Michael Hutchings. Floer theory.

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Morse theory.

Suppose X is a finite-dimensional smooth manifold and $f: X \to \mathbf{R}$ is a Morse function. Morse theory relates the topology of X and critical points and gradient flow of f. For example, if f is a Morse function, then the number of critical points is at least $\sum_i \dim H_i(X)$. We can also describe the cup product, characteristic classes, Reidemeister torsion etc.

We are interested in this because this theory has infinite-dimensional generalizations. A classical example is the loop space of X: LX. Choose a Riemannian metric on X. Then we can define a function $f: LX \to \mathbf{R}$: $f(\gamma) = \int_{S^1} |\gamma'(t)|^2 dt$. This function is called energy. Critical points of f are closed geodesics parametrized at constant speed. It follows that for any metric on S^2 there are at least three closed geodesics.

Floer theory has many versions: symplectic Floer theory studies invariants of symplectomorphisms of a symplectic manifold.

Theorem. *(Arnold Conjecture.) If M is a closed symplectic manifold and ϕ is a Hamiltonian symplectomorphism with nondegenerate fixed points, then the number of fixed points is at least $\sum_i \dim H_i(M)$.*

Much weaker Lefschetz theorem says that the number of fixed points is at least $\left|\sum_i (-1)^i \dim H_i(M)\right|$.

The strategy for proving Arnold conjecture is to define Floer homology of a symplectomorphism as the homology of some chain complex generated by fixed points.

The second step is to show invariance under Hamiltonian isotopies. And the third step is to show that Floer homology of id_M is $H_*(M)$.

First we shall do finite-dimensional Morse theory. Then we shall discuss symplectic Floer homology. Then we shall apply this theory to obtain some topological invariants of 3-manifolds. We take a 3-dimensional manifold, construct a functional on some infinite-dimensional space and take Floer homology of it. An interesting example is Seiberg-Witten homology. There is a (mostly equivalent) Hegaard-Floer homology. A third example is embedded contact homology.

Symplectic field theory is related to this stuff. SFT gives invariants of contact manifolds and Legendrian knots. A contact form on a closed oriented 3-manifold is a 1-form λ such that $\lambda \wedge d\lambda \neq 0$. The kernel of λ is called a contact structure. Associated to λ is the Reeb vector field R defined by $d\lambda(R, x) = 0$ and $\lambda(R) = 1$.

Weinstein Conjecture. *For any closed oriented 3-manifold Y and any contact form R has a closed orbit. Proved by Taubes using Floer homology ideas.*

End of overview.

Finite-dimensional Morse theory.

Suppose that X is a closed smooth manifold, $f: X \to \mathbf{R}$ is a smooth function. A critical point of f is a point $p \in X$ such that $df(p) = 0$. Denote the set of all critical points by $\mathrm{Crit}(f)$. If p is a critical point, then we define the Hessian $H(f,p): T_pX \otimes T_pX \to \mathbf{R}$ as follows: if ∇ is a any connection on cotangent bundle, then $H(f,p) = \nabla(df)_p: T_pX \to T_p^*X$. Equivalently, we can regard df as a section $df: X \to T^*X$. Then $H(f,p)$ is the composition $T(df)_p: T_pX \to T_{(p,0)}(T^*X) = T_pX \oplus T_p^*X \to T_p^*X$. Critical point p is called nondegenerate if $H(f,p)$ is nondegenerate. Equivalently, the graph of df is transverse to the zero section at $(p,0)$. The index of p is the number of negative eigenvalues counted with multiplicities. Note that nondegenerate critical points are isolated. Morse lemma says that if p is a nondegenerate critical point of index i, then there is a chart centered at p such that $f = f(p) + H(f,p)$ in this chart.

A function f is called a Morse function if all of its critical points are nondegenerate.

Proposition. *The set of Morse functions is open and dense in the space of smooth functions $X \to \mathbf{R}$ in the smooth topology.*

Example. *Height function on torus.*

Non-example. *Height function on torus lying horizontally.*

A Morse-Bott function is a function whose critical points have nondegenerate Hessian and the set of critical functions is a nice submanifold.

Classical Morse theory. *If f is a Morse function and the values of f at critical points are all distinct. If a is not a critical value, define $X_a = f^{-1}(-\infty, a]$. This is a smooth manifold with boundary $\partial X_a = f^{-1}(a)$.*

Basic lemmas. *If the interval $[a, b]$ contains no critical values, then X_a is diffeomorphic to X_b. If the interval $[a, b]$ contains a single critical value and its index is i, then X_b is obtained from X_a by attaching an i-handle $D^i \times D^{n-i}$ along $S^{i-1} \times D^{n-i}$.*

The simplest application of this theory is Morse inequalities: if c_i is the number of critical points of index i and $b_i = \dim H_i(X)$, then for each i we have $c_i - c_{i-1} + \cdots + (-1)^i c_0 \geq b_i - b_{i-1} + \cdots + (-1)^i b_0$.

Morse complex and Morse homology.

Suppose that X is a closed smooth manifold and $f\colon X \to \mathbf{R}$ is a Morse function. Choose a generic Riemannian metric on X. Hence we have ∇f, the gradient of f. Idea: Define a chain complex which is generated by critical points of f and whose differential counts flow lines of $-\nabla f$ between critical points. This complex leads to Morse homology $H_*^M(X, f, g)$.

Fundamental theorem. $H_*^M(X, f, g) = H_*^s(X)$.

Exercise. *If C is a chain complex, $c_i = \dim C_i$ and $b_i = \dim H_i$, then $c_i - \cdots + (-1)^i c_0 \geq b_i - \cdots + (-1)^i b_0$ for all i.*

Let $\xi = -\nabla f$, let $\phi_s\colon X \to X$ for $s \in \mathbf{R}$ denote the flow generated by ξ. If p is a critical point, define the descending manifold of p as $D(p) = \{x \in X \mid \lim_{s\to-\infty} \phi_s(x) = p\}$ and the ascending manifold of p as $A(p) = \{x \in X \mid \lim_{s\to\infty} \phi_s(x) = p\}$.

Proposition. *Let p be an index i critical point. Then $D(p)$ is an embedded open disc of dimension i. Also $T_pD(p)$ is the negative eigenspace of $H(f, p)\colon T_pX \to T_p^*X \to T_pX$. Likewise, $A(p)$ is an embedded open disc of dimension $\dim(X) - i$ and $T_pA(p)$ is the positive eigenspace.*

Definition. *The pair (f, g) is called Morse-Smale if f is a Morse function and for any two critical points p and q the manifold $D(p)$ is transversal to $A(q)$.*

Proposition. *Fix a Morse function f. If g is generic, then (f, g) is Morse-Smale. Generic means open and dense.*

From now on we assume that (f, g) is Morse-Smale.

If p and q are critical points a flow line of $\xi = -\nabla f$ from p to q is a path $\gamma\colon \mathbf{R} \to X$ such that $\lim_{s\to-\infty} \gamma(s) = p$ and $\lim_{s\to\infty} \gamma(s) = q$ and $\gamma' = \xi$. Flow lines are not unique.

Note: Morse cohomology is obtained by replacing $-\nabla f$ with ∇f.

Note that $\mathbf{R}$ acts on the set of flow lines from p to q by precomposition with translations, Define $m(p, q)$ as the set of all flow lines modulo this action.

The map $\gamma \to \gamma(0)$ identifies the set of all flow lines from p to q with $D(p) \cap A(q)$. The Morse-Smale condition implies that if $p \neq q$ then $m(p, q)$ is a manifold of dimension $\mathrm{ind}(p) - \mathrm{ind}(q) - 1$.

Orientation of $m(p, q)$. Choose an orientation of $D(p)$ for each p. Given $[\gamma] \in m(p, q)$ let $x = \gamma(0) \in D(p) \cap A(q)$. We have $T_xD(p) = T_x(D(p) \cap A(q)) \oplus (T_xX/T_xA(q))$ (follows from transversality). The last space is isomorphic to $T_{[\gamma]}m(p, q) \oplus T_x\Im(\gamma) \oplus T_qD(q)$. Orient $T_{[\gamma]}m(p, q)$ so that this isomorphism is orientation preserving.

Definition of Morse complex $C_*(X, f, g)$. *Denote by C_i the free abelian group generated by index i critical points. A differential $\partial_i\colon C_i \to C_{i-1}$ is defined as follows: $\partial p = \sum_q q \cdot \#m(p, q)$. Note that $m(p, q)$ has dimension 0. The number $\#m(p, q)$ counts points with orientation (positive orientation gives 1, negative gives -1).*

Theorem. *If X is closed and (f, g) is Morse-Smale then for any two critical points p and q the moduli space $m(p, q)$ has a natural compactification to a manifold with corners $\bar{m}(p, q)$ whose codimension k stratum is $\bar{m}(p, q)_k = \cup_{r_i \in \mathrm{Crit}(f)} m(p, r_1) \times m(r_1, r_2) \times \cdots m(r_{k-1}, r_k) \times m(r_k, q)$. Here p, r_1, …, r_k, and q must be all different.*

Corollary. *If $\mathrm{ind}(p) - \mathrm{ind}(q) = 1$, then $m(p, q)$ is compact and has dimension 0, hence it is finite and the Morse differential is well-defined.*

Corollary. *If* $\mathrm{ind}(p) - \mathrm{ind}(q) = 2$*, then* $\bar{m}(p,q)$ *is a compact oriented manifold with boundary*

$$\bigcup_{r:\mathrm{ind}(p)-\mathrm{ind}(r)=1} m(p,r) \times m(r,q).$$

Now we shall prove that $\partial^2 = 0$. Suppose that $\mathrm{ind}(p) - \mathrm{ind}(q) = 2$. Then $\bar{m}(p,q)$ is a compact oriented manifold with boundary $\cup_{r:\mathrm{ind}(p)-\mathrm{ind}(r)=1} m(p,r) \times m(r,q)$. Then $\langle \partial^2 p, q\rangle = \sum_{r:\mathrm{ind}(p)-\mathrm{ind}(r)=1} \langle \partial p, r\rangle \cdot \langle \partial r, q\rangle = \#\partial \bar{m}(p,q) = 0$.

In general, in good case we can get a compactified moduli space of flow lines by adding broken flow lines.

Decomposition into (compactifications of) descending boundary manifolds gives us a CW-decomposition and the Morse boundary map is the cellular boundary map. Note that the closure of descending manifold need not to be a closed ball. The sign of the differential depends on choice of orientations of descending manifolds.

First we shall show that Morse homology does not depend on f and g. If (f', g') is another Morse-Smale pair, then there is canonical isomorphism between two Morse homologies.

We can define Morse complex without making any orientation choices. For every critical point we add two different elements in the set of generators, which correspond to different orientations. Then we say that their sum is zero. Henceforth we shall usually omit orientation choices from the notation.

There are two ways to prove that Morse homology is independent of metric and Morse function. One of them is called continuation maps. Suppose (f_0, g_0) and (f_1, g_1) are two Morse-Smale pairs on X. Let $T = \{(f_t, g_t) \mid t \in [0,1]\}$ be a generic smooth path from (f_0, g_0) to (f_1, g_1). Define the continuation map between two Morse complexes as follows. Fix a nonnegative smooth function $\beta\colon [0,1] \to \mathbf{R}$ such that $\beta^{-1}(0) = \{0, 1\}$, $\beta'(0) > 0$ and $\beta'(1) < 0$.

Define a vector field V on $[0,1] \times X$ by $V = \beta(t)\partial_t + \zeta_t$, where ζ_t is the negative of g_t-gradient of f_t. Note that $\mathrm{Crit}(V) = \{0\} \times \mathrm{Crit}(f_0) \cup \{1\} \times \mathrm{Crit}(f_1)$.

If P and Q are critical points of V, then denote by $m^V(P,Q)$ the moduli space of flow lines of V (modulo reparametrization) from P to Q. Orient descending manifolds of critical points of V as follows: $(0,p)$: use $[0,1]$ direction first, then chosen orientation of $D(p)$ in X, $(1,q)$: use chose orientation of $D(q)$ in X. If (f_t, g_t) is generic, then $m^V(P,Q)$ is an oriented manifold. Note that $m^V((0,p),(0,q)) = (-1)^{\mathrm{ind}(p)+\mathrm{ind}(q)} m_0(p,q)$, $m^V((1,p),(1,q)) = m_1(p,q)$ and $\dim m^V((0,p),(1,q)) = \mathrm{ind}(p) - \mathrm{ind}(q)$. Now for $p \in \mathrm{Crit}_i(f_0)$ define $\Phi(p) = \sum_{q \in \mathrm{Crit}_i(f_1)} \# m^V((0,p),(1,q)) q$.

Lemma. Φ *is a well-defined chain map.*

Proof. *Usual arguments show that if* $p \in \mathrm{Crit}_i(f_0)$ *and* $q \in \mathrm{Crit}_i(f_1)$*, then* $m^V((0,p),(1,q))$ *is compact (finite), hence* Φ *is well-defined. If* $p \in \mathrm{Crit}_i(f_0)$ *and* $q \in \mathrm{Crit}_{i-1}(f_1)$*, then* $m^V((0,p),(1,q))$ *has a compactification to a 1-manifold with boundary* $\partial \overline{m^V((0,p),(1,q))} = \cup_{r \in \mathrm{Crit}_{i-1}(f_0)} m^V((0,p),(0,r)) \times m^V((0,r),(1,q)) \cup \cup_{r \in \mathrm{Crit}_i(f_1)} m^V((0,p),(1,r)) \times m^V((1,r),(1,q))$.

One can now easily see that chain map conditions are satisfied. Another way to look at this: V has a well-defined Morse complex with $C_i = C_{i-1}(X, f_0, g_0) \oplus C_i(X, f_1, g_1)$ and $\partial = \begin{pmatrix} -\partial_0 & 0 \\ \Phi & \partial_1 \end{pmatrix}$, therefore $0 = \partial^2 = \begin{pmatrix} \partial_0^2 & 0 \\ -\Phi\partial_0 + \partial_1\Phi & \partial_1^2 \end{pmatrix}$.

Example. *If* (f_t, g_t) *does not depend on* t*, then* $\Phi = \mathrm{id}$.

Proof. *Since* $V = \beta(t)\partial_t + \zeta$*, any flow line of* V *on* $[0,1] \times X$ *projects to a flow line of* ζ *on* X*. If* $p, q \in \mathrm{Crit}_i(f)$*, then* $m_\zeta(p,q) = \emptyset$ *when* $p \neq q$ *and one-element set consisting of constant map if* $p = q$*. It follows that* $\Phi = \pm\,\mathrm{id}$ *and it is easy to check that* $\Phi = \mathrm{id}$.

Lemma. *Suppose we have two generic paths between a pair of Morse-Smale pairs. Then associated continuation maps* Φ_0 *and* Φ_1 *are chain homotopic.*

Proof. *The space of Morse-Smale pairs is contractible, hence we can choose a generic homotopy between two given generic paths. Hence we have a map of* $[0,1]^2$ *to Morse-Smale pairs. We define a vector field* $\tilde{V}$ *on*

$[0,1]^2 \times X$ *by* $\tilde{V} = \beta(t)\partial_t + \zeta_{s,t}$*. Define* $K\colon C_*^M(X, f_0, g_0) \to C_*^M(X, f_1, g_1)$ *by counting flow lines of* $\tilde{V}$ *with appropriate signs. If* $p \in \mathrm{Crit}_i(f_0)$*, then* $K(p) = \sum_{q \in \mathrm{Crit}_{i+1}(f_1)} \# \cup_{s \in [0,1]} m^{\tilde{V}}(((s,0),p),((s,1),q))q$*. Trivial computations show that* K *is a chain homotopy.*

Corollary. *The map* $\Phi_*\colon H_*^M(X, f_0, g_0) \to H_*^M(X, f_1, g_1)$ *does not depend on generic path* T *from* (f_0, g_0) *to* (f_1, g_1) *because the space of Morse-Smale pairs is contractible.*

Warning/Cool Thing. *For other kinds of Floer theory continuation map depends on homotopy class of path.*

Lemma. Φ *is a functor from fundamental groupoid of piecewise-smooth Morse-Smale pairs to chain complexes. (Composition of paths gets mapped to the composition of maps.)*

Theorem. *Given* (f_0, g_0) *and* (f_1, g_1)*, let* Γ *be a generic path from* (f_0, g_0) *to* (f_1, g_1) *and let* Δ *be the reverse path. Obviously* $\Phi_\Delta \Phi_G amma \sim \mathrm{id}_{C_*^M(X,f_0,g_0)}$*. Likewise for* $\Phi_\Gamma \Phi_\Delta$*. Hence* Φ_Γ *induces isomorphism on Morse homology. Hence we can define* $H_*^M(X) = H_*^M(X, f, g)$*.*

Note that if f_t is Morse for all t, then we can identify critical points of f_0 and f_1. If (f_t, g_t) is Morse-Smale for all t, this identification is an isomorphism of chain complexes. This isomorphism is chain homotopic to continuation map. If we replace β by $\epsilon\beta$ for sufficiently small ϵ, then they become equal. Continuation maps for different β are chain homotopic.

