield localization of (C, W) with respect to S is a homotopy equivalence of relative categories.

53 Sheaf cohomology

Sheaf cohomology

Further topics

54 Quasicategories

Stable homotopy theory

55 Generalized homology theories

generalized homology theory generalized cohomology theory

56 K-theory

K-theory

57 Spectra

spectrum

58 Further topics

tensored

homotopy equivalence of relative categories

Quillen–Kan–Serre–Milnor equivalence. Universal coefficient theorem.

Intersection product in homology. Homotopy limits and colimits of simplicial sets. Homotopy groups. The Hurewicz isomorphism. Chain complexes and their homotopy (co)limits. The Dold–Kan correspondence. Eilenberg–MacLane spaces. Interaction of homotopy (co)limits with simplicial (co)homology. Homology and cohomology theories as (co)continuous functors. Eilenberg–Steenrod axioms.

Pushforward and pullback of local systems. Verdier duality. Long exact sequence of a fibration. Fiber and cofiber sequences of homotopy groups. Blakers–Massey theorem. Freudenthal suspension theorem.

Topological K-theory. Model categories. The Smith recognition theorem. Example: simplicial sets, chain complexes, spectra. Simplicial symmetric spectra. Representability of homology and cohomology theories by spectra. Smash product and internal hom of spectra. Multiplicative cohomology theories. Spanier–Whitehead duality. Thom spectra. Atiyah duality. Simplicial presheaves. Sheaf cohomology. De Rham cohomology, de Rham theorem.

Prerequisites

59 Appendix: sets and functions

Supplementary sources: Lawvere and Rosebrugh [SETS].

59.1. ***Relations and maps of sets*** Used in 8.6, 8.8, 8.14*, 8.17, 11.4, 11.14, 11.16, 11.17, 12.12, 12.15, 18.0*, 59.1*, 59.2*, 59.3*, 59.4*, 59.5*, 59.6*, 59.7*, 60.12.

The *ordered pair* (a,b) is defined as $\{\{\{a\}\},\{\{b\},\{\emptyset\}\}\}$. The reasoning behind this definition is that $(a,b)=(a',b')$ if and only if $a=a'$ and $b=b'$. The *product of sets* A and B is the set $A\times B=\{z\mid \exists a\in A, b\in B\colon z=(a,b)\}$. An *ordered triple* (a,b,c) can now be defined as $((a,b),c)$ and likewise for n-tuples.

A *relation* is a triple (A,B,R), where A and B are sets and $R\subset A\times B$. We emphasize that A and B form a part of the data of a relation. We also say that R is a relation from A to B. We often write aRb instead of $(a,b)\in R$. Relations can be composed: if R is a relation from A to B and S is a relation from B to C, then $S\circ R$ is a relation from A to C for which $(a,c)\in S\circ R$ if and only if there is $b\in B$ such that aRb and bSc. We have $(T\circ S)\circ R=T\circ(S\circ R)$, for which we simply write $T\circ S\circ R$. The *identity relation* id_A from A to A satisfies $(a,a')\in\mathrm{id}_A \iff a=a'$. We have $\mathrm{id}_B\circ R=R=R\circ\mathrm{id}_A$. An *equivalence relation* on a set A is a relation R from A to A such that aRa for all $a\in A$, aRb implies bRa for all $a,b\in A$, and aRb and bRc implies aRc for all $a,b,c\in A$. The *equivalence class* of an element $a\in A$ with respect to an equivalence relation R on A is the set $[a]:=\{x\in A\mid aRx\}$.

A *map of sets* from A to B is defined as a *functional relation* from A to B, namely, a relation f from A to B with an additional property that for any $a\in A$ there is exactly one $b\in B$ (denoted by $f(a)$) such that $(a,b)\in f$. We refer to A as the domain of f (denoted by $\operatorname{dom} f$) and B as the codomain of f (denoted by $\operatorname{codom} f$). The composition of two functional relations is again functional, which allows us to define compositions of maps via compositions of relations.

Remark 59.2. In the modern mathematical parlance, the word "*function*" is exactly synonymous with "map" (of sets). Historically, though, a very different meaning was used: "E is a function of x" meant that x is a variable, and substituting some value for x in the expression E would give us various values, denoted by $E(x)$, which therefore are "functions" of x. Of course, the historical meaning is closely related to the modern meaning: if $f\colon A\to B$ is a map of sets, then $f(x)$ is a function of $x\in A$. Vice versa, if E is a function of $x\in A$ and we are given a set B such that $E(x)\in B$ for all $x\in A$, then the set of pairs $\{(x,E(x))\mid x\in A\}$ defines a functional relation from A to B, i.e., a map of sets $A\to B$. The passage from functions in the old sense to maps is ambiguous: B has to be given separately. Sometimes, A is also omitted and must be guessed from the context. Occasionally, even x is suppressed, which may be quite confusing: is x^2+y a function of x, of y, or both x and y? Even more confusing is the situation when the old and new meanings are freely mixed together and both of them referred to as "function". This is the case for many high school and lower-division undergraduate mathematics textbooks, which is an endless source of frustration for students. Used in 11.6, 42.7.

We denote $2^A = \{S \subset A\}$, the set of all subsets of A. Given a map of sets $f: A \to B$, we have two induced maps: the *pushforward* $f_*: 2^A \to 2^B$ and *pullback* $f^*: 2^B \to 2^A$. Sometimes the notation $f^{-1} = f^*$ is used, however, it conflicts with a totally different notion of an inverse map, which is also denoted by f^{-1}. Given $A' \in 2^A$ (i.e., $A' \subset A$), we set $f_*(A') = \{b \in B \mid \exists a \in A': f(a) = b\}$. Given $B' \in 2^B$ (i.e., $B' \subset B$), we set $f^*(B') = \{a \in A \mid f(a) \in B'\}$.

