This class will be about a single result:

Theorem (Atiyah-Singer). If X is a closed smooth manifold and D is an elliptic operator, then the analytic index of D is equal to the topological index of D.

Plan for the course.
(1) Define the analytic and the topological indices.
(2) Special cases: Gauß-Bonnet, Riemann-Roch, Hirzebruch signature theorem, Dirac operator.
(3) Applications to geometry and number theorem
(4) Bordism proof, K-theory proof, heat equation proof (supersymmetry).
(5) Generalizations: Index theorem for manifolds with boundary, G-equivariant index theorem (equality of virtual representations of G), B-family index theorem (equality of classes in $K(B)$), equivariant family version, local index theorem.

Today we start with (1) and (2).

Suppose E and F are smooth vector bundles (complex) and $D\colon C^\infty(E) \to C^\infty(F)$ is a complex-linear elliptic operator. Since D is elliptic, it is Fredholm, which means that its kernel and cokernel are finite-dimensional. The analytic index is the difference of their dimensions: $\text{a-ind}(D) := \dim\ker(D) - \dim\operatorname{coker}(D)$.

Now $\operatorname{ch}(D) \in \mathrm{H}^*(X)$ is the result of the Thom isomorphism applied to the Chern character of the difference bundle associated to the symbol of D and $\operatorname{Td}(X) \in \mathrm{H}^{4*}(X)$ is the Todd class of $TX \otimes \mathbf{C}$. Using the fact that $\operatorname{ch}(D)$ and $[X]$ are twisted by the same character, the orientation character of X, one defines the topological index as $\text{t-ind}(D) := \langle \operatorname{Td}(X) \cup \operatorname{ch}(D), [X]\rangle$.

In most cases we use a Riemannian metric g on X and hermitian metrics on E and F to define D and then $\operatorname{ch}(D)$ and $\operatorname{Td}(X)$ are represented by closed differential forms $\widetilde{\operatorname{ch}}(D)$ and $\widetilde{\operatorname{Td}}(X)$. Then the topological index of D is $\int_X \widetilde{\operatorname{ch}}(D) \wedge \widetilde{\operatorname{Td}}(X) \in \mathbf{R}$. Thus the topological index is an integral of a local expression in X, whereas the analytic index depends on the global structure of X.

Example: Gauß-Bonnet on a Riemann surface (X, g). The topological index of a certain elliptic operator D is $(2\pi)^{-1} \int_X \text{scalar curvature}(g)$. The analytic index is the Euler characteristic of X, which was originally defined in terms of a triangulation of X. The Gauß-Bonnet theorem implies that this integer depends neither on the metric nor on the triangulation!

Project 1. Dig into the history of the index theorem.

Definition. $D\colon C^\infty(E) \to C^\infty(F)$ is a differential operator if it is locally given by $P_{i,j} \in C^\infty(U)[\partial_1, \ldots, \partial_n]$. Here we identify $C^\infty(E|_U) = C^\infty(U, \mathbf{R}^p)$ and likewise for F. The order of D is the maximum degree of $P_{i,j}$. The individual components of the symbol $Q_{i,j}$ are obtained from $P_{i,j}$ by substituting $i\xi_k$ for ∂_k and taking the top order degree. Observation: $Q_{i,j}$ fit together on X to give the symbol $\sigma(D)\colon \pi^*E \to \pi^*F$ as a homomorphism of vector bundles on the total space of the cotangent bundle T^*X. Here $\pi\colon T^*X \to X$ and $\sigma_x(D)(\xi) = (Q_{i,j}(\xi_1, \ldots, \xi_n))$. The symbol depends polynomially on ξ.

Definition. D is elliptic if $\sigma(D)(\xi)$ is invertible for all $\xi \neq 0$.

Example. $X = \mathrm{pt}$. $D\colon E \to F$ is a morphism of finite-dimensional vector spaces. The analytical index is equal to $\dim\ker D - \dim\operatorname{coker} D$. The topological index is $\langle \operatorname{ch}(D) \cup \operatorname{Td}(X), [X]\rangle = \operatorname{ch}^0(D) = \dim E - \dim F$. Hence the index of a morphism of vector spaces is invariant under deformations and this will stay true for arbitrary elliptic operators.

Example. $X = S^1$, $E = F = X \times \mathbf{C}$, $D = -i\partial$. We have $\sigma = \xi$. There is a basis of eigenvectors: $-i\partial(\exp(inx)) = n\exp(inx)$. We have $\dim\ker D = \dim\operatorname{coker} D = 1$, hence $\operatorname{ind}(D) = 0$. We can compute the index more easily by a deformation: $\operatorname{ind}(D) = \operatorname{ind}(D + \lambda) = 0$, because $D + \lambda$ is invertible for non-integer λ.

Example. (Elaboration of the previous example.) $X = S^1 = \mathbf{R}/\mathbf{Z}$, $E = F = \mathbf{C}$, $D = -i\partial\colon C^\infty(S^1) \to C^\infty(S^1)$. Consider $L^2(S^1)$: $(f, g) = \int_{S^1} f\bar{g}\,dx$. Orthonormal basis: $\exp(inx)$ for $n \in \mathbf{Z}$. There is an isomorphism $L^2(S^1) \to l^2(\mathbf{Z})$. The inner product on $l^2(\mathbf{Z})$ is given by $(a, b) = \sum_{n\in\mathbf{Z}} a_n \bar{b}_n$. The isomorphism is given by the formulas $a \mapsto \sum_{n\in\mathbf{Z}} a_n \exp(inx)$ and $f \mapsto (n \mapsto (f, \exp(inz)) = \int_{S^1} f \exp(-inx)\,dx$.

The space $L^2(S^1)$ admits a filtration $L^2 \supset C^0 \supset C^1 \supset \cdots C^\infty = \cap_k C^k \supset \mathrm{Pol}$. Likewise $l^2(\mathbf{Z})$ admits a (Sobolev) filtration $l^2(\mathbf{Z}) = W^0 \supset W^1 \supset W^2 \supset \cdots W^\infty = \cap_k W^k \supset \mathbf{C}[\mathbf{Z}]$. Here W^k is the completion of

$\mathbf{C}[\mathbf{Z}]$ with respect to $(a,b)_k = \sum_{n\in\mathbf{Z}} a_n \bar{b}_n (1+n^2)^k$. Thus $a \in l^2$ belongs to W^k if and only if $(n \mapsto a_n n^r) \in l^2$ for all $0 \le r \le k$ if and only if $\|a\|_k^2 < \infty$.

Lemma 1. $C^k \subset W^k$.

Proof. Suppose $f \in C^k$, then for any $0 \le r \le k$ we have $D^r f \in L^2$, hence $n^r \hat{f} \in l^2$, i.e., $\hat{f} \in W^k$.

Lemma 2. $W^{k+1} \subset C^k$.

Proof. $k = 0$: $f \in W^1$; $\hat{f} = a$. We have $\sum_{n\in\mathbf{Z}} |a_n|^2 (1+n^2) < \infty$. Cauchy's inequality: $\left(\sum_{n\in\mathbf{Z}} |a_n|\right)^2 \le \left(\sum_n |a_n|^2 (1+n^2)\right)\left(\sum_n (1+n^2)^{-1}\right) < \infty$. Now $\|f\|_\infty = \sup_{x\in S^1} |f(x)| \le \sum_n |a_n| < \infty$, hence f is continuous.

Now let's do the example more carefully. D sends C^k to C^{k-1}. Moreover, it sends W^k to W^{k-1}. Elliptic regularity: The operator $D: W^k \to W^{k-1}$ is Fredholm and the kernel and the cokernel of $D: W^k \to W^{k-1}$ do not depend on k and equal their partners in C^∞.

Remark. W^k is the completion of C^k with respect to the norm $f \mapsto \sum_{|r|\le k} \|\partial^r f\|$.

Remark. $C^k \subset W^k$ for any smooth manifold M. Moreover, $W^{k+\lfloor d/2\rfloor} \subset C^k$, where $d = \dim M$.

One way to define Sobolev spaces on a compact manifold M is to choose a partition of unity ψ indexed by I together with embeddings $\phi_i: U_i \to \mathbf{T}^d$. and complete C^k in the norm $\|f\|_k := \sum_{i\in I} \|(\psi_i f) \circ \phi_i^{-1}\|$. (We define Sobolev spaces on a torus T^r via its Pontrjagin dual $\mathbf{Z}^r$ as for $r = 1$.) The chain rule implies that the Sobolev norms obtained from different choices are equivalent, see homework.

Remark. Given a metric on M, the Laplacian picks out a canonical inner product on W^k.

Example. $X = S^1$, $E = F = S^1 \times \mathbf{C}$, $D: C^\infty(S^1) \to C^\infty(S^1)$, $D = \sum_{0\le k\le r} f_k(x)(-i\partial)^k$ (arbitrary order r differential operator on the trivial bundle).

D is elliptic if $\sigma(D)(x,\xi) = f_r(x)\xi^r$ is invertible for $\xi \ne 0$, in other words $f_r(x) \ne 0$ for all $x \in S^1$.

Example. Constant coefficient operators $D = \sum_{0\le k\le r} f_k(-i\partial)^k$. D extends to $W^r(S^1) \to L^2(S^1)$ and then to $l^2(\mathbf{Z}) \to l^2(\mathbf{Z})$. The last operator has the form $a \mapsto a \cdot p$ (pointwise multiplication), where $p(n) = \sum_{0\le k\le r} f_k n^k$. Hence $\dim\ker(D)$ is the number of integers n such that $p(n) = 0$. By the index theorem this is also the dimension of the cokernel of D. (We will later prove the lemma that if X is odd-dimensional, then the topological index of D is zero.)

$\mathrm{K}(X) := \mathrm{KU}(X)$ is the group completion of the commutative monoid of isomorphism classes of finite-dimensional complex vector bundles over X with the direct sum as the monoid structure. Thus $\mathrm{K}(X)$ consists of formal differences of vector bundles module certain equivalence relation: $[E_0]-[E_1] = [F_0]-[F_1]$ if and only if there is G such that $E_0 \oplus F_1 \oplus G = E_1 \oplus F_0 \oplus G$. Remark: The universal property of the group completion allows us to define pullbacks in K-theory: If we have a map $f: X \to Y$, then the pullback $f^*: \mathrm{K}(Y) \to \mathrm{K}(X)$ is given by extending pullbacks of vector bundles to their group completions. Homotopic maps induce identical maps on K-theory. Example: $\mathrm{K}(\mathrm{pt}) = \mathbf{Z}$. Also $\mathrm{K}(S^1) = \mathbf{Z}$ because every complex bundle on S^1 is trivial. Likewise n-dimensional vector bundles on S^k are in bijection with $\pi_{k-1}(\mathrm{GL}_n(\mathbf{C}))$. We don't know how to compute these groups, however, for K-theory we only need to compute $\pi_{k-1}(\mathrm{GL}_\infty(\mathbf{C}))$. Bott periodicity states that these groups alternate between 0 and $\mathbf{Z}$ for k odd respectively even. Thus $K(S^k) = \mathbf{Z}$ for k odd and $K(S^k) = \mathbf{Z} \oplus \mathbf{Z}$ for k even.

Definition. The reduced K-theory of X is $\tilde{\mathrm{K}}(X) := \mathrm{coker}(\mathrm{K}(\mathrm{pt}) \to \mathrm{K}(X))$.

Lemma 1: $\tilde{\mathrm{K}}(X) \oplus \mathbf{Z} = \mathrm{K}(X)$. Lemma 2: $\tilde{\mathrm{K}}(X)$ can be obtained by identifying E and F if $E \oplus m = F \oplus n$ for some trivial bundles m and n.

For a closed subset $Y \subset X$ of a compact Hausdorff space, we define $K(X,Y)$ as follows: Representatives are triples (E_0, E_1, α), where $E_i \to X$ and $\alpha: E_0/Y \to E_1/Y$ is an isomorphism. The equivalence relation is the same as before: $(E_0, E_1, \alpha) \sim (F_0, F_1, \beta)$ if there is (G, G, γ) such that $E_0 \oplus F_1 \oplus G = E_1 \oplus F_0 \oplus G$. Note: $\mathrm{K}(X, \emptyset) = \mathrm{K}(X)$. The sequence $\mathrm{K}(X,Y) \to \mathrm{K}(X) \to \mathrm{K}(Y)$ is exact.

Definition. If Z is a locally compact, then $\mathrm{K}_{\mathrm{cs}}(Z) := \mathrm{K}(Z \cup \infty, \infty)$. (Remark: If Z is Hausdorff, then Z is locally compact if and only if $Z \cup \infty$ is Hausdorff.)

Classes in $\mathrm{K}_{\mathrm{cs}}(Z)$ are represented by (E_0, E_1, ϕ), where $\phi\colon E_0/(Z \setminus K) \to E_1/(Z \setminus K)$. Note: $(Z \cup \infty) \setminus K$ are the open neighborhoods of ∞ in $Z \cup \infty$.

Remark. Existence of partitions of unity implies that any vector bundle G embeds into a trivial bundle, moreover one can find a bundle H such that $G \oplus H$ is trivial using a hermitian metric (in this case, on the trivial bundle). Any bundle that pulls back from the universal bundle on the (infinite) Graßmannian $B\mathrm{U}(n)$ will come with an embedding into the trivial bundle, since the universal bundle does. So $\mathrm{K}(X)$ can only coincide with the homotopy theoretic definition $[X, B\mathrm{U} \times \mathbf{Z}]$ if we somehow require this property. For simplicity, we will work with compact Hausdorff spaces X in the following.

Serre-Swan theorem: The category of vector bundles over X is equivalent to the category of finitely generated projective $C^0(X)$-modules.

Definition. If $D\colon C^\infty(E) \to C^\infty(F)$ is an elliptic differential operator, then $[\sigma(D)] \in \mathrm{K}_{\mathrm{cs}}(T^*X)$ is represented by the triple $(\pi^*E, \pi^*F, \sigma(D))$ (the compact set off of which $\sigma(D)$ is invertible is the image of the zero section, which is homeomorphic to X) Here $\pi\colon T^*X \to X$ is the canonical projection.

To obtain the (cohomological description of the) topological index from the symbol, we apply the Chern character to $[\sigma(D)]$, then we apply the (twisted) Thom isomorphism, then we multiply the result by $\mathrm{Td}(X)$ and evaluate it on the fundamental class.

There is also a K-theoretic description of the topological index of D which goes as follows: Pick an embedding $f\colon X \to \mathbf{R}^n$, then the composition of $g\colon T^*X \cong TX$ and $Tf\colon TX \to T\mathbf{R}^n$ is proper. Moreover, there is *always* a complex structure on the normal bundle of this map, even if X is not oriented. For proper maps with complex normal bundle we will define a pushforward in K-theory: $[\sigma(D)] \in Tf_!\colon \mathrm{K}_{\mathrm{cs}}(T^*X) \to \mathrm{K}_{\mathrm{cs}}(\mathbf{R}^{2n}) = \tilde{\mathrm{K}}(S^{2n}) \to \tilde{\mathrm{H}}^{2*}(S^{2n}, \mathbf{Q}) = \mathbf{Q}$. By Bott periodicity, we can also identify $\tilde{\mathrm{K}}(S^{2n}) = \mathbf{Z}$ directly and this gives us a direct proof of the integrality of the topological index.

Books on K-theory: Atiyah, Karoubi, Rosenberg, ...

Fredholm operators on Fréchet spaces. A Fréchet space is a metrizable complete locally convex Hausdorff topological vector space. Alternatively, it is a complete topological vector space whose topology is defined by an increasing sequence of seminorms. Examples: Banach spaces, $C^\infty(X)$ for compact manifolds X (take Sobolev norms), $C^0(X)$ if X is σ-compact (norms are suprema on compact subsets), $C^\infty(X)$ for non-compact manifolds X (norms are Sobolev norms on compact submanifolds).

Fact. The dual space V' has many topologies. One of them (the strong topology) agrees with the norm topology if V is Banach. But V' in this topology is Fréchet if and only if V is Banach. So we will be very careful when working with topologies on mapping spaces of Fréchet spaces.

The open mapping theorem. If V and W are Fréchet spaces and a continuous linear map $A\colon V \to W$ is surjective, then A is open.

Definition. An operator $A\colon V \to W$ is Fredholm if its kernel and cokernel are finite-dimensional.

Lemma. The image of a Fredholm operator is closed.

Proof. We can assume that $A\colon V \to W$ is injective. Pick a complement Z for the image of A in W. Since A is Fredholm, Z is finite-dimensional and Hausdorff, in particular it is Fréchet. The map $A \oplus \mathrm{id}_Z\colon V \oplus Z \to W$ is continuous, linear, bijective, and therefore a homeomorphism (by the open mapping theorem). Since V is closed in $V \oplus Z$, its image is closed in W.

Lemma. $A\colon V \to W$ is Fredholm if and only if it is invertible up to finite-rank operators if and only if it is invertible up to compact operators.

Proof. The first equivalence follows from the above lemma. For the second equivalence we observe that the 1-eigenspace (i.e., the invariant subspace) of a compact operator is finite-dimensional. Moreover, if K is compact then $\mathrm{id} + K$ always has a finite-dimensional cokernel.

