

## Spring 2011, Math 276: Index Theory, Homework 4

Please submit by whatever date you deem appropriate.

**Problem 9: Chern-Weil theory.** Recall that a connection on a vector bundle  $E$  over a manifold  $M$  can be seen as a degree 1 endomorphism  $\nabla$  of the complex  $\Omega(M) \otimes_{C^\infty(M)} C^\infty(E)$  that satisfies Leibniz identity. The endomorphism  $\nabla^2$  is  $C^\infty(M)$ -linear and therefore can be seen as an element of  $\Omega^2(M) \otimes_{C^\infty(M)} \text{End}(E)$ , which we denote by  $R(\nabla)$  and call the *curvature* of  $\nabla$ . The  $n$ th power of  $R(\nabla)$  can be defined as the element of  $\Omega^{2n}(M, \text{End}(E))$  that corresponds to  $\nabla^{2n}$ . We equip  $\Omega(M, \text{End}(E))$  with the obvious trace and Lie bracket. Finally we observe that we can substitute  $R(\nabla)$  into any  $f \in \mathbf{k}[[x]]$  in the obvious way, where  $\mathbf{k}$  is the field of coefficients (i.e., real or complex numbers).

- (a) Prove that for any  $f \in \mathbf{k}[[x]]$  the form  $\text{tr}(f(R(\nabla)))$  is closed. Prove that if  $\nabla'$  is a different connection on  $E$ , then the difference  $\text{tr}(f(R(\nabla))) - \text{tr}(f(R(\nabla')))$  is exact. Conclude that any  $f \in \mathbf{k}[[x]]$  gives a cohomology class that does not depend on the choice of  $\nabla$ . Hint: Connections form an affine space and any two connections can be connected by a path.
- (b) Prove that the formal powers series  $\exp(ix/2\pi) \in \mathbf{C}[[x]]$  gives the Chern character. Prove that the total Chern class is the exponent of the class given by the formal power series  $\log(1 + ix/2\pi) \in \mathbf{C}[[x]]$ . Prove that the total Pontryagin class is the exponent of the class given by the formal power series  $\log((1 - (x/2\pi)^2)^{1/2}) \in \mathbf{R}[[x]]$ . Is there a power series that gives the Euler class?

**Problem 10: Chern characters and the topological index of operators on trivial line bundles.**

- (a) Define the *odd Chern character* as the composition of maps  $K^{\text{odd}}(X) \rightarrow \tilde{K}^{\text{even}}(SX) \rightarrow H^{\text{even}}(SX) \rightarrow H^{\text{odd}}(X)$ . Prove that Chern characters combine into a homomorphism of graded rings  $K^{\text{even/odd}}(X) \rightarrow H^{\text{even/odd}}(X)$ . Extend Chern character to the relative setting and prove that the diagram consisting of the long exact sequences for K-theory and ordinary cohomology connected by Chern character is commutative.
- (b) Use part (a) to prove that the topological index of every elliptic operator from any trivial line bundle to itself vanishes.