59.3. ***Injective and surjective maps of sets*** Used in 11.8, 16.6*, 29.8*, 59.2*, 59.3*, 59.4*, 60.7.

A map of sets $f: A \to B$ is *injective* if $f(a) = f(a')$ implies $a = a'$, *surjective* if for any $b \in B$ there is $a \in A$ such that $f(a) = b$, and *bijective* if it is injective and surjective. Bijective maps are precisely *invertible maps of sets*: a map of sets $f: A \to B$ is invertible if there is a map $g: B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. The map g is always unique. It is known as the *inverse map* of f and is denoted by f^{-1}.

An *inclusion of sets* is an (automatically injective) map $f: A \to B$ such that $f(a) = a$ for all $a \in A$.

A *quotient map of sets* is an (automatically surjective) map $f: A \to B$ such that $\emptyset \notin B$ and for any $b \in B$ we have $b = f^*(\{b\})$. Such maps can be identified with equivalence relations on A: a quotient map f yields an equivalence relation R on A such that $aRa' \iff f(a) = f(a')$. Vice versa, an equivalence relation R on A gives rise to a quotient map $f: A \to B$, where $B = \{P \subset A \mid \exists a \in A: P = \{a' \in A \mid aRa'\}\}$ and $f(a) = \{a' \in A \mid aRa'\}$. Equivalently, such maps can be identified with partitions into disjoint nonempty subsets of A: a quotient map f yields a partition whose elements are $f^*(\{b\})$ for all $b \in B$.

Many maps of sets that seem to be inclusions of sets are in fact merely injective. For instance, one could say that any integer number is also a rational number. Naively, such a claim could be formalized as $\mathbf{Z} \subset \mathbf{Q}$. However, this is false for the most common construction of $\mathbf{Q}$ as a quotient set of $\mathbf{Z} \times (\mathbf{Z} \setminus \{0\})$ with respect to the equivalence relation $(p, q) \sim (p', q') \iff pq' = p'q$. (Instead of (p, q) one could write $\frac{p}{q}$, in which case the above relation reads

$$\frac{p}{q} \sim \frac{p'}{q'} \iff pq' = p'q,$$

a fundamental property of fractions that is taken as a definition here.) However, we do have a canonical injective map $\mathbf{Z} \to \mathbf{Q}$, which sends $n \in \mathbf{Z}$ to the set $\{(nk, k) = \frac{nk}{k} \mid k \in \mathbf{Z} \setminus \{0\}\}$. Thus, although $\mathbf{Z} \not\subset \mathbf{Q}$, we can still pretend that $\mathbf{Z} \subset \mathbf{Q}$ by implicitly applying the injective map $\mathbf{Z} \to \mathbf{Q}$ whenever necessary. We will often use injective maps as if they were inclusions of sets. One important difference, however, is that an inclusion of sets is completely determined by its domain, whereas for an injective map we must know the map of sets itself, not just the domain.

Given a set X the intersection of a family of equivalence relations on X is again an equivalence relation. In particular, any subset $S \subset X \times X$ is contained in the smallest equivalence relation R generated by it, namely, the intersection of all equivalence relations on X that contain S. Two elements $x, y \in X$ are equivalent in R if there is a finite sequence of elements $x_0 = x, x_1, x_2, \ldots, x_n = y$ such that for each i we have $(x_i, x_{i+1}) \in S$ or $(x_{i+1}, x_i) \in S$. Given two maps of sets $f, g: W \to X$, we can take $S = \{(f(w), g(w)) \mid w \in W\}$, and the resulting quotient map $q: X \to X/R$, where R is generated by S as above, has the following universal property: if $t: X \to T$ is a map of sets such that $t(f(w)) = t(g(w))$ for all $w \in W$, then there is a unique map of sets $t/R: X/R \to T$ such that $t/R \circ q = t$. Thus the map q identifies $f(w)$ and $g(w)$ for all $w \in W$.

Similar reasoning applies to quotient maps of sets. For instance, the map $\exp: i\mathbf{R} \to \mathrm{U}(1) = \{z \in \mathbf{C} \mid |z| = 1\}$ is not a quotient map of sets (or groups) because elements of $\mathrm{U}(1)$ are not subsets of $i\mathbf{R}$, but the difference is superficial: the group quotient map $i\mathbf{R} \to i\mathbf{R}/2\pi i\mathbf{Z}$ (i.e., $i\mathbf{R}/\sim$, where $x \sim y \iff x - y = 2\pi ik$ for some $k \in \mathbf{Z}$) is a quotient map of sets, and there is a canonical isomorphism $i\mathbf{R}/2\pi i\mathbf{Z} \to \mathrm{U}(1)$, so we can pretend that $\mathrm{U}(1)$ is a quotient of $i\mathbf{R}$.

59.4. ***Restrictions and corestrictions of maps of sets*** Used in 16.6*, 59.4*.

Suppose $f: A \to B$ is a map of sets and $A' \subset A$. The *restriction* of f to A' is the composition $f \circ \iota: A' \to B$, where $\iota: A' \to A$ is an inclusion of sets. We also write $f|_{A'}$ for $f \circ \iota$. More generally, if $f: A \to B$ is a map of sets and $\iota: A' \to A$ is an injective map, then the restriction of $f: A \to B$ along ι is the composition $f \circ \iota$.