Remark. The property of ellipticity is important because the index of a non-elliptic operator may vary if we compute it in C^∞ versus in L^2. Think of the Fredholm operator $C^\infty(S^1) \to C^\infty(S^1)$ given by the multiplication by $z-1$. Its extension to L^2 is not Fredholm because its image is not closed. In fact, the image is dense in L^2 but the operator cannot be invertible since zero lies in the spectrum, which is the image of the map $z-1\colon S^1 \to \mathbf{C}$. This is related to the fact that the evaluation map $C^\infty(S^1) \to \mathbf{C}$ at $1 \in S^1$ does *not* extend to L^2. You should contrast that with the elliptic operator of differentiation that we studied above. There the cokernel is detected by the integration map $C^\infty(S^1) \to \mathbf{C}$, which *does* extend to L^2. In this sense the *unbounded* differentiation operator is simpler than the *bounded* multiplication operator. One just has to be precise about its domain, which we can take to be the first Sobolev space $W^1(S^1)$.

Chern-Gauß-Bonnet theorem. If X is a closed smooth n-manifold then $\chi(X) = \langle e(TX), [X]\rangle$. If X comes with a cell structure, then the *Euler characteristic* $\chi(X)$ is $\sum_{0\le k\le n}(-1)^k n_k$, where n_k is the number of k-dimensional cells. We can also compute it as the alternating sum $\sum_k(-1)^k b_k$ of Betti numbers (for any ordinary cohomology theory). Now $\sum_k(-1)^k \dim \mathrm{H}^k(X,\mathbf{R})$ is also the analytic index of the de Rham differential $d_\bullet\colon \to \Omega^{k-1}X \to \Omega^k X \to \Omega^{k+1}X \to$, thought of as an elliptic complex (see below).

The right hand side of the Chern-Gauß-Bonnet theorem $\langle e(TX), [X]\rangle$ is the topological index of $d_\bullet$, where $e(TX)$ is the *Euler class* of the tangent bundle of X. By Chern-Weil theory $\langle e(TX), [X]\rangle = \int_X \tilde{e}(TX,g)d\mathrm{vol}_g$ for any metric g on X.

Definition. $D_\bullet\colon 0 \to C^\infty(E_0) \to C^\infty(E_1) \to \cdots \to C^\infty(E_k) \to 0$ is an elliptic complex if it is a complex of differential operators and the associated complex of symbols is exact.

An elliptic complex with one differential operator is an elliptic differential operator (if a sequence $0 \to A \to B \to 0$ is exact, then the map $A \to B$ is an isomorphism).

Index theorem generalizes to elliptic complexes: The analytic index of an elliptic complex equals its topological index. The analytic index is $\sum_k(-1)^k \dim \mathrm{H}^i(D_\bullet)$. The topological index comes from the K-theory class $[\sigma(D_\bullet)] \in \mathrm{K_{cs}}(T^*X)$ represented by the corresponding complex of symbols (see Problem 4d in the homework).

To translate elliptic complexes back into elliptic operators, we choose a Riemannian metric on X and hermitian metrics on E_i. If $D\colon C^\infty(E) \to C^\infty(F)$ is a differential operator, then there is a unique differential operator $D^*\colon C^\infty(F) \to C^\infty(E)$ such that $\langle D(\phi), \psi\rangle_X = \langle \phi, D^*\psi\rangle_X$. This operator can be constructed locally via integration by parts (using compactly supported sections for the defining property) and is then glued together to an operator on X via partition of unity (and uniqueness). We have $\sigma(D^*) = \sigma(D)^*$ so D is elliptic if and only if D^* is elliptic. Note that $D^{**} = D$.

It follows from elliptic regularity (see below) that the natural map $\ker D^* \to \mathrm{coker} D$ is an isomorphism. Note that this is easy to see on Banach spaces but in general not true on Fréchet spaces: The multiplication operator $m_{(z-1)}$ on $C^\infty(S^1)$ has adjoint $m_{(\bar{z}-1)}$ and both have trivial kernel and 1-dimensional cokernel. As we have discussed before, this operator is not elliptic and the problem doesn't arise by elliptic regularity. It follows that $\text{a-ind}(D^*) = -\,\text{a-ind}(D)$.

To pass from an elliptic complex $D_\bullet$ to an elliptic differential operator, set $E_{\mathrm{even}} = \oplus_{i \text{ even}} E_i$ and $E_{\mathrm{odd}} = \oplus_{i \text{ odd}} E_i$ and $D\colon C^\infty(E_{\mathrm{even}}) \to C^\infty(E_{\mathrm{odd}})$ is given by $D = \sum_{i \text{ even}}(D_i + D_{i+1}^*)$.

Lemma. We have $\text{a-ind}(D_\bullet) = \text{a-ind}(D)$ and $\text{t-ind}(D_\bullet) = \text{t-ind}(D)$.

Proof. If we have an elliptic complex $0 \to V_0 \overset{D_0}{\to} V_1 \overset{D_1}{\to} V_2 \to \cdots$, we consider its adjoint $0 \leftarrow V_0 \overset{D_0^*}{\leftarrow} V_1 \overset{D_1^*}{\leftarrow} V_2 \leftarrow \cdots$. Since the inner product is positive definite, it follows that $\ker D_i^* D_i = \ker D_i$ and from $D_i D_{i-1} = 0$ we see that $\operatorname{im} D_i \cap \ker D_i^* = 0$ and $\operatorname{im} D_i^* \cap \ker D_i = 0$. Moreover, the fact that $\ker D_i^* = \mathrm{coker} D_i$ implies the decomposition $V_i = \operatorname{im} D_{i-1} \oplus \operatorname{im} D_i^* \oplus (\ker D_1^* \cap \ker D_2)$. It follows that the operators D_i and D_i^* restrict to isomorphisms $\operatorname{im} D_i^* \to \operatorname{im} D_i$ and $\operatorname{im} D_i \to \operatorname{im} D_i^*$ and that they are zero on the other summands in V_i. This easily implies the claim.

In the homework we saw how to associate a K-theory class to a complex of vector bundles and in the case of the symbol complex of an elliptic complex $D_\bullet$ it follows from that construction that $[\sigma(D_\bullet)] = [\sigma(D)] \in \mathrm{K}(T^*X)$. We have thus shown that the index theorem for elliptic operators is equivalent to the following result:

Index Theorem for elliptic complexes. If $D_\bullet\colon 0 \to C^\infty(E_0) \to C^\infty(E_1) \to \cdots \to C^\infty(E_k) \to 0$ is an elliptic complex, then $\text{a-ind}(D_\bullet) = \text{t-ind}(D_\bullet)$.

Hodge Theory. Given an elliptic operator $D\colon C^\infty(E) \to C^\infty(F)$, one can construct the self-adjoint elliptic operator $\tilde{D} = \left(\begin{smallmatrix} 0 & D^* \\ D & 0 \end{smallmatrix}\right)\colon C^\infty(E \oplus F) \to C^\infty(E \oplus F)$. From the self-adjointness it follows that its index is zero but we have $\ker \tilde{D} = \ker D \oplus \ker D^*$. If we define a $\mathbf{Z}/2$-grading on $E \oplus F$ by declaring E to be even and F to be odd, then $\tilde{D}$ is an *odd* operator and $\operatorname{sdim} \ker(\tilde{D}) = \text{a-ind}(D)$. Here $\operatorname{sdim} V := \dim V^{\text{even}} - \dim V^{\text{odd}}$ if $V = V^{\text{even}} \oplus V^{\text{odd}}$. One also associates to a $\mathbf{Z}/2$-graded vector bundle E an element in K-theory via $[E] := E^{\text{even}} - E^{\text{odd}} \in \mathrm{K}(X)$.

Now starting with an elliptic complex $D_\bullet$, we can construct the associated operators D and $\tilde{D}$. It is not hard to see that $\tilde{D} = \sum_i (D_i + D_i^*)^2\colon C^\infty(\oplus_i E_i) \to C^\infty(\oplus E_i)$. In particular, for the de Rham complex $D_\bullet = d$ we obtain the *Hodge Laplacian* and for the Dolbeault complex $D_\bullet = \bar{\partial}$ we get the *Laplace-Beltrami* operator. It follows from the above considerations that $\ker \tilde{D} \cong \oplus_i \mathrm{H}^i(D_\bullet)$ and sections in $\ker \tilde{D}$ are called *harmonic*. Note that $\operatorname{sdim} \ker \tilde{D} = \text{a-ind}(D) = \text{a-ind}(D\bullet)$.

Thom isomorphism. Let $E \to X$ be a real n-dimensional vector bundle over a paracompact space X with metric and orientation. Then there is a class $u(E) \in \mathrm{H}^n(DE, SE) = \tilde{\mathrm{H}}^n(DE/SE)$ such that the map $\mathrm{H}^k(X) \to \mathrm{H}^{k+n}(DE, SE)$ $(a \mapsto u(E) \cup \pi^*(a))$ is an isomorphism. Moreover, $u(E)$ is uniquely determined by the requirement that its restriction to all fibres gives the orientation class of E.

Remark. If X is compact, then $E \cup \infty \approx DE/SE$ and hence $\mathrm{H}^*_{\text{cs}}(E) := \mathrm{H}^*(E \cup \infty, \infty) \cong \mathrm{H}^*(DE, SE)$.

Example. For the Hopf bundle H over $\mathbf{CP}^n$ we have homeomorphisms

$$\mathrm{Th}(H) = DH/SH \approx DH/S^{2n+1} \approx D\nu(\mathbf{CP}^n, \mathbf{CP}^{n+1})/S^{2n+1} \approx D\nu(\mathbf{CP}^n, \mathbf{CP}^{n+1}) \cup_{S^{2n+1}} D^{2n} \approx \mathbf{CP}^{n+1}.$$

It follows that the map $\mathrm{H}^{2k}(\mathbf{CP}^n) \to \mathrm{H}^{2k+2}(\mathbf{CP}^{n+1})$ $(a \mapsto \pi^*(a) \cup u)$ is an isomorphism. This allows us to compute the cohomology ring of $\mathbf{CP}^n$ inductively: $\mathrm{H}^*(\mathbf{CP}^n) = \mathbf{Z}[x]/(x^{n+1})$.

Definition. (a) $e(E) := i_0^* u(E) \in \mathrm{H}^n(X)$ (the Euler class of E). (b) If L is complex r-dimensional bundle, then $c_r(L) := e(L_{\mathbf{R}}) \in \mathrm{H}^{2r}(X)$ (rth Chern class).

Proof of Thom isomorphism. Use induction on the number of trivializing charts (works for compact spaces). For one chart (the trivial bundle) we have $U \times \mathbf{R}^n \to U$. Now $(DE, SE) = (U \times D^n, U \times S^{n-1}) = U \times (D^n, S^{n-1})$. Künneth isomorphism: $a \in \mathrm{H}^k(U) \mapsto a \otimes \pm u_0 \in \mathrm{H}^k(U) \otimes \mathrm{H}^n(D^n, S^{n-1}) \mapsto a \times u_0 \in \mathrm{H}^k(U \times (D^n, S^{n-1}))$. To fix the sign of u_0 we use the orientation $\mathrm{H}^n(DE, SE) \to \mathrm{H}^n(DE_x, SE_x) \cong \mathbf{Z}$. The result follows from a Mayer-Vietoris argument (and the 5-lemma) that uses the existence and uniquess of u inductively.

Definition. The orientation sheaf $o(E)$ is a locally constant sheaf on X with stalks $\mathrm{H}_n(E_x, E_x \setminus 0)$.

With this definition we extend Thom isomorphism to non-oriented bundles: We get a unique (twisted) Thom-class $u(E) \in \mathrm{H}^n(DE, SE, o(E))$, which restricts on each fibre to the identity map in the twisted cohomology groups $\mathrm{H}^n(E_x, E_x \setminus 0; \mathrm{H}_n(E_x, E_x \setminus 0)) \cong \mathrm{Hom}(\mathrm{H}_n(E_x, E_x \setminus 0), \mathrm{H}_n(E_x, E_x \setminus 0))$. The Thom isomorphism $\mathrm{H}^k(X; o(E)) \cong \mathrm{H}^{k+n}(DE, SE)$ is still given by $a \mapsto u(E) \cup \pi^*(a)$.

Remark. As a consequence, the (twisted) Euler class lives in $\mathrm{H}^n(X, o(E))$.

A quick trip through characteristic classes.

The uniqueness of the Thom class readily implies the following multiplicative properties:

(0) $u(E, -\theta) = -u(E, \theta)$ and $e(E, -\theta) = -e(E, \theta)$, where θ is an orientation on E.
(1) $u(E_1 \times E_2, \theta_1 \times \theta_2) = u(E_1, \theta_1) \times u(E_2, \theta_2)$.
(2) $e(E_1 \times E_2) = e(E_1) \times e(E_2)$ with orientations suppressed.
(3) $e(E_1 \oplus E_2) = e(E_1) \cup e(E_2)$.

Remark. Since all sections of a vector bundle are homotopic one can get the Euler class via pullback from the Thom class using an arbitrary section (rather then the zero section as in the definition). If a vector bundle E admits a non-vanishing section it follows that $e(E) = 0$ because the Thom class is relative to the complement of the zero-section. It follows from this and property (3) above that e is *not* stable: $e(E \oplus \mathbf{R}) = 0$.

Corollary to the Thom isomorphism. (Gysin sequence.) If $E \to X$ is an oriented n-dimensional vector bundle, then the sequence

$$\cdots \to \mathrm{H}^{i+n-1}(E^0) \to \mathrm{H}^{i+n}(E, E^0) \to \mathrm{H}^{i+n}(E) \to \mathrm{H}^{i+n}(E^0) \to \mathrm{H}^{i+n+1}(E, E^0) \to \cdots$$

is exact and it is isomorphic to

$$\cdots \to \mathrm{H}^{i+n-1}(E^0) \to \mathrm{H}^{i}(X) \to \mathrm{H}^{i+n}(X) \to \mathrm{H}^{i+n}(E^0) \to \mathrm{H}^{i+1}(X) \to \cdots$$

The map $\mathrm{H}^i(X) \to \mathrm{H}^{i+n}(E, E^0)$ is given by cup product with $u(E)$ and the map $\mathrm{H}^{i+n}(X) \to \mathrm{H}^{i+n}(E)$ is π^*. Hence the map $\mathrm{H}^i(X) \to \mathrm{H}^{i+n}(X)$ is given by cup product with $e(E)$.

Definition. Suppose $E \to X$ is a complex n-dimensional vector bundle. Define $c_n(E) := e(E_{\mathbf{R}}, \theta_{\mathbf{C}}) \in \mathrm{H}^{2n}(X)$. Here $E_{\mathbf{R}}$ is the underlying oriented real bundle of E. By induction define $c_i(E) \in \mathrm{H}^{2i}(X)$ for $i < n$ via the Gysin sequence for $E_1 \to SE$, where the fiber of E_1 over $v \in SE_x$ is the orthogonal complement $v^\perp \subset E_x$. For $k \le n-1$ the image of $c_k(E) \in \mathrm{H}^{2k}(X)$ under π^* is $c_k(E_1) \in \mathrm{H}^{2k}(SE)$.

Remark. Chern classes are natural and for the Hopf bundle one can normalize $c_1(H) \in \mathrm{H}^2(\mathbf{CP}^1)$ as the generator that evaluates as -1 on the complex orientation of $\mathbf{CP}^1$.

Theorem. Let $c(E) := 1 + c_1(E) + c_2(E) + \cdots$ be the total Chern class. Then $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$. Together with naturality and normalization properties this characterizes Chern classes.

Consider universal bundles γ_n over the n-dimensional Graßmannian $\mathrm{Gr}_n := \cup_{m \ge 0} \mathrm{Gr}_n(\mathbf{C}^{m+n})$.

Theorem. $\mathrm{H}^*(\mathrm{Gr}_n) = \mathbf{Z}[c_1, \ldots, c_n]$ where $c_i := c_i(\gamma_n)$ are the *universal Chern classes.*

Proof. Induction on n. The case $n = 1$ has been done above via the Thom isomorphism. Lemma: Consider the map $p\colon S(\gamma_n) \to \mathrm{Gr}_{n-1}$ $((V, v) \mapsto v^\perp)$. This map is a fiber bundle with the fiber over $W \subset \mathbf{C}^\infty$ being equal to $S(W^\perp)$, which is contractible (since it is an *infinite dimensional* sphere). Now we use the Gysin sequence, which shows that the map $\mathrm{H}^i(\mathrm{Gr}_n) \to \mathrm{H}^{i+2n}(\mathrm{Gr}_n)$ given by cup product with c_n is injective and by induction on n we conclude that its cokernel is isomorphic to $\mathrm{H}^{i+2n}(\mathrm{Gr}_{n-1})$. Another induction on i finishes the proof.

Lemma. $p_{m,n} := c(\gamma_m \times \gamma_n) = (1 + c_1 + \cdots + c_m)(1 + c'_1 + \cdots + c'_n) =: q_{m,n}$.

Proof. Induction on m and n. Restricting to $\mathrm{Gr}_m \times \mathrm{Gr}_{n-1}$ respectively $\mathrm{Gr}_{m-1} \times \mathrm{Gr}_n$ shows by induction $p_{m,n} \equiv q_{m,n} \pmod{c_m c'_n}$. Then the result follows from property (3) above, which says that for any two vector bundles E_i of dimensions m and n one has $c_{m+n}(E_1 \oplus E_2) = c_m(E_1) \cup c_n(E_2)$.