Bifurcations. For a generic family (f_t, g_t) there are times $t_1 < \cdots < t_k$ such that at each t_m one of the following happens: (1) Descending and ascending manifolds of two critical points whose indices differ by 1 do not intersect transversally. Morse complex does not change. (2) Handleslide: if $q, q' \in \mathrm{Crit}_i$ and $q \neq q'$. A flow line from q' to $r \in \mathrm{Crit}_{i-1}$ gets glued to a new flow line from q to r either before or after the bifurcation. Thus, $\partial_+ q = \partial_- q \pm \partial q'$. Also $\partial_+ p = \partial_- p \mp \langle \partial_- p, q \rangle q'$ for $p \in \mathrm{Crit}_{i+1}$. Define an isomorphism $\Phi\colon C_- \to C_+$ as $\Phi(q) = q \pm q'$ and $\Phi(s) = s$ for all other critical points s. (3) Birth or death: two critical points $p \in \mathrm{Crit}_i$ and $q \in \mathrm{Crit}_{i-1}$ cancel. Algebraically: $\partial_+ = \partial_- p' \pm \langle \partial_- p', q \rangle \partial_- p$ for all $p' \neq p$. Define chain homotopy $\Phi\colon C_- \to C_+$ as $\Phi(p) = 0$, $\Phi(q) = \pm \partial_- p \mp \langle \partial_- p, q \rangle q$ and $\Phi(r) = r$ for all $r \neq p, q$.

Exercise. *With corrections as necessary,* Φ *induces an isomorphism on homology.*

Disadvantages of this approach: Analysis gets more complicated; Not clear that the isomorphism on homology is canonical.

Conjecture. *For a given family* (f_t, g_t) *if we replace* β *by* $\epsilon\beta$ *for sufficiently small* ϵ*, then* $\Phi' = \Phi$*.*

Morse homology versus singular homology.

Idea: Define a map $C_*^M \to C_*$ by sending a critical point to its descending manifold.

Proposition. *For each critical point* p *the descending manifold* $D(p)$ *has a compactification to a manifold* $\bar{D}(p)$ *with corners such that codimension* k *stratum is a union of* $m(p, q_1) \times m(q_1, q_2) \times \cdots \times m(q_{k-1}, q_k) \times D(q_k)$ *over all distinct* $q_i \in \mathrm{Crit}(f)$ *distinct from* p*. The endpoint map to* $D(q_k)$ *is continuous.*

Claim. *In general* $\bar{D}(p)$ *is homeomorphic to a closed ball with dimension* $\mathrm{ind}(p)$*. Hence the compactified manifolds* $\bar{D}(p)$ *together with the maps* $e\colon \bar{D}(p) \to X$ *give* X *the structure of CW-complex. Now the cellular complex coincides with Morse complex, hence Morse homology is isomorphic to cellular (therefore, singular) homology.*

Other approach. *Define chain map* $C_*^M(X, f, g) \to C_*(X)$*. Roughly* $p \to \bar{D}(p)$*.*

Another approach. *If things are sufficiently smooth, we can use currents.*

We shall not take these approaches.

One more approach. *Pseudocycles. We shall not take this approach, but it is probably the best one.*

Proof of the Proposition. *We have*

$$\partial \bar{D}(p) = \bigcup_{q \in \mathrm{Crit}(f) \setminus \{p\}} (-1)^{\mathrm{ind}(p) + \mathrm{ind}(q) + 1} m(p, q) \times D(q)$$

and

$$\partial\bar{m}(p,q) = \bigcup_{p\neq q\neq r\neq p} (-1)^{\mathrm{ind}(p)+\mathrm{ind}(r)+1} m(p,r)\times m(r,q).$$

Lemma. *For each $p\neq q$ there is a cubical singular chain $m_{p,q}\in C^s_{|p|-|q|-1}(\bar{m}(p,q))$ such that $m_{p,q}$ represents the relative fundamental chain in $H_{|p|-|q|-1}(\bar{m}(p,q),\partial\bar{m}(p,q))$ and at the chain level $\partial m_{p,q} = \sum_{p\neq q\neq r\neq p}(-1)^{|p|+|r|+1} m_{p,r}\times m_{r,q}$. Here $\times$ is the cross product on cubical chains.*

Proof. *By induction on $|p|-|q|$. Given p and q the right hand side represents the fundamental class of $\partial\bar{m}(p,q)$. It is easy to see that ∂ of right hand side equals zero. So right hand side is a cycle, hence it clearly represents the fundamental class of ∂.*

Lemma. *For each critical point p we can choose $d_p\in C^s_{|p|}(\bar{D}(p))$ such that d_p represents the relative fundamental class in $H_*(\bar{D}(p),\partial\bar{D}(p))$ and $\partial d_p = \sum_{p\neq q}(-1)^{|p|+|q|+1} m_{p,q}\times d_q$.*

Proof. *User previous lemma and induction.*

Rest of the proof of the Proposition. *Define a map $D\colon C^M_*(X,f,g)\to C^s_*(X)$ by $D(p)=e_\# d_p$. Now D is a chain map: $\partial^s D = D\partial^M$. Proof: Let $p\in \mathrm{Crit}_i(f)$. We have $\partial^s d_p = \sum_{q\in\mathrm{Crit}(f)\setminus\{p\}}(-1)^{i+|q|+1} e_\#(m_{p,q}\times d_q)$. If $\mathrm{ind}(q)\geq i$, then $m(p,q)=\emptyset$, if $\mathrm{ind}(q)\leq i-2$, then the corresponding cube is degenerate. Hence we have $\sum_{q\in\mathrm{Crit}_{i-1}(f)} \#m(p,q)e_\#(dq) = D(\partial^M p)$.*

Hence we have a map $H^M_*(X,f,g)\to H^s_*(X)$. It is easy to show that this map is independent on the choice of f and g.

Now we show that this map is an isomorphism. Fix (f,g) as before. A singular cube $\sigma\colon[-1,1]^i\to X$ will be called admissible if it is smooth and transverse to the ascending manifolds of the critical points. The subcomplex generated by admissible cubes is a deformation retract, hence induces canonical isomorphism on homology. Given an admissible chain σ and $p\in\mathrm{Crit}(f)$, define $m(\sigma,p)=\{(t,\gamma)\mid t\in\operatorname{dom}\sigma\wedge\gamma\colon[0,\infty)\to X\wedge\gamma(0)=\sigma(t)\wedge\gamma'(s)=\xi(\gamma(s))\wedge\lim_{s\to\infty}\gamma(s)=q\}$. Now $T_t[-1,1]^i$ is isomorphic to $T_\gamma m(\sigma,p)\oplus T_pD(p)$. We have $\dim m(\sigma,p)=i-\mathrm{ind}(p)$.

Define $A\colon C^a_*(X)\to C^M_*(X,f,g)$ as $A(\sigma)=\sum_{p\in\mathrm{Crit}_i(f)}\#m(\sigma,p)p$. We can compactify $m(\sigma,p)$. The boundary consists of the boundary of the cube times the broken lines. We have $\partial\bar{m}(\sigma,p) = m(\partial\sigma,p)\cup\bigcup_{q\in\mathrm{Crit}(f)\setminus\{p\}}(-1)^{i+\mathrm{ind}(q)}m(\sigma,q)\times m(q,p)$. Lemma: A is a chain map: $A\partial^s=\partial^M A$. Proof: If $p\in\mathrm{Crit}_{i-1}(f)$, then $\bar{m}(\sigma,p)$ is a compact 1-manifold with boundary. Obvious computation completes the proof of the lemma. Exercise: The induced map $A_*\colon H^s_*(X)\to H^M_*(X)$ is defined. Lemma: If the metric g is nice near the critical points (so that moduli spaces are smooth manifolds with corners), then $AD=\mathrm{id}_{C^M_*}$ and DA is chain homotopic to $\mathrm{id}_{C^a_*(X)}$. Proof: $AD=\mathrm{id}$ is clear because if $p\in\mathrm{Crit}_i$ and $q\in\mathrm{Crit}_i$, then $D(p)\cap A(q)$ is empty if $p\neq q$ and $\{p\}$ if $p=q$. Chain homotopy $K\colon C^a_*(X)\to C^a_{*+1}(X)$ is defined by flowing a cube down. More precisely, define the forward orbit of σ to be the space $F(\sigma)=[-1,1]^i\times[0,\infty)$ with the map $e\colon F(\sigma)\to X$: $e(t,s)=\phi_s(\sigma(t))$ where ϕ_s is the gradient flow. Compactify to $e\colon\bar{F}(\sigma)\to X$, choose appropriate singular chains on $\bar{F}(\sigma)$.

More algebraic topology (on closed smooth manifolds) using Morse theory.

Poincaré duality. *If X^n is oriented, then $H^M_i(X)=H^{n-i}_M(X)$.*

Definition of Morse cohomology. *Replace chains by additive functions on chains. Replace the differential by its dual.*

Proof of Poincaré duality. *Replace f with $-f$. Note that orientations for descending manifolds of f together with orientation of X induce orientations of descending manifolds of $-f$ (which are the same as ascending manifolds of f) because $T_pX=T_pD(p,f)\oplus T_pA(p,f)$ and $A(p,f)=D(p,-f)$.*

Definition. *A local coefficient system on a topological space X is a functor from fundamental groupoid of X to the category of abelian groups. On a locally path connected X this is the same as a flat G-bundle.*

Now we define a chain complex C generated by pairs (σ,g) where σ is a cube (or a simplex) in X and $g\in G_{\sigma(t_0)}$, where t_0 is the center of the cube. The differential is defined in an obvious way (use obvious path from the center of the cube to the center of its face).

If X is path connected and simply connected, then any local coefficient system is constant. If X^n is a manifold, define a local coefficient system O_x on X by $O_x = H_n(X, X \setminus \{x\})$ (an orientation of X at x is a generator of O_x).

Theorem. *If X is a closed manifold and G is any local coefficient system on X, then*

$$H_i(X, G) = H^{n-i}(X, G \otimes O).$$

Suppose X is a closed smooth manifold and (f, g) is a Morse-Smale pair and L is a local coefficient system on X. Define $C_i^M(X, f, g, L) = \oplus_{p \in \mathrm{Crit}_i(f)} L_p$ and $\partial(p, l) = \sum_{q \in \mathrm{Crit}_{i-1}(f)} \sum_{\gamma \in m(p,q)} \epsilon(\gamma)(q, \phi_\gamma(l))$.

Proof. *We have $\partial^2(p, l) = \sum_{r \in \mathrm{Crit}_{i-2}}(r, \sum_{q \in \mathrm{Crit}_{i-2}} \sum_{\substack{\gamma_1 \in m(p,q) \\ \gamma_2 \in m(q,r)}} \epsilon(\gamma_1)\epsilon(\gamma_2)\phi_{\gamma_2}(\phi_{\gamma_1}(l))) = 0$, because two corresponding paths are homotopic.*

Poincaré duality. *$C_i^M(X, f, g, L) = C_M^{n-i}(X, -f, g, L \otimes O)$ is an isomorphism of chain complexes.*

Cup product. *Let X be a closed smooth manifold. Choose a generic triple of Morse-Smale pairs (f_1, g_1), (f_2, g_2), (f_3, g_3). Define $*\colon C_M^i(X, f_1, g_1) \otimes C_M^j(X, f_2, g_2) \to C_M^{i+j}(X, f_3, g_3)$ as follows: $\langle p * q, r\rangle$ is the number of flow lines from r to s, from s to p and from s to q, where s is an arbitrary point of X. Codimension counting and transversality and compactness shows that there is only a finite number of such points. Why is $*$ a chain map? Want $\delta(p * q) = (\delta p) * q + p * (\delta q)$. Trivial configuration counting show that this is true.*

We can also do this for general coefficient systems: $H_M^i(X, G_1) \otimes H_M^j(X, G_2) \to H_M^{i+j}(X, G_1 \otimes G_2)$.

Theorem. *Under the canonical isomorphism $H_M^* = H_s^*$ the product $*$ agrees with the cup product.*

Proof. *Idea: $\langle p * q, r\rangle = \#(A(p) \cap A(q) \cap D(r))$.*

Spectral sequences in Morse theorem.

Let $F \to E \to B$ be a smooth fiber bundle where F, E, and B are closed smooth manifolds. Define $E_{i,j}^2 = H_i(B, H_j(E_b))$. Here $H_j(E_b)$ is a local coefficient system. For $k \geq 2$ we have maps $\partial_k\colon E_{i,j}^k \to E_{i-k,j+k-1}^k$ such that $\partial_k^2 = 0$ and $E^{k+1} = H(E^k)$. We define $E_{i,j}^\infty = E_{i,j}^k$ for $k > i$ and $k > j$.

$H_*(E)$ has a filtration: $\alpha \in F_i H_*(E)$ iff α can be represented by a sum of singular cubes such that their projections to B depend on only i of the coordinates on $[-1, 1]^*$.

We have associated graded groups $G_i H_*(E) = F_i H_*(E)/F_{i-1} H_*(E)$. Note that $G_m H_n(E) = E_{m,n-m}^\infty$.

Example: $\mathrm{SU}(2) \to \mathrm{SU}(3)$ is a fibration with base S^5 and fiber S^3. We have $E^2 = E^\infty$. Hence $H_*(\mathrm{SU}(3)) = H_*(S^5 \times S^3)$.

Construction of Leray-Serre spectral sequence for Morse theory.

Suppose B, E, and F are closed manifolds. Choose a generic family of pairs $\{(f_b, g_b) \mid b \in B\}$ where $f_b\colon E_b \to \mathbf{R}$ is a smooth function on b, and g_b is a metric on E_b. Choose a Morse-Smale pair (f^B, g^B) on B such that (f_b, g_b) is Morse-Smale whenever $b \in \mathrm{Crit}(f^B)$. Choose a connection ∇ on E. For $b \in B$ denote by ξ_b the negative gradient of f_b with respect to g_b. Denote by ξ^B the negative gradient of f^B with respect to g^B. Define a vector field V on E by $V(b, e) = \xi_b + H(\xi^B)$, where $H(\xi^B)$ is the horizontal lift with respect to ∇. Now $\mathrm{Crit}(V) = \cup_{b \in \mathrm{Crit}(f^B)} \mathrm{Crit}(f_b)$. Define a chain complex $C_* = \oplus_{i+j=*} \oplus_{b \in \mathrm{Crit}_i(f^B)} \mathrm{Crit}_j(f_b)$. Define $\partial\colon C_* \to C_{*-1}$ by counting flow lines of V in the usual manner. Usual arguments show that ∂ is well defined, $\partial^2 = 0$, and $H_*(C_*, \partial) = H_*(E)$.

In general, this chain complex has a filtration defined by $F_i C_* = \oplus_{\substack{b \in \mathrm{Crit}(f^B) \\ \mathrm{ind}(b) \leq i}} \mathbf{Z}\{\mathrm{Crit}_{*-\mathrm{ind}(b)}\}(f_b)$.

This filtered chain complex gives us a spectral sequence. We have $E_{i,j}^0 = G_i C_{i+j} = \oplus_{b \in \mathrm{Crit}_i(f^B)} \mathrm{Crit}_j(f_b)$. The differential $\partial_0\colon E_{i,j}^0 \to E_{i,j-1}^0$ is induced by ∂. We have $E_{i,j}^1 = \oplus_{b \in \mathrm{Crit}_i(f^B)} H_j^M(E_b, f_b, g_b)$.

It is easy to see that $E_{i,j}^2 = H_i^M(B, f^B, g^B, \{H_j^M(E_b, f_b, g_b)\})$.

Theorem. *For $k \geq 2$ Morse theory spectral sequence agrees with Leray-Serre spectral sequence.*

Idea of proof. *Leray-Serre spectral sequence comes from filtration on $C_*^s(E)$ defined by $F_i C_*(E)$ spanned by cubes whose projection depends on at most i coordinates.*

A chain map that preserves filtration defines a map of spectral sequences. If it is an isomorphism on E^2 term, then it is an isomorphism on all higher terms.

The idea is to define $D\colon C_*^M \to C_*(E)$ by sending a critical point of V to its descending manifold, compactified and made into a singular chain, taking care to use cubes that project to cubes of the correct dimension in B. Previous discussion shows that this is an isomorphism on E^2 terms.