We also need the dual concept, which makes the codomain of a map smaller. Unfortunately, there is no widely accepted name for this operation, only the rather obscure names like “astriction” and “corestriction” can be found in the literature. If $f: A \to B$ is a map of sets and $B' \subset B$, then the *corestriction* of f to B' is the unique map $f|^{B'}: A \to B'$ such that $\kappa \circ f|^{B'} = f$, where $\kappa: B' \to B$ is an inclusion of sets. Such a map exists if and only if $f(A) \subset B'$. The corestriction of f along an arbitrary injective map $\kappa: B' \to B$ is defined as the unique map $f|^{\kappa}: A \to B'$ such that $\kappa \circ f|^{\kappa} = f$. It exists if and only if $f(A) \subset \kappa(B')$.

The corestriction of f along κ is also known as the base change of f along κ. The latter name is far more common than “astriction” or “corestriction”, but it refers to a rather more general concept: the map κ need not be injective, and the base change is always defined, i.e., there is no requirement that $f(A) \subset \kappa(B')$, because in the latter case the domain of the base change will be different from the domain of f. Additionally, the canonical map from the domain of the base change to A need not be an inclusion of sets, but only an injection, which makes base changes different from corestrictions.

59.5. ***Disjoint unions and products of sets*** Used in 12.15.

The *disjoint union* of sets A and B is defined as $A \sqcup B = A \times \{\emptyset\} \cup B \times \{\{\emptyset\}\}$. We have canonical injection maps $\iota_A: A \to A \sqcup B$ ($a \mapsto (a, \emptyset)$) and $\iota_B: B \to A \sqcup B$ ($b \mapsto (b, \{\emptyset\})$). The sets $\{\emptyset\}$ and $\{\{\emptyset\}\}$ could be replaced by any pair of distinct singleton sets. The point of this construction is that A and B are replaced by isomorphic copies of themselves that happen to be disjoint (hence the name “disjoint union”). In particular, we have a canonical map of sets $A \sqcup B \to A \cup B$ ($(a, \emptyset) \mapsto a$, $(b, \{\emptyset\}) \mapsto b$), which is an isomorphism if and only if $A \cap B = \emptyset$. A confusing point of the above definition is that it makes use of a specific pair of sets, $\{\emptyset\}$ and $\{\{\emptyset\}\}$, which seems to be quite random. What matters is not a specific construction of the disjoint union, but rather its universal property: the disjoint union of A and B is a set (denoted by $A \sqcup B$) together with two maps $\iota_A: A \to A \sqcup B$ and $\iota_B: B \to A \sqcup B$ such that for any set Z and a pair of maps $f: A \to Z$, $g: B \to Z$, there is exactly one map (denoted by $[f, g]: A \sqcup B \to Z$) such that $[f, g] \circ \iota_A = f$ and $[f, g] \circ \iota_B = g$. This property should really be taken as a definition of disjoint union. One can then prove the existence of the disjoint union of A and B using the above construction with sets $\{\emptyset\}$ and $\{\{\emptyset\}\}$, or any other pair of disjoint singleton sets. While different choices of disjoint singleton sets will give different disjoint unions, they will all be canonically isomorphic to each other. More precisely, if $A \sqcup' B$, $\iota'_A: A \to A \sqcup' B$, $\iota'_B: B \to A \sqcup' B$ is another disjoint union, then the maps $[\iota'_A, \iota'_B]: A \sqcup B \to A \sqcup' B$ and $[\iota_A, \iota_B]': A \sqcup' B \to A \sqcup B$ form a mutually inverse pair of isomorphisms between $A \sqcup B$ and $A \sqcup' B$ that is compatible with the inclusion maps: $[\iota'_A, \iota'_B] \circ \iota_A = \iota'_A$, $[\iota'_A, \iota'_B] \circ \iota_B = \iota'_B$, $[\iota_A, \iota_B] \circ \iota'_A = \iota_A$, $[\iota_A, \iota_B] \circ \iota'_B = \iota_B$. By the universal property of coproduct (namely, the “exactly one” part) isomorphisms with such properties are unique, and it is in this sense that the coproduct of sets is unique.

At this point we should remark that the entire discussion applies equally well to products of sets. Recall that we defined $A \times B = \{z \mid \exists a \in A, b \in B: z = (a, b)\}$, where the ordered pair (a, b) was defined as $\{\{\{a\}\}, \{\{b\}, \{\emptyset\}\}\}$. There is nothing special about the last formula; one could just as easily use $\{\{\{a\}\}, \{\{b\}, \emptyset\}\}$, since it also satisfies the fundamental property that $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$. We could reformulate the definition of product in such a way that this ambiguity goes away. The definition is entirely analogous to the definition of coproduct given above: the product of A and B consists of a set (denoted by $A \times B$) and maps $p_A: A \times B \to A$ and $p_B: A \times B \to B$ such that for any set Z and maps $f: Z \to A$, $g: Z \to B$ there is exactly one map (denoted by $(f, g): Z \to A \times B$) such that $p_A \circ (f, g) = f$ and $p_B \circ (f, g) = g$. The rest of the discussion about coproduct carries over in an entirely analogous fashion.