Theorem 1. If X is paracompact, then $[X, \mathrm{Gr}_n] \approx \mathrm{Vect}_n(X)$, via $f \mapsto f^*(\gamma_n)$. In other words, Gr_n is the classifying space for n-dimensional complex vector bundles.

Theorem 2. There are unique Chern classes $c = 1 + c_1 + \cdots + c_n\colon \mathrm{Vect}_n(X) \to \mathrm{H}^{2*}(X)$ characterized by (i) naturality; (ii) normalization: $\langle c_1(H \to \mathbf{CP}^1), [\mathbf{CP}^1] \rangle = -1$; (iii) Whitney sum formula: $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$.

Proof of Theorem 2. We already constructed Chern classes satisfying (i) and (ii). The sum formula follows from the Lemma above and the fact that $E_1 \times E_2$ is the pullback of $\gamma_{n_1} \times \gamma_{n_2}$ via $f_1 \times f_2$. Uniqueness follows from Theorem 1.

Outline of the proof of Theorem 1. (Assuming X is compact.) The map α is surjective: Any n-dimensional bundle $E \to X$ embeds into some trivial bundle $X \times \mathbf{C}^N \to X$ and therefore is a pullback of γ_n for the appropriate map $f\colon X \to \mathrm{Gr}_n(\mathbf{C}^N) \to \mathrm{Gr}_n$. The map α is injective: If $f_1^*(\gamma_n) \cong f_0^*(\gamma_n)$, one uses this isomorphism to construct a bundle E over $X \times [0, 1]$ whose pullbacks along the two boundary embeddings are $f_i^*(\gamma_n)$. The bundle E gives us a homotopy between f_i just as in the surjectivity part of the argument, if we are careful to construct the embedding into the trivial bundle relative to the given embeddings on $X \times \{0, 1\}$.

Corollary: The Splitting Principle. Consider the map $i\colon \mathrm{Gr}_1^n \to \mathrm{Gr}_n$ given by taking direct sums of subspaces, or given by the classifying map of the sum of line bundles $\pi_i^*(\gamma_1)$. The induced map in cohomology is $i^*\colon \mathrm{H}^*(\mathrm{Gr}_n) = \mathbf{Z}[c_1,\ldots,c_n] \to \mathrm{H}^*(\mathrm{Gr}_1^n) = \mathbf{Z}[x_1,\ldots,x_n]$. We have $x_i = \pi_i^*(c_1(\gamma_1))$. The class $c_k = c_k(\gamma_n)$ maps to $\sigma_k(x_i) = c_k\left(\bigoplus \pi_i^*(\gamma_1)\right)$, which by the Whitney sum formula is the kth elementary symmetric polynomial. It follows that the map i^* is injective and its image consists of symmetric polynomials, i.e., polynomials that are invariant under the action of the symmetric group S_n.

Generalization. $\mathrm{Vect}_n(X) = \mathrm{Vect}_n^h(X) = \mathrm{Bun}_{\mathrm{U}(n)}(X)$, where Vect^h denotes (isomorphism classes of) vector bundles with hermitian metric and Bun_G denotes (isomorphism classes of) G-principal bundles on X. Recall that BG is the classifying space of G, i.e., the base space of the universal principal G-bundle $EG \to BG$, which is characterized up to homotopy by the fact that EG is contractible. For example, $B\mathrm{U}(n)$ is Gr_n and $E\mathrm{U}(n)$ is the bundle of orthonormal frames over Gr_n. Characteristic classes for principal G-bundles are elements of $\mathrm{H}^*(BG)$.

Theorem (Borel). If G is a compact Lie group and $T \subset G$ is a maximal torus with Weyl group $W := N(T)/T$, then $\mathrm{H}^*(BG,\mathbf{Q}) \cong \mathrm{H}^*(BT,\mathbf{Q})^W = \mathbf{Q}[x_1,\ldots,x_{\dim T}]^W$. For example, for $\mathrm{U}(n)$ the maximal torus is $\mathrm{U}(1)^n$ and the Weyl group is S_n.

Sometimes, one can also make similar statements for other coefficients. For example, $\mathrm{H}^*(B\mathrm{O}(n),\mathbf{Z}/2) = \mathbf{Z}/2[w_1,\ldots,w_n]$, where $w_i = w_i(\gamma_n) \in \mathrm{H}^i(B\mathrm{O}(n),\mathbf{Z}/2)$ is the Stiefel-Whitney class. These can be defined in exactly the same way we defined Chern classes, starting with $w_n(E) := e(E)$ modulo 2 for a real n-dimensional vector bundle E.

For example, $\mathrm{H}^*(B\mathrm{O}(2n),\mathbf{Q}) = \mathrm{H}^*(B\mathbf{T}^n,\mathbf{Q})^W = \mathbf{Q}[y_1,\ldots,y_n]^W = \mathbf{Q}[\sigma_k(y_i^2)]$ because the Weyl group of $\mathrm{O}(2n)$ consists of reflections and permutations: $W = (\mathbf{Z}/2)^n \tilde{\times} S_n$. For the oriented case we have in addition the Euler class $e(\gamma_{2n}^{\mathbf{R}}) = y_1\cdots y_n \in \mathrm{H}^*(B\mathrm{SO}(2n),\mathbf{Q}) = \mathbf{Q}[y_1,\ldots,y_n]^W = \mathbf{Q}[\sigma_k(y_i^2), y_1\cdots y_n] = \mathbf{Q}[p_k, e]$. This follows from the fact that only pairs of reflections lie in the Weyl group of $\mathrm{SO}(2n)$ and hence the product of all y_i is invariant.

Pontrjagin classes. If $E \to X$ is a real vector bundle, define $p_i(E) := (-1)^i c_{2i}(E\otimes\mathbf{C}) \in \mathrm{H}^{4i}(X)$. It is not hard to see that the universal Pontrjagin class $p_k(\gamma_2^{\mathbf{R}}n) = \sigma_k(y_i^2)$ in rational cohomology.

Lemma. (a) $c_k(\bar{E}) = (-1)^k c_k(E)$; (b) $c_1(E) = c_1(\det E)$; (c) $c_1(L_1\otimes L_2) = c_1(L_1) + c_1(L_2)$.

Proof. By the above splitting principle it suffices to check (a) and (b) on sums of line bundles. (a) Since $c_k = \sigma_k$ it suffices to check $c_1(\bar{L}) = -c_1(L)$. Since the endomorphism bundle $\mathrm{End}(L)$ has a non-vanishing (identity) section, it follows that

$$0 = c_1(\mathrm{End}(L)) = c_1(L^*\otimes L) = c_1(L^*) + c_1(L) = c_1(\bar{L}) + c_1(L).$$

(b) By part (c) we have $c_1(L_1\oplus\cdots\oplus L_n) = c_1(L_1)+\cdots+c_1(L_n) = c_1(L_1\otimes\cdots\otimes L_n) = c_1(\det(L_1\oplus\cdots\oplus L_n))$. (c) It suffices to check this on $\mathrm{Vect}_2(\mathbf{CP}^1) = [S^1,\mathrm{U}(2)]$ where it follows from the isomorphism for the Hopf bundle H: $(H\otimes H)\oplus\mathbf{C} \cong H\oplus H$.

Hirzebruch genera. We have the isomorphism $\mathrm{H}^*(\mathrm{Gr}_n) = \mathbf{Z}[c_1,\ldots,c_n] = \mathbf{Z}[x_1,\ldots,x_n]^{S_n} = \mathrm{H}^*(\mathrm{Gr}_1^n)^{S_n}$ and similarly, the completions $\hat{\mathrm{H}}^* := \prod_i \mathrm{H}^i$ are formal power series rings in variables c_k respectively x_k. Start with a formal power series g in one variable and define $g_n^a := \sum_{1\le i\le n} g(x_i) \in \Lambda_n^{S_n}$ and $g_n^m := \prod_{1\le i\le n} g(x_i) \in \Lambda_n^{S_n}$. Here $\Lambda_n = \mathbf{Z}[[x_1,\ldots,x_n]]$.

Lemma. There are characteristic classes $\hat{g}^a, \hat{g}^m\colon \mathrm{Vect}(X) \to \hat{\mathrm{H}}^{2*}(X)$ such that (i) $\hat{g}^a(L) = g(c_1(L)) = \hat{g}^m(L)$; (ii) $\hat{g}^a(E_1\oplus E_2) = \hat{g}^a(E_1) + \hat{g}^a(E_2)$ and $\hat{g}^m(E_1\oplus E_2) = \hat{g}^m(E_1)\cup\hat{g}^m(E_2)$. Moreover, (i) and (ii) determine $\hat{g}^a$ and $\hat{g}^b$ uniquely.

Proof. By the splitting principle we have $\hat{g}^m(E) := \hat{g}^m(f^*(\gamma_n)) = f^*(\hat{g}^m(\gamma_n)) = f^*(g_n^m)$. Thus (i) can be proved as follows: $\hat{g}^m(L) = f^*(g_1^m) = f^*(g(x)) = g(f^*(x)) = g(c_1(L))$. Case (ii) works similarly.

Remark. (i) $\hat{g}^a$ extends uniquely to a group homomorphism $\hat{g}^a\colon \mathrm{K}(X) \to \hat{\mathrm{H}}^*(X)$. (ii) $\hat{g}^m$ is stable, i.e., $\hat{g}^m(E\oplus\mathbf{C}) = \hat{g}^m(E)$, if and only if $g(0) = 1$. If $g(0) = 1$, then $\hat{g}^m$ extends uniquely to a group homomorphism $\hat{g}^m\colon \tilde{\mathrm{K}}(X) \to \hat{\mathrm{H}}^{2*}(X)^\times$. Here $\hat{\mathrm{H}}^{2*}(X)^\times$ is the group (under cup product) of classes $1 + a_1 + \cdots$.

Example. Take $g(x) = \exp(x) = 1 + x + x^2/2 + \cdots \in \mathbf{Q}[[x]]$.

Lemma. The Chern character $\mathrm{ch} := \hat{g}^a\colon \mathrm{K}(X) \to \hat{\mathrm{H}}^{2*}(X, \mathbf{Q})$ is a ring homomorphism.

Proof. Need to check $\mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1) \cup \mathrm{ch}(E_2)$ for vector bundles E_i over X. It suffices to check this statement for direct sums of line bundles. In fact, by additivity, it is enough to check it on line bundles. Now $\mathrm{ch}(L_1 \otimes L_2) = \mathrm{ch}(L_1) \cup \mathrm{ch}(L_2)$ is implied by $\exp(c_1(L_1 \otimes L_2)) = \exp(c_1(L_1) + c_1(L_2)) = \exp(c_1(L_1)) \cup \exp(c_1(L_2))$.

If X is a finite-dimensional CW-complex, then the map $\mathrm{ch}\colon \mathrm{K}(X, \mathbf{Q}) \to \mathrm{H}^{2*}(X, \mathbf{Q})$ is an isomorphism.

Definition. The Todd character $\mathrm{Td} := \hat{g}^m\colon \tilde{\mathrm{K}}(X) \to \hat{\mathrm{H}}^{2*}(X, \mathbf{Q})^\times$ is induced by $g(x) = x/(1 - \exp(-x)) = 1 + x/2 + x^2/12 + \cdots$. The Todd class of a vector bundle E is $\mathrm{Td}([E])$ and the Todd class of a real manifold X is $\mathrm{Td}(X) := \mathrm{Td}(TX \otimes_{\mathbf{R}} \mathbf{C})$. The Todd genus of X is $\langle \mathrm{Td}(X), [X] \rangle \in \mathbf{Q}$.

Now the topological index is fully defined because the Chern character for relative K-theory can be defined using the isomorphism $\mathrm{K}(X, Y) = \tilde{\mathrm{K}}(X/Y)$.

Lemma. We have $e(TX) \cup \mathrm{Th}^{-1}(\mathrm{ch}(\sigma(D))) = \mathrm{ch}(E) - \mathrm{ch}(F) \in \mathrm{H}^*(X, \mathbf{Q})$ for any elliptic differential operator $D\colon E \to F$, or more generally, for any symbol class.

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{K}(T^*X, T^*X \setminus 0) & \longrightarrow & \mathrm{K}(T^*X) \\ \big\downarrow \mathrm{ch} & & \big\downarrow \mathrm{ch} \\ \mathrm{H}^*(T^*X, T^*X \setminus 0) & \longrightarrow & \mathrm{H}^*(T^*X) \\ {\scriptstyle \cup u(T^*X)} \big\uparrow \cong & & {\scriptstyle \pi^*} \big\uparrow \cong \\ \mathrm{H}^{*-d}(X, \mathbf{Q}) & \longrightarrow & \mathrm{H}^*(X, \mathbf{Q}) \end{array}$$

This lemma is most useful when cup product with the Euler class is injective. Therefore, we will study elliptic operators whose symbols come from representation theory.

Definition. Let G be a Lie group. A G-structure on a manifold X consists of a principal G-bundle P over X, a real representation V of G, and an isomorphism $P \times_G V \cong TX$ over X. G-structures form a category, whose isomorphism classes we can usually compute.

Examples. (0) Every manifold has a canonical $\mathrm{GL}(\mathbf{R}^d)$-structure. (1) An $\mathrm{O}(d)$-structure is a Riemannian metric. Here $V = \mathbf{R}^d$. (2) An $\mathrm{SO}(d)$-structure is a Riemannian metric with an orientation. (3) A $\mathrm{GL}(\mathbf{C}^{d/2})$-structure ($d$ is even) is an almost complex structure. (4) A $\mathrm{Spin}(d)$-structure is a Spin-structure.

The topological index $\text{t-ind}\colon \mathrm{K}_{\mathrm{cs}}(T^*X) \to \mathbf{Q}$, where $\sigma \mapsto \langle \mathrm{Td}(X) \cup \mathrm{Th}^{-1}(\mathrm{ch}(\sigma)), [X] \rangle$ can be computed most easily for those σ that come (via G-representations) from a G-structure on X. Let $[\sigma] \in \mathrm{K}_{\mathrm{cs}}(T^*X)$ be represented by $(E, F, \pi^*E/(T^*X \setminus 0) \overset{\cong}{\to} \pi^*F/(T^*X \setminus 0))$. Assume there are complex representations M and N of G such that $E = P \times_G M$ and $F = P \times_G N$. If $f\colon X \to BG$ classifies P, then we form vector bundles on BG: $\tilde{V} := EG \times_G V$, $\tilde{M} := EG \times_G M$ and $\tilde{N} := EG \times_G N$.

Running assumptions. There is an isomorphism $\tilde{\pi}^*(\tilde{M})/(\tilde{V} \setminus 0) \overset{\cong}{\to} \tilde{\pi}^*(\tilde{N})/(\tilde{V} \setminus 0)$ and the Euler class is nontrivial: $0 \neq e(V^*) \in \mathrm{H}^*(BG, \mathbf{Q})$. This is what we mean when we say that the symbol comes from G-representations.

Remark. $\mathrm{H}^*(BG, \mathbf{Q}) = \mathrm{H}^*(BT, \mathbf{Q})^W$ has no zero divisors and hence one can uniquely divide by any nonzero class. In particular, the above lemma shows that under our assumptions the class $\mathrm{Th}^{-1}(\mathrm{ch}(\sigma))$ only depends on the vector bundles in question (and not on the symbol class σ):

Lemma. $\mathrm{Th}^{-1}(\mathrm{ch}(\sigma)) = (\mathrm{ch}(\tilde{M}) - \mathrm{ch}(\tilde{N}))e(\tilde{V}^*)^{-1} \in \hat{\mathrm{H}}^*(BG, \mathbf{Q})$.

Example: Chern-Gauß-Bonnet. Let $G = \mathrm{SO}(2n)$, $V = \mathbf{R}^{2n}$, $M = \Lambda^{\mathrm{even}} V^* \otimes \mathbf{C}$, $N = \Lambda^{\mathrm{odd}} V^* \otimes \mathbf{C}$. We will show that $\text{t-ind}(X, P, G, V, M, N) = \langle e(TX), [X] \rangle$. For any elliptic operator $D\colon C^\infty(E) \to C^\infty(F)$ whose symbol comes from G-representations we therefore have a-ind$(D) = \langle e(TX), [X]\rangle$. For example, we can take $D = (d + d^*)/(\Lambda^{\text{even}} V \otimes \mathbf{C})$ and a-ind$(D) = \chi(X)$.

Example: Hirzebruch-Riemann-Roch. Let $G = \mathrm{U}(\mathbf{C}^m)$, $V = \mathbf{C}^m$ (X is almost Hermitian), $M = \Lambda^{\text{even}} V$, $N = \Lambda^{\text{odd}} V$. We will show that in this case t-ind$(X) = \langle \mathrm{Td}(TX), [X]\rangle$. Note that TX is a complex bundle, hence we don't need to tensor it with $\mathbf{C}$ as we usually do. For example, we can take the Dolbeault operator $(\bar\partial + \bar\partial^*)/\Lambda^{\text{even}} V$ whenever X is a complex manifold (integrability is needed to define the Dolbeault operator). Thus its analytic index is the holomorphic Euler characteristic $\chi_h(X)$, also known as arithmetic genus.

Claim. $[\mathrm{Td}(\tilde V \otimes \mathbf{C}) \cup (\mathrm{ch}(\tilde M) - \mathrm{ch}(\tilde N)) e(\tilde V^*)^{-1}]^{\deg = 2n} = (-1)^n e(\tilde V) \in \mathrm{H}^{2n}(B\mathrm{SO}(2n), \mathbf{Q})$.