Definition. *A 1-form ω on X is Morse if it is closed and is locally a differential of Morse function.*

Let p and q be critical points of ω. Let γ_n be a sequence of flow lines from p to q. Let $r_0, \ldots, r_{k+1}$ be critical points such that $r_0 = p$ and $r_{k+1} = q$. Let η_i be a flow line from r_i to r_{i+1}. Say that γ_n converges to $(\eta_0, \ldots, \eta_k)$ if there are real numbers $s_{n,0} < \cdots < s_{n,k}$ such that γ_n restricted to corresponding interval converges to η_i in C^∞ on compact sets as $n \to \infty$. Also we require that homology class of $\gamma_n - \sum_i \eta_i$ is zero for sufficiently large n. Define the energy of a flow line γ by $E(\gamma) = \int |\gamma'^2| = \int \gamma^*\omega$.

Proposition. *If γ_n is a sequence of flow lines with $E(\gamma_n) < C$, then there is a subsequence converging to a broken flow line.*

Lemma. *Let $\gamma\colon \mathbf{R} \to X$ such that $\gamma'(s)$ is dual to $-\omega(\gamma(s))$ (a) $E(\gamma) \geq 0$ and equality holds iff γ is a constant map to a critical point. (b) If $E(\gamma) < \infty$, then there are critical points p and q such that γ is a flow line from p to q: $\lim_{s\to-\infty} \gamma(s) = p$ and $\lim_{s\to\infty} \gamma(s) = q$. (c) There is $\delta > 0$ such that if γ is nonconstant then $E(\gamma) > \delta$.*

Proof of Proposition. *Suppose γ_n is a sequence of flow lines from p to q with energy less than C. Without loss of generality we can assume that $\lim_n E(\gamma_n) = D$. Choose $\epsilon > 0$ such that ϵ-balls around critical points are disjoint. Define $s_{n,0} = \inf\{s \in \mathbf{R} \mid d(\gamma_n(s), \omega^{-1}(0)) > \epsilon\}$. Pass to a subsequence so that γ_n restricted to $[s_{n,0}, s_{n+1,0}]$ converges to η_9 in C^∞ on compact sets. Must have $E(\eta_0) \leq C_0$. If $E(\eta_0) = C_0$, then $\gamma_n \to (\eta_0)$ in the defined sense. If $E(\eta_0) < C_0$, do the following. Let $t_0 = \sup\{t \mid d(\eta_0(t), \omega^{-1}(0)) > \epsilon\}$. Define $s_{n,1} = \inf\{s \mid s > s_{n,0} + t_0 + 1 \wedge d(\gamma_n(s)\omega^{-1}(0)) > \epsilon\}$. Pass to a subsequence so that γ_n restricted to $[s_{n,1}, s_{n+1,1}]$ converges to η_1. And so on. Note that γ_n is homotopic rel endpoints to $\eta_0 \ldots \eta_k$.*

Morse theory for $f\colon X \to S^1$. *(Simplest version.) Assume f is Morse, choose generic metric. Let $\Sigma \subset X$ be a level set of f not containing any critical points. Define C_* over $\Lambda = \{\sum_{n\geq n_0} a_n t^n\}$ for $a_n \in \mathbf{Z}$. Let C_i be a free abelian group generated by index i critical points. If $p \in \mathrm{Crit}_i(f)$, then $\partial p = \sum_{q\in \mathrm{Crit}_{i-1}(f)} \sum_{n\geq 0} t^n z_n$, where z_n is the number of flow lines from p to q that cross Σ n times. Usual argument shows that $\partial^2 = 0$, homology as a Λ-module is a topological invariant of X and the homotopy class of $f\colon X \to S^1$ in $H^1(X, \mathbf{Z})$.*

Let X be a closed smooth manifold, ω be a Morse 1-form ($d\omega = 0$ and locally ω is a differential of a Morse function), g be a generic metric on X. Choose $K \subset H_1(X)$ such that $\int \omega = 0$ for all $\alpha \in K$. (Usual choices: $K = \{0\}$ or $K = \ker(\int)$.) Let $H = H_1(X)/K$.

Idea: Classify flow lines modulo the following equivalence relation: If γ and γ' are elements of $m(p,q)$, then $\gamma \sim \gamma'$ iff $[\gamma - \gamma'] \in K$. Note: if $\gamma \sim \gamma'$, then $E(\gamma) = E(\gamma')$. So if $\mathrm{ind}(p) - \mathrm{ind}(q) = 1$, then each equivalence class contains only finitely many flow lines.

Novikov ring: Let H be an abelian group and $N\colon H \to \mathbf{R}$ be a homomorphism. Define $\mathrm{Nov}(H, N)$ as formal sums of elements of H with integral coefficients such that for any neighborhood of 0 there are only finitely many elements $h \in H$ with nonzero coefficient and $N(h)$ in the neighborhood. Multiplication is defined as in group ring. There is an obvious injection $\mathbf{Z}[H] \to \mathrm{Nov}(H, N)$. If $N = 0$, then is an iso.

Definition of Novikov complex. *Fix a base point $x_0 \in X$. An anchored critical point is a pair (p, η) where $p \in \omega^{-1}(0)$ and η is a path from p to x_0, modulo $\sim$. Define the energy $E(p, \eta) = \int_\eta -\omega$. (Alternatively: Let $\pi\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be the covering space determined by K. The pair (p, η) corresponds to a zero of $\pi^*\omega$ on $\tilde{X}$. We have $\pi^*\omega = d\tilde{f}$, where $\tilde{f}(\tilde{x}_0) = 0$ and $E(p, \eta) = \tilde{f}(\tilde{p})$.*

Let C_i be a submodule of Novikov ring generated by all points $\tilde{p}$ with index i such that only finite number of terms have nonzero coefficients and energy lying outside given neighborhood of 0.

Exercise. *C_i is a module over $\Lambda = \mathrm{Nov}(H, -\omega)$ where H acts by $h(p, \eta) = (p, \eta + h)$. Moreover this is a free module with one generator for each index i critical point of ω in X.*

Define $\partial\colon C_i \to C_{i-1}$ by $\partial(p, \eta) = \sum_{q\in \mathrm{Crit}_{i-1}(\omega)} \sum_{\gamma \in m(p,q)} \epsilon(\gamma)(q, \eta - \gamma)$.

Exercise. *∂ is well-defined.*

Theorem. *$H_*^N(X, x_0, \omega, g, K)$ depends only on X, $[\omega]$, and K. (The isomorphisms are canonical only up to multiplication by elements of H.)*

Usually we cannot vary $[\omega]$ except by positive scaling.

Interpretation in classical topology: Chose a cell decomposition of X, lift it to a cell decomposition of $\tilde{X}$. Then $C_*^c(\tilde{X})$ is a module over $\mathbf{Z}[H]$.

Theorem. *(Novikov.)* $H_*^N = H_*(C_*^c(\tilde{X}) \otimes_{\mathbf{Z}[H]} \Lambda) = H_*(C_*^{\infty/2}(\tilde{X}))$. *Here* $C_*^{\infty/2}(\tilde{X})$ *is the half-infinite singular chain complex of* $\tilde{X}$ *(formal sums of simplices such that for any real* c *only finitely many simplices intersect* $\{\tilde{x} \in \tilde{X} \mid \tilde{f}(\tilde{x}) > c\}$.

Often H_*^N is torsion. Suppose H has no torsion so that the quotient ring of Λ is a field. Suppose $H_*^N \otimes Q(\Lambda) = 0$.

Theorem. *The Reidemeister torsion of* $C_*^c(\tilde{X}) \otimes_{\mathbf{Z}[H]} Q(\Lambda)$ *is equal to the Reidemeister torsion of* $C_*^N \otimes_\Lambda Q(\Lambda))$ *times a certain count of closed orbits of the gradient flow.*

Pseudoholomorphic curves in symplectic manifolds.

Note: we shall call them holomorphic curves.

Definition. *A symplectic manifold is a smooth manifold with a closed nondegenerate 2-form (nondegenerate means that it defines an isomorphism* $TM \to T^*M$, *equivalently* $\omega^n \neq 0$ *everywhere, where* $2n$ *is the dimension of the manifold, which must be even).*

Definition. *A morphism of symplectic manifolds is a smooth map such that 2-form of the second manifold pulls back to the 2-form of the first manifold. An isomorphism of symplectic manifolds is called a symplectomorphism.*

Darboux's Theorem. *Every symplectic manifold has an open cover such that restriction to each element is symplectomorphic to* R^{2n} *with standard symplectic form.*

Note. *Every symplectomorphism preserves volume form* ω^n, *hence it preserves volume.*

Question. *How different is a symplectomorphism from a volume-preserving diffeomorphism?*

Answer. *For* $n = 1$ *these notions are the same.*

Gromov nonsqueezing theorem. *Suppose there is a symplectic embedding* $\phi: B^{2n}(r) \to B^2(R) \times \mathbf{R}^{2n-2}$. *Then* $r \leq R$.

Recognition of $\mathbf{R}^4$. *Suppose* M *is a symplectic manifold such that* $\tilde{H}_*(M) = 0$ *and* M *is asymptotically symplectomorphic to* $\mathbf{R}^4$, *more precisely, there are compact sets* $K_1 \subset M$ *and* $K_2 \subset \mathbf{R}^4$ *and a symplectomorphism* $\phi: M \setminus K_1 \to \mathbf{R}^4 \setminus K_2$. *Then* M *is symplectomorphic to* $\mathbf{R}^4$, *by a symplectomorphism agreeing with* ϕ *outside a compact set.*

Symplectomorphisms of $S^2 \times S^2$. *Let* ω_1 *and* ω_2 *be two symplectic forms on* S^2 *with* $\int \omega_1 = \int \omega_2$. *Then* $\mathrm{Symp}(S^2 \times S^2, \omega_1 \oplus \omega_2)$ *has two connected components. The identity component is homotopy equivalent to* $\mathrm{SO}(3) \times \mathrm{SO}(3)$.

Definition. *An* almost complex structure *on* M *is a bundle map* $J: TM \to TM$ *such that* $J^2 = -1$. *Example:* M *is a complex manifold, and* J *is multiplication by* i. *In such a case* J *is called* integrable.

Definition. *An almost complex structure is* ω*-tame if* $\omega(v, Jv) > 0$ *whenever* $v \neq 0$. *It is* ω*-compatible if it is* ω*-tame and* $\omega(Jv, Jw) = \omega(v, w)$.

Note. *If* J *is* ω*-tame, then it defines a Riemannian metric* g *on* M *by* $g(v, w) = (\omega(v, Jw) + \omega(w, Jv))/2$. *If* J *is* ω*-compatible, then* $g(v, w) = \omega(v, Jw)$.

Idea. *Lots of complex geometry generalizes to symplectic manifolds with tame almost complex structure.*

Proposition. *There is no obstruction to finding or extending tame or compatible almost complex structures. More precisely, let* Ω *be the space of symplectic forms on* $\mathbf{R}^{2n}$, J *be the space of complex structures on* $\mathbf{R}^{2n}$, C *be the set of compatible pairs, and* T *be the set of tame pairs. Then* C *and* T *are fibrations over* Ω *with contractible fibers.*

Proof. *We only show that these fibrations have contractible, more precisely, convex fibers. This is obvious.*

Exercise. *Let* ω *be a linear symplectic form and* g *a linear metric on* $\mathbf{R}^4$. *Then* g *comes from* ω *and* ω*-compatible* J *iff* ω *is self-dual with respect to* g *and* $|\omega| = \sqrt{2}$.

Definition. *A Riemann surface is a closed surface σ is a closed surface Σ with an almost complex structure J (necessarily integrable). $m_{g,n}$ is the space of genus g Riemann surfaces with n marked points.*

Fact. *$m_{g,n}$ is a smooth orbifold with dimension $3g-3+n$ for $g \geq 2$ or $g=1$ and $n \geq 1$ or $g=0$ and $n \geq 3$. For $g=1$ and $n=0$ the dimension is 1 and for $g=0$ and $n \leq 2$.*

Definition. *Let M be an almost complex manifold. A (pseudo)holomorphic curve in M is a smooth map $u: (\Sigma, j) \to (M, J)$ such that $J \circ du = du \circ j$. (This means that du respects almost complex structure.) Two curves are equivalent if there is a biholomorphic map from (Σ, j) to (Σ', j') such that the obvious diagram commutes.*

Note. *If u is an embedding then the equivalence class of u is determined by its image in M. Hence embedded holomorphic curves are the same as embedded surfaces Σ such that J maps $T\Sigma$ to itself.*

Proposition. *If $u: \Sigma \to M$ is a holomorphic curve, then its area is equal to $\int u^*\omega = \langle [\omega], u_*[\Sigma] \rangle$.*

Proof. *To prove the first part it is enough to show that $\mathrm{Area}(v, Jv) = \omega(v, Jv)$ for all $v \in TM$. This follows from the fact that $\mathrm{Area}(v, Jv) = \omega(v, Jv)$. To prove the second part it is enough to show that for all v and w in T_pM we have $\mathrm{Area}(v, w) \geq \omega(v, w)$ with equality iff w is in the span of v and Jv. Since J is ω-compatible, we can find a basis e_i and f_i for T_pM such that at p we have $\omega = \sum_i e_i^* \wedge f_i^*$ and $J(e_i) = f_i$. Wlog $v = e_1$. Write $w = \sum_i a_i e_i + b_i f_i$. Now $\mathrm{Area}(v, w) = \left(\sum_i (a_i^2 + b_i^2) - a_1^2\right)^{1/2} \geq b_1$. We have equality iff $a_i = b_i = 0$ for $i \neq 1$.*

Trivial example of holomorphic curves.

1. The zero set of a homogeneous polynomial over $\mathbf{C}$ in 3 variables is a holomorphic curve in $\mathbf{C}P^3$.
2. If $u: (\Sigma, j) \to (M, J)$ is J-holomorphic (we assume that J is ω-tame) and $u_*[\Sigma] = 0 \in H_2(M)$, then u is constant. (Proof: $\mathrm{Area}(u) = 0$.)
3. Suppose $M = \Sigma \times V$, where Σ is a Riemann surface and V is a symplectic manifold. Take ω and J be product symplectic and almost-complex structure. For any $p \in V$ the map $\Sigma \to \Sigma \times V$ sending $x \to (x, p)$ is holomorphic. Claim: These are they only holomorphic curves in the homology class $[\Sigma] \times [\cdot]$. Proof: Let $u': (\Sigma', j') \to (\Sigma \times V, J)$ be another holomorphic curve in this homology class. The projection $\Sigma' \to \Sigma \times V \to V$ is J_V-holomorphic. Its homology class is 0 in $H_2(V)$. So it is constant and the image of u is contained in $\Sigma \times \{p\}$ for some $p \in V$. The projection $\Sigma' \to \Sigma \times V \to \Sigma$ is also holomorphic. A non-constant holomorphic map $\Sigma' \to \Sigma$ is a branched cover of degree at least 1. Since homology class is $[\Sigma]$ the degree of this map is 1 and it is an isomorphism.

Gromov Non-Squeezing Theorem. *If $\phi: B^{2n}(r) \to B^2(R) \times \mathbf{R}^{2n-2}$ is symplectomorphism, then $r \leq R$.*

Proof. *(Modulo some stuff.) Choose $c > 0$ such that $\Im(\phi) \subset B^2(R) \times [-c, c]^{2n-2}$. We get a map $\phi: B^{2n}(r) \to S^2(R+\epsilon) \times \mathbf{T}^{2n-2}$. Choose an ω-tame almost complex structure on the last manifold in such a way that on $\Im(\phi)$ it agrees with ϕ-pushforward of standard complex structure on $\mathbf{R}^{2n}$. The key is to show the existence of J-holomorphic sphere $u: S^2 \to S^2(R+\epsilon) \times \mathbf{T}^{2n-2}$ such that $\Im(u) \ni \phi(0)$ and $u_*[S^2] = [S^2(R+\epsilon)] \times [\cdot]$. Let Σ denote $\phi^{-1}(\Im(u)) \subset B^{2n}(r)$. We have $0 \in \Sigma$. If $t < r$ and if we replace $\Sigma \cap B^{2n}(t)$ with another surface with the same boundary, area does not decrease. Monotonicity lemma for minimal surfaces: Above conditions on Σ imply $\mathrm{Area}(\epsilon) \geq \pi r^2$. Proof: Define $\Sigma_t = \Sigma \cap B^{2n}(t)$, $\theta(t) = \mathrm{Area}(\Sigma_t)/(\pi t^2)$, and $l(t)$ as the length of $\Sigma \cap S^{2n}(t)$. We have $\lim_{t \to 0} \theta(t) = 1$ and $\theta'(t) \geq 0$. Also $\mathrm{Area}'(\Sigma_t) \geq l(t)$ and $\mathrm{Area}(\Sigma_t) \leq tl(t)/2$. We have $\theta(r) \geq \theta(0) = 1$. (Monotonicity.) Now $\pi r^2 \leq \mathrm{Area}(\sigma) \leq \mathrm{Area}(u) = \pi(R+\epsilon)^2$. This is true for any $\epsilon > 0$, hence $r \leq R$.*

How do we show that the desired holomorphic curve exists? (1) Define some kind of count of holomorphic spheres containing a given point. (2) Show that this count is a topological invariant, (3) In the case of interest, this invariant equals 1.