59.6. ***Families of sets***

If I is a set, then an I-indexed *family of sets* is a map of sets $f: T \to I$. The underlying idea of this definition is that we assign to $i \in I$ the set $f^*(\{i\})$. An equivalent definition: an I-indexed family of sets is a surjective map of sets $g: I \to W$. The underlying idea of this definition is that we assign to $i \in I$ the set $g(i)$. In order to construct g from f, we define $W = \{S \subset T \mid \exists i \in I: S = f^*(\{i\})$ and set $g(i) = f^*(\{i\})$. In order to construct f from g, we use the family version of the disjoint union construction discussed above. Set $T = \bigcup_{i \in I} g(i) \times \{i\}$ and for any $t \in T$ set $f(t) = i$, where $t = (x, i)$ for some $i \in I$ and $x \in g(i)$.

59.7. ***Ordered sets*** Used in 11.17, 11.18, 25.2, 43.4, 59.7*.

A *preorder* or a *preordered set* is a pair $(S, \leq)$, where S is a set and $\leq \subset S \times S$ is a relation on S that is reflexive ($x \leq x$ for all $x \in S$) and transitive ($x \leq y$ and $y \leq z$ imply $x \leq z$ for all $x, y, z \in S$). A *poset* or a *partially ordered set* is a preordered set $(S, \leq)$ that is antisymmetric ($x \leq y$ and $y \leq x$ imply $x = y$ for all $x, y \in S$). A *totally ordered set* or simply an ordered set is a poset $(S, \leq)$ that is total ($x \leq y$ or $y \leq x$ for all $x, y \in S$).

60 Appendix: abelian groups

Supplementary sources: Aluffi [ZERO].

Informally, abelian groups are "vector spaces over integers".

Definition 60.1. (Cayley, 1854.) An *abelian group* is a tuple $(S, +, -, 0)$, where S is a set, $+\colon S \times S \to S$, $-\colon S \to S$, and $0 \in S$ are such that the following properties are satisfied for all $a, b, c \in S$: $(a+b)+c = a+(b+c)$ (associativity), $a+0 = 0+a = a$ (unitality), $-a+a = 0$ (existence of inverses), $a+b = b+a$ (commutativity). Used in 20.0*, 20.7, 23.1, 33.1, 33.7, 35.8*, 60.5, 60.12, 60.14, 60.15*, 61.1.

Example 60.2. The (additive) abelian group of *integers* is $(\mathbf{Z}, +, -, 0)$, where $\mathbf{Z}$ denotes the set of integer numbers and the three operations are the familiar operations on integers. We denote this group simply by $\mathbf{Z}$.

Example 60.3. Analogously to the previous example, we have the (additive) abelian groups of *rationals* $\mathbf{Q}$, *reals* $\mathbf{R}$, and *complex numbers* $\mathbf{C}$.

Example 60.4. We can also define the *multiplicative groups* for the above sets of numbers. Their elements are *invertible numbers*, i.e., x is *invertible* if there is y such that $x \cdot y = 1$. For $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$, the sets of invertible elements are $\{-1, 1\}$, $\mathbf{Q} \setminus \{0\}$, $\mathbf{R} \setminus \{0\}$, and $\mathbf{C} \setminus \{0\}$. In all four cases, the three operations are given respectively by multiplication, reciprocal, and the element 1. The multiplicative groups are typically denoted by a superscript $\times$: $\mathbf{Z}^\times$, $\mathbf{Q}^\times$, $\mathbf{R}^\times$, $\mathbf{C}^\times$. Used in 60.4.

Definition 60.5. Suppose $A = (S, +, -, 0)$ and $A' = (S', +', -', 0')$ are abelian groups. A *homomorphism of abelian groups* from A to A' is a map of sets $f\colon S \to S'$ such that the following properties are satisfied for all $a, b \in S$: $f(a+b) = f(a) +' f(b)$ (additivity), $f(-a) = -'f(a)$ (preservation of inverses), $f(0) = 0'$ (preservation of zeros). Used in 16.5, 20.0*, 26.16, 35.9, 35.10*, 60.6, 60.7, 60.8, 60.9, 60.10, 60.12, 61.5, 61.8.

Example 60.6. The following maps are homomorphisms of abelian groups.

- $\mathbf{Z} \to \mathbf{C}$, $n \mapsto an$ for some fixed $a \in \mathbf{C}$.
- $\mathbf{Z} \to \mathbf{C}^\times$, $n \mapsto a^n$ for some fixed $a \in \mathbf{C}^\times$.
- $\mathbf{C} \to \mathbf{C}^\times$, $z \mapsto \exp(az)$ for some fixed $a \in \mathbf{C}$.
- $\mathbf{C}^\times \to \mathbf{C}^\times$, $z \mapsto z^n$ for some fixed $n \in \mathbf{Z}$.
- $\mathbf{C}^\times \to \mathbf{C}$, $z \mapsto a \log |z|$ for some fixed $a \in \mathbf{C}$.

Definition 60.7. Suppose $f\colon A \to A'$ is a homomorphism of abelian groups. If f is an inclusion of sets, we say that A is a *subgroup* of A'. If f is a quotient map of sets, we say that A' is a *quotient group* of A. If f is an bijection, we say that A is isomorphic to A'. Used in 12.3*, 16.3, 60.8, 61.8, 61.9.

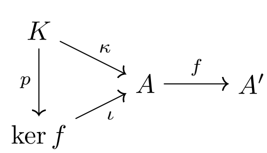
Definition 60.8. Suppose $f\colon A \to A'$ is a homomorphism of abelian groups.