Proof. Suppose M is a complex representation of a compact Lie group G. Denote by T the maximal torus of G. Now M as a representation of T splits as a direct sum of M_r, where each M_r corresponds to some weight w_r of T. For a weight w_r an element $t \in T$ acts on $m \in M_r$ by $t(m) = \exp(2\pi i w_r(t)) m$. Now $\mathrm{Hom}(T, S^1) = \mathrm{H}^2(BT)$. To be continued.

A distraction: Rigid and integrable G-structures. An example of a G-structure is the *flat G-structure* on a vector space V, which is given by $P := V \times G \to V$ and the isomorphism is given by $P \times_G V = (V \times G) \times_G V \cong V \times V \cong TV$ via translation in V.

Definition. A G-structure on X is *integrable* if it is locally flat.

Examples. (i) An integrable $\mathrm{GL}_n(\mathbf{C})$-structure ($V = \mathbf{C}^n$) is a complex structure: There are complex charts. (ii) For $G = \mathrm{Sp}(2n)$ and $V = \mathbf{R}^{2n}$ integrable structures are symplectic structures by Darboux's theorem. (iii) For $G = \mathrm{O}(n)$ and $V = \mathbf{R}^n$ integrable structures are flat metrics. (iv) For $G = \mathrm{U}(n)$ and $V = \mathbf{C}^n$ integrable structures are flat Kähler structures.

Example. The total space of the cotangent bundle T^*X is always symplectic: $\omega = d\alpha$. A metric on X induces a metric on T^*X. This metric is integrable if and only if the original metric is flat. It turns out that T^*X is always Kähler but with respect to a different metric. See http://mathoverflow.net/questions/26776/.

Chart versions of integrable G-structures. Choose a covering collection of charts on a manifold X with codomain being an open subset of a vector space V. We can now say that an integrable G-structure is a lift of the derivatives of transition functions along the map $\rho\colon G \to \mathrm{GL}(V)$ that satisfies the usual cocycle condition.

Definition. A G-structure on X is *rigid* if the transition functions are restrictions of the action of the semidirect product of G and the additive group of V on V. For $G = \mathrm{O}(n)$ and $V = \mathbf{R}^n$ we get the notion of a rigid Euclidean manifold. For general rigid geometry we should generalize the model space from a vector space to an arbitrary (homogenous) G-space.

Hirzebruch-Riemann-Roch theorem. We have a-ind$(\bar\partial_\bullet) = \chi_{\text{hol}}(X) := \sum_{0 \le i \le n} (-1)^i \dim_{\mathbf{C}} \mathrm{H}^{0,i}(X) = \langle \mathrm{Td}(TX), [X]\rangle =:$ t-ind$(\bar\partial_\bullet)$.

Observation. If this holds for $X = \mathbf{CP}^n$ for $n \ge 0$, then the formal power series that defines $\mathrm{Td}(X)$ is uniquely determined.

Proof. We have $\chi_{\text{hol}}(\mathbf{CP}^n) = 1$ because $\mathbf{CP}^n$ is Kähler with the Kähler form representing a non-trivial class in $\mathrm{H}^{1,1}(\mathbf{CP}^n)$. Lemma: $T\mathbf{CP}^n \oplus \mathbf{C} = \oplus_{1 \le i \le n+1} H^*$. Observe that $\mathbf{C}^{n+1} = H \oplus H^\perp$ and $T\mathbf{CP}^n = \mathrm{Hom}(H, H^\perp)$. Now the total Chern class of $T\mathbf{CP}^n$ is $c(\oplus^{n+1} H^*) = \prod^{n+1} c(H^*) = (1+a)^{n+1}$, where $a = c_1(H^*) = -c_1(H)$ is a "complex" generator of $\mathrm{H}^2(\mathbf{CP}^n)$. Recall that the Todd genus is given by the power series $x/(1-\exp(-x))$. Hence $\widetilde{\mathrm{Td}}^m(T\mathbf{CP}^n) = \prod^{n+1} \mathrm{Td}(H^*) = \mathrm{Td}(c_1(H^*))^{n+1} = \mathrm{Td}(a)^{n+1} \in \mathrm{H}^*(\mathbf{CP}^n)$. Thus $\langle \mathrm{Td}(T\mathbf{CP}^n), [\mathbf{CP}^n]\rangle$ is the coefficient of a^n in $\mathrm{Td}(a)^{n+1}$. By Cauchy formula this coefficient equals $(2\pi i)^{-1} \int_{S^1} z^{-n-1} (z/(1-\exp(-z)))^{n+1} dz = 1$.

Suppose X is a compact $2d$-dimensional oriented manifold and $D\colon C^\infty(E) \to C^\infty(E')$ is an elliptic operator. Then t-ind$(D) := (-1)^d \langle \mathrm{Td}(X) \cup \mathrm{Th}^{-1}(\mathrm{ch}(\sigma(D))), [X]\rangle$. We have so far ignored the additional sign. Recall from homework that the topological index always vanishes on odd-dimensional manifolds.

Theorem. If X has a G-structure and $\sigma(D)$ comes from G-representations, then

$$\text{t-ind}(D) = (-1)^d \left\langle \prod_{1\le k\le d} (\mathrm{Td}(\eta_k)\mathrm{Td}(-\eta_k)/\eta_k) \sum_{1\le k\le m} (\exp(w_k) - \exp(w'_k)), a \right\rangle$$

for any a such that $f_*([X]) = i_*(a)$. Here $\mathrm{H}_{2n}(BT, \mathbf{Q}) \xrightarrow{i_*} \mathrm{H}_{2d}(BG, \mathbf{Q}) \xleftarrow{f_*} \mathrm{H}_{2d}(X)$.

The assumption about representations means the following: $TX \cong P \times_G V^{2d}$, $E \cong P \times_G M$, $E' \cong P \times_G M'$ for some complex G-modules M and M'. The w_k respectively w'_k are the complex weights of M respectively M' and the η_k are the real weights of V. All of these live in $H^2(BT)$. Our running assumption is that

$$\begin{array}{ccccccc} P \times_G (V^* \times M) & \cong & \pi^* E & \xrightarrow{\sigma(D)} & \pi^* E' & = & P \times_G (V^* \times M') \\ \downarrow & & \downarrow & & \downarrow & & \\ P \times_G V^* & \cong & T^*X & = & T^*X & & \end{array}$$

comes from a G-equivariant map $\tilde{\sigma}: V^* \times M \to M'$ and that $0 \ne \eta_k \in \mathrm{H}^2(BT, \mathbf{Q})$.

Definition. (a) Suppose G is a compact connected group and $T \le G$ is a maximal torus. If M is a complex G-module, then $M|_T \cong M_1 \oplus \cdots \oplus M_m$ and for $m \in M_k$ we have $t(m) = \exp(2\pi i w_k(t))m$ where $w_k \in \mathrm{Hom}(T, S^1) = \mathrm{H}^2(BT)$ are "complex weights of M". (b) If V is a real oriented $2d$-dimensional G-module, i.e., we have a map $\rho: G \to \mathrm{SO}(V)$, then its real weights η_k are defined by $V|_T = V_1 \oplus \cdots \oplus V_d$, where $\dim V_k = 2$. Here $\eta_k \in \mathrm{Hom}(T, S^1) = \mathrm{H}^2(BT)$.

Recall that we have an isomorphism $i^*: \mathrm{H}^*(BG, \mathbf{Q}) \to \mathrm{H}^*(BT, Q)^W \subset \mathbf{Q}[[z_1, \ldots, z_r]]$. In particular, the cohomology ring does not have zero divisors, hence the formula for t-ind(D) makes sense because the product $\eta_1 \cdots \eta_d$ is non-zero.

Remark about notation: For $\rho: G \to \mathrm{GL}(M)$ the *character* is defined by $\mathrm{cha}(\rho)(t) := \mathrm{tr}_M(\rho(t)) = \sum_{1\le k\le m} \exp(2\pi i w_k(t))$. Compare this to $\hat{\mathrm{H}}^{2*}(BT) \ni i^*\mathrm{ch}(M) := i^*\mathrm{ch}(EG \times_G M) = \sum_{1\le k\le m} \exp(w_k)$.

Lemma. (a) If M has complex weights w_k then $\bar{M}$ has complex weights $-w_k$. (b) If V has real weights $\eta_1, \ldots, \eta_d$, then $V \otimes_{\mathbf{R}} \mathbf{C}$ has complex weights $\pm\eta_1, \ldots, \pm\eta_d$. (c) If M has complex weights $w_1, \ldots, w_m$, then the underlying real representation of M has real weights $w_1, \ldots, w_m$.

Proof of Theorem. $\langle (TX \otimes \mathbf{C}) \cup \mathrm{Th}^{-1}(\mathrm{ch}(\sigma(D))), [X]\rangle = \langle i^*(\mathrm{Td}(V \otimes \mathbf{C}) \cup (\mathrm{ch}(M) - \mathrm{ch}(M'))e(V^*)^{-1}), a\rangle$. Here $\tilde{V} = EG \times_G V$. We have the following commutative diagram:

$$\begin{array}{ccccccc} V & \in & R(G) & \longrightarrow & R(T) & = & \mathbf{Z}[\hat{T}] \\ \downarrow & & \downarrow & & \downarrow & & \\ EG \times_G V = \tilde{V} & \in & \mathrm{K}(BG) & \longrightarrow & \mathrm{K}(BT) & & \\ & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \\ & & \hat{\mathrm{H}}^*(BG, \mathbf{Q}) & \xrightarrow{i^*} & \hat{\mathrm{H}}(BT, \mathbf{Q}) & \cong & \mathbf{Q}[[y_1, \ldots, y_k]] \end{array}$$

Remark: Atiyah-Segal completion theorem. Consider the augmentation ideal $I := \ker(\dim: R(G) \to \mathbf{Z})$. Define $R(G)_{\hat{I}} := \lim_n R(G)/I^n$. The map $R(G)_{\hat{I}} \to K(BG) = \lim_n K((BG)^{(n)})$ is an isomorphism.

Theorem. Suppose X is a $2d$-manifold with G-structure (P, V, α) and the elliptic complex $(C^\infty(E), D) = 0 \to C^\infty(E_0) \xrightarrow{D_0} C^\infty(E_1) \xrightarrow{D_1} \cdots \to C^\infty(E_n) \to 0$ comes from G-representations, i.e., $E_i \cong P \times_G M_i$ for some complex G-representation M_i satisfying our running assumptions: $0 \ne e(V) \in \mathrm{H}^{2d}(BG)$ and there is a G-equivariant (linear for each $v \in V^*$) sequence of maps $\tilde{\sigma}_i: V^* \times M_i \to M_{i+1}$ such that $\sigma(D) = \mathrm{id}_P \times_G \tilde{\sigma}_\bullet$. Then $\text{t-ind}(D) = (-1)^d \langle \mathrm{Td}(V \otimes \mathbf{C})\mathrm{ch}(M)e(V^*)^{-1}, f^P_*[X]\rangle = (-1)^d \langle \prod_{1\le k\le d} \mathrm{Td}(\eta_k)\mathrm{Td}(-\eta_k)\eta_k^{-1} i^*\mathrm{ch}(M), a\rangle$ for any $a \in \mathrm{H}_{2d}(BT, \mathbf{Q})$ such that $i_*(a) = f^P_*[X]$. Here $f^P: X \to BG$ classifies P, η_k are $\mathbf{R}$-weights of V, and

$\operatorname{ch}(M) := \sum_{0\le i\le n}(-1)^i \operatorname{ch}(M_i) \in \hat{\mathrm{H}}^{2*}(BG,\mathbf{Q})$ pulls back via i to $\sum_{0\le i\le n}(-1)^i \sum_{1\le k\le m_i}\exp(w_i^k)$, where w_i^k are the $\mathbf{C}$-weights of M_i.

Example: Chern-Gauß-Bonnet theorem. We have $G = \mathrm{SL}(\mathbf{R}^{2d})$, $V = \mathbf{R}^{2d}$, $M = \Lambda(V^*)\otimes\mathbf{C}$ (complexified exterior algebra). The $\mathbf{R}$-weights of V are $y_k \in \mathrm{H}^*(B\mathrm{SO}(2d),\mathbf{Q})$. Now we compute $\mathrm{H}^*(B\mathrm{SO}(n),\mathbf{Q}) \cong \mathrm{H}^*(BT,\mathbf{Q})^W = \mathbf{Q}[y_1,\ldots,y_{\lfloor n/2\rfloor}]^W$, where W is the semidirect product of $(\mathbf{Z}/2)^{\lfloor n-1/2\rfloor}$ with $S_{\lfloor n/2\rfloor}$. We have the following polynomial generators:

$n=1$	2	3	4	5	6
	y_1	y_1^2	$y_1^2+y_2^2$	$y_1^2+y_2^2$	$y_1^2+y_2^2+y_3^2$
			y_1y_2	$y_1^2y_2^2$	$y_1^2y_2^2+y_2^2y_3^2+y_1^2y_3^2$
					$y_1y_2y_3$

It is easy to check from our definition that $i^*e(V) = y_1\cdots y_d$ and $i^*p_i(V) = \sigma_i(y_k)$. Therefore we have isomorphisms $\mathrm{H}^*(B\mathrm{SO}(2k),\mathbf{Q}) \cong \mathbf{Q}[p_1,\ldots,p_{k-1},e]$ with $p_k = e^2$ and $\mathrm{H}^*(B\mathrm{SO}(2k+1),\mathbf{Q}) \cong \mathbf{Q}[p_1,\ldots,p_k]$. In fact, these isomorphisms also hold for any coefficient ring that is an integral domain containing $1/2$.

Lemma. If M has complex weights w_k, then $i^*\operatorname{ch}(\Lambda^*M) = \prod_{1\le k\le m}(1-\exp(w_k)) \in \hat{\mathrm{H}}^{2*}(BT,\mathbf{Q})$.

Proof. $M|_T = \bigoplus_{1\le k\le m} M_k$, where M_k are 1-dimensional with weights w_k. Then $i^*\operatorname{ch}(\Lambda M) = \operatorname{ch}(\Lambda^*(M_1\oplus\cdots\oplus M_m)) = \operatorname{ch}(\Lambda^*(M_1))\otimes\cdots\otimes\operatorname{ch}(\Lambda^*(M_m)) = \prod_{1\le k\le m}(1-\exp(w_k))$ because $\operatorname{ch}(\Lambda^*(M_k)) = \operatorname{ch}(1)-\operatorname{ch}(M_k)$.

Proof of last steps in Chern-Gauß-Bonnet theorem. The $\mathbf{R}$-weights of V are $y_k \in \mathrm{H}^*(BT,\mathbf{Q})^W$, hence the $\mathbf{C}$-weights of $V\otimes\mathbf{C}$ are $\pm y_k$, where $1\le k\le d = \dim T$. By Lemma $\operatorname{ch}(\Lambda^*(V^*)\otimes\mathbf{C}) = \prod_{1\le i\le d}(1-\exp(y_k))(1-\exp(-y_k))$. Thus $\text{t-ind}(D) = (-1)^d\langle\prod_{1\le k\le d}\operatorname{Td}(y_k)\operatorname{Td}(-y_k)(1-\exp(y_k))(1-\exp(-y_k))y_k^{-1}, a\rangle$. We have $\operatorname{Td}(y_k) = y_k/(1-\exp(-y_k))$ and therefore $\text{t-ind}(D) = (-1)^d\langle\prod_{1\le k\le d}(-y_k), a\rangle = \langle\prod_{1\le k\le d} y_k, a\rangle = \langle e(V), f_*^P[X]\rangle = \langle e(TX),[X]\rangle$.

Example: Hirzebruch-Riemann-Roch. $G = \mathrm{GL}(\mathbf{C}^m)$, $V = \mathbf{C}^m$, $M^k = \Lambda^k(\bar{V}^*)$, i.e., X is almost complex. Complex weights of V are $x_k \in \mathrm{H}^2(BT)$, hence real weights of $V_{\mathbf{R}}$ are also x_k. We have $\operatorname{ch}(M) = \prod_{1\le k\le m}(1-\exp(x_k))$ by Lemma. Note that the Dolbeault operator D exists only for complex manifolds. We have $\text{t-ind}(D) = (-1)^m\langle\prod_{1\le k\le m}\operatorname{Td}(x_k)\operatorname{Td}(-x_k)(1-\exp(x_k))x_k^{-1}, a\rangle = \langle\prod_{1\le k\le m} x_k/(1-\exp(-x_k)), a\rangle = \langle\prod_{1\le k\le m}\operatorname{Td}(x_k), a\rangle = \langle i^*\operatorname{Td}(V), a\rangle = \langle\operatorname{Td}(V), f^P[X]\rangle = \langle\operatorname{Td}(TX),[X]\rangle$.

Twisted Dolbeault operator: $G = \mathrm{GL}(\mathbf{C}^m)\times\mathrm{GL}(\mathbf{C}^n)$ and $M = \Lambda(V)\otimes W$. We have $\operatorname{ch}(M) = \prod_{1\le k\le m}(1-\exp(x_k))\times\operatorname{ch}(W)$ by Lemma. Thus the factor $\operatorname{ch}(W)$ appears in all formulas above and we obtain $\text{t-ind}(D) = \langle\operatorname{Td}(TX)\cup\operatorname{ch}(P\times_{\mathrm{GL}(\mathbf{C}^n)}W),[X]\rangle$. Here $P\times_{\mathrm{GL}(\mathbf{C}^n)}W$ is the twisting holomorphic bundle.