Let (X, ω) be a symplectic manifold, J an ω-tame almost complex structure, $m_{g,n}(X, A)$ be the space of genus g J-holomorphic curves in X with n marked points in the homology class A. There are evaluation maps $\mathrm{ev}_i: m_{g,n}(X, A) \to X$ that send a J-holomorphic curve to its corresponding marked point.

Last time: deduced Gromov non-squeezing theorem from the following statement: Let (V, ω) be a compact symplectic manifold with $\pi_2(V) = 0$. Let $X = (S^2 \times V, \omega_0 \oplus \omega)$. Let $A = [S^2] \times [\cdot] \in H_2(X)$.

Let J be any ω-tame almost complex structure on X. Then $\mathrm{ev}_1\colon m_{0,1}(X,A) \to X$ is surjective. Outline of the proof: Suppose V has dimension $2n-2$. We prove that if J is generic, then $m_{0,1}(X,A)$ is a compact oriented manifold of dimension $2n$. If $\{J_t \mid t \in [0,1]\}$ is a generic homotopy then $\bigcup_t \{t\} \times m^{J_t}_{0,1}(X,A)$ is a cobordism from $m^{J_0}_{0,1}(X,A)$ to $m^{J_1}_{0,1}(X,A)$. If J is product almost complex structure, then $m^{J}_{0,1}(X,A) = X$. This implies that for generic J the degree of ev_1 is 1.

Genericity and transversality.

Definition. *A J-holomorphic curve is multiply covered if it can be factored through a branched cover of surface of degree greater than 1. Otherwise it is called simple.*

Let $m^*_{g,n}(X,A)$ be the space of simple curves in $m_{g,n}(X,A)$. Terminology: generic means Baire (countable intersection of open dense sets).

Theorem. *For generic J the manifold $m^*_{g,k}(X,A)$ is oriented (not necessarily compact) and we have* $\dim m^*_{g,k}(X,A) = (n-3)(2-2g) + 2\langle c_1(TX), A\rangle + 2k$.

Sard-Smale theorem. *Let $f\colon X \to Y$ be a smooth map of separable Banach manifolds whose differential at each point is Fredholm and has index l. Assume the map is C^k, where $k \geq 1$ and $k \geq l+1$. Then a generic $y \in Y$ is a regular value of f so that $f^{-1}(y)$ is a manifold of dimension l.*

Theorem. *Let Z be a smooth finite-dimensional manifold. Let $k \geq 2$. Then a generic C^k function $f\colon Z \to \mathbf{R}$ is Morse.*

Proof. *Let $Y = C^k(Z,\mathbf{R})$. This is a smooth Banach manifold (a Banach space). Let E be a vector bundle over Y such that $E_{f,z} = T^*_z Z$. Define a section $\psi\colon Y \times Z \to E$ by $\psi(f,z) = df_z$. Then $\psi^{-1}(0) = \bigcup_f \{f\} \times \mathrm{Crit}(f)$. Claim: For $(f,z) \in \psi^{-1}(0)$ we have $d\psi\colon T_{f,z}Y \times Z \to T_{f,z,0}E$ is surjective, therefore $\psi^{-1}(0)$ is a Banach manifold. Actually we show that $T_{f,z}Y \times Z \to T_{f,z,0}E \to E_{f,z} = T^*_z Z$ is surjective. If $f_1 \in C^k(Z,\mathbf{R})$ and $v \in T_z Z$, then $\nabla\psi(f_1,v) = df_1(z) + \nabla_v(df)$ can be anything. Claim: The projection $\psi^{-1}(0) \to Y$ has $d\pi_{f,z}$ Fredholm so that Sard-Smale applies. We have $\ker(d\pi_{f,z}) = \ker(\nabla\psi\colon T_z Z \to E_{f,z}$. There is an automorphism $\nabla\psi\colon \mathrm{coker}(d\pi\colon T_{f,z}\psi^{-1}(0) \to T_f Y) \to \mathrm{coker}(\nabla\psi\colon T_z Z \to E_{f,z})$. It follows that a generic $f \in C^k(Z,\mathbf{R})$ is a regular value of $\pi\colon \psi^{-1}(0) \to C^k(Z,\mathbf{R})$. If f is a regular value of π then for every $z \in \mathrm{Crit}(f)$ we have $d\pi\colon T_{f,z}\psi^{-1}(0) \to T_f C^k(Z,\mathbf{R})$ is surjective.*

We say that $E \to X$ is a Banach vector bundle if E and X are Banach manifolds, $\pi^{-1}(x)$ is a Banach space for any $x \in X$, and X has an open cover such that the corresponding restrictions are trivial bundles.

Theorem. *If $\nabla\psi\colon T_x X \to E_x$ is surjective for all $x \in \psi^{-1}(0)$, then $\psi^{-1}(0)$ is a Banach submanifold of X.*

Proposition. *Let Y and Z be separable Banach manifolds, $\pi\colon E \to Y \times Z$ a Banach vector bundle, $\psi\colon Y \times Z \to E$ a smooth section. Suppose that for all $(y,z) \in \psi^{-1}(0)$ the following holds: (1) $\nabla\psi\colon T_{y,z}(Y \times Z) \to E_{y,z}$ is surjective. (2) $\nabla\psi\colon T_z Z \to E_{y,z}$ is Fredholm of index l. Then for generic $y \in Y$ the set $\{z \in Z \mid \psi(y,z) = 0\}$ is an l-dimensional submanifold of Z (and moreover, at each point in this set $\nabla\psi$ is surjective on tangent space to Z).*

Proof. *It follows from (a) that $\psi^{-1}(0) \subset Y \times Z$ is a Banach manifold (by implicit function theorem). Let $\pi\colon \psi^{-1}(0) \to Y$ denote the projection. By Sard-Smale theorem, it is enough to show that for all $(y,z) \in \psi^{-1}$ the differential of π, $D = d\pi\colon T_{y,z}\psi^{-1}(0) \to T_y Y$ is Fredholm. By (b) it is sufficient to prove these two statements: (1) $\ker(D) = \ker(\nabla\psi\colon T_z Z \to E_{y,z})$ and (2) The map $\nabla\psi\colon T_y Y \to E_{y,z}$ induces an isomorphism $\mathrm{coker}(D) \to \mathrm{coker}(\nabla\psi\colon T_z Z \to E_{y,z})$. Proof of (1): $\ker(D) = \{(0,\dot z) \mid (0,\dot z) \in T_{y,z}\psi^{-1}(0)\} = \ker(\nabla\psi\colon T_z Z \to E_{y,z})\}$. Proof of (2): Check that $\nabla\psi\colon T_y Y \to E_{y,z}$ sends $\mathrm{im}(D) \to \mathrm{im}(\nabla\psi\colon T_z Z \to E_{y,z})$. Let $(\dot y,\dot z) \in T_{y,z}\psi^{-1}(0)$. Need to show $\nabla\psi(D(\dot y,\dot z)) \in \mathrm{im}(\nabla\psi\colon T_z Z \to E_{y,z})$. Check $\nabla\psi\colon \mathrm{coker}(D) \to \mathrm{coker}(\nabla\psi\colon T_z Z \to E_{y,z})$ is injective. Have $\nabla\psi(\dot y,0) = \nabla\psi(0,\dot z)$ for some $\dot z \in T_z Z$. Then $(\dot y,-\dot z) \in T_{y,z}\psi^{-1}(0)$ and $\dot y = D(\dot y,-\dot z)$. Check $\nabla\psi\colon \mathrm{coker}(D) \to \mathrm{coker}(\nabla\psi\colon T_z Z \to E_{y,z})$ is surjective. Let $e \in E_{y,z}$. By hypothesis (a), $e = \nabla\psi(\dot y,\dot z)$ for some $(\dot y,\dot z) \in T_{y,z}Y \times Z$. We have $e = \nabla\psi(\dot y,0) + \nabla\psi(0,\dot z) = 0$ in $\mathrm{coker}(\nabla\psi\colon T_z Z \to E_{y,z})$.*

Example. *Suppose Z is a closed smooth manifold, $Y = C^k(Z,\mathbf{R})$, $E_{f,z} = T^*_z Z$, and $\psi(f,z) = df_z$. It follows that for generic $f \in C^k(\mathbf{R})$ and for any $z \in df^{-1}(0) = \mathrm{Crit}(f)$ we have $\nabla\psi\colon T_z Z \to E_{f,z}$ is surjective.*

Spectral flow. *Reference: Robbin and Salaman, Spectral flow and the Maslov index. Let H be a Hilbert space and let $A_s\colon H \to H$ be a continuous family of unbounded operators parametrized by $s \in \mathbf{R}$. Assume there are invertible self-adjoint operators A_+ and A_- such that $\lim_{s\to\infty} A_s = A_+$ and $\lim_{s\to-\infty} A_s = A_-$ in the norm topology. More technical assumptions. (All technical assumptions are satisfied if H is finite dimensional.) Can define spectral flow of A (an integer number). Idea: count the number of eigenvalues of A_s that cross 0 as s goes from $-\infty$ to ∞. If H is finite-dimensional, then $\mathrm{SF}(A)$ is the difference between the number of positive eigenvalues of A_+ and A_-.*

Theorem. *Under certain assumptions $\partial_s + A_s\colon L^2_1(\mathbf{R}, H) \to L^2(\mathbf{R}, H)$ is Fredholm and its index is the spectral flow of A.*

Proof. *Assume H is finite-dimensional. For each $h \in H$ by fundamental theorem of ODE's there is a function $f\colon \mathbf{R} \to H$ such that $(\partial_s + A_s)f_h(s)$ and $f_h(0) = h$. Define $H^+ = \{h \in H \mid \lim_{s\to\infty} f_h(s) = 0\}$ and $H^- = \{h \in H \mid \lim_{s\to-\infty} f_h(s) = 0\}$. Then $\ker(\partial_s + A_s)$ is isomorphic to $H^+ \cap H^-$. Claim: H^+ is the negative eigenspace of A^+ and H^- is the positive eigenspace of A^-. Claim: $\partial_s + A_s$ is bounded and has closed image. Then $\operatorname{coker}(\partial_s + A_s) = \ker(-\partial_s + A_s^*) = \ker(\partial_s - A_s^*)$. Similarly to above, $\ker(\partial_s - A_s^*) = H^{+\perp} \cap H^{-\perp}$. We have $\operatorname{ind}(\partial_s + A_s) = \dim(H^+ \cap H^-) - \dim(H^{+\perp} - H^{-\perp}) = \dim(H^+ \cap H^-) + \dim(H^+ + H^-) - \dim(H) = \dim H^+ + \dim H^- - \dim H = \mathrm{SF}(A)$.*

Proposition. *If X is a closed smooth manifold, $f\colon X \to \mathbf{R}$ is a Morse function. then for a generic C^k-metric g on X, the pair (f, g) is Morse-Smale.*

Proof. *Fix distinct critical points p and q of f. Let Y be the space of C^k-metrics g on X. Let Z be the space of locally L^2_1 maps $\gamma\colon \mathbf{R} \to X$ such that $\lim_{s\to-\infty} \gamma(s) = p$ and $\lim_{s\to\infty} \gamma(s) = q$, and $\gamma(-\infty, -R]$ and $\gamma[R, \infty)$ are L^2_1 for sufficiently large R.*

Proposition. *Let Y and Z be C^k separable Banach manifolds, $E \to Y \times Z$ a Banach space bundle, $\psi\colon Y \times Z \to E$ a C^k section such that $k \geq 1$ and $k \geq l + 1$. Suppose for all $(y, z) \in \psi^{-1}(0)$ we have (a) $\nabla\psi\colon T_{(y,z)}(Y \times Z) \to E_{(y,z)}$ is surjective and (b) restriction $\nabla\psi\colon T_z Z \to E_{(y,z)}$ is Fredholm of index l. Then for generic $y \in Y$ the set $\{z \in Z \mid \psi(y, z) = 0\}$ is an l-dimensional C^k submanifold of Z (and on this set $\nabla\psi$ is surjective onto tangent space to Z).*

Proposition. *Let X be a closed smooth manifold, $f\colon X \to \mathbf{R}$ a Morse function, and k a sufficiently large integer. Then for a generic C^k metric g on X, the pair (f, g) is Morse-Smale.*

Proof. *Fix p and q in $\mathrm{Crit}(f)$. Let Y be the space of C^k metrics on X. Let Z be the space of locally L^2_1 maps $\gamma\colon \mathbf{R} \to X$ such that $\lim_{s\to-\infty} \gamma(s) = p$, for R sufficiently small so that $\gamma(-\infty, R] \subset U_p$, where U_p is a coordinate chart around p, we have $\gamma\colon (-\infty, R] \to \mathbf{R}^n$ is L^2_1, Also we must have $\lim_{s\to\infty} = q$ and an analogous statement for U_q. Exercise: Z is a C^∞ Banach manifold, and $T_\gamma Z = L^2_1(\gamma^* TX)$. Let $E_{g,\gamma} = L^2(\gamma^* TX)$. This is a C^∞ Banach space bundle. Let ψ be a section of E such that $\psi(g, \gamma)(s) = \gamma'(s) - \xi(\gamma(s))$. Thus $\psi(g, \gamma) = 0$ iff γ is a C^{k+1} flow line with respect to g from p to q. Claim: hypotheses of previous proposition are satisfied. Fix some torsion free connection on $TX \to X$. If $\psi(g, \gamma) = 0$ then $\nabla\psi(\dot{g}, \dot{\gamma}) = \nabla_{\gamma'}\dot{\gamma} - \nabla_{\dot{\gamma}}\xi - \dot{\xi}$. Given (g, γ), choose a trivialization of $\gamma^* TX$, which is parallel with respect to connection on TX. We have $\nabla\psi(\dot{g}, \dot{\gamma}) = \partial_s\dot{\gamma} - A_s\dot{\gamma} - \dot{\xi}$, $A_s = \nabla\xi\colon T_{\gamma(s)}X \to T_{\gamma(s)}X$. We also have $\lim_{s\to-\infty} A_s = H(f, p)$ and $\lim_{s\to\infty} A_s = H(f, q)$. (a) We check $\nabla\psi$ is surjective. Can show $\nabla\psi$ has closed range. (Skip.) Enough to show if $\omega \in L^2(\gamma^* TX)$ is orthogonal to $\operatorname{im}(\nabla\psi)$ then $\omega = 0$. Such an ω satisfies $\int_{\mathbf{R}} \langle \dot{\xi}, \omega \rangle ds = 0$ for every $\dot{g}$. At any given point in the image of γ can find a $\dot{g}$ such that $\dot{\xi} = \omega$ there. Choosing $\dot{g}$ supported near that point gives $\omega = 0$ there. (b) $L^2_1(\gamma^* TX) \to L^2(\gamma^* TX)$ sends $\dot{\gamma}$ to $\partial_s\dot{\gamma} - A_s\dot{\gamma}$ and is a Fredholm operator by theory from last time. The index is equal to the spectral flow, which is equal to $\operatorname{ind}(p) - \operatorname{ind}(q)$. Previous proposition says for generic g the manifold $m(p, q)$ has dimension $\operatorname{ind}(p) - \operatorname{ind}(q)$ and for all $\gamma \in m(p, q)$, operators $\partial_s - A_s$ is surjective. Observe $H_+ = T_{\gamma(0)}D(p)$ and $H_- = T_{\gamma(0)}A(q)$. Recall $\operatorname{coker}(\partial_s - A_s)$ is the intersection of orthogonal complements to H_+ and H_-, which is zero, hence the span of H_+ and H_- is $\mathbf{R}^n$, hence $D(p)$ intersects $A(q)$ transversally. This completes the proof.*

Now let (M, ω) be a symplectic manifold, J be an ω-tame almost complex structure on M, A be an element of $H_2(X)$, $m_{g,k}(X, A)$ be the space of genus g J-holomorphic curves in X in class A with n marked points, $m^*_{g,k}(X, A)$ be the space of simple (not multiply covered) curves in $m_{g,k}(X, A)$.