- The *kernel* of f is the subgroup $\ker f$ of A with the underlying set $\{a \in A \mid f(a) = 0\}$ and all operations induced from A.
- The *image* of f is the subgroup $\operatorname{im} f$ of A' with the underlying set $\{a' \in A' \mid \exists a \in A\colon f(a) = a'\}$.
- The *cokernel* of f is the quotient group $\operatorname{coker} f$ of A' whose underlying set is the quotient of A' by the equivalence relation $x \sim y \iff x - y \in \operatorname{im} f$ and all operations induced from A'.

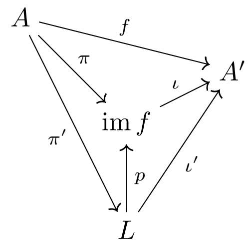
Used in 15.12, 16.1, 17.3, 17.5, 17.15, 18.3, 20.4, 60.8, 60.9, 60.10, 61.8.

Proposition 60.9. The three groups defined above for $f\colon A \to A'$ can be equivalently characterized by the following universal properties.

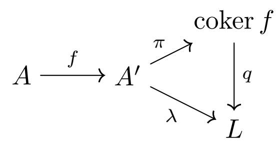
- The homomorphism $\iota\colon \ker f \to A$ satisfies $f\iota = 0$. Furthermore, if $\kappa\colon K \to A$ is another homomorphism such that $f\kappa = 0$, then there is a unique homomorphism $p\colon K \to \ker f$ such that $\iota p = \kappa$.



- The homomorphisms $\pi\colon A \to \operatorname{im} f$ and $\iota\colon \operatorname{im} f \to A'$ satisfy $\iota\pi = f$. Furthermore, if $\pi'\colon A \to L$ and $\iota'\colon L \to A'$ is another pair of homomorphisms such that $\iota'\pi' = f$, then there is a unique $p\colon L \to \operatorname{im} f$ such that $p\pi' = \pi$, hence also $\iota' = \iota p$.



- The homomorphism $\pi\colon A' \to \operatorname{coker} f$ satisfies $\pi f = 0$. Furthermore, if $\lambda\colon A' \to L$ is another homomorphism such that $\lambda f = 0$, then there is a unique homomorphism $q\colon \operatorname{coker} f \to L$ such that $q\pi = \lambda$.



Used in 15.5, 16.6*.

Remark 60.10. Consider a homomorphism of abelian groups $f\colon A \to B$ with the associated homomorphisms

$$\ker f \to A \to \operatorname{im} f \to A' \to \operatorname{coker} f.$$

- The cokernel of the homomorphism $\ker f \to A$ is isomorphic to $\operatorname{im} f$.
- The kernel of the homomorphism $A' \to \operatorname{coker} f$ is isomorphic to $\operatorname{im} f$.
- The kernel of $A \to \operatorname{im} f$ is isomorphic to $\ker f$.
- The cokernel of $\operatorname{im} f \to A'$ is isomorphic to $\operatorname{coker} f$.

Definition 60.11. Suppose $\{A_i\}_{i\in I}$ is a family of abelian groups indexed by a (possibly infinite) set I. The *direct sum* of A is the abelian group $\bigoplus_i A_i = (S, +, -, 0)$, where $S \subset \prod_{i\in I} A_i$ is the set of all elements $f \in \prod_{i\in I} A_i$ such that $\{i \in I \mid f(i) \neq 0\}$ is a finite set. The operations are defined indexwise. The *direct product* $\prod_i A_i$ is defined in the same way, but with the finiteness condition dropped.

We now cover some elementary facts about bilinear maps and tensor products.

Definition 60.12. If A, B, and C are abelian groups, then a *bilinear map* (alias *biadditive map*) from A and B to C is a map of sets $f\colon \mathsf{U}(A) \times \mathsf{U}(B) \to \mathsf{U}(C)$ with the following properties: for any $a \in \mathsf{U}(A)$ the map of sets $\mathsf{U}(B) \to \mathsf{U}(C)$ $(b \mapsto f(a,b))$ is a homomorphism of abelian groups, and likewise, for any $b \in \mathsf{U}(B)$ the map of sets $\mathsf{U}(A) \to \mathsf{U}(C)$ $(a \mapsto f(a,b))$ is also a homomorphism. Used in 60.13, 60.14, 60.15*.

In other words, f is bilinear if $f(a+a',b) = f(a,b) + f(a',b)$ and $f(a,b+b') = f(a,b) + f(a,b')$.

Notation 60.13. We denote bilinear maps from A and B to C using a comma:

$$A, B \to C.$$

We do not use the more obvious choice of notation $A \times B \to C$, because it can be easily confused with a homomorphism of abelian groups from the product of A and B to C, which satisfies a very different property: $f(a+a', b+b') = f(a,b) + f(a',b')$, whereas for a bilinear map as defined above we would have $f(a+a',b+b') = f(a,b) + f(a,b') + f(a',b) + f(a',b')$. Thus, unless $f(a,b') + f(a',b)$ is always zero, which holds if and only if f is the constant function with value 0, these two concepts are completely different.

Definition 60.14. (Hassler Whitney, 1938.) If A and B are abelian groups, then the *universal bilinear map* from A and B is a bilinear map $A, B \to A \otimes B$ $(a, b \mapsto a \otimes b)$ with the following universal property: for any abelian group C and for any bilinear map $f\colon A, B \to C$ there is a unique homomorphism of abelian groups $h\colon A \otimes B \to C$ such that $f(a, b) = h(a \otimes b)$ for all $a \in A$ and $b \in B$. The abelian group $A \otimes B$ is known as the *tensor product* of A and B. Used in 60.15.