Summary: Examples of Index Theorem.

	Chern-Gauß-Bonnet	Hirzebruch-Riemann-Roch	Hirzebruch Signature Theorem
operator	d	$\bar\partial$	$(d+d^*)\vert_{\Omega^*(X,\mathbf{C})^+}$
G	$\mathrm{SL}_{2d}(\mathbf{R})$	$\mathrm{GL}_d(\mathbf{C})$	SO_{2d}
M^*	$\Lambda^*(V^*)\otimes\mathbf{C}$	$\Lambda^*(\bar{V}^*)$	$(\Lambda^*(V^*)\otimes\mathbf{C})^\pm$
t-ind	$\langle e(TX),[X]\rangle$	$\langle\operatorname{Td}(TX),[X]\rangle$	$\langle L(TX),[X]\rangle$
a-ind	$\chi(X)$	$\chi_{\mathrm{hol}}(X)$	$\sigma(X)$

Recall that the symbol of the de Rham differential evaluated at a point $\xi\in\mathbf{T}_x^*$ is the exterior multiplication by ξ from the left: $\sigma(d_i)\colon \mathbf{T}_x^*\times\Lambda^i(T_x^*)\to\Lambda^{i+1}(T_x^*)$ is hence given by $(\xi,w)\mapsto\xi\wedge w$ for all i. Since this map is $\mathrm{SL}(V)$-equivariant and linear in w, we see that our running assumption is satisfied: The symbol of the de Rham operator comes from $\mathrm{SL}(V)$-representations. Moreover, the resulting sequence of operators is exact whenever $\xi\ne 0$, thus the resulting complex is elliptic. A geometric proof can be immediately obtained from the geometric interpretation of the exterior algebra. This finally completes our derivation of the Chern-Gauß-Bonnet theorem from the Atiyah-Singer index theorem.

For the Dolbeault operator $\bar\partial$ its symbol evaluated at $\xi\in\overline{T_x^*X} = \Lambda_x^{0,1}(X)$ is the exterior multiplication by ξ followed by the multiplication by the imaginary unit. Hence again our running assumptions for computing t-ind($\bar\partial$) via representations are satisfied and the Dolbeault complex is elliptic. Hence the Hirzebruch-Riemann-Roch theorem follows from the Atiyah-Singer index theorem.

We recall the "wrapping" operation of elliptic complexes: After choosing a Riemannian metric on the manifold X and hermitian inner products on the vector bundles E_i, we can pass from an elliptic complex $D_\bullet\colon 0 \to C^\infty(E_0) \xrightarrow{D_0} C^\infty(E_1) \xrightarrow{D_1} C^\infty(E_2) \cdots \to C^\infty(E_n) \to 0$ to an elliptic operator $D := \sum_i D_{2i} + D^*_{2i-1}$ going from the even part of $C^\infty\left(\bigoplus_i E_i\right)$ to its odd part. The analytic index is preserved under this operation.

If we apply wrapping to the de Rham complex, we obtain an operator $d + d^*\colon \Omega^{\mathrm{even}}(X) \to \Omega^{\mathrm{odd}}(X)$. The analytic index of $d+d^*$ is $\chi(X)$. However, we can also think of $d+d^*$ as an operator $\Omega^+(X) \to \Omega^-(X)$, where $\Omega^\pm(X)$ are the ± 1-eigenspaces of the Hodge star on $\Omega^*(X, \mathbf{C})$ (adjusted in such a way that it is its own inverse). The Hodge star operator depends on an orientation of X. The analytic index of $(d+d^*)|_{\Omega^+(X)}$, the so-called *signature operator*, is the signature of X, which is defined as the signature of the non-degenerate symmetric bilinear form $\mathrm{H}^{2d}_{\mathrm{dR}}(X) \otimes \mathrm{H}^{2d}_{\mathrm{dR}}(X) \to \mathbf{R}$, where $\dim X = 4d$ (otherwise the signature is zero).

To compute the topological index of the signature operator recall that the L-genus is given by the power series $L(x) = x/\tanh(x) = 1 + x^2/3 - x^4/45 + \cdots$. Note that only even powers are present. Recall that we have an isomorphism $i^*\colon \mathrm{H}^*(B\mathrm{SO}(2d), \mathbf{Q}) = \mathbf{Q}[p_1, \ldots, p_{d-1}, e] \to \mathrm{H}^*(B\mathbf{T}^d, \mathbf{Q})^W = \mathbf{Q}[y_1, \ldots, y_d]^W$, where $i^*p_k = \sigma_k(y_i^2)$. Thus $\prod_{1\le i\le d} L(y_i) = 1 + L_1(p_1) + L_2(p_1, p_2) + \cdots$ and for a real vector bundle $E \to X$ we have $L(E) = 1 + L_1(p_1(E)) + L_2(p_1(E), p_2(E)) + \cdots$.

Example. We have $c(T\mathbf{CP}^n)c(\overline{T\mathbf{CP}^n}) = c(T\mathbf{CP}^n \otimes \mathbf{C})$, i.e., $(1+a)^{n+1}(1-a)^{n+1} = (1-a^2)^{n+1}$. Here $a \in \mathrm{H}^2(\mathbf{CP}^n)$ is the "complex" generator. Thus $p(T\mathbf{CP}^n) = (1+a^2)^{n+1}$ since $p_k(E) = (-1)^k c_{2k}(E \otimes_{\mathbf{R}} \mathbf{C})$. Hence $\langle L(T\mathbf{CP}^n), [\mathbf{CP}^n]\rangle = \langle (a/\tanh(a))^{n+1}, [\mathbf{CP}^n]\rangle$. This is the coefficient of a^{-1} in $1/\tanh(a)^{n+1}$, which can be computed via the Cauchy formula to be equal to 1 if n is even and 0 if n is odd. Thus the signature theorem is true for all complex projective spaces (and this fact determines the formula for the L-genus). Hirzebruch showed that this implies the signature theorem for all manifolds using Thom's computation of the rational bordism ring of oriented manifolds. This bordism proof was later generalized to all elliptic operators by Atiyah-Singer, that's why we'll explain Thom's results next.

Signature theorem (Hirzebruch, Thom). If X is a smooth oriented closed $4d$-manifold, then a-ind$(D) = \sigma(X) = \langle L(TX), [X]\rangle =$ t-ind(D), where $D = (d + d^*)|_{\Omega^*(X,\mathbf{C})^+}$ and the L-genus is given by $L(x) = x/\tanh(x) = 1 + x^2/3 - x^4/45 + \cdots$.

Corollary (Milnor). (a) There is a smooth structure on S^7 not diffeomorphic to the standard smooth structure. (b) There is a closed PL-manifold T^8 without a smooth structure.

Lemma 1. For any $k \equiv 2 \pmod 4$ there is a 4-dimensional vector bundle $E_k \to S^4$ such that $e(E_k) = u$, where u is the generator of $\mathrm{H}^4(S^4)$ and $p_1(E_k) = ku \in \mathrm{H}^4(S^4)$.

Proof of Lemma 1. For the tangent bundle we have $e(TS^4) = 2u$, $p_1(TS^4) = 0$. Consider now the 4-dimensional tautological bundle $\gamma_{\mathbf{H}} \to \mathbf{HP}^1$, where $\mathbf{HP}^1 = \mathbf{H} \cup \infty = S^4$ is the quaternion projective line. The sphere bundle $\gamma_{\mathbf{H}} \to \mathbf{HP}^1$ is isomorphic to $S^{4+3} \to S^4$. By Gysin sequence $e(\gamma_{\mathbf{H}}) = u$ and $p_1(\gamma_{\mathbf{H}}) = -c_2(\gamma_{\mathbf{H}} \otimes \mathbf{C}) = -c_2(\gamma_{\mathbf{H}} \oplus \bar\gamma_{\mathbf{H}}) = -c_2(\gamma_{\mathbf{H}} \oplus \gamma_{\mathbf{H}}) = -2c_2(\gamma_{\mathbf{H}}) = -2u$. We have $\mathrm{Vect}^4_{\mathbf{R}}(S^4) \cong \pi_4(B\mathrm{O}(4)) = \pi_3(\mathrm{O}(4)) = \pi_3(\mathrm{Spin}(4)) = \pi_3(\mathrm{SU}(2) \times \mathrm{SU}(2)) = \mathbf{Z} \times \mathbf{Z}$. We observe that $\mathrm{Vect}^4_{\mathbf{R}}(S^4)$ is a group and (p_1, e) is a homomorphism from this group to $\mathbf{Z} \times \mathbf{Z}$.

Remark. The condition $k \equiv 2 \pmod 4$ is necessary because $p_1 \equiv p(w_2) + 2e \pmod 4$, where the cohomology operation $p\colon \mathrm{H}^2(X, \mathbf{Z}/2) \to \mathrm{H}^4(X, \mathbf{Z}/4)$ is known as the *Pontrjagin square*.

Lemma 2. For any $k \equiv 2 \pmod 4$ we have $S(E_k) \simeq S^7$.

Proof. We have $\pi_i(S(E_k)) = 0$ for $0 \le i \le 2$. By Gysin sequence $\mathrm{H}^*(S(E_k)) \cong \mathrm{H}^*(S^7)$, hence $\mathrm{H}_*(S(E_8)) = \mathrm{H}_*(S^7)$. By Hurewicz $\pi_i(S(E_k)) = \pi_i(S^7)$ for all i. Thus by Whitehead theorem the obvious degree one map $S(E^k) \to S^7$ is a homotopy equivalence.

Proof of Corollary. By Smale's h-cobordism theorem $S(E_k)$ is PL-equivalent to S^7. This also follows from the existence of smooth function $f\colon S(E_k) \to \mathbf{R}$ with exactly two critical points, which Milnor wrote down in his short Fields medal paper. If we cut out two 7-balls, the rest is diffeomorphic to $S^6 \times [0,1]$. We can then glue one ball to obtain D^7 and the manifold $S(E_k)$ is diffeomorphic to the union of two smooth 7-balls, glued along a diffeomorphism of S^6. Any such diffeomorphism extends to a PL-homeomorphism to D^7 (via coning) but not necessarily to a diffeomorphism. That's why $S(E_k)$ is PL-homeomorphic to S^7 but not necessarily diffeomorphic.

Recall that $L(x)/\tanh(x) = 1+x^2/3-x^4/45+\cdots$. Thus $L(V) = 1+L_1(p_1(V))+L_2(p_1(V), p_2(V))+\cdots = 1 + p_1(V)/3 + p_2(V)/9 - (p_1(V)^2 - 2p_2(V))/45 + \cdots$. Let $T_k^8 := D(E_k) \cup_{S(E_k)\cong S^7} D^8$. If T_k^8 had a smooth structure (e.g., if $S(E_k)$ was diffeomorphic to S^7) then we would get the following contradiction for $k = 6$: T_6 does not satisfy the signature theorem: $1 = \sigma(T_k) \neq \langle L(T(T_k)), [T_k]\rangle = \langle 7p_2(T(T_k)) - p_1(T(T_k))^2, [T_k]\rangle/45$. If $\sigma = \langle L, [T]\rangle$ then $45 = 7\langle p_2, [T]\rangle - k^2$ would imply that $k \equiv \pm 2 \pmod 7$.

Original (bordism) proof of signature theorem uses results of Serre (1951): Stable homotopy groups of spheres are finite (except π_0) and Thom (1952), who computed the rational oriented bordism ring. Steps in the proof of signature theorem: (1) It is true for $\mathbf{CP}^n$ for all n; (2) Both sides have the following properties: (a) They are additive under coproduct of manifolds; (b) They are multiplicative under product of manifolds; (c) They are invariant under oriented bordism: End of proof: Denote by Ω_* the oriented bordism ring (Ω_n consists of closed oriented smooth n-manifolds modulo compact oriented smooth $(n+1)$-cobordisms). We just proved that σ and L are ring homomorphisms $\Omega_* \to \mathbf{Q}$.

Theorem (Thom). $\Omega_* \otimes \mathbf{Q} = \mathbf{Q}[\mathbf{CP}^2, \mathbf{CP}^4, \mathbf{CP}^6, \ldots]$.

Proof of (c): We need to show that σ and L vanish on ∂W^{n+1}. Observe that $TW|_M \cong TM \oplus \mathbf{R}$. Hence $\langle L(TM \oplus \mathbf{R}), [M]\rangle = \langle L(TW), i_*[M] = 0\rangle = 0$. Remark: If $\mathrm{H}_n(M^{2n}, \mathbf{Q})$ contains a Lagrangian (for the intersection form), then $\sigma(M) = 0$. Lemma: $L := \ker(\mathrm{H}_n(M, \mathbf{Q}) \to \mathrm{H}_n(W, \mathbf{Q}))$ is a Lagrangian. Here $\dim M = 2n$ and $\dim W = 2n+1$. Proof: (a) The intersection form vanishes on L because the intersection points of two n-cycles in M bound intersection arcs of the bounding $(n+1)$-cochains in W. (b) $2\dim_{\mathbf{Q}} L = \dim_{\mathbf{Q}} \mathrm{H}_n(M, \mathbf{Q})$ by Lefschetz duality.

Proof of Thom's theorem. The first step is the Pontrjagin-Thom construction: Consider an embedding of manifolds $M \to S^{d+n}$ (here $\dim M = d$ and n is large). Take the normal bundle νM of this embedding and construct the collapse map $S^{d+n} \to \mathrm{Th}(\nu M)$ (map a tubular neighborhood of M diffeomorphically to the normal bundle νM and map everything outside the tubular neighborhood to the basepoint). The embedding gives a classifying map $M \to B\mathrm{O}(n, n+d) \subset B\mathrm{O}(n)$ for the normal bundle νM. It is the pullback of the universal bundle γ^n via this map. Thus we get a map $\mathrm{Th}(\nu)\colon \mathrm{Th}(\nu M) \to \mathrm{Th}(\gamma^n)$, which we compose with the collapse map to arrive at a map $t_M\colon S^{d+n} \to \mathrm{Th}(\gamma^n)$, associated to our manifold M.

Theorem. (a) For the unoriented bordism groups we have $\Omega_d^{\mathrm{un}} \cong \pi_{d+n}(\mathrm{Th}(\gamma_{\mathrm{O}}^n))$ as vector spaces over $\mathbf{Z}/2$. The forward map is given by sending a manifold M to the map t_M. The backward map is given by sending a map $S^{d+n} \to \mathrm{Th}(\gamma_{\mathrm{O}}^n)$ to the preimage of the zero section. (First we deform the map to make it smooth and transversal.) (b) $\Omega_d \cong \pi_{d+n}(\mathrm{Th}(\gamma_{\mathrm{SO}}^n))$ and more generally: (c) $\Omega_d^\xi \cong \pi_{d+n}(\mathrm{Th}(\xi_n))$ for stable normal structures $\xi\colon B \to B\mathrm{O}$ like Spin, complex, symplectic. Here we define spaces B_n as pullbacks of a fibration $\xi\colon B \to B\mathrm{O}$ under the inclusions $B\mathrm{O}(n) \to B\mathrm{O}$ and hence they come equipped with an n-dimensional bundle ξ_n. In particular, for framed bordism we can use the contractible spaces $B_n = E\mathrm{O}(n)$ and hence $\mathrm{Th}(\xi_n \to B_n) \simeq \mathrm{Th}(\mathbf{R}^n \to \mathrm{pt}) \simeq S^n$. We thus obtain $\Omega_d^{\mathrm{fr}} \cong \pi_{d+n}\mathrm{Th}(\xi_n \to E\mathrm{O}(n)) \cong \pi_{d+n}(S^n)$ for large n. These are the stable homotopy groups of spheres; Serre's theorem states that these groups π_d^s are finite for $d > 0$.

Remark. The spaces $\mathrm{Th}(\xi_n)$ for $n \geq 0$ form a *spectrum* $M\xi$, i.e., these are pointed spaces together with maps $\mathrm{Th}(\xi_n) \wedge S^1 \to \mathrm{Th}(\xi_{n+1})$, arising from the fact that $S^1 = \mathrm{Th}(\mathbf{R})$. In the oriented case $\xi\colon B\mathrm{SO} \to B\mathrm{O}$ we have the following sequence of isomorphisms: $\Omega_d \otimes \mathbf{Q} \to \pi_d(M\mathrm{SO}) \otimes \mathbf{Q} \to \mathrm{H}_d(M\mathrm{SO}, \mathbf{Q}) = \mathrm{H}_d(B\mathrm{SO}, \mathbf{Q}) = \mathrm{Hom}(\mathrm{H}^d(B\mathrm{SO}, \mathbf{Q}), \mathbf{Q})$. This composition is given by Pontrjagin numbers, i.e., evaluation of products of Pontrjagin classes on the fundamental class of a closed oriented d-manifold. The rational Hurewicz homomorphism, used here for the spectrum $M\mathrm{SO}$, is an isomorphism for any spectrum because of Serre's finiteness theorem. If one looks at all degrees together, then the above maps are actually ring isomorphisms if one uses the H-space structure on $B\mathrm{SO}$ given by direct sum of vector bundles. It induces the structure of a ring spectrum on $M\mathrm{SO}$. The last step in Thom's rational computation of the bordism ring is to show that the $\mathbf{CP}^{2n}$, $n \geq 0$ do not satisfy any polynomial identities and hence can be used as polynomial generators. This can be seen by computing their Pontrjagin numbers as in Milnor and Stasheff.