Theorem. *For generic J the manifold $m^*_{g,n}(X, A)$ is oriented. Its dimension is*

$$(n-3)(2-2g) + 2\langle c_1(TX), A\rangle + 2k.$$

Theorem. *Fix a complex structure j on Σ_g. Let $\Sigma = (\Sigma_g, j)$, $m(\Sigma, X, A)$ be the space of J-holomorphic maps $(\Sigma_g, j) \to (X, J)$ in class A, $m^*(\Sigma, X, A)$ the space of simple curves in $m(\Sigma, X, A)$. For generic J the manifold $m^*(\Sigma, X, A)$ is oriented and has dimension $n(2-2g) + 2\langle c_1(TX), A\rangle$.*

Proof Sketch. *Ignore the issue of Banach space completions. Let Y be the space ω-tame almost complex structures J on X, Z be the space of smooth maps $u: \Sigma \to X$ representing the class A. Let $E_{J,u} = T(T^{0,1}\Sigma \otimes_{\mathbf{C}} u^*TX)$ is the space of $\mathbf{C}$-antilinear bundle maps $T\Sigma \to u^*TX$. Let $\psi(J, u)$ be the map $T\Sigma \to u^*TX$ defined by $\psi(J, u) = du + J \circ du \circ j$. We have $\psi(J, u) = 0$ iff $J \circ du = du \circ j$, i.e., u is holomorphic. We say that u is somewhere injective whenever there is a $z \in \Sigma$ such that $u^{-1}(u(z)) = \{z\}$ and $du: T_z\Sigma \to T_{u(z)}X$ is injective. Fact: If u is holomorphic then u is simple iff u is somewhere injective (McDuff-Salamon, Chapter 2). Need to show: for $(J, u) \in \psi^{-1}(0)$: (a) $\nabla\psi: T_{J,u}Y \times Z \to E_{J,u}$ is simple; (b) $\nabla\psi: T_uZ \to E_{J,u}$ is Fredholm of index $n(2-2g)+2\langle c_1(TX), A\rangle$. We see that T_JY is the space of bundle maps $\dot{J}: TX \to TX$ such that $\dot{J}J + J\dot{J} = 0$. and $T_uZ = T(u^*TX)$. We have $\psi(J, u) = du + J \circ du \circ j$. (b) We have $\nabla\psi(0, \dot{u}) = (\partial_s u + J\partial_t u)(ds - idt)$ plus zero order term. We can deform the zeroth order term so that this is $\bar{\partial}$ operator on holomorphic vector bundle u^*TX over Σ. Riemann-Roch theorem implies that the index is $n(2-2g) + 2c_1(u^*TX)$. To complete the proof, we use somewhere injective to prove (a).*

Suppose we have a symplectic manifold (M, ω), J is ω-tame almost complex structure. Fix (Σ, j) and $u: \Sigma \to M$. We have $\bar{\partial}(u) = du + J \circ du \circ j \in \Omega^{0,1}(\Sigma, u^*TM) = \Gamma(T^{0,1}\Sigma \otimes_{\mathbf{C}} u^*TM)$. The map u is holomorphic iff $\bar{\partial}(u) = 0$. If u is holomorphic, derivative of $\bar{\partial}$ defines an operator $D_u: T(u^*TM) \to \Gamma(T^{0,1}\Sigma \otimes_{\mathbf{C}} u^*TM)$.

Definition. *u is transverse if D_u is surjective (on C^∞ on appropriate Banach space completions).*

Last time we proved that if J is generic then all simple holomorphic curves are transverse. Note: If u is transverse, then $m(\Sigma, M)$ is a manifold near u, and $T_u m(\Sigma, M) = \ker(D_u)$. Also $\dim = \mathrm{ind}(D_u) = n(2-2g)+2\langle c_1(TM), A\rangle$. To complete the proof we need to show that for any pair (J, u) where u is J-holomorphic, the following operator is surjective: $T_J g \oplus \Gamma(u^*TM) \to \Gamma(T^{0,1}\Sigma \otimes u^*TM)$, $(\dot{J}, \xi) \to D_u\xi + \dot{J} \circ du \circ j$. Can show closed range. More precisely, suppose η is perpendicular to image. Since u is somewhere injective, find $z \in \Sigma$ such that u embeds a neighborhood of z into M, disjoint from the rest of $u(\Sigma)$. It follows that $\eta = 0$ in a neighborhood of z. So we have $\langle D(\dot{J}, 0), \eta\rangle = \int_\Sigma \langle \dot{J} \circ du \circ j, \eta\rangle = \int_{u^{-1}(U)} \langle \dot{J} \circ du \circ j, \eta\rangle = \int_N \langle \dot{J} \circ du \circ j, \eta\rangle > 0$. Since $\eta \perp \mathrm{im}\, D_u\rangle$ we have $D_u^*\eta = 0$. Unique continuation theorem shows that if $D_u^*\eta = 0$ and η vanishes to infinite order at a point, then $\eta = 0$.

Back to the Gromov nonsqueezing theorem. *Take $M = S^2 \times V$, $\omega = \omega_{S^2} \oplus \omega_V$, $J = J_{S^2} \oplus J_V$. Claim: The holomorphic spheres $S^2 \times \{v\}$ are transverse.*

Proof. *Let u be the map $S^2 \to S^2 \times \{v\}$. We have the operator $D_u: \Gamma(u^*TM) \to \Gamma(T^{0,1}S^2 \otimes u^*TM)$. We have splitting $u^*TM = TS^2 \oplus T_vV = TS^2 \oplus \mathbf{C}^{n-1}$. D_u respects this splitting. D_u is a sum of operators of the form $\Gamma(L) \to \Gamma(T^{0,1}S^2 \otimes L)$ where L is a line bundle. Suppose an operator of the latter type has a nonzero cokernel. Carleman Similarity Principle: If you have a solution of Cauchy-Riemann type equation with a zero-order perturbation, then you can perform a change of coordinates such that the solution becomes holomorphic. In our case $\eta \in \Omega^{0,1}(S^2, L)$ satisfies such an equation. It follows that any zero of η has negative multiplicity. From Carleman it follows that $\deg(T^{0,1}S^2 \otimes L) \leq 0$.*

Remark. *Consider $M = \Sigma_g \times V$. Are the curves $\Sigma \times \{v\}$ transverse? No, if $g > 0$.*

Also: moduli spaces are canonically oriented.

Gromov Compactness Theorem. *(Simplest version.) M, ω, J as usual. Let $A \in H_2(M)$. Suppose there is no $B \in \pi_2(M)$ such that $0 < \int_B \omega < \int_A \omega$. Then $m(S^2, M, A)$ is compact. Moreover, if J_t is a family of ω-tame almost complex structures, then $\cup_t \{t\} \times m_{J_t}(S^2, M, A)$ is compact.*

Idea of proof. *Consider a sequence of maps $u_k: S^2 \to M$. The energy $E(u_k) = \int_{S^2} |du_k|^2 = \int_A C\omega$. If you also have $|du_k| < c$ then we can pass to a convergent subsequence. If not, reparametrize and pass to a* *subsequence so that $|du_k(0)| > k$. Rescale the maps near 0. A holomorphic sphere magically appears with energy less than $\int_A \omega$.*

Intersection property. *Let (M,ω) be a 4-dimensional symplectic manifold, C_1 and C_2 two distinct simple connected J-holomorphic curves. Then the intersection of C_1 and C_2 are isolated. Also each intersection point has positive multiplicity. Multiplicity is 1 iff intersection is transverse.*

Easy part: transverse intersection must have positive sign.

Adjunction formula. *Let C be a simple J-holomorphic curve in X. Define $c_1(C) = c_1(TX)$, where TX is restricted to C. Let $\chi(C)$ be the Euler characteristic of domain of C. Then $c_1(C) = \chi(C) + C\cdot C - 2\sum \delta(p)$, where the sum is taken over all singular points of C, and $\delta(p)$ is a positive integer, $\delta(p) = 1$ iff C has a transverse self-intersection at p.*

Theorem. *(Recognition of $\mathbf{R}^4$.) Let (M,ω) be (noncompact) symplectic manifold, $\tilde{H}_*(M) = 0$. Suppose there is a compact subset K of M such that $(M\setminus K,\omega)$ is symplectomorphic to $(\mathbf{R}^4\setminus B,\omega_s)$. Then (M,ω) is symplectomorphic to $\mathbf{R}^4$ with standard symplectic structure.*

Proof. *We see that $\tilde{M}\setminus K$ is symplectomorphic to $S^2\times S^2\setminus B$. Also $H_*(\tilde{M}) = H_*(S^2\times S^2)$. Choose ω-tame J on $\tilde{M}$ which is product structure outside of K. Take $A = (1,0)\in H_*(\tilde{M})$ and $B = (0,1)\in H_*(\tilde{M})$. Look at $\mathrm{ev}\colon (\tilde{M},A)\to \tilde{M}$. Similarly to proof of nonsqueezing, this evaluation map has degree 1. Consider (p,q) as shown. There is a holomorphic sphere $S^2\times\{q\}\subset m_{0,1}$. We claim that $S^2\times\{q\}$ is the only holomorphic sphere in the class $(1,0)$ containing (p,q). Suppose c' is another one. Then it is simple as above. By intersection positivity $c\cdot c' > 0$. But $(1,0)\cdot(1,0) = 0$. Contradiction. Also C is transverse by arguments from last time. So (p,q) is a regular value of ev, and $\#\,\mathrm{ev}^{-1}(p,q) = 1$. Claim: The holomorphic spheres in the class $(1,0)$ give a foliation of $\tilde{m}$. There is a unique such sphere through each point. Any such sphere is embedded: $2 = c_1(C) = \chi(C) + C\cdot C - 2\delta(C) = 2+0-0$. Different spheres don't intersect.*

Now we know that every point in $\tilde{M}$ is in a unique embedded holomorphic sphere in class $(1,0) = A$ or $(0,1) = B$ respectively. Define $\phi\colon S^2\times S^2\to\tilde{M}$ as follows. Given $(x,y)\in S^2\times S^2$ let C_1 be the sphere through (p,x) in class A, let C_2 be the sphere through (y,q) in class B. Define $\phi(x,y)$ to be the intersection of C_1 with C_2. (Unique by intersection positivity.) Claim: ϕ is a diffeomorphism. Surjective because every point in $\tilde{M}$ is contained in an A sphere and a B sphere. Injective because different A spheres are disjoint and different B spheres are disjoint. Smoothness omitted.

Also ϕ is the identity at infinity. Claim: ϕ is isotopic to a symplectomorphism that is still identity at infinity. Proof: Let ω denote symplectic forms. Claim: $\omega\wedge\phi^*\omega > 0$. We have $(\omega\wedge\phi^*\omega)(\partial_{x_1},\partial_{y_1},\partial_{x_2},\partial_{y_2}) = \omega(\partial_{x_1},\partial_{y_1})\phi^*\omega(\partial_{x_2},\partial_{y_2}) + \omega(\partial_{x_2},\partial_{y_2})\phi^*\omega(\partial_{x_1},\partial_{y_1})$. Let $\omega_t = t\omega + (1-t)\phi^*\omega$ on $S^2\times S^2$ for $t\in[0,1]$. Since $\omega\wedge\phi^*\omega > 0$ it follows that $\omega_t\wedge\omega_u > 0$ for all t and u. Hence all ω_t's are all in the same cohomology class. Choose 1-form β_t with $d\beta_t = (d/dt)\omega_t$ and $\beta_t = 0$ at infinity. Look for $\phi_t\colon S^2\times S^2\to\tilde{M}$ such that $\phi_t^*\omega = \omega_t$ and $\phi_0 = \phi$ and ϕ_t depends smoothly on t. Then $\phi_1^*\omega = \omega_1$. To get $\phi_t^*\omega = \omega_t$ for all t it is enough to get $(d/dt)\phi_t^*\omega = (d/dt)\omega_t$. We have $(d/dt)\phi_t^*\omega = L_{X_t}\omega = (d/dt)\omega_t = d\beta_t$. X_t is a vector field on $S^2\times S^2$ determining the isotopy. Since ω is nondegenerate there is X_t with $I_{X_t} = \beta_{t'}$ $X_t = 0$ at infinity so $\phi_1 = \phi_0$ at infinity. Conclusion: we have a symplectic diffeomorphism between $S^2\times S^2$ and $\tilde{M}$, which is identity at infinity.

Theorem. *The group of symplectic diffeomorphisms of $S^2\times S^2$ is $\mathrm{SO}(3)\times\mathrm{SO}(3)$.*

Proof. *Key point is that for any ω-tame J on $S^2\times S^2$ we have two foliations by A-spheres and B-spheres. Note: The claim is false for $S^2\times S^2$ with symplectic structure $\omega_1\oplus\omega_\lambda$, where $\int_{S^2}\omega_1 = 1$, $\int_{S^2}\omega_\lambda = \lambda\in(1,2]$. Trouble is that $(0,1) = (1,0) + (-1,1)$. So in constructing B foliation the compactness argument fails.*

Suppose (X,ω) is a closed symplectic manifold. A symplectic isotopy is a smooth family of symplectomorphisms of X. We say that isotopy ϕ is generated by vector fields X_t if $(d/dt)\phi_t(x) = X_t(\phi(x))$. Differentiating $\phi_t^*\omega = \omega$ we get $0 = L_{X_t}\omega = di_{X_t}\omega + i_{X_t}d\omega$. Conclusion: X_t generates a symplectic isotopy iff $\omega(X_t,\cdot)$ is a closed 1-form.

Definition. *The isotopy ϕ_t is Hamiltonian if $\omega(X_t,\cdot)$ is exact, i.e., $\omega(X_t,\cdot) = dH_t$, where $H_t\colon X\to\mathbf{R}$.*

Degenerate Arnold Conjecture. *If $\phi\colon(X,\omega)\to(X,\omega)$ is Hamiltonian isotopic to id_X then $|\operatorname{Fix}(\phi)|\geq \min\{|\operatorname{Crit}(f)|\mid f\colon X\to\mathbf{R}\}$. It is still open!*

Nondegenerate Arnold Conjecture. *If* $\phi: (X,\omega) \to (X,\omega)$ *is Hamiltonian isotopic to* id_X *then* $|\operatorname{Fix}(\phi)| \geq \sum_i \dim(H_i(X,\mathbf{Q}))$. *Proved using Floer homology.*

A fixed point p of f is nondegenerate if $1 - df_p: T_pX \to T_pX$ is invertible. Equivalently, graph $T(f) = \{(x, f(x))\} \subset X \times X$ is transverse to the diagonal $\Delta = \{(x,x)\} \subset X \times X$ at (p,p).

Lefschetz Fixed Point Theorem. *If X is a closed smooth manifold and $f: X \to X$ is a smooth map with nondegenerate fixed points then* $\sum_{p\in\operatorname{Fix}(f)} \operatorname{sign}\det(1 - df_p) = \sum_i (-1)^i \operatorname{tr}(f_*: H_i(X,\mathbf{Q}) \to H_i(X,\mathbf{Q}))$. *It follows that if* $f \sim \mathrm{id}_X$ *then* $|\operatorname{Fix}(f)| \geq \left|\sum_i (-1)^i \dim H_i(X,\mathbf{Q})\right| = |\chi(X)|$.

Note. *Arnold conjecture is false if we only require ϕ to be symplectically isotopic to* id_X.

Counterexample. *The map $\phi: (T^2,\omega) \to (T^2,\omega)$ such that $\phi(x,y) = (x + 1/2, y)$ is symplectically isotopic to identity via the map $\phi_t(x,y) = (x + t/2, y)$. This symplectic isotopy is not Hamiltonian because it is generated by $X_t = \partial_x/2$ and $i_{X_t}\omega = dy/2$, which is not exact. In fact, there is no Hamiltonian isotopy from ϕ to id. If ϕ is a symplectic isotopy, define the flux of ϕ to be an element of $H^1(X,\mathbf{R})$ such that its value on $\gamma: S^1 \to X$ is $\int_{[0,1]\times S^1} f^*\omega$, where $f: [0,1] \times S^1 \to X$ is the map such that $f(t,0) = \phi_t(\gamma(0))$.*

Fact. *This gives a well-defined element of $H^1(X,\mathbf{R})$ and ϕ is homotopic relative to endpoints to a Hamiltonian isotopy iff flux of ϕ is zero.*

If ϕ is symplectically isotopic to identity, then the flux of ϕ is defined and is zero iff ϕ is Hamiltonian isotopic to identity. If $X = T^2$, the flux of ϕ in the previous example is $(0, 1/2) \neq 0$.

Now we enter the world of Floer homology. A. Floer, Symplectic fixed points and holomorphic spheres, Communications in Mathematical Physics.