In other words, bilinear maps $A, B \to C$ are the "same thing" as homomorphisms of abelian groups $A \otimes B \to C$. Given a homomorphism $h\colon A \otimes B \to C$, we produce a bilinear map $f\colon A, B \to C$ by setting $f(a, b) = h(a \otimes b)$. Given a bilinear map $f\colon A, B \to C$, there is exactly one homomorphism $h\colon A \otimes B \to C$ such that $h(a \otimes b) = f(a, b)$.

Proposition 60.15. The universal bilinear map exists for any abelian groups A and B.

Proof. Denote by F the free abelian group on the set $\mathsf{U}(A) \times \mathsf{U}(B)$. An element of the latter set is a pair (a, b), where $a \in A$ and $b \in B$, and denote the corresponding basis element in F by $a\hat{\otimes}b$. Now consider the abelian subgroup R of F generated by the following two families of elements:

$$(a + a')\hat{\otimes}b - a\hat{\otimes}b - a'\hat{\otimes}b, \qquad a\hat{\otimes}(b + b') - a\hat{\otimes}b - a\hat{\otimes}b'.$$

Set $A \otimes B = F/R$. We define a bilinear map $A, B \to A \otimes B$ by sending $(a, b) \in \mathsf{U}(A) \times \mathsf{U}(B)$ to the equivalence class of $a\hat{\otimes}b$, which we denote by $a \otimes b$. We verify that this map is indeed bilinear: for all $a \in A$ and $b \in B$ we must have $(a + a') \otimes b - a \otimes b - a' \otimes b = 0$. Indeed, lifting these elements of F/R to F, we get $(a + a')\hat{\otimes}b - a\hat{\otimes}b - a'\hat{\otimes}b$, which is an element of R, hence its image in F/R vanishes. The other identity is verified in the same way.

We now prove the universality property: for any abelian group C and for any bilinear map $f\colon A, B \to C$ there is a unique homomorphism of abelian groups $h\colon A \otimes B \to C$ such that $f(a, b) = h(a \otimes b)$ for all $a \in A$ and $b \in B$. Homomorphisms $h\colon A \otimes B \to C$ are in bijection with homomorphisms $h'\colon F \to C$ that vanish on the subgroup R. Since $f(a, b) = h(a \otimes b)$, we get $h'(a\hat{\otimes}b) = h(a \otimes b) = f(a, b)$. Thus h' is specified on all basis elements of F, so it is uniquely defined. It remains to verify that h' vanishes on R. The latter is generated by $(a + a')\hat{\otimes}b - a\hat{\otimes}b - a'\hat{\otimes}b$ and its symmetric cousin. We compute:

$$h'((a + a')\hat{\otimes}b - a\hat{\otimes}b - a'\hat{\otimes}b) = h'((a + a')\hat{\otimes}b) - h'(a\hat{\otimes}b) - h'(a'\hat{\otimes}b) = f(a + a', b) - f(a, b) - f(a', b) = 0. \blacksquare$$

Remark 60.16. A necessary, but not sufficient, condition for a bilinear map $A, B \to C$ to be the universal bilinear map is that its image spans C. A necessary and sufficient condition can be formulated as follows: the image of f spans C and if $\sum_i f(a_i, b_i) = 0$, then the set of pairs (a_i, b_i) can be transformed into the empty set by repeatedly applying bilinearity relations.

Remark 60.17. If $\{a_i\}$ spans A and $\{b_i\}$ spans B, then $\{a_i \otimes b_i\}$ spans $A \otimes B$.

Examples 60.18. We have
- $A \otimes B \cong B \otimes A$;
- $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$;
- $(A \oplus B) \otimes C \cong A \otimes C \oplus B \otimes C$;
- $A \otimes \mathbf{Z} \cong A$;
- $A \otimes \mathbf{Z}/n\mathbf{Z} \cong A/nA$;
- $\mathbf{Z}/m\mathbf{Z} \otimes \mathbf{Z}/n\mathbf{Z} \cong \mathbf{Z}/\gcd(m, n)\mathbf{Z}$;
- $\mathbf{Z}/m \otimes \mathbf{Z}/m \cong \mathbf{Z}/m$;
- $A \otimes \mathbf{Q} \cong (A/T) \otimes \mathbf{Q}$, where T is the torsion subgroup of A;
- $\mathbf{Z}/m\mathbf{Z} \otimes \mathbf{Q} \cong 0$;
- $\mathbf{Q} \otimes \mathbf{Q} \cong \mathbf{Q}$;
- $\mathbf{Q} \otimes \mathbf{R} \cong \mathbf{R}$;
- $\mathbf{Q}/\mathbf{Z} \otimes A \cong 0$ for any A such that $A = nA$ for all $n \geq 1$;
- $\mathbf{Q}/\mathbf{Z} \otimes \mathbf{Q} \cong 0$;
- $\mathbf{Q}/\mathbf{Z} \otimes \mathbf{Q}/\mathbf{Z} \cong 0$;
- $\mathbf{Z}^a \otimes \mathbf{Z}^b \cong \mathbf{Z}^{ab}$.

61 Appendix: rings

Supplementary sources: Aluffi [ZERO, Chapter III].

Definition 61.1. (Emmy Noether, 1921.) A *ring* is a triple $(R, \mu, 1)$, where R is an abelian group, $\mu\colon R \otimes R \to R$ is the multiplication map, and $1 \in R$ is the unit element, with the following properties: multiplication is associative ($(xy)z = x(yz)$ for all $x, y, z \in R$) and unital ($1x = x1 = x$ for all $x \in R$). A *commutative ring* is a ring R such that $xy = yx$ for all $x, y \in R$. Used in 61.8.