The Pontrjagin-Thom construction sends elements of $\pi_{n+d}(S^n)$ to d-dimensional closed smooth submanifolds of S^{n+d} with a framing of the normal bundle modulo framed bordism in $S^{n+d} \times I$. This can be discussed for arbitrary (and fixed) n, not just in the stable case where $n \to \infty$ as in the previous lecture. For example, $\pi_3(S^2)$ consists of framed 1-manifolds in S^3 modulo bordism. Knotting and linking is not an issue because of the existence of Seifert surfaces. So one obtains normal framings on S^1 modulo bordism, which are isomorphic to $\pi_1(\mathrm{SO}(2)) \cong \mathbf{Z}$.

We have suspension morphism $\pi_3(S^2) \to \pi_4(S^3)$, the elements of the latter group correspond to framed 1-manifolds in S^4 modulo bordism, which are isomorphic to $\pi_1(\mathrm{SO}(3)) = \mathbf{Z}/2$. The induced map $\mathbf{Z} \to \mathbf{Z}/2$ is non-trivial because of the fibration $\mathrm{SO}(2) \to \mathrm{SO}(3) \to S^2$. By transversality, the sequence stabilizes starting from $\pi_4(S^3)$.

Recall that $\Omega_d^\xi \cong \pi_{n+d}(\mathrm{Th}(\xi_n)) = \pi_d(M\xi)$ for all $n \gg d$.

Corollaries of Pontrjagin-Thom construction. (a) $\Omega_d \otimes \mathbf{Q} \cong \pi_d(M\mathrm{SO}) \otimes \mathbf{Q} \cong \mathrm{H}_d(M\mathrm{SO}, \mathbf{Q}) \cong \mathrm{H}_d(B\mathrm{SO}, \mathbf{Q}) \cong \mathrm{Hom}(\mathrm{H}^d(B\mathrm{SO}, \mathbf{Q}), \mathbf{Q})$. Thus Pontrjagin numbers detect elements of bordism group. (b) For non-oriented bordism ring we have $\mathbf{Z}/2[x_i] = N_* = \Omega_*^{\mathrm{O}} \to \mathrm{H}_*(B\mathrm{O}, \mathbf{Z}/2)$ via Stiefel-Whitney numbers. Here $i \neq 2^k - 1$. The tool for computing $\pi_d(E) \to \mathrm{H}_d(E, R)$ is Adams spectral sequence. (c) Unitary bordism: $\mathbf{Z}[a_{2n}] \cong \Omega_*^{\mathrm{U}} \to \mathrm{H}_*(B\mathrm{U})$. Milnor did the computation first and later Quillen explained it via the relation to formal group laws. Rationally we have $\mathbf{Q}[\mathbf{CP}^n] \cong \Omega_*^{\mathrm{U}} \otimes \mathbf{Q}$. (d) For Spin we have K-theory Hurewicz map: $\Omega_*^{\mathrm{Spin}} \to \mathrm{KO}_*(B\mathrm{Spin})$. Together with Stiefel-Whitney numbers, it detects spin bordism.

Spectra and (co)homology theories. Examples of spectra: (1) Thom spectra $M\xi$; (2) Suspension spectra $\Sigma^\infty X$, for example $\mathbf{S} := \Sigma^\infty(S^0)$. (3) Eilenberg-Mac Lane spectra $\mathrm{H}A$, where A is an abelian group; (4) K-theory spectra: $K\mathrm{U}$, $K\mathrm{O}$, $K\mathbf{Q}$.

Theorem. Any spectrum E gives homology and cohomology theories as follows:

$$E_d(X) := [S^d, X \wedge E] = \pi_d(X \wedge E) = \operatorname*{colim}_n \pi_{n+d}(X \wedge E_n)$$

and

$$E^d(X) := [X, E]_{-d} = \operatorname*{colim}_n [X \wedge S^n, E_{n+d}]$$

satisfying homotopy axiom, Mayer-Vietoris axiom, and the wedge axiom. For example, we have

$$\begin{array}{ccccc} \Omega_*^{\mathrm{Spin}} & \cong & \pi_*(M\mathrm{Spin}) & = & M\mathrm{Spin}_*(S^0) \\ & & \downarrow & & \downarrow \\ \pi_*(\mathrm{KO} \wedge M\mathrm{Spin}) & = & \mathrm{KO}_*(M\mathrm{Spin}) & = & M\mathrm{Spin}_*(KO) \end{array}$$

Pushforwards in cohomology theories. We can assume that our cohomology theory comes from an Ω-spectrum, i.e., the map $E_n \to \Omega E_{n+1}$ (the adjoint to $E_n \wedge S^1 \to E_{n+1}$) is a homeomorphism. For example, $E = \mathrm{H}A$, $E = \mathrm{K}$, $E = \mathrm{KO}$. For $E = \mathrm{K}$ observe that $\Omega(BU \times \mathbf{Z}) = \Omega BU \cong U$. If E is an Ω-spectrum, then $E^n(X) = [X, E_n]$. We have $E^{n+k}(X \wedge S^k) = [X \wedge S^k, E_{n+k}] = [X, \Omega^k E_{n+k} \cong E_n]$. Thus we can define $E^{-k}(X) = E^0(X \wedge S^k)$, where $k \geq 0$. For K-theory we have $\mathrm{K}^0(X) \cong \mathrm{K}^{2n}(X)$ as before and $\mathrm{K}^1(X) \cong \mathrm{K}^{2n+1}(X) = [X, \mathrm{U}]$. We have $\mathrm{H}^0(\mathrm{pt}, A) = A$ and $\mathrm{H}^k(\mathrm{pt}, A) = 0$ for $k \neq 0$, which characterizes *ordinary* cohomology theories. K-theory is *extra-ordinary* because we have $\mathrm{K}^n(\mathrm{pt}) = \mathrm{K}^0(\mathrm{pt}) = \mathbf{Z}$ for even n and $\mathrm{K}^n(\mathrm{pt}) = \mathrm{K}^1(\mathrm{pt}) = 0$ for odd n. As a consequence, K-theory is a 2-periodic cohomology theory.

If E is an Ω-spectrum, then a ring spectrum consists of maps $E_m \wedge E_n \to E_{m+n}$ and $1 \in E_0$ that are associative up to coherent higher homotopies. For example, we have a concrete model for $(\mathrm{H}A)_n = K(A, n)$ (points in S^n marked with elements of A). Multiplication is given by multiplying points and their labels.

Definition. Given a fibration $\xi\colon B \to BO$, a ring spectrum E is called ξ-oriented if it is equipped with a natural multiplicative Thom isomorphism $E^{n+k}(DV, SV) \cong E^k(X)$ given by $a \in E^k(X) \mapsto \pi^*(a) \cup u_E(V) \in E^{n+k}(DV, SV)$ for some Thom classes $u_E(V, \tilde{c}_V) \cong E^n_{\mathrm{cs}}(V)$. Here $\pi\colon V \to X$ is a vector bundle with classifying map $c_V\colon X \to B\mathrm{O}(n)$ that is lifted to a ξ-structure $\tilde{c}_V\colon X \to B_n$.

Remark. A ξ-orientation is the same structure as a ring spectrum map $u\colon M\xi \to E$. In one direction, the maps $u_n\colon \mathrm{Th}(\xi_n) = (M\xi)_n \to E_n$ are the universal Thom classes.

Examples of orientations. $M\mathrm{U} \to M\mathrm{SO} \to \mathrm{H}\mathbf{Z}$, $M\mathrm{O} \to \mathrm{H}\mathbf{Z}/2$, $M\mathrm{Spin} \to K\mathrm{O}$, $M\mathrm{U} \to M\mathrm{Spin}^c \to K$.

Lemma. Suppose E is ξ-orientable. If M and N are smooth manifolds and $f\colon M \to N$ is a proper embedding with a ξ-structure on the normal bundle of f, then define $f_!\colon E^k_{\mathrm{cs}}(M) \to E^{k+n-m}_{\mathrm{cs}}(N)$ as the composition of maps $E^k_{\mathrm{cs}}(M) \to E^{k+n-m}_{\mathrm{cs}}(\nu(f))$ (Thom isomorphism, i.e., cup product with $u_E(\nu(f))$) and $E^{k+n-m}_{\mathrm{cs}}(\nu(f)) \to E^{k+n-m}_{\mathrm{cs}}(N)$ (extend by zero). Here we use the fact that $E^n_{\mathrm{cs}}(Z)$ is π_0 of the space of maps $Z \to E_n$ that are equal to the basepoint of E_n outside of a compact set.

Lemma. Every morphism of manifolds $f\colon M \to N$ can be decomposed as a composition of a proper embedding and a projection: $M \to N \times \mathbf{R}^s \to N$, where $i\colon M \to \mathbf{R}^s$ is a proper embedding of M into $\mathbf{R}^s$ for some large s. We now define $f_! := (\pi_1)_! \circ (f \times i)_!\colon E^k_{\mathrm{cs}}(M) \to E^{k+s+n-m}_{\mathrm{cs}}(N \times \mathbf{R}^n) \to E^{k+n-m}_{\mathrm{cs}}(N)$. Here $m = \dim M$, $n = \dim N$, and $(\pi_1)_!$ is defined as the inverse of the Thom isomorphism (for the trivial bundle). This map is independent of the choice of i by the multiplicativity of universal Thom classes and the homotopy invariance of cohomology. It depends on a ξ-structure on the stable vector bundle $-TM \oplus f^*TN$. The dependence comes from the fact that a ξ-structure on $\nu(f \times i)$ is the same thing as a stable ξ-structure on $-TM \oplus f^*TN$.

Example. If $M \to N$ is a fiber bundle, then the pushforward map is sometimes called "integration over the fibers" (literally the case for de Rham cohomology). The stable normal bundle is in this case the inverse to the tangent bundle along the fibres.

Theorem. K-theory is complex oriented, i.e., Thom classes exist for complex vector bundles.

Proof. For a complex n-dimensional vector bundle $V \to X$ we need a class $u_{\mathrm{K}}(V) \in \mathrm{K}^{2n}_{\mathrm{cs}}(V)$ and these classes must behave multipicatively (as for ordinary cohomology).

Theorem. K-theory is complex oriented, i.e., there are Thom isomorphisms $\mathrm{K}_{\mathrm{cs}}(X) \xrightarrow{\cong} \mathrm{K}_{\mathrm{cs}}(V)$ for any complex vector bundle $p\colon V \to X$ given by the map $a \mapsto p^*(a) \cup u_{\mathrm{K}}(V)$, where $u_{\mathrm{K}}(V) \in \mathrm{K}_{\mathrm{cs}}(V)$ is the Thom class. Here $\mathrm{K}_{\mathrm{cs}}(Y) = \tilde{\mathrm{K}}(Y^\infty)$.

Proof. The Thom class $u_{\mathrm{K}}(v)$ is represented by $0 \to \Lambda^0_{\mathbf{C}} p^*V \to \Lambda^1_{\mathbf{C}} p^*V \to \cdots \to \Lambda^n_{\mathbf{C}} p^*V \to 0$. All maps are exterior products with a base vector $v \in V$. Recall that this complex is exact on $V \setminus 0$. Bott periodicity implies that $i_x^*(u_{\mathrm{K}}(v)) \in \mathrm{K}_{\mathrm{cs}}(V_x) \cong \mathbf{Z}$ is a generator. We have the following isomorphism given by the tensor product: $\bigotimes^n \mathrm{K}_{\mathrm{cs}}(\mathbf{C}) \to \mathrm{K}_{\mathrm{cs}}(V_x)$, which is isomorphic to $\bigotimes^n \tilde{\mathrm{K}}(S^2) \to \tilde{\mathrm{K}}(S^{2n})$, which is isomorphic to $\bigotimes^n \mathbf{Z} \to \mathbf{Z}$. Here $\tilde{\mathrm{K}}(S^2) \cong \mathbf{Z}$ is generated by $1 - H$, which is the Thom class $u_{\mathrm{K}}(\mathbf{C})$.

Definition of shriek/Gysin/pushforward/integration-over-the-fibers/wrong-way/Umkehr maps in K-theory. Let $f\colon X \to Y$ be a complex oriented morphism of smooth manifolds, i.e., the stable normal bundle $\nu(f) := (-TX) \oplus f^*(TY)$ has a complex structure. Then we get a pushforward map $f_!\colon \mathrm{K}_{\mathrm{cs}}(X) \to \mathrm{K}_{\mathrm{cs}}(Y)$. Case 1: f equals p or i, where $p\colon V \to X$ is the projection map of a vector bundle and i is a section. The $p_!$ and $i_!$ come from Thom isomorphism. Note that p is not proper. Case 2: An arbitrary map $f\colon X \to Y$ can be decomposed as a section $i\colon X \to \nu(f \times j)$ followed by an open inclusion $\nu(f \times j) \to Y \times \mathbf{C}^N$ followed by a projection $p_1\colon Y \times \mathbf{C}^N \to Y$. The pushforward map is given by the composition of individual pushforward maps. This map is independent of the choice of a proper embedding $j\colon X \to \mathbf{C}^N$ because u_{K} is multiplicative and K is a homotopy functor.

Henceforth we denote $\mathrm{K}(X) := \mathrm{K}_{\mathrm{cs}}(X)$ for a locally compact space X. This functor is contravariant for proper continuous maps, covariant for smooth maps with complex normal bundle, and satisfies twisted Bott periodicity (Thom isomorphism): $\mathrm{K}(X) \cong \mathrm{K}(V)$ for complex vector bundles $p\colon V \to X$, where $a \mapsto p^*(a) \otimes u_{\mathrm{K}}(V)$.

Compare this to $\mathrm{H}^*_{\mathrm{cs}}(X)$. We have a pushforward map $\mathrm{H}^k_{\mathrm{cs}}(X) \to \mathrm{H}^{k+\dim Y - \dim X}_{\mathrm{cs}}(Y)$ for a smooth (or even continuous) map $f\colon X \to Y$ of oriented manifolds, which is given by taking the Poincaré dual of the pushforward in homology. Still works if only the normal bundle $\nu(f)$ is (stably) orientable.

Remark. For any oriented d-manifold X we have $\mathrm{H}^k(X) \cong \mathrm{H}^{\mathrm{lf}}_{d-k}(X)$ (locally finite or Borel-Moore homology). The isomorphism is given by the pairing with the fundamental class $[X]_{\mathrm{lf}}$.

Lemma 1. If $p\colon V \to X$ is a complex vector bundle of dimension n, then

$$\mathrm{ch}(u_{\mathrm{K}}(V)) = (-1)^n u_{\mathrm{H}}(V) \cup p^*(\mathrm{Td}(\bar{V})^{-1}) \in \mathrm{H}^{\mathrm{even}}_{\mathrm{cs}}(V, \mathbf{Q}).$$

Proof. Enough to show for $\gamma^n \to B\mathrm{U}(n) \leftarrow B\mathbf{T}^n$. The direct sum $L_1 \oplus \cdots \oplus L_n$ maps to γ^n and $B\mathbf{T}^n$. The map $i_0^* \circ i^*$ maps $\mathrm{ch}(u_{\mathrm{K}}(\gamma^n))$ to $\prod_{1\le i\le n}(1 - \exp(x_i))$, where $x_i := c_1(L_i)$. The same map maps $u_{\mathrm{H}}(\gamma^n) \cup p^*(\mathrm{Td}(\bar{\gamma}^{n-1}))$ to $x_1 \cdots x_n \cup \prod_{1\le i\le n}(1 - \exp(x_i))/(-x_i)$. Hence the result follows from the fact that the Todd genus corresponds to the power series $x/(1 - \exp(-x))$.

Lemma 2. $\tilde{\mathrm{K}}(S^{2n}) \cong \mathrm{K}(\mathbf{C}^n) \xrightarrow{\mathrm{ch}} \mathrm{H}^{\mathrm{even}}_{\mathrm{cs}}(\mathbf{C}^n, \mathbf{Q})$ is given by the inclusion $\mathbf{Z} \to \mathbf{Q}$. The isomorphism with $\mathbf{Q}$ is given by the pairing with the fundamental class $[\mathbf{C}^n] = [\mathbf{C}^n]_{\mathrm{lf}}$.

Proof. We have $\prod_{1 \le i \le n}(1 - H_i) \in \mathrm{K}_(\mathbf{C}^n) \cong \mathbf{Z} \ni 1$ and $1 - H_i \mapsto S^2$. Hence

$$\mathrm{ch}\left(\prod_{1\le i\le n}(1-H_i)\right) = \prod_{1\le i\le n}(1-\exp(x_i)) = (-1)^n \prod_{1\le i\le n}(x_i + x_i^2/2 + x_i^3/6 + \cdots) = (-1)^n \prod_{1\le i\le n} x_i,$$

which is a generator of $\mathrm{H}^{2n}(S^{2n})$. Here $x_i \in \mathrm{H}^2(S^2)$ is a generator.