Symplectic action functional. *Let L be the space of contractible smooth loops in X. Let $\tilde{L}$ be the space of pairs $(\gamma, [u])$, where $\gamma \in L$ and $[u]$ is homotopy class of $u: D^2 \to X$ relative boundary such that γ is the restriction of u to the boundary. Consider the map $A: \tilde{L} \to \mathbf{R}$ such that $A(\gamma,[u]) = \int_{[0,1]} H_t(\gamma(t))dt + \int_{D^2} u^*\omega$.*

Lemma. $(\gamma,[u]) \in \operatorname{Crit}(A)$ *iff* $\gamma'(t) = X_{H_t}(\gamma(t))$ *iff* $\gamma(t) = \phi_t(\gamma(0))$. *Hence* $\operatorname{Crit}(A) = \operatorname{Fix}(\phi)$.

Proof. *Let* $\xi \in T_{(\gamma,[u])}\tilde{L} = \Gamma(\gamma^*TX)$. *Now*

$$\begin{aligned} dA_\gamma(\xi) &= (d/ds)\left(\int_{[0,1]} H_t(\gamma_s(t))dt + \int_{u_s} \omega\right) = \int_{[0,1]} dH_t(\xi(t))dt + \int_{[0,1]} \omega(\xi, \gamma'(t))dt \\ &= \int_{[0,1]} (\omega(X_{H_t}, \xi(t)) + \omega(\xi(t), \gamma'(t))dt = \int_{[0,1]} \omega(\xi(t), \gamma'(t) - X_{H_t})dt. \end{aligned}$$

$(\gamma,[u]) \in \operatorname{Crit}(A)$ *iff the above integral is zero for all ξ iff $\gamma'(t) - X_{H_t} = 0$ for all t.*

Key example. *Suppose $H_t = H: X \to \mathbf{R}$ is a Morse function, $X_{H_t} = X$ and $\omega(X,\cdot) = dH$. Now* $\operatorname{Crit}(H) \subset \operatorname{Fix}(\phi)$. *Arnold conjecture is trivial in this case.*

Let J_t, where $t \in S^1$ be a family of ω-compatible almost complex structures on X. Each J_t defines a metric g_t on X by $g_t(v,w) = \omega(v, J_tw)$. This defines a metric on $\tilde{L}$ as follows: If $(\gamma,[u]) \in \tilde{L}$ and $\xi_1, \xi_2 \in T_{(\gamma,[u])}$, then $\langle \xi_1, \xi_2 \rangle = \int_{S^1} g_t(\xi_1(t), \xi_2(t))dt$. Consider a path $\tilde{u}: \mathbf{R} \to \tilde{L}$ such that $\tilde{u}(s) = (u(s,\cdot), [D_s])$. Lemma: $\tilde{u}$ is an upward gradient flow line of A iff $\partial_s u + J_t(\partial_t u - X_{H_t}) = 0$. Note: This is almost the equation for u to define a pseudoholomorphic map $\mathbf{R} \times S^1 \to X$. (Only almost because J depends on t and there is X_{H_t} term.) Proof: Need to check that $(d/ds)\tilde{u}(s) = \nabla A(u(s))$, i.e., fix s, let $\gamma(t) = u(s,t)$, want for any $\xi \in T_\gamma L = \Gamma(\gamma^*TX)$ the following: $dA_\gamma(\xi) = \langle (d/ds)\tilde{u}(s), \xi \rangle$, i.e., $\int_{[0,1]} \omega(\xi, \gamma'(t) - X_{H_t}) = \langle J_t(-\gamma'(t) + X_{H_t}), \xi \rangle$, i.e., $\int_{[0,1]} \omega(\xi, \gamma'(t) - X_{H_t}) = \int_{[0,1]} \omega(J_t(-\gamma'(t) + X_{H_t}), J_t\xi)dt$. (Tame ok.)

Key example. *$H_t = H: X \to \mathbf{R}$ is a Morse function, $J_t = J$ is an ω-compatible almost complex structure, therefore $g_t = g$, $g(v,w) = \omega(v, Jw)$. Suppose $\eta: \mathbf{R} \to X$ is an upward gradient flow line of H, i.e., $\eta'(s) =$*

$\nabla H(\eta(s))$. *Define* $u(s,t) = \eta(t)$. *This satisfies* $\partial_s \eta = JX_H$ *because* $JX_H = \nabla H$ *because* $\langle JX_H, v\rangle = dH(v)$ *because* $\langle JX_H, v\rangle = \omega(JX_H, Jv) = \omega(X_H, v) = dH(v)$.

Conclusion. *When* H_t *and* J_t *do not depend on* t *we have* $\mathrm{Crit}(H) \subset \mathrm{Crit}(A)/\pi_2(X)$. *All gradient flow lines of* H *are gradient flow lines of* A*, therefore Morse homology of* A *is isomorphic to the Morse homology of* H.

Recall that (M,ω) is a closed symplectic manifold, H is a 1-periodic Hamiltonian: $H: S^1 \times M \to \mathbf{R}$, X_{H_t} is the corresponding vector field, and ϕ_t is a family of symplectomorphisms such that $\phi_0 = \mathrm{id}_M$, $(d/dt)\phi_t(x) = X_{H_t}(\phi_t(x))$. Let $\phi = \phi_1$. We say that ϕ_t is a Hamiltonian isotopy from id_M to ϕ.

Arnold Conjecture: ϕ has at least as many fixed points as the minimum number of critical points of a function $M \to \mathbf{R}$. Nondegenerate version: If fixed points of ϕ are nondegenerate then the number of fixed points is at least $\sum_i \dim H_i(M,\mathbf{Q})$.

Now let L be the space of contractible loops in M. Define a symplectic action functional $A: L \to \mathbf{R}$ (if $\langle \omega, \pi_2(M)\rangle = 0$) $A(\gamma) = \int_{S^1} H_t(\gamma(t))dt + \int_D \omega$, where D is a disc in M with boundary γ. We have $\mathrm{Crit}(A) = \mathrm{Fix}(\phi) = \{\gamma: S^1 \to M \mid \gamma'(t) = X_{H_t}(\gamma(t))$. If J_t is an ω-compatible almost complex structure for $t \in S^1$ then we have a metric on L.

An upward gradient flow line of A from γ_- to γ_+ is a map $u: \mathbf{R}_s \to S^1_t \to M$ satisfying $\partial_s u + J_t(\partial_t u - X_{H_t}) = 0$. We have $\lim_{s\to-\infty} u(s,t) = \gamma_-(t)$ and $\lim_{s\to\infty} u(s,t) = \gamma_+(t)$.

Floer homology: define Morse homology for A generated by fixed points of ϕ differential counts flow lines as above.

If H_t does not depend on t, this should recover ordinary Morse homology of H.

Technical issues: transversality, grading on chain complex (dimension of moduli space of flow lines), compactness (counting), gluing ($\partial^2 = 0$), orientations (counting with signs), removing symplectically aspherical assumption.

Let $\phi: (M,\omega) \to (M,\omega)$ be any symplectomorphism (not necessarily Hamiltonian isotopic to identity). Define the mapping torus $Y_\phi = [0,1] \times M/(1,x) \sim (0,\phi(x))$. Y_ϕ fibers over S^1 with fiber M. There is a vector field ∂_t on Y_ϕ. Fixed points of ϕ correspond to circles in Y that are tangent to ∂_t and go once around the S^1 direction. $\mathbf{R}\times Y_\phi$ has a symplectic form $\Omega = \omega + ds \wedge dt$. Choose an Ω-tame almost complex structure J on $\mathbf{R}\times Y$ such that $J: TM \to TM$, $J(\partial_s) = \partial_t$ and J does not depend on s. Equivalently, choose ω-tame almost complex structure J_t on M for each $t \in \mathbf{R}$ such that $J_{t+1} = \phi_* \circ J_t \circ \phi_*^{-1}$. If X_+ and X_- are fixed points of ϕ corresponding to circles γ_+ and γ_- in Y_ϕ, define a flow line from X_+ to X_- to be a J-holomorphic cylinder $C \subset \mathbf{R}\times Y_\phi$ such that C is asymptotic to $\mathbf{R}\times Y_t$ as $s \to \infty$.

This is more general version of Floer homology for any $\phi \in \mathrm{Symp}(M,\omega)$. Chain complex is generated by $\mathrm{Fix}(\phi)$. Differential counts holomorphic cylinders in $\mathbf{R}\times Y_\phi$. Any flow line as above is a section of $\mathbf{R}\times Y_\phi \to \mathbf{R}\times S^1$.

Why this generalizes previous setup? Suppose $\phi = \phi_1$ comes from $H: S^1 \times M \to \mathbf{R}$. $Y_\phi \leftrightarrow S^1 \times M$, $(t,x) \leftrightarrow (t, \phi_t(x))$, $\gamma: S^1 \to Y_\phi \leftrightarrow \gamma: S^1 \to M$.

Grading. *Given two fixed points* x_+ *and* x_-*, what is the dimension of the moduli space* $m(x_+,x_-)$ *of flow lines from* x_+ *to* x_-*? Dimension may be different for different components of* $m(x_+,x_-)$*. Let* u *be a flow lines from* x_+ *to* x_-*. We have* $u: \mathbf{R}\times\mathbf{R} \to M$ *such that* $u(s,t+1) = \phi^{-1}(u(s,t))$, $\partial_s u + J_t \partial_t u = 0$ *and* $\lim_{s\to\pm\infty} u(s,t) = \gamma_\pm(t)$*. What is the dimension of* $m(x_+,x_-)$ *near* u*? Assume everything is transverse.* u *corresponds to a cylinder* $C \subset \mathbf{R}\times Y$.

Deformation operator $D: L^2_1(C,TM) \to L^2(C, T^{0,1}C \otimes_{\mathbf{C}} TM)$. If we choose some trivialization of TM over C, then D has the form $D: L^2_1(\mathbf{R}\times S^1, \mathbf{R}^{2n}) \to L^2(\mathbf{R}\times S^1, \mathbf{R}^{2n})$ such that $D\xi = \partial_s\xi + J_0\partial_t\xi + A(s,t)\xi)$. J_0 is the standard complex structure on $\mathbf{R}^{2n}$.

$\dim m(x_+,x_-)$ near u is in $d(D)$. What is the index of D? $A_\pm(t) = \lim_{s\to\pm\infty} A(s,t)$ is a symmetric matrix. We have trivialized TM over $\gamma_\pm$. Linearization of the flow ∂_t along $\gamma_\pm$ from $t=0$ to a given t defines a pair of symplectic linear maps $\psi^\pm_t: T_xM \to T_xM$ such that $\psi^\pm_0 = \mathrm{id}_{T_xM}$ and $\psi^\pm_1 = d\phi: T_xM \to T_xM$. With respect to trivialization we get two paths of symplectic matrices $\psi^\pm_t$. Assume fixed points are nondegenerate, so 1 is not an eigenvalue of $\psi^\pm_1$. $A_\pm(t)$ is given by $(d/dt)\psi^\pm_t = J_0 A_t \psi^\pm_t$.

Theorem. *Let* $A(s,t)$ *be matrices parametrized by* $(s,t) \in \mathbf{R}\times S^1$ *such that* $\lim_{s\to\pm\infty} A(s,t) = A_\pm(t)$ *is symmetric and 1 is not an eigenvalue of* $\psi^p m_1$*. Then the operator* $D: L^2_1(\mathbf{R}\times S^1, \mathbf{R}^{2n}) \to L^2(\mathbf{R}\times S^1, \mathbf{R}^{2n})$

defined by $D\xi = \partial_s\xi + J_0\partial_t\xi + A(s,t)\xi$ *is Fredholm and* $\operatorname{ind}(D) = \mathrm{CZ}(\psi_t^+) - \mathrm{CZ}(\psi_t^-)$. *CZ is the integer Conley-Zehnder index associated to a path of symplectic matrices.*

Recall that we have a symplectic manifold (M,ω) with an automorphism ϕ. Consider the mapping torus of ϕ: $Y_\phi = [0,1] \times /(1,x) \sim (0, \phi(x) = \mathbf{R} \times M/(t+1,x) \sim (t, \phi(x))$. Fixed points of ϕ correspond to parallel sections. Choose an ω-tame almost complex structure on E, or equivalently J_t on M such that $J_{t+1} = \phi_* J_t \phi_*^{-1}$.

Floer homology of ϕ: Chain complex generators are fixed points of ϕ, regarded as circles in Y_ϕ, differential counts holomorphic sections of $\mathbf{R} \times Y_\phi$: If γ_+ and γ_- are circles in Y_ϕ corresponding to x_+ and x_- in $\mathrm{Fix}(\phi)$, then $\langle \partial x_+, x_- \rangle$ counts maps $u\colon \mathbf{R} \times \mathbf{R} \to M$ such that $u(s,t+1) = \phi^{-1}(u(s,t))$, $\partial_s u + J_t \partial_t u = 0$ and $\lim_{s\to\pm\infty} u(s,t) = \gamma_\pm(t) = x_\pm$.

The difference between gradings of x_+ and x_- equals expected dimension of $m(x_+,x_-)$. Compute this. Given $u \in m(x_+,x_-)$ compute $\operatorname{ind}(D_u)$. We have $D_u\colon L^2_1(\mathbf{R}\times S^1, u^*E) \to L^2(\mathbf{R}\times S^1, u^*E)$. and $D_u\xi = \partial_s\xi + J_t\partial_t\xi + A(s,t)\xi$. Choose Hermitean trivialization τ of u^*E over $\mathbf{R}\times S^1$. We have $u^*E = \mathbf{R}\times S^1 \times \mathbf{R}^{2n}$ and $D_u\xi = \partial_s\xi + J_0\partial_s\xi + A(s,t)\xi$, where J_0 is the standard complex structure on $\mathbf{R}^{2n}$. τ restricts to a trivialization $\tau_\pm$ of E over $\gamma_\pm$. For $|s| \gg 0$ we have $\partial_s\xi + J_0\partial_t\xi + A(s,t)\xi \approx \partial_s\xi + J_0\nabla_t\xi$. With respect to $\tau_\pm$ parallel transport along $\gamma_\pm$ from 0 to t defines a symplectic map $\psi_t^\pm \in \mathrm{Symp}(\mathbf{R}^{2n}, \omega_0)$. The path $\psi_t^\pm$ of symplectic matrices is equivalent to a path $A_t^\pm$ of symmetric matrices via $(d/dt)\psi_t^\pm = J_0 A_t^\pm \psi_t$. Conclusion: $D_u\xi = \partial_s\xi + J_0\partial_t\xi + A(s,t)\xi$ where $\lim_{s\to\pm\infty} A(s,t) = A_t^\pm$. Theorem: If $1 \notin \mathrm{Spec}(\psi_1^\pm)$ then D is Fredholm and $\operatorname{ind}(D) = \mathrm{CZ}(\psi_t^+) - \mathrm{CZ}(\psi_t^-)$.

Let $\{\psi_t \mid t \in [0,1]\}$ be a path of symplectic matrices on $\mathbf{R}^{2n}$ with $\psi_0 = \mathrm{id}$ and $1 \notin \mathrm{Spec}(\psi_1)$. Define the Conley-Zehnder index $\mathrm{CZ}(\psi_t) \in \mathbf{Z}$ as follows. Define the Maslov cycle $M = \{A \in \mathrm{Sp}_{2n}(\mathbf{R}) \mid 1 \in \mathrm{Spec}(A)\}$. Roughly speaking, $\mathrm{CZ}(\psi_t)$ is the signed count of ψ_t with M. Note: $\mathrm{U}(n) = \mathrm{Sp}(2n) \cap O(n)$ is a maximal compact subgroup of $\mathrm{Sp}(2n)$. $H^1(\mathrm{Sp}(2n), \mathbf{Z}) = \mathbf{Z}$ is generated by a continuous extension of $\det\colon U(n) \to S^1$. M is a co-oriented codimension 1 subvariety of $\mathrm{Sp}(2n)$, Poincaré dual to the generator of $H^1(\mathrm{Sp}(2n),\mathbf{Z})$. So if $\{\psi_t \mid t \in S^1\}$ is an arbitrary loop in $\mathrm{Sp}(2n)$ then $\{\psi_t\} \cap M \in \mathbf{Z}$ is defined. To define CZ index, declare $\mathrm{CZ}\left(t \to \bigoplus_n \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\right) = 0$. To defined $\mathrm{CZ}(\psi_t)$, let γ be a path from $\bigoplus_n \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ to ψ_1, which is transverse to M such that $\eta + \gamma - \{\psi_t\} = 0 \in H_1(\mathrm{Sp}(2n))$. Define $\mathrm{CZ}(\psi_t) = \#(\gamma \cap M)$.

