

Spring 2011, Math 276: Index Theory, Homework 4

Please submit by whatever date you deem appropriate.

Problem 9: Chern-Weil theory. Recall that a connection on a vector bundle E over a manifold M can be seen as a degree 1 endomorphism ∇ of the complex $\Omega(M) \otimes_{C^\infty(M)} C^\infty(E)$ that satisfies Leibniz identity. The endomorphism ∇^2 is $C^\infty(M)$ -linear and therefore can be seen as an element of $\Omega^2(M) \otimes_{C^\infty(M)} \text{End}(E)$, which we denote by $R(\nabla)$ and call the *curvature* of ∇ . The n th power of $R(\nabla)$ can be defined as the element of $\Omega^{2n}(M, \text{End}(E))$ that corresponds to ∇^{2n} . We equip $\Omega(M, \text{End}(E))$ with the obvious trace and Lie bracket. Finally we observe that we can substitute $R(\nabla)$ into any $f \in \mathbf{k}[[x]]$ in the obvious way, where \mathbf{k} is the field of coefficients (i.e., real or complex numbers).

- (a) Prove that for any $f \in \mathbf{k}[[x]]$ the form $\text{tr}(f(R(\nabla)))$ is closed. Prove that if ∇' is a different connection on E , then the difference $\text{tr}(f(R(\nabla))) - \text{tr}(f(R(\nabla')))$ is exact. Conclude that any $f \in \mathbf{k}[[x]]$ gives a cohomology class that does not depend on the choice of ∇ . Hint: Connections form an affine space and any two connections can be connected by a path.
- (b) Prove that the formal powers series $\exp(ix/2\pi) \in \mathbf{C}[[x]]$ gives the Chern character. Prove that the total Chern class is the exponent of the class given by the formal power series $\log(1 + ix/2\pi) \in \mathbf{C}[[x]]$. Prove that the total Pontryagin class is the exponent of the class given by the formal power series $\log((1 - (x/2\pi)^2)^{1/2}) \in \mathbf{R}[[x]]$. Is there a power series that gives the Euler class?

Problem 10: Chern characters and the topological index of operators on trivial line bundles.

- (a) Define the *odd Chern character* as the composition of maps $K^{\text{odd}}(X) \rightarrow \tilde{K}^{\text{even}}(SX) \rightarrow H^{\text{even}}(SX) \rightarrow H^{\text{odd}}(X)$. Prove that Chern characters combine into a homomorphism of graded rings $K^{\text{even/odd}}(X) \rightarrow H^{\text{even/odd}}(X)$. Extend Chern character to the relative setting and prove that the diagram consisting of the long exact sequences for K-theory and ordinary cohomology connected by Chern character is commutative.
- (b) Use part (a) to prove that the topological index of every elliptic operator from any trivial line bundle to itself vanishes.