Lemma 3. If $E \to X$ is a real oriented vector bundle over an oriented manifold X, then for any $\alpha \in \mathrm{H}^d_{\mathrm{cs}}(X)$ we have $\langle p^*(\alpha) \cup u_{\mathrm{H}}(E), [E]_{\mathrm{lf}}\rangle = \langle \alpha, [X]_{\mathrm{lf}}\rangle$. Here we use the pairing $\mathrm{H}^d_{\mathrm{cs}}(X) \otimes \mathrm{H}^{\mathrm{lf}}_d(X) \to \mathbf{Z}$.

For de Rham cohomology this follows from Fubini's theorem, for singular cohomology one needs the the homological Thom isomorphism. We will not spell this out here.

Theorem. We have $\text{t-ind}(\sigma) = \pi_!(\sigma)$, where $\sigma \in \mathrm{K}(T^*X) \xrightarrow{\pi_!} \mathrm{K}(\mathrm{pt})$ and π is the constant map $T^*X \to \mathrm{pt}$.

Proof. We identify TX and T^*X using the metric. We choose a proper embedding $X \subset \mathbf{R}^n$ with the normal bundle ν_0 and obtain a proper embedding of almost complex manifolds $TX \subset \mathbf{C}^n$ with the normal bundle $\nu \cong q^*(\nu_0 \otimes \mathbf{C})$, where $q\colon TX \to X$ is the projection map of the tangent bundle. We also denote by $p\colon \nu \to TX$ the projection map of the normal bundle. We have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{K}(TX) & \xrightarrow{\pi_!} & \mathrm{K}(\mathrm{pt}) & \cong & \mathbf{Z} \\
\cong\big\downarrow \mathrm{Th} & & \cong\big\downarrow \mathrm{Th} & & \\
\mathrm{K}(\nu) & \xrightarrow{\text{extend by } 0} & \mathrm{K}(\mathbf{C}^n) & & \\
\big\downarrow \mathrm{ch} & & \big\downarrow \mathrm{ch} & & \\
\mathrm{H}^{\mathrm{even}}_{\mathrm{cs}}(\nu, \mathbf{Q}) & \xrightarrow{\text{extend by } 0} & \mathrm{H}^{\mathrm{even}}_{\mathrm{cs}}(\mathbf{C}^n, \mathbf{Q}) & &
\end{array}$$

Therefore $\pi_!(\sigma) = \langle \mathrm{ch}(p^*(\sigma) \cup u_{\mathrm{K}}(\nu)), [\mathbf{C}^n]\rangle = \langle (p^*(\mathrm{ch}(\sigma)) \cup \mathrm{ch}(u_{\mathrm{K}}(\nu)), [\nu]\rangle = \pm\langle p^*(\mathrm{ch}(\sigma)) \cup u_{\mathrm{H}}(\nu) \cup p^*(\mathrm{Td}(\bar\nu)^{-1}), [\nu]\rangle$ and hence $\pi_!(\sigma) = \pm\langle \mathrm{ch}(\sigma) \cup \mathrm{Td}(q^*(\nu_0 \otimes \mathbf{C})^{-1}), [TX]\rangle = \pm\langle q^*(\mathrm{Th}^{-1}(\mathrm{ch}(\sigma))) \cup u_{\mathrm{H}}(TX) \cup q^*(\mathrm{Td}(TX \otimes \mathbf{C})), [TX]\rangle = \pm\langle \mathrm{Th}^{-1}(\mathrm{ch}(\sigma)) \cup \mathrm{Td}(TX \otimes \mathbf{C}), [X]\rangle = \text{t-ind}(\sigma)$ by the above three Lemmas. In particular, Lemma 3 was applied to both bundles, p and q.

***G*-index theorem.** Suppose a compact Lie group G acts on a closed smooth manifold X and D is a G-invariant elliptic complex. Then $\text{a-ind}_G(D) = \text{t-ind}_G(D) \in \mathrm{R}(G)$.

Here $\text{a-ind}_G(D) := [\ker D] - [\mathrm{coker} D]$ is a virtual representation of G and hence an element in $\mathrm{R}(G)$ and $\text{t-ind}_G(D) := \pi_!(\sigma(D))$ for the equivariant push-forward $\pi_!\colon \mathrm{K}_G(T^*X) \to \mathrm{K}_G(\mathrm{pt}) = \mathrm{R}(G)$. To define $f_!\colon \mathrm{K}_G(Y) \to \mathrm{K}_G(Z)$ for maps with complex stable normal bundle, we first embed $Y \subset \mathbf{C}^n$ G-equivariantly, where $\mathbf{C}^n$ has a linear G-action and do the rest as before.

We note that every element $g \in G$ gives a homomorphism $\mathrm{R}(G) \to \mathbf{C}$ by taking the trace of g in the given (virtual) representation. We denote the resulting complex numbers by $\text{a-ind}_G(D, g)$ respectively $\text{t-ind}_G(D, g)$.

Examples. Take $D := d$ (the de Rham operator). We have $\text{a-ind}_G(D, g) = \sum_{0 \le i \le n}(-1)^i \,\mathrm{tr}(g|_{\mathrm{H}^i(X)})$, where $\mathrm{H}^i(X)$ is the complex de Rham cohomology. Also $\text{t-ind}_G(D, g) = \chi(X^g)$, where X^g is the g-fixed set of X. This is Lefschetz fixed point formula.

Application to $X = S^n$. If $\chi(X^g) = 0$, then g reverses/preserves orientation if and only if n is even/odd.

Corollary. If G is finite, then $\chi(X/G) = |G|^{-1} \sum_{g\in G} \chi(X^g) = \chi(X)/|G| + r$, where r denotes contributions from non-trivial fixed sets.

Example. For $G = \mathbf{Z}/2$ and $X = S^2$ with reflection in the equator we have $1 = \chi(S^2/(\mathbf{Z}/2)) = 2^{-1}(2 + 0)$. For $X = \mathbf{CP}^2$ with complex conjugation we have $\chi(\mathbf{CP}^2/(\mathbf{Z}/2)) = 2^{-1}(3 + 1) = 2$. Arnold showed that $\mathbf{CP}^2/(\mathbf{Z}/2)$ is homeomorphic to S^4.

Proof of Corollary. By a result of Grothendieck, the projection map induces an isomorphism $\mathrm{H}^i(X/G) = \mathrm{H}^i(X)^G$ (we are using $\mathbf{C}$-coefficients here). Moreover, $\dim \mathrm{H}^i(X)^G = |G|^{-1} \sum_{g\in G} \mathrm{tr}(g|_{\mathrm{H}^i(X)})$ by the lemma below. Summing over all g, the Lefschetz fixed point formula gives: $\chi(X/G) = \sum_{0\le i\le n}(-1)^i \dim \mathrm{H}^i(X/G) = \sum_{0\le i\le n}(-1)^i \dim \mathrm{H}^i(X)^G = \sum_{0\le i\le n}(-1)^i \left(|G|^{-1}\sum_{g\in G}\mathrm{tr}(g|_{\mathrm{H}^i(X)})\right) = |G|^{-1}\sum_{g\in G}\chi(X^g)$.

Lemma. If a finite group G acts on a finite dimensional vector space V (over a field of characteristic zero) then $\dim V^G = \mathrm{tr}(N)$, where $N(v) := |G|^{-1}\sum_{g\in G} g(v)$ is the norm element.

Proof. Consider the inclusion $i\colon V^G \to V$, which is a right inverse to N. We have $\dim(V^G) = \mathrm{tr}(\mathrm{id}_{V^G}) = \mathrm{tr}(N_G \circ i) = \mathrm{tr}(i \circ N_G) = |G|^{-1}\sum_{g\in G}\mathrm{tr}(g|_V)$.

Example. Take $D := d + d^*$ (the signature operator). We have

$$\text{t-ind}_G(D, g) = \sum_F \langle L(TF) \prod_{0<\theta\le\pi} L_\theta(N_k^\theta), [F]\rangle,$$

where F runs through all connected components of X^g, the normal bundle of F in X is written as $\bigoplus_{0<\theta\le\pi} N_k^\theta$, where g acts on N_k^θ by multiplication with $\exp(i\theta)$ and $L_\theta(x) := (\exp(i\theta)\exp(2x)+1)/(\exp(i\theta)\exp(2x)-1) \in \mathbf{C}[[x]]$.

Lemma. The homomorphisms $\pi_! = \pi_G^X\colon \mathrm{K}_G(T^*X) \to \mathrm{R}(G)$ are characterized by the following properties: (A0) Functoriality for homomorphisms $\phi\colon G' \to G$ with respect to restriction maps; (A1) $\pi_G^{\mathrm{pt}} = \mathrm{id}_{\mathrm{R}(G)}$; (A2) Functoriality for G-embeddings of closed manifolds: If $f\colon X' \to X$ is a G-embedding, then $Tf\colon TX' \to TX$ is proper and the composition $\mathrm{K}_G(TX') \overset{Tf_!}{\to} \mathrm{K}_G(TX) \overset{\pi_G^X}{\to} \mathrm{R}(G)$ equals $\pi_G^{X'}$.

Proof. Pick a G-embedding $X \to V$, where V is a real orthogonal G-representation. We have an embedding $j\colon TX \to TV \cong V \otimes_{\mathbf{R}} \mathbf{C}$. If there are homomorphisms $a_G^X\colon \mathrm{K}_G(TX) \to \mathrm{R}(G)$ satisfying (A1) and (A2) then the following diagram commutes:

$$\begin{array}{ccccccc}
 & & \mathrm{K}_G(TV) & & & & \\
 & \overset{j_!}{\nearrow} & \downarrow & & \overset{i_!}{\nwarrow} & & \\
\mathrm{K}_G(TX) & \overset{j_!}{\longrightarrow} & \mathrm{K}_G(TV^\infty) & & \overset{i_!^\infty}{\longleftarrow} & & \mathrm{K}_G(\mathrm{pt}) = \mathrm{R}(G) \\
 & \underset{a_G^X}{\searrow} & \downarrow a_G^{V^\infty} & & \underset{a_G^{\mathrm{pt}}=\mathrm{id}_{\mathrm{R}(G)}}{\swarrow} & & \\
 & & \mathrm{R}(G) & & & &
\end{array}$$

The map $i_!$ is an isomorphism by Bott periodicity and hence a_G^X can be computed by going clockwise around the diagram. By definition, this is the map $\pi_!$.

Theorem. There is an index function $a_G^X\colon \mathrm{K}_G(TX) \to \mathrm{R}(G)$ for any closed Riemannian manifold X and compact Lie group G such that the properties (A0), (A1) and (A2) in the Lemma above are satisfied and $a_G^X(\sigma(D)) = \text{a-ind}_G(D)$ for any G-invariant elliptic operator D on X.

Corollary. We have $a_G^X = (\pi_!)_G^X$, where $\pi\colon TX \to \mathrm{pt}$ and therefore $\text{a-ind}_G(D) = \text{t-ind}_G(D)$ for any G-equivariant elliptic complex D.

Reminder: Consider the map $a_1^X\colon \mathrm{K}(TX) \to \mathrm{K}(\mathrm{pt})$. $\mathrm{K}(TX)$ is generated by triples (E^0, E^1, α), where $E_i \to X$ are vector bundles and $\alpha\colon \pi^*E^0 \to \pi^*E^1$ is a morphism of bundles that is an isomorphism outside $D_\epsilon(TX)$. Relations: (1) isomorphism of triples; (2) homotopies of α; (3) additions of triples where α extends to an isomorphism on all of TX.

For $G = 1$ we have $\mathrm{Diff}(E^0, E^1) \to \mathrm{Symb}(E^0, E^1) = \mathrm{Hom}_{S(TX)}(\pi^*E^0, \pi^*E^1) \ni \alpha$. We embed $\mathrm{Diff}^m(E^0, E^1) \subset \Psi^m(E^0, E^1)$. The symbol map extends to Ψ^m: $\sigma\colon \Psi^m(E^0, E^1) \to \mathrm{Symb}(E^0, E^1)$. The kernel of σ is $\Psi^{m-1}(E^0, E^1)_S$.

Lemma 1. If D is elliptic, then $D_s\colon W_s(E^0) \to W_{s-m}(E^1)$ is Fredholm, i.e., invertible up to compact operators.

Proof. Pick $P \in \Psi_s^{-m}(E^1, E^0)$ such that $\sigma(PD - \mathrm{id}) = 0$. Thus $PD - \mathrm{id}$ is compact. Use Fourier transform to express D as the multiplication by the total symbol of D. For any symbol p we define a pseudo-differential operator P_p with symbol p as the Fourier transform of the multiplication by p.

Lemma 2. The G-invariant index is a locally constant function from G-invariant Fredholm operators to representations of G.

Lemma. If $p(x,\xi)$ satisfies $|D_x^\beta D_\xi^\alpha| \le C_{\alpha,\beta,K}(1+|\xi|)^{m-|\alpha|}$ for all $x\in K$ (K is compact), then for each $s\in \mathbf{Z}$ we have $P_p \in L(W_s(E^0), W_{s-m}(E^1))$ on closed manifolds X.

Definition. $P^m(E^0,E^1)\subset \mathrm{Hom}(C^\infty E^0, C^\infty E^1)$ are those operators that can be locally written as P_p with $p(x,\xi)$ satsfying the above growth condition and $\sigma_p(x,\xi) := \lim_{\lambda\to\infty}\lambda^{-m}p(x,\lambda\xi)$ exists fro all $\xi\ne 0$.

The map $P^m(E^0,E^1)\to \mathrm{Symb}(E^0,E^1)$ is continuous.

Recall that $P\colon C^\infty(E^0)\to C^\infty(E^1)$ is a pseudodifferential operator of order $m\in\mathbf{R}$ if it can be locally (in X) written as $P(u)(x) = (2\pi)^{-n}\int_{\mathbf{R}^n}\exp(i(x,\xi))p(x,\xi)\hat{u}(\xi)d\xi$, where p lies in the symbol class $\mathrm{Symb}^m(\mathbf{R}^n, E^0, E^1) = \{p\in \mathrm{Hom}(\pi^*E^0, \pi^*E^1) \mid \forall\alpha,\beta \quad \forall K\subset \mathbf{R}^n \quad \exists C\ge 0 \quad \forall x\in K\colon |D_x^\alpha D_\xi^\beta p(x,\xi)| \le C(1+|\xi|)^{m-|\beta|}\}$ (K is compact).

Properties: (a) The symbol map $\sigma\colon (P^m/P^{m-1})(X,E^0,E^1)\to (\mathrm{Symb}^m/\mathrm{Symb}^{m-1})(X,E^0,E^1)$ is an isomorphism that preserves compositions. (More generally, σ is an equivalence of the associated graded categories of P and Symb.) (b) For any $s\in\mathbf{Z}$ (more generally, $s\in\mathbf{R}$) and for any $P\in P^m$ we get an induced bounded operator $P_s\colon W_s\to W_{s-m}$. In particular, the intersection of all P^m is the space of smoothing operators $P^{-\infty}$. Every element of this space induces an operator $P\colon W_s(E^0)\to C^\infty(E^1)$.

Lemma. If $D\in \mathrm{Diff}^m\subset P^m$ is elliptic, then $\sigma(D)$ is invertible modulo $\mathrm{Symb}^{-\infty}$.

Definition. $P\in P^m$ is elliptic if $\sigma(P)$ is invertible modulo $\mathrm{Symb}^{-\infty}$.

Proof of Lemma. Outside some compact set around $X\subset TX$ the inverse $\sigma(D)^{-1}$ exists and lies in Symb^{-m}. We extend this to q_0 such that $\sigma(D)q_0 - 1\in \mathrm{Symb}^{-\infty}$ and $q_0\sigma(D)-1\in\mathrm{Symb}^{-\infty}$.

Elliptic regularity. If P is an elliptic pseudodifferential operator, then for any $s\in\mathbf{R}$ we have $\ker(P_s)\subset C^\infty(E^0)$ and $P_s(u)\in C^\infty(E^1)$ implies $u\in C^\infty(E^0)$. Similarly for the cokernel of P.

Corollary. The index of P_s does not depend on s and equals the index of P.

Proof of elliptic regularity. Denote by m the order of P. Let $Q_0\in P^{-m}$ satisfy $\sigma(PQ_0)-1\in\mathrm{Symb}^{-\infty}$ and $\sigma(Q_0P)-1\in\mathrm{Symb}^{-\infty}$. Consider $R := 1-PQ_0$. We know that $\sigma_0(R)=0$, hence $R\in P^{-1}$. Denote by T_n the sum $1+R+\cdots+R^{n-1}$. We have $(1-R)T_n = 1-R^n\equiv 1 \pmod{P^{-n}}$. The operator $Q_n := Q_0T_n\in P^{-m}$. We have $PQ_n - 1 = PQ_0T_n - 1 = (1-R)T_n - 1\in P^{-n}$. Let $u\in W_s$ satisfy $P_s(u)=0$. Suppose $u\in W_s$ satisfies $P_s(u)=0$. Then $u = ((Q_n)_{s-m}P_s - 1)(u)\in W_{s-n}$. This is true for any n, hence $u\in C^\infty(E^0)$. To prove the other statement we need to construct the actual parametrix, i.e., consruct T_∞ and Q_∞.

Definition. For a class $(E^0,E^1,\alpha)\in \mathrm{K}_G(TX)$ define $a_G^X(E^0,E^1,\alpha) := \mathrm{ind}_G(P_s)$, where $P\in P^m$ is G-invariant satisfies $\sigma_m(P)(x,\xi) = |\xi|^m\alpha(x,\xi/|\xi|)$ for a fixed $m\in\mathbf{R}$.