General properties: Naturality: If $\phi\colon [0,1] \to \mathrm{Sp}(2n)$ is an arbitrary path, then $\mathrm{CZ}(\phi\psi\phi^{-1}) = \mathrm{CZ}(\psi)$. Inverses: $\mathrm{CZ}(\psi^{-1}) = -\mathrm{CZ}(\psi)$. Change of trivialization: If $\phi\colon[0,1] \to \mathrm{Sp}(2n)$ is a path such that $\phi(0) = \phi(1) = \mathrm{id}$, then $\mathrm{CZ}(\phi\psi) = \mathrm{CZ}(\psi) + 2\deg\phi$. Signature: If A is a symmetric matrix with $\det(A) \neq 0$ and $\|A\| < 2\pi$ then $\mathrm{CZ}(\exp(J_0 A t)) = \sigma(A)/2$.

Why is $\operatorname{ind}(\partial_s + J_0\partial_t + A(s,t)) = \mathrm{CZ}(\psi_t^+) - \mathrm{CZ}(\psi_t^-)$? We omit the proof of Fredholm property. To compute index we assume that $A(s,t)$ is symmetric. For each $s \in \mathbf{R}$ take a family of symplectic matrices $\psi_{s,t}$ defined by $\psi_{s,0} = \mathrm{id}$, $(d/dt)\psi_{s,t} = J_0 A_{s,t}\psi_{s,t}$. Claim: For a given s we have $0 \in \mathrm{Spec}(J_0\partial_t + A(s,t))$ iff $1 \in \mathrm{Spec}(\psi_{s,1})$. $\mathrm{CZ}(\psi_t^+) - \#(\psi_{s,1}) \cap M - \mathrm{CZ}(\psi_t^-) = 0$.

Theorem. *If the almost complex structures J_t on M are generic, then for any $m \in m(x_+,x_-)$, the operator D_u is surjective, so $m(x_+,x_-)$ is a manifold near u of dimension* $\operatorname{ind}(u)$.

Proof. *One needs to show that if u is constant, then D_u is always surjective. If u is nonconstant, then the projection of u to $Y_\emptyset$ is somewhere injective.*

Index theorem for Cauchy-Riemann operators on Riemann surfaces with cylindrical ends. *Assumptions: C is a Riemann surface with ends identified with $[0,\infty)\times S^1$, E is a rank n complex vector bundle on C (with Hermitean metric), $D\colon L^2_1(E) \to L^2(T^{0,1}C \otimes E)$ such that in local coordinates and trivialization* $D = \partial_s + i\partial_t + \text{zeroth order term}$, *on each end, for some trivialization of E, $D = \partial_s + i\partial_t + A(s,t)$ where* $\lim_{s\to\infty} A(s,t) = A(t)$ *symmetric, and if* $\psi_0 = \mathrm{id}$ *and* $(d/dt)\psi_t = J_0 A(t)\psi_t$, *then* $1 \notin \mathrm{Spec}(\psi)$.

Theorem. *D is Fredholm and* $\operatorname{ind}(D) = n\chi(C) + 2c_1(E,\tau) + \sum_{\text{ends}} \mathrm{CZ}_\tau$.

To compute $c_1(E,\tau)$: take a generic section s of $\Lambda^n E$ such that on each end, s is nonvanishing and constant with respect to trivialization τ. Then $c_1(E,\tau) = \#s^{-1}(0)$. This depends only on E and homotopy class of τ. CZ_τ is the CZ index of the path ψ_t obtained as above. (Only depends on homotopy class of τ.) We can identify some ends with $(-\infty, 0] \times S^1$ instead, in which case you subtract the corresponding CZ terms instead of adding them. Granted that D is Fredholm, prove index formula as follows: the right hand side of formula is well defined, i.e., does not depend on τ; if C is a cylinder, then the theorem is true; if C has no ends, then the theorem is true (by Riemann-Roch); index is additive under gluing.

Glue some ends of C_1 to some ends of C_2. If the operators agree on the global ends, then we can glue E_1 and E_2 (using this on ends) to a bundle $E_1 \# E_2$ over $C_1 \# C_2$ and glue D_1 and D_2 to $D_1 \# D_2$ over $C_1 \# C_2$. Then $\operatorname{ind}(D_1 \# D_2) = \operatorname{ind}(D_1) + \operatorname{ind}(D_2)$. Idea of additivity: neck stretching.

Why is $n\chi(C) + 2c_1(E, \tau) + \sum_{\text{ends}} \mathrm{CZ}_\tau$ independent of τ? For any given end, the set of homotopy classes of trivializations is an affine space over $\pi_1 U(n) = \mathbf{Z}$. If you shift the trivialization by 1, then c_1 changes by ± 1. If s is any generic section of E which is nonvanishing on ends, then $c_1(E, \tau) = \# s^{-1}(0) - \sum$. If you shift trivialization by 1, CZ_τ shifts by ∓ 2. We now know: index formula is true for cylinders and closed surfaces and both sides of index formula are additive under gluing. Next step: deduce index formula when C is a disc. For an arbitrary C, cap off the ends with discs. Since formula is true for closed surface and for discs, by additivity it is true for C.

Back to Floer homology. *$H: S^1 \times M \to \mathbf{R}$ generates $\phi: (M, \omega) \to (M, \omega)$ and Hamiltonian isotopy ϕ_t from 1 to ϕ. Assume M is symplectically aspherical: $\langle c_1(TM), \pi_2(M) \rangle = \langle \omega, \pi_2(M) \rangle = 0$. $\mathrm{CF}_*(H)$ is the free $\mathbf{Z}$-module generated by contractible loops $\gamma: S^1 \to M$ with $\gamma'(t) = X_{H_t}$. (These are some of the fixed points of ϕ.) Grading: Let $\gamma: S^1 \to M$ be a generator. Define $\mu(\gamma) \in \mathbf{Z}$ as follows. Choose a map $\eta: D^2 \to M$ such that η restricted to ∂D^2 is γ. Trivialize $\eta^* TM$ use this to trivialize $\gamma^* TM$. Homotopy class of trivialization of $\gamma^* TM$ does not depend on η because $\langle c_1(TM), \eta - \eta' \rangle = 0$. Linearization of equation, $\gamma'(t) = X_{H_t}$ defines a family of symplectic matrices $\psi(t): TM_{\gamma(0)} \to TM_{\gamma(t)}$.*

Floer homology. *We use $\mathbf{Z}/(2)$ coefficients. Assume ϕ has nondegenerate fixed points. Choose a generic family of ω-tame almost complex structures J_t on M for $t \in S^1$. Define $(\mathrm{CF}_*(H, J), \partial)$ as follows. $\mathrm{Fix}(\phi) = \{\gamma: S^1 \to M \mid \gamma'(t) = X_{H_t}(\gamma(t))\}$. CF_* is the free $\mathbf{Z}/(2)$-module generated by fixed points corresponding to contractible γ, with $\mathbf{Z}$-grading. Given a generator γ, let $u: D^2 \to M$ be a map such that its restriction to S^1 is γ. This determines a homotopy class of trivialization of $\gamma^* TM$, independent of u because $\langle c_1(TM), u - u' \rangle = 0$. With this trivialization, $\{d\phi_t: T_{\gamma(0)} M \to T_{\gamma(0)} M\}_{t \in [0,1]}$ is a path of symplectic matrices from 1 to $\mu(x) = \mathrm{CZ}$.*

Given fixed points x_+ and x_- corresponding to γ_+ and γ_- let $m(x_+, x_-) = \{u: \mathbf{R}_s \times S^1 \to M \mid \partial_s u + J_t(\partial_t u - X_{H_t}) = 0 \wedge \lim_{s \to \pm\infty} u(s, \cdot) = \gamma_\pm\}$. By previous theorem, if J is generic, $\dim m(x_+, x_-) = \mu(x_+) - \mu(x_-)$. $\mathbf{R}$ acts on $m(x_+, x_-)$ by translating s.

Definition. *$\partial: \mathrm{CF}_* \to \mathrm{CF}_{*-1}$: $\partial x_+ = \sum_{x_-: \mu(x_+) - \mu(x_-) = 1} \#(m(x_+, x_-)/\mathbf{R}) x_-$.*

Lemma. *∂ is well defined, i.e., $m(x_+, x_-)/\mathbf{R}$ is finite when $\mu(x_+) - \mu(x_-)$.*

Gromov compactness. *For any closed (M, ω) and any x_+ and x_- a sequence in $m(x_+, x_-)$ has a subsequence which "converges" to a "broken flow line" with "bubble trees" attached. If (M, ω) is symplectically aspherical, then we do not have any bubbles because any holomorphic sphere has $\int \omega > 0$.*

Theorem. *$\partial^2 = 0$.*

Theorem. *$\mathrm{HF}_*(H, J)$ does not depend on (H, J).*

Proof idea. *Consider generic family $\{(H_s, J_s) \mid s \in \mathbf{R}\}$. Assume $(H_s, J_s) = (H_+, J_+)$ for large positive s, $(H_s, J_s) = (H_-, J_-)$ for large negative s. Define $\Phi: \mathrm{HF}_*(H_+, J_+) \to \mathrm{HF}_*(H_-, J_-)$. Choose x_+ and x_- in $\mathrm{Fix}(\phi_\pm)$ corresponding to $\gamma_\pm: S^1 \to M$. Let $\Phi(x_+) = \sum_{x_-: \mu(x_+) = \mu(x_-)} \# m(x_+, x_-) x_-$. Similarly to Morse theory case, Φ is a chain map, induces an isomorphism on homology depending only on the homotopy class of the path from (H_+, J_+) to (H_-, J_-).*

Remark. *$\mathrm{HF}_*(H, J)$ does depend on ϕ in the sense that if (H_+, J_+) and (H_-, J_-) has ϕ_+ and ϕ_- the map $\Phi: \mathrm{HF}_*(H_+, J_+) \to \mathrm{HF}_*(H_-, J_-)$ might be nontrivial.*

Theorem. *$\mathrm{HF}_*(H, J) = H_{*+n}(M, \mathbf{Z}/(2))$.*

Proof. *Take $H_t: M \to \mathbf{R}$ independent of t, Morse function $H: M \to \mathbf{R}$. (May have to replace H by ϵH, $\epsilon > 0$ small.) Take $J_t = J$ independent of t, let g be the corresponding metric. Can arrange that (H, g) is Morse-Smale.*

Claim. *If we replace H by ϵH for $\epsilon > 0$ sufficiently small, then $(\mathrm{CF}_*(H,J),\partial)$ is well-defined and equal to $(C_*^M(H,g),\partial)$. Need: Generators are the same. Gradings. Floer differential is defined. Differentials agree. (1) $\mathrm{Crit}(H) \subset \mathrm{Fix}(\phi)$. This is an equality if ϵ is sufficiently small. (2) Recall $\mathrm{CZ}\{\exp(J_0At) \mid t \in [0,1]\} = \sigma(A)/2$. If $x \in \mathrm{Crit}(H)$, then $d\phi_t\colon T_xM \to T_xM$ is $\exp(t\nabla X_H) = \exp(-tJ \cdot \mathrm{Hess}(H,x))$, therefore $\mu(x) = -\sigma(\mathrm{Hess})/2 = -n + \mathrm{ind}(f,x)$. (3) and (4): If $\mu(x_+) - \mu(x_-) = 1$, then every $u \in m(x_+,x_-)$ is independent of t. If $u \in m(x_+,x_-)$ is independent of t, then D_u is surjective. $D_u\colon L_1^2(\mathbf{R} \times S^1, \mathbf{R}^{2n}) \to L^2(\mathbf{R} \times S^1, \mathbf{R}^{2n})$. $D_u\xi = \partial_s\xi + J_0\partial_t\xi + A(s)\xi$, where $\partial_s + A(s)$ is the Morse theory deformation operator. Show: If ϵ is small enough, then any $\xi \in \ker(D_u)$ is independent of t. Then the same argument implies that anything in $\mathrm{coker}(D_u) = \ker(D_u^*)$ is t-independent. Let $\xi \in \ker(D_u)$. Write $\xi = \xi_0 + \xi_1$, where $\xi_0(s,t) = \int_{\tau \in S^1} \xi(s,\tau)d\tau$. Wlog $\xi = \xi_1$. We see that $\|\xi\|_{L^2} \le c\epsilon\|xi\|_{L^2}$.*

Suppose (M,ω) is symplectically aspherical, $H\colon S^1 \times M \to \mathbf{R}$, J_t ($t \in S^1$) is a family of almost complex structures. Then $\mathrm{HF}_*(H,J)$ is independent of (H,J) and $\mathrm{HF}_*(H,J) = \mathrm{HF}_{*+n}(M, \mathbf{Z}/(2))$.

Take $H\colon M \to \mathbf{R}$ independent of t, $H \to \epsilon H$, $\epsilon > 0$ small, J independent of t corresponding to g. Then $\mathrm{CF}_i(H,J) = C_{i+n}^M(f,g) \otimes \mathbf{Z}/(2)$. Now t-independent J-holomorphic curve corresponds to gradient flow line. Transversality as a holomorphic cylinder corresponds to transversality as a gradient flow line.

Last step: If ϵ is sufficiently small, then every solution to the equation $\partial_s u + J(\partial_t u - X_H) = 0$ $(*)$, where $\lim_{s\to\pm\infty} u(s,t) = x_\pm$ and $\mathrm{ind}(x_+) - \mathrm{ind}(x_-) = 1$ is t-independent. Morally S^1 acts on the space of solution by rotating t. If J is regular then all solutions are S^1-independent, otherwise the dimension of moduli space is too big. If J is not regular, then "localization" works. Direct argument in this case: Suppose that for any $\epsilon > 0$ there is a t-dependent solution. Start with $\epsilon_0 > 0$ sufficiently small that all previous steps work. For every positive integer n there is a t-dependent solution to the equation for ϵ_0/n: $\partial_s u_n + J(\partial_t u_n - X_H/n) = 0$. Define $v_n(s,t) = u_n(ns,nt)$. Then $\partial_s v_n + J(\partial_t v_n - X_H) = 0$ and $v_n(s,t+1/n) = v_n(s,t)$. By Gromov compactness, a subsequence of v_n converges to a solution of $(*)$. By previous equation, v_∞ is S^1-invariant. Since v_∞ is transverse, $v_n = v_\infty$ for large n up to $\mathbf{R}$-translation.

Definition. *A symplectic manifold (M,ω) is called monotone if there is a $\lambda > 0$ such that $\langle c_1(TM), A\rangle = \lambda\langle\omega, A\rangle$ for all $A \in \pi_2(M)$. Definition of $\mathrm{HF}_*(H,J)$ in the monotone case: Again, $\mathrm{CF}_*(H,J)$ is the free $\mathbf{Z}/(2)$-module generated by contractible loops $\gamma\colon S^1 \to M$ such that $\gamma'(t) = X_{H_t}$.*

Grading is only defined in $\mathbf{Z}/N$, where $N = 2\min\{\langle c_1(TM), A\rangle \mid A \in \pi_2(M) \wedge \langle c_1(TM), A\rangle > 0\}$. If $\gamma\colon S^1 \to M$ is a generator, let $u\colon D^2 \to M$ be a map such that it restriction to the boundary is γ. Let τ be a trivialization of γ^*TM that extends to a trivialization of u^*TM with respect to τ, $d\phi_t\colon TM_{\gamma(0)} \to TM_{\gamma(t)}$ is a symplectic matrix ψ_t. Define grading $\mu(\gamma) = \mathrm{CZ}\{\psi_t \mid t \in [0,1]\}$. If u' is another disk, then $\mathrm{CZ}\{\psi_t\} = \mathrm{CZ}\{\psi'_t\} = \pm 2\langle c_1(TM), u - u'\rangle$. Differential: $\partial\gamma = \sum_{\mu(\gamma)-\mu(\gamma')=1} \#m_1(\gamma,\gamma')\gamma'$, where $m_1(\gamma,\gamma') = \{u \in m(\gamma,\gamma') \mid \mathrm{ind}(D_u) = 1\}$. Claim: ∂ is well-defined, $\partial^2 = 0$. Compactness argument. Suppose $u_n \in m(\gamma,\gamma')$, $\mathrm{ind}(D_{u_*}) \in \{1,2\}$. Subsequence converges to a cylinder with bubbles. We have $\mathrm{ind}(u_n) = \sum_i \mathrm{ind}(v_i) + \sum_j 2\langle c_1(TM), [S_j]\rangle$, hence there are no bubbles and previous argument applies. As before, $\mathrm{HF}_*(H,J) = \oplus_{i \equiv *+n \pmod N} H_i(M, \mathbf{Z}/(2))$.