Example 61.2. Some of the most important commutative rings include
- the ring of integers $\mathbf{Z}$;
- the ring of rationals $\mathbf{Q}$;
- the ring of real numbers $\mathbf{R}$;
- the ring of complex numbers $\mathbf{C}$;
- the ring of p-adic integers $\mathbf{Z}_p$ and p-adic rationals $\mathbf{Q}_p$.

Example 61.3. An important example of a noncommutative ring is the *group algebra* (known also as the *group ring* in the case $R = \mathbf{Z}$). Given a ring R and a group G (or even a monoid G, in which case it is known as the *monoid algebra*), the group algebra $R[G]$ is the abelian group

$$\bigoplus_{g \in G} R,$$

whose typical element is denoted by

$$\sum_{g \in G} x_g \cdot g,$$

equipped with the multiplication

$$\left(\sum_{g \in G} x_g \cdot g\right) \cdot \left(\sum_{h \in G} y_h \cdot h\right) \mapsto \sum_{g,h \in G} (x_g y_h) \cdot (gh)$$

and the unit element $1 \cdot 1$. Used in 61.4.

Example 61.4. The *polynomial ring* $R[x]$ can be defined as the monoid algebra $R[\mathbf{N}]$, whose typical element is denoted by

$$\sum_{n \geq 0} r_n x^n.$$

The polynomial ring in n variables $R[x, y, \ldots]$ is defined as the monoid algebra $R[\mathbf{N}^n]$. Used in 61.4.

Definition 61.5. A *homomorphism of rings* $f\colon (R, \mu, 1) \to (R', \mu', 1')$ is a homomorphism of abelian groups $f\colon R \to R'$ such that $f(x \cdot y) = f(x) \cdot' f(y)$ and $f(1) = 1'$. Used in 61.6, 61.8, 61.9*.

Example 61.6. Given any $a \in R$ we construct a homomorphism of rings $R[x] \to R$ by sending $\sum_{n \geq 0} r_n x^n$ to $\sum_{n \geq 0} r_n a^n$. (The left side uses operations in $R[x]$, whereas the right side uses operations in R.) All such homomorphism can be assembled together in a single homomorphism $R[x] \to R^{\mathsf{U}(R)}$, where the right side denotes the product of copies of R indexed by the elements of $\mathsf{U}(R)$.

Example 61.7. Consider the polynomial $x^p - x$ in the ring $\mathbf{F}_p[x]$. This polynomial is by definition a nonzero element of this ring. However, its value on any element of $\mathbf{F}_p$ is zero. Thus, the combined evaluation map $\mathbf{F}_p[x] \to \mathbf{F}_p^{\mathsf{U}(\mathbf{F}_p)}$ is a homomorphism of rings that is not injective. In particular, polynomials should be distinguished from the maps of sets that they induce through evaluation.

Definition 61.8. An *ideal* in a ring R is a subgroup $I \subset R$ that is the kernel of the underlying homomorphism of abelian groups of some homomorphism of rings $R \to Q$, denoted by $\mathsf{U}_{\mathsf{Ab}}(R) \to \mathsf{U}_{\mathsf{Ab}}(Q)$. (We use the subscript Ab to indicate that the forgetful functor discards only the multiplication and unit, and not the abelian group structure, as opposed to the forgetful functor $\mathsf{U}\colon \mathsf{Ring} \to \mathsf{Set}$, which discards everything.) Used in 61.9.

Proposition 61.9. A subgroup I of a ring R is an ideal if and only if it is closed under multiplication by elements of R, i.e., if $i \in I$ and $r \in R$, then ir and ri belong to I.

Proof. If $I \to \mathsf{U}_{\mathsf{Ab}}(R)$ is the kernel of $\mathsf{U}_{\mathsf{Ab}}(R) \to \mathsf{U}_{\mathsf{Ab}}(Q)$ for some homomorphism of rings $q\colon R \to Q$, $i \in I$, and $r \in R$, then $q(ir) = q(i)q(r) = 0q(r) = 0$ and $q(ri) = q(r)q(i) = q(r)0 = 0$, so ir and ri belong to I. Vice versa, if an abelian subgroup $I \subset \mathsf{U}_{\mathsf{Ab}}(R)$ is closed under multiplication by elements of R, then the quotient map of abelian groups $q = [-]\colon \mathsf{U}_{\mathsf{Ab}}(R) \to \mathsf{U}_{\mathsf{Ab}}(R)/I$ can be promoted to a homomorphism of rings by equipping the quotient with an induced multiplication and unit (i.e., $[a][b] = [ab]$ and $1 = [1]$), which is well-defined: if $[a] = [a']$, then $a - a' \in I$ and $[a'][b] = [a + (a' - a)][b] = [a][b] + [a' - a][b] = [a][b] + 0[b] = [a][b]$. The quotient map preserves multiplication and unit by definition. ▮

Notation 61.10. Given an ideal I in a ring R, the resulting *quotient ring* is denoted by R/I. Given a subset S of $\mathsf{U}(R)$, the intersection of all ideals containing S is again an ideal, which is denoted by (S). Used in 61.11, 61.12.

Example 61.11. The quotient ring $\mathbf{Z}/(n) = \mathbf{Z}/nZ$ has n elements. The ideal (n) consists of all integers divisible by n.

Example 61.12. The quotient ring $R[x]/(x^n)$ can be identified with finite sums of the form $\sum_{0 \le i \le n} r_i x^i$, where $r_i \in R$, which are multiplied like polynomials, but throwing away terms of degree n or higher.