Proof of Definition. Step 1: The G-index of P_s is independent of $P\in P^m$. If $\sigma_m(P) = \sigma_m(P')$, then $(P-P')_s\colon W_s\to W_{s-m}$ is compact. Hence $\mathrm{ind}(P_s) = \mathrm{ind}(P'_s)$.

Fix a closed G-manifold X, G-vector bundles E^0 and E^1 on X with G-invariant metrics. We assume that G and X are compact. For any $m\in\mathbf{R}$ we have a map from the set of isomorphisms from π^*E^0 to π^*E^1 (where $\pi\colon STX\to X$) to $\mathrm{Symb}^m(E^0,E^1)$ that sends an isomorphism α to α_m, where $\alpha_m(x,\xi) = |\xi|^m\alpha(x,\xi/|\xi|)$.

The index function a_G^X is the composition of the above map, the map that lifts a symbol to a pseudodifferential operator, and the composition $P_e^m(E^0,E^1)\to \mathrm{Fred}(W_s(E^0), W_{s-m}(E^0))\to \mathrm{R}(G)$.

Lemma. The above construction is independent of the choice of m, P, and s.

Proof. Elliptic regularity implies that the index of P_s is independent of s, hence the above construction is independent from P. The symbol map σ_m is surjective because pseudodifferential operators modulo smoothing pseudodifferential operators form a sheaf (the Schwarz kernel of a pseudodifferential operator has singularities only on the diagonal). Suppose $P\in P^m$ and $Q\in P^n$, then there is $R\in P^{m-n}$ such that $R = R^*$ and $\sigma_m(P) = \sigma_n(Q)\sigma_{m-n}(R)$. (We construct R by restricting to the unit sphere bundle.) The index vanishes for self-adjoint operators. Now $\mathrm{ind}(P) = \mathrm{ind}(QR) = \mathrm{ind}(Q)+\mathrm{ind}(R) = \mathrm{ind}(Q)$. The index function also respects equivalence relations on triples $[E^0,E^1,\alpha]$.

We now look at some generalizations of the index theorem. For simplicity we assume $G=1$.

Example. There is an operator $P\colon C^\infty(S^1) \to C^\infty(S^1)$ such that $P(z \mapsto z^n)$ is $z \mapsto nz^n$ for $n \le 0$ and nz^{n-1} for $n > 0$ (here we identify $S^1 = \mathrm{U}(1)$). This is an elliptic pseudodifferential operator of index 1. Recall that $\mathrm{K_{cs}}(TS^1) \cong \mathbf{Z}$.

Proof. Observe that $P_0 = -\partial$ is a self-adjoint differential operator of index 0. Its spectrum consists of all integers with multiplicity 1. Its symbol satisfies $\sigma_{P_0}(z, \xi) = \xi$. The operator P_0^2 has positive spectrum (zero has multiplicity 1, squares of other positive integers have multiplicity 2). Now we take $P_1 = (P_0^2)^{1/2}$ has eigenvalue 0 with multiplicity 1 and other positive integers with multplicity 2). The symbol of P_1 satisfies $\sigma_{P_1}(z, \xi) = |\xi|$. Now define $P := (z \mapsto z^{-1})(P_0 + P_1)/2 + (P_0 - P_1)/2$. The symbol of P satisfies $\sigma_P(z, \xi) = z^{-1}\xi$ for $\xi \ge 0$ and $\sigma_P(z, \xi) = \xi$ for $\xi \le 0$. Hence P is elliptic and its index is 1. The kernel of P is one-dimensional and P is surjective.

If X is almost complex then $\mathrm{K}(X) \cong \mathrm{K_{cs}}(TX)$. This isomorphism pulls back a bundle to TX and multiplies it by the Thom class. Composing this isomorphism with the index map we get a map $\mathrm{K}(X) \to \mathbf{Z}$, which sends $W \in \mathrm{K}(X)$ to the index of P such that $\sigma_m(P) = q^*W \otimes u_\mathrm{K}(TX)$. For complex X we can use $P = \bar\partial \otimes (W, \nabla)$.

Corollary. If X is complex then the general Atiyah-Singer index theorem follows from twisted Hirzebruch-Riemann-Roch.

Remark. The same is true if X has a spin$^\mathbf{C}$-structure: The map $W \in \mathrm{K}(X) \mapsto \sigma_{D\otimes(W,\nabla)} = q^*W \otimes \sigma_D \in \mathrm{K_{cs}}(TX)$ is an isomorphism, where $D\colon C^\infty(S^+) \to C^\infty(S^-)$ is the Dirac operator.

Corollary. As above, but for twisted Dirac operators.

Let $X \to F \to B$ be a bindle of smooth closed manifolds over a compact Hausdorff space B. The analytic index is now a map $\mathrm{K}(T_|F) \to [B, \mathrm{Fred}(H)] \cong \mathrm{K}(B)$. The topological index is a map $\mathrm{K}(T_|F) \to \mathrm{K}(B)$. The isomorphism $[B, \mathrm{Fred}(H)] \to \mathrm{K}(B)$ is given by the index bundle construction. If $T_|F$ has a spin$^\mathbf{C}$-structure, then we have an isomorphism $\mathrm{K}(F) \to \mathrm{K}(T_|F)$.

For spin$^\mathbf{C}$-families the family index theorem says that two pushforward maps are equal.

The analog of the index theorem for ordinary cohomology is the following diagram, whose commutativity boils down to Fubini's theorem:

$$\begin{array}{ccc} \mathrm{H}^*_{\mathrm{dR}}(F) & \xrightarrow{\int_X} & \mathrm{H}^{*-d}_{\mathrm{dR}}(B) \\ \downarrow\cong & \# & \downarrow\cong \\ \mathrm{H}^*(F, \mathbf{R}) & \xrightarrow{p_!} & \mathrm{H}^{*-d}(B, \mathbf{R}) \end{array}$$

Thus Fubini's theorem is the index theorem for ordinary cohomology.

This point of view can be generalized to K-theory. Feynman-Kac formula: The value of the heat kernel $\exp(-tD^2(b))(x, y)$ is the integral with respect to the Wiener measrure of the parallel transport homomorphism $\|^S(\gamma)$, where $\gamma\colon [0, t]\colon X$ is any path in X. In terms of Euclidean field theories this amounts to going from dimension $0|1$ to dimension $1|1$.

Non-existent index theorem: If $X \to F \to B$ is family of string manifolds ($p_1/2 = 0$), then there is a non-existent push-forward for $2|1$-dimensional EFTs over F to $2|1$-dimensional EFTs over B using the two-dimensional Feynman integral. The index theorem should say that this push-forward is the same as the push-forward for TMF.

Theorem. The function $a^X_G\colon \mathrm{K}_G(TX) \to \mathrm{R}(G)$ satisfies conditions (A0), (A1), and (A2).

Corollary. We have $a^X_G = \pi_!$, where $\pi\colon TX \to \mathrm{pt}$ and hence a-ind$_G$ = t-ind$_G$.

Proof. Need to show (A2), i.e., for every embedding $i\colon X \to Y$, where X and Y are closed G-manifolds, the composition $\mathrm{K}_G(TX) \xrightarrow{(Ti)_!} \mathrm{K}_G(TY) \xrightarrow{a^Y_G} \mathrm{R}(G)$ equals the map $a^X_G\colon \mathrm{K}_G(TX) \to \mathrm{R}(G)$. Step 1: Excision: If $U \subset Y$ is open and $U \subset Y'$ is open, then the two obvious triagles commute. Step 2: If $p\colon V \to X$ is a real vector bundle with the zero section $i_0\colon X \to V$ and $V = P \times_{\mathrm{O}(n)} \mathbf{R}^n$ for some principal $\mathrm{O}(n)$-bundle $P \to X$. For $\alpha \in \mathrm{K}_G(TX)$ we have $a^V_G((Ti_0)_!\alpha) = a^V_G(\alpha \otimes u_\mathrm{K}(V \otimes \mathbf{C})) = a^X_G(\alpha) a^{\mathbf{R}^n}_{G\times \mathrm{O}(n)}(\beta) = a^X_G(\alpha) = a^X_G(\alpha) a^{\mathbf{R}^n}_{\mathrm{O}(n)}(\beta)$. Here β is the image of 1 under the map $(Tj)_!\colon \mathrm{R}(\mathrm{O}(n)) \to \mathrm{K}_{\mathrm{O}(n)}(T\mathbf{R}^n)$. Fact: If H acts freely on Z, then $\mathrm{K}_H(Z) \cong \mathrm{K}(Z/H)$. By normalization (Step 3) $a^{\mathbf{R}^n}_{\mathrm{O}(n)}(\beta) = 1$. The normalization axiom says that 1 is mapped to 1 under the composition $\mathrm{R}(\mathrm{O}(n)) \xrightarrow{(Tj)_!} \mathrm{K}_{\mathrm{O}(n)}(T\mathbf{R}^n) \xrightarrow{a^{\mathbf{R}^n}_{\mathrm{O}(n)}} \mathrm{R}(\mathrm{O}(n))$. Lemma: Normalization follows from multiplicativity and the following computation of $a^{S^n}_{\mathrm{O}(n)}(S^n)$. We have $a^X_G(\rho_X = \sigma_1(d_X)) = \oplus_{0\le i\le n}(-1)^i \mathrm{H}^i_{\mathrm{dR}}(X) \in \mathrm{R}(G)$. Multiplicativity: Given an action of G on an H-principal bundle $P \to X$ with connection and a left H-manifold F we have $a^W_G(\alpha\beta) = a^X_G(\alpha) \otimes a^F_{G\times H}(\beta) \in \mathrm{R}(G)$ if $a^F_{G\times H}(\beta) \in \mathrm{R}(G) \le \mathrm{R}(G\times H)$.

Knot concordance and L^2-index theorem for 4-manifolds with boundary.

Consider the knot K_n (the n-twist knot) consisting of n full twists. K_0 is the unknot, for $n > 0$ we get a non-trivial knot.

Definition. A knot in S^3 is smoothly slice if it bounds a smooth embedded disc in S^4. It is topologically slice if it bounds a disc with a normal bundle.

There are topologically sliced knots that are not smoothly slice (Freedman, Donaldson).

Theorem. (a) K_n is algebraically slice if and onlu if $4n+1$ is a square (Casson-Gordon). (b) K_n is slice if and only if $n = 0$ or $n = 2$. (Fintushel-Stern, Cochran-Orr).

Remark. In higher dimensions a knot is slice if and only if it is algebraically slice.

Definition. Given a Seifert surface F for a knot K consider the Seifert pairing $S_F\colon \mathrm{H}_1(F)\otimes\mathrm{H}_1(F) \to \mathbf{Z}$ given by computing the linking number of a and $b^\uparrow$. Here the arrow denotes the "pushing" operation.

Definition. A knot K is algebraically slice if there is a Seifert surface F such that S_F has a Lagrangian subspace.

Theorem (Levine, 1960s). The factormonoid of the monoid of oriented knots by the submonoid of (topologically or smoothly) slice knots is an abelian group.

Proof. Consider a knot K and its inverse $-K$ (reversed mirror). Then $K\# -K$ is slice (actually a ribbon knot).

The groups defined above are still unknown. However, for algebraically sliced knots the group was compute by Levine: $\mathbf{Z}^\infty \times \mathbf{Z}/2^\infty \times \mathbf{Z}/4^\infty$.

Lemma. If K is slice, then it is algebraically slice.

Proof. Just draw some pictures.

Levine-Tristran signatures. For a Seifert surface F of K and $\omega \in S^1$ define the hermitian form $h_\omega = (1-\bar\omega)S_F + (1-\omega)S^t_F$ and denote by $\sigma_\omega(K)$ the signature of h_ω. This is a piecewise-constant element of $\mathrm{L}^\infty(S^1,\mathbf{C})$. (Note: The determinant of h_ω corresponds to the Alexander polynomial of K.)

Lemma. The function σ_ω does not depend on F, in fact it only depends on $S^0(K)$, the 0-surgery of K. We have $\sigma_\omega = \sigma(N, \mathbf{C}_\omega)$, where N is a 4-manifold with boundary $S^0(K)$ and $\pi_1(N) \cong \mathbf{Z}$ and $\mathbf{C}_\omega$ denotes the flat line bundle with holonomy ω.

Lemma. The Lagrangian in $\mathrm{H}_1(F)$, where $\partial F = K_{m(m+1)}$, is generated by $\gamma_m = mh_1 + h_2$, where $S_F = \begin{pmatrix} -1 & 0 \\ -1 & m(m+1)\end{pmatrix}$. Remark: $4n+1 = (2m+1)^2$ if and only if $n = m(m+1)$.

Proof. Just draw some pictures.

Theorem. If K is topologically slice then any genus 1 Seifert surface contains an embedded circle γ such that (1) $\langle\gamma\rangle \subset \mathrm{H}_1 F$ is a Lagrangian; (2) $\int_{S^1}\sigma_\omega(\gamma) = 0$.

Lemma. The Lagrangian γ_m in $\mathrm{H}_1(F)$, where $\partial F = K_{m(m+1)}$, is an $(m, m+1)$-torus knot. In particular, (Julia Bergner) $\int_{S^1}\sigma_\omega(\gamma_m) \ne 0$ and hence $K_{m(m+1)}$ is not topologically slice.

Signature theorem. If (W, g) is a compact Riemannian 4-manifold with product metric near $\partial W = M$, then (Atiyah-Patodi-Singer) $\sigma(W) = (1/3)\int_W p_1(W,g) - \eta(M,g)$. Here $\sigma(W)$ equals the signature of $(\mathrm{H}_2(W,\mathbf{C}),\lambda)$.

Proof. If $D := *d - d*\colon \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{even}}(M)$ is the self-adjoint signature operator, then define $\eta(s) := \sum_{\lambda\in\mathrm{Spec}(D)\setminus\{0\}\subset\mathbf{R}} \mathrm{sign}(\lambda)|\lambda|^{-s}$. This is a holomorphic function for $\Re s > -1/2$. Set $\eta(M,g) := \eta(0)$.

Twisted signature theorem. If W is as above and $\rho\colon \pi_1(W) \to \mathrm{U}(n)$, then $\sigma(W,\rho) = (1/3)\int_W p_1(W,g) - \eta(M,g,\rho)$. Here $\sigma(W,\rho)$ is the signature of $(\mathrm{H}_2(\tilde{W}) \otimes_{\pi_1 W,\rho} \mathbf{C}^n, \tilde{\lambda})$.

Corollary. The value of $\sigma(W,\rho) - \sigma(W) = \eta(M,g,\rho|_M) - \eta(M,g)$ only depends on $(M,\rho|_M)$.

Example. For $M = S^0K$ and $\rho_\omega\colon \pi_1 \to \mathrm{H}_1 M = \mathbf{Z} \to \mathrm{U}(1)$ we have $\tilde{\eta}(M,\rho_\omega) = \sigma_W(F) = \sigma((1-\bar{\omega})S_F + (1-\omega)S_F^t)$.

$\mathbf{L^2}$-signature theorem. (Atiyah and Ramachandran.) If $\rho\colon \pi_1 W \to \Gamma$ is a representation, then $\sigma_\Gamma^{(2)}(W,\rho) = (1/3)\int_W p_1(W,g) - \eta_\Gamma(M,g_M,\rho_M)$. Here $\sigma_\Gamma^{(2)}$ is the $N\Gamma$-signature of $(\mathrm{H}_2(\tilde{W} \otimes_{\pi_1 W,\rho} l^2\Gamma), \tilde{\lambda})$.

We look at the representation of the metabelian quotient $\pi_1(S^0K)/\pi_1(S^0K)''$, which maps to the group Γ, which is the crossed product of $\mathbf{Q}(t)/\mathbf{Q}[t^{\pm 1}]$ and $\mathbf{Z}$. Now we look at the right-translation action of Γ on $l^2(\Gamma)$. The closure of the algebra generated by this action in the σ-weak topology is the group von Neumann algebra $N\Gamma$. Now we do everything as before except that we twist by an infinite-dimensional bundle and the dimension now refers to the real-valued dimension over type II_1 factor.

Example. For $\Gamma = \mathbf{Z}$ we obtain $\sigma_{\mathbf{Z}}^{(2)}(W,\rho) = \int_{S^1} \sigma(W,\rho_\omega) \in \mathbf{R}$ because the trace on $N\mathbf{Z} = \mathrm{L}^\infty(S^1)$ is the integration. Hence $\int_{S^1} \sigma_\omega = \tilde{\eta}_{\mathbf{Z}}^{(2)}(S^0K, \mathrm{id}_{\mathbf{Z}})$.

Final steps. (1) In the setting of the theorem $\int_{S^1} \sigma_\omega(\gamma) = \tilde{\eta}_\Gamma^{(2)}(S^0K,\rho_\gamma)$, where $\rho_\gamma\colon \pi_1(S^0K) \to \Gamma := \mathbf{Q}(t)/\mathbf{Q}[t^{\pm 1}] \times \mathbf{Z}$. (2) (Cochran, Orr, Teichner): If K is topologically slice, then there is $L \subset \mathrm{H}_1 F$ such that $\tilde{\eta}_\Gamma^{(2)}(S^0K,\rho_\gamma) = 0$ for all $\gamma \in L$.