Bad case: there are holomorphic spheres S with $\langle c_1(TM), [S]\rangle < 0$. Multiple covers of S have very negative c_1. Not so bad case: $\langle c_1(TM), A\rangle = 0$ for all $A \in \pi_2(M)$. Novikov rings.

Recall that a Lagrangian in (M^{2n},ω) is a closed submanifold $L^n \subset M$ such that ω restricted to L is 0. Let L_1 and L_2 be two Lagrangians intersecting transversally. Idea: define $\mathrm{HF}_*(L_1,L_2)$ and $\mathrm{CF}_*(L_1,L_2)$ generated by intersection points. Choose J_t, an ω-tame almost complex structure for $t \in [0,1]$. Differential $\langle\partial x_+, x_-\rangle$ counts $u\colon \mathbf{R} \times [0,1] \to M$ such that $u(s,0) \in L_1$, $u(s,1) \in L_2$, $\lim_{s\to\pm\infty} u(s,t) = x_\pm$ and $\partial_s u + J_t\partial_t u = 0$. Under favorable circumstances, ∂ is well defined, $\partial^2 = 0$, $\mathrm{HF}_*(L_1,L_2)$ is invariant under appropriate isotopy of L_1 and L_2. Bad stuff: bubbling of holomorphic spheres and bubbling of holomorphic discs with boundary on L_1 and L_2. Let L be the manifold of all Lagrangian liner subspaces of $(\mathbf{R}^{2n},\omega)$. We claim that $\pi_1(L) = \mathbf{Z}$. The generator comes from $\{\exp(i\pi t)\mathbf{R} \subset \mathbf{C} \mid t \in [0,1]\}$. Relative grading: If x_+ and x_- belong to $L_1 \cap L_2$ and there is u satisfying conditions above, define $\mu(x_+) - \mu(x_-) \in \mathbf{Z}$. Given u, trivialize u^*TM such that $TL_1 = \mathbf{R}^n \oplus \{0\} \subset \mathbf{R}^{2n}$ over $\mathbf{R} \times \{0\}$ and $TL_2 = \{0\} \oplus \mathbf{R}^n$ over $\{0\} \times [0,1]$. Along $\mathbf{R} \times \{1\}$, TL_2 defines a loop of Lagrangians starting and ending at $\{0\} \oplus \mathbf{R}^n$. Then $\mu(x_+) - \mu(x_-)$ is the integer corresponding to the given element of the fundamental group.

Last time we learned Lagrangian Floer homology. If L_0 and L_1 are two Lagrangian submanifolds of (M,ω) intersecting transversally and $\mathrm{CF}_*(L_0,L_1)$ is a free $\mathbf{Z}/(2)$-module generated by intersection points with relative grading given by Maslov index. Relative grading lies inside $\mathbf{Z}/(n)$. If L_0 and L_1 are oriented, then we have absolute $\mathbf{Z}/(2)$ grading by intersection sign. Differential counts holomorphic strips $u: \mathbf{R} \times [0,1] \to M$ such that $u(s,0) \in L_0$, $u(s,1) \in L_1$, $\lim_{s\to\pm\infty} u(s,t) = x_\pm$, $\partial_s u + J_t \partial_t u = 0$. In good cases, $\mathrm{HF}_*(L_0, L_1)$ is well-defined and invariant under Hamiltonian isotopy of L_0 or L_1, e.g., if two noncontractible circles on a surface are Hamiltonian isotopic then they must intersect.

Why does it matter that L_0 and L_1 are Lagrangian? If two strips are homotopic as such, then the corresponding integrals coincide. Why is $\mathrm{HF}_*(L_0, L_1)$ invariant only under Hamiltonian and not symplectic isotopy?

Example. *Suppose $f: (M, \omega) \to (M, \omega)$ is a symplectomorphism. Its graph Γ is Lagrangian. The diagonal Δ is also Lagrangian. The intersection $\Gamma \cap \Delta$ is the set of fixed points of f.*

Theorem. $\mathrm{HF}_*(\Gamma, \Delta) = \mathrm{HF}_*(f)$. *To define $\mathrm{HF}_*(f)$, choose J_t, $t \in \mathbf{R}$, $J_{t+1} = f_* J_t f_*^{-1}$.*

Floer homology for symplectomorphisms of surfaces. *Suppose we have a surface of genus greater than 1. Let $f: \Sigma$*

Nielsen-Thurston classification of surfaces diffeomorphism. Properties. Every diffeomorphism f is isotopic to one of the following: finite order: ($f^n = f$ for some m); reducible: there is an essential arc $\gamma \subset \Sigma$ such that $f(\gamma) = \gamma$; Pseudo-Anosov: two transverse singular foliations f_1 and f_2 on Σ.

Mapping torus: $Y_f = [0,1] \times \Sigma/(1,x)\tilde{(}0, f(x))$. ω on Σ induces a closed form ω on $Y_f \to [\omega] + H^2(\Upsilon, \mathbf{R})$.

Definition. *f is monotone if $[\omega] = \lambda c_1(E)$ in $H^2(Y, \mathbf{R})$ for some $\lambda \in \mathbf{R}$. If f is monotone, define $\mathrm{HF}_*(f)$ as follows. $\mathrm{CF}_*(f) = \mathbf{Z}/(2)\,\mathrm{Fix}(f)$. For J_t on Σ we have $J_{t+1} = F_* J_t F_*^{-1}$. We obtain a $\mathbf{Z}/(2)$-grading by sign of fixed points. Now $\langle \partial x_+, x_- \rangle = \# m_1(x_+, x_-)/\mathbf{R}$. For compactness part of proof that ∂ is well-defined, $\partial^2 = 0$, need that u_n is a sequence in $m_1(x_+, x_-)$ then $\int_{u_n} \omega < c$. If u and u' are in $m_1(x_+, x_-)$, then $\langle c_1(E), u - u' \rangle = 0$, therefore $\langle [\omega], u - u' \rangle = 0$.*

Example. *If f is isotopic to identity then f is monotonic iff f is Hamiltonian isotopic to identity. More generally, suppose $\{f_t \mid t \in [0,1]\}$ is a symplectic isotopy. This induces a diffeomorphism $\phi: Y_{f_0} \to Y_{f_1}$, $[\omega_0] \neq \phi^*[\omega_1]$ and $c_1(E_0) = \phi^* c_1(E_1)$.*

$[\omega_0]$ relates to $\phi^*[\omega_1]$ as follows: $0 \to \mathbf{Z} \to H_2(Y_f) \to \ker(1 - f_*) \to 0$.

Conclusion. *$\{f_t\}$ preserves monotonicity iff flow $\{f_t\}$ iff $(\Sigma, \mathbf{R})$ annihilates ker. Also, any f is isotopic to a monotonic symplectomorphism.*

Suppose Σ is a closed surface, ω is an area form on Σ, $\phi: (\Sigma, \omega) \to (\Sigma, \omega)$ is a symplectomorphism. The mapping torus of ϕ is $Y_\phi = [0,1] \times \Sigma/(1,x) \sim (0, \phi(x))$. Suppose that ω extends to a closed 2-form on Y and $\mathbf{R}^2 \to E \to Y$ is the vertical tangent bundle of $\Sigma \to Y \to S^1$. ϕ is monotone if $[\omega] = \lambda c_1(E)$ in $H^2(Y, \mathbf{R})$.

Fact. *(Seidel, Pacific Journal of Mathematics.) This map from monotone symplectomorphisms to orientation-preserving diffeomorphisms of Σ is a homotopy equivalence.*

Assume ϕ is monotone and has nondegenerate fixed points. Define $\mathrm{HF}_*(\phi)$ as follows: C_* is the $\mathbf{Z}/(2)$-module generated by fixed points graded by Lefschetz sign. Choose J_t on Σ as usual. Now $\partial p = \sum_q q \cdot \# m_1(p,q)/\mathbf{R}$. Monotonicity implies that ∂ is well-defined.

Lemma. *If J is generic then for any $p, q \in \mathrm{Fix}(\phi)$ the set $m_1(p,q)/\mathbf{R}$ is finite.*

Proof. *Key is that if $c, c' \in m_1(p,q)$ then $\int_c \omega = \int_{c'} \omega$. Because if $\mathrm{ind}(c) = \mathrm{ind}(c')$ then $\lambda \langle c_1(E), [c - c'] \rangle = 0 = \langle [\omega], [c - c'] \rangle = \int_c \omega = \int_{c'} \omega$.*

Gromov compactness: if $\{c_n\}$ is a sequence in $m(p,q)$ with $\int_{c_n} \omega < R$ then there is a subsequence converging to a "broken trajectory" from p to q. No bubbling because $\pi_2(\Sigma) = 0$.

Suppose $m_1(p,q)/\mathbf{R}$ is infinite. Let $\{c_n\}$ be a sequence of distinct elements in $m_1(p,q)/\mathbf{R}$. Then a subsequence converges to a broken trajectory $(\hat{c}_0, \ldots, \hat{c}_k)$. If $k = 0$ then $\hat{c}_0 \in m_1(p,q)/\mathbf{R}$ is not isolated, contradicting transversality. If $k > 0$ then $\sum_{0 \le i \le k} \mathrm{ind}(c_i) = 1$. So some $\hat{c}_i$ has $\mathrm{ind}(\hat{c}_i) \le 0$, again contradicting transversality. Similarly, $\mathrm{HF}_*(\phi)$ depends only on mapping class of ϕ.

Examples. *(1)* $\phi = \mathrm{id}$, $Y = S^1 \times \Sigma$. $\mathrm{HF}_*(\mathrm{id}) = H_*(\Sigma)$. *(2)* ϕ *is finite order* ($\phi^n = 1$). *All fixed points have the same* $\mathbf{Z}/(2)$*-grading, hence* $\mathrm{HF}_*(\phi) = \oplus_{p \in \mathrm{Fix}(\phi)} \mathbf{Z}/(2)$. *(3)* ϕ *is a Dehn twist. Let* $\gamma \subset \Sigma$ *be an embedded circle,* N *be a neighborhood of* γ, $N \cong [0,1] \times S^1 \supset N' \cong [\epsilon, 1-\epsilon] \times S^1$. $\phi = \mathrm{id}$ *on* $\Sigma \setminus N$. *On* N, $\psi(x, y) = (x, y - x)$. *We have* $\mathrm{HF}_*(\phi) = H_*(\Sigma \setminus \gamma)$. *Proof: Do Hamiltonian isotopy so that* ϕ *has the following form. On* N', $\phi(x, y) = (x, y - x)$. *On* $\Sigma \setminus N'$, ϕ *is the time 1 flow of* X_H, *where* $H \colon \Sigma \setminus N' \to \mathbf{R}$ *is a Morse-Smale.* $\mathrm{Fix}(\phi) = \mathrm{Crit}(H)$. *Choose* J *as usual. Lemma: If* $c \in m(p, q)$, *then* c *does not intersect* N'. *This lemma implies that* $\mathrm{HF}_*(\phi) = H_*^M(H) = H_*(\Sigma \setminus N') = H_*(\Sigma \setminus \gamma)$. *Proof: Let* $c \in m_1(p, q)$. *Let* $x \in [\epsilon, 1-\epsilon]$. *Let* T *be the mapping tows of* ϕ *restricted to* $\{x\} \times S^1$. *Want to show that* $C \cap T = \phi$, $\mathbf{R} \times T = \mathbf{R}_s \times S^1_t \times S^1_y$. $J(\partial/\partial s) = \partial/\partial t - x\partial/\partial y$. *Let* F *be the foliation of* T *generated by* $\partial/\partial t - x\partial/\partial y$. *Then* $\mathbf{R} \times F$ *is a holomorphic foliation of* $\mathbf{R} \times T$. *Wlog* x *is rational and* c *is transverse to* T. $[c \cap (\mathbf{R} \times T)] = (a, b) \in H_1(\mathbf{R} \times T)$. *Since* c *has positive intersection with the holomorphic cylinders in* $\mathbf{R} \times F$, *we have* $ax - b \geq 0$, *equality only if* $c \cap (\mathbf{R} \times T) = \emptyset$. *In particular, if* $c \cap (\mathbf{R} \times T) \neq \emptyset$, *then* $[c \cap (\mathbf{R} \times T)] \neq 0$. *To prove lemma, show* $[c \cap (\mathbf{R} \times T)] = 0$. *Write* $[c] = z_0 + z$, *where* z_0 *is the real homology class of Morse cylinder from* p *to* q *and* $z \in H_2(Y)$. *Need to show* $z \cap (\mathbf{R} \times T) = 0$. $\mathrm{ind}(c) = \mathrm{ind}(H, p) - \mathrm{ind}(H, q) + 2\langle c_1(E), z\rangle$. $H_2(Y) = (S^1 \otimes \{\alpha \in \Sigma \mid \alpha \cdot \gamma = 0\}) \oplus H_2(\Sigma)$. *Write* $z = (z_1, z_2)$. $1 = \mathrm{ind}(c) = \mathrm{ind}(H, p) - \mathrm{ind}(H, q) + 2(2 - 2g)z_2$, *therefore* $z_2 = 0$.

Contact geometry. *Let* Y *be a closed oriented 3-manifold. A* contact form *on* Y *is a 1-form* λ *such that* $\lambda \wedge d\lambda > 0$. *Let* $\xi = \ker(\lambda)$. *This is an oriented 2-plane field on* Y. ξ *is called a* contact structure. *Note:* ξ *is totally nonintegrable. (The kernel of* λ *is a foliation iff* $\lambda \wedge d\lambda = 0$.*)* λ *and* λ' *determine the same* ξ *iff* $\lambda' = f\lambda$ *for some* $f \colon Y \to \mathbf{R}_{>0}$. *(If* $\dim(Y) = 2n - 1$, *we require* $\lambda \wedge (d\lambda)^{n-1} > 0$.*) Example: Standard contact form on* $\mathbf{R}^3$: $\lambda = dz - ydx$. *Darboux-type theorem: Any contact structure is locally isomorphic to this one. Gray stability theorem: If* ξ_t *is a family of contact structures for* $t \in [0,1]$, *then there is a family of diffeos* $\phi_t \colon Y \to Y$ *such that* $\phi_0 = \mathrm{id}$ *and* $\phi_{t*}\xi_0 = \xi_t$.

Reference: John Etnyre, Introductory Lectures in Contact Geometry.

An overtwisted contact form on $\mathbf{R}^3$: $\lambda = \cos(r)dz + \sin(r)d\theta$. This contact structure is not diffeomorphic to the previous one.

Why do we care? Contact manifolds are natural odd-dimensional counterparts of symplectic manifolds. Information from contact geometry can give topological invariants of 3-manifolds.

If Y is a 3-manifold with a contact form λ, define the *symplectization* $(\mathbf{R}_s \times Y, d(\exp(s)\lambda))$. Check symplectic: $d(\exp(s)\lambda) = \exp(s)(ds \wedge \lambda + d\lambda)$.

Definition. *Let* (Y_+, ξ_+) *and* (Y_-, ξ_-) *be contact 3-manifolds. A* symplectic cobordism *from* (Y_+, ξ_+) *to* (Y_-, ξ_-) *is a compact symplectic 4-manifold* (X, ω) *such that* $\partial X = Y_+ \sqcup -Y_-$ *and there are contact forms* $\lambda_\pm$ *with* $\xi_\pm = \ker(\lambda_\pm)$ *such that* $\omega|_{Y_+} = d\lambda_+$ *and* $\omega|_{Y_-} = d\lambda_-$.

Example. $(\mathbf{R}^4, \omega) \supset (U, \omega|_U)$. *Under appropriate convexity conditions,* ∂U *has a contact form* λ *with* $\omega|_{\partial U} = d\lambda$.

Definition. (Y, ξ) *is symplectically fillable if there exists a symplectic cobordism from* (Y, ξ) *to* ϕ.

Example. *There is a "functor" from differential topology to contact geometry. Let* M *be any smooth manifold. Choose a metric on* M. *Let* ST^*M *be the unit cotangent bundle of* M. *This has an obvious canonical contact form, which is obtained by tautological mapping of tangent bundle of* ST^*M *into* T^*M. *This contact manifold does not depend on a metric on* M.

Let (Y^{2n-1}, ξ) be a contact manifold. Let $\xi = \ker(\lambda)$ and $\lambda \wedge (d\lambda)^{n-1} > 0$. A *Legendrian submanifold* of (Y, ξ) is a submanifold $L^{n-1} \subset Y$ such that $TL \subset \xi|_L$. ($\mathbf{R} \times L$ is Lagrangian in $\mathbf{R} \times Y$.)

Example. *Legendrian knots in* $\mathbf{R}^3$. *Tangent vector cannot