References

[SSCSH] Samuel Eilenberg, Joseph A. Zilber. Semi-simplicial complexes and singular homology. Annals of Mathematics 51:3 (1950), 499–513. 7.5, 10.4.

[KanCSS] Daniel M. Kan. On c.s.s. complexes. American Journal of Mathematics 79:3 (1957), 449–476. 7.6.

[Approx] Paul Alexandroff. Simpliziale Approximationen in der allgemeinen Topologie. Mathematische Annalen 96:1 (1927), 489–511. doi:10.1007/bf01209183. 17.14*.

[ZH] Leopold Vietoris. Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen. Mathematische Annalen 97:1 (1927), 454–472. 17.14*.

[SHTe] Samuel Eilenberg. Singular homology theory. Annals of Mathematics (Second Series) 45:3 (1944), 407–447. MR0010970, Zbl:0061.40603, doi:10.2307/1969185. 4.8.

[HGR] Clifford H. Dowker. Homology groups of relations. Annals of Mathematics 56:1 (1952), 84–95. 17.17*.

[CFHT] Peter Gabriel, Michel Zisman. Calculus of fractions and homotopy theory. Ergebnisse der Mathematik und ihrer Grenzgebiete 35 (1967). 2.0*.

[SOAT] J. Peter May. Simplicial objects in algebraic topology. Van Nostrand Mathematical Studies 11 (1967). 2.0*.

[SAT] Klaus Lamotke. Semisimpliziale algebraische Topologie. Grundlehren der mathematischen Wissenschaften 147 (1968). 2.0*.

[SHTc] Edward B. Curtis. Simplicial homotopy theory. Advances in Mathematics 6 (1971), 107–209. 2.0*.

[ATs] Edwin H. Spanier. Algebraic topology. McGraw–Hill (1966). 2.0*, 22.0*.

[LAT] Albrecht Dold. Lectures on Algebraic Topology. Die Grundlehren der mathematischen Wissenschaften 200 (1972), Springer. Second Edition (1980). 2.0*, 22.0*.

[ATHH] Robert M. Switzer. Algebraic Topology—Homotopy and Homology. Die Grundlehren der mathematischen Wissenschaften 212 (1975), Springer. 2.0*.

[EAT] James R. Munkres. Elements of Algebraic Topology. Addison–Wesley (1984). 2.0*.

[ATf] William Fulton. Algebraic Topology. Graduate Texts in Mathematics 153 (1995). 2.0*.

[LNAT] James F. Davis, Paul Kirk. Lecture Notes in Algebraic Topology. 2.0*.

[Homol] Saunders Mac Lane. Homology. Grundlehren der mathematischen Wissenschaften 114 (1975). 22.0*.

[EIISS] Greg Friedman. An elementary illustrated introduction to simplicial sets. arXiv:0809.4221v5, doi:10.1216/rmj–2012–42–2–353. 2.0*, 4.0*, 5.0*, 6.0*, 7.0*, 8.0*, 10.0*, 17.7*, 17.19*.

[ICHT] Francis Sergeraert. Introduction to Combinatorial Homotopy Theory. July 7, 2013. https://www-fourier.ujf-grenoble.fr/∼sergerar/Papers/Trieste-Lecture-Notes.pdf. 2.0*, 4.0*, 5.0*, 6.0*, 7.0*, 8.0*, 10.0*.

[WFS] André Joyal. Weak factorisation systems. Joyal's CatLab. January 21, 2013. https://ncatlab.org/joyalscatlab/published/Weak+factorisation+systems. 44.0*.

[NSHT] André Joyal, Myles Tierney. Notes on simplicial homotopy theory. March 7, 2008. http://mat.uab.cat/∼kock/crm/hocat/advanced-course/Quadern47.pdf. 2.0*, 31.0*, 32.0*.

[ATh] Allen Hatcher. Algebraic Topology. Cambridge University Press (2001). http://pi.math.cornell.edu/∼hatcher/AT/AT.pdf. 2.0*, 23.0*, 24.0*.

[ATd] Tammo tom Dieck. Algebraic Topology. EMS Textbooks in Mathematics (2008). doi:10.4171/048. 2.0*, 22.0*, 23.0*, 24.0*.

[CCAT] J. Peter May. A Concise Course in Algebraic Topology. Chicago Lectures in Mathematics (1999). https://math.uchicago.edu/∼may/CONCISE/ConciseRevised.pdf. 2.0*.

[SHTgj] Paul G. Goerss, John F. Jardine. Simplicial homotopy theory. Progress in Mathematics 174 (1999). 1.2*, 2.0*.

[VPD] Lucien Clavier. Visualizing Poincaré duality. arXiv:1201.5150v1. 37.0*.

[ZERO] Paolo Aluffi. Algebra: Chapter 0. Graduate Studies in Mathematics 104 (2009). 11.0*, 12.0*, 13.0*, 21.0*, 60.0*, 61.0*.

[SETS] F. William Lawvere, Robert Rosebrugh. Sets for Mathematics. Cambridge University Press (2003). 11.0*, 12.0*, 13.0*, 21.0*, 59.0*.

[CATS] Dmitri Pavlov. Category Theory. https://dmitripavlov.org/notes/2018s-6325.pdf. 11.0*, 12.0*, 13.0*, 21.0*.

[ICT] V. G. Boltyanskiĭ, V. A. Efremovich. Intuitive Combinatorial Topology. Springer, Universitext, 2001. 16.0*.

[VGT] Anatolij Fomenko. Visual Geometry and Topology. Springer, 1994. 16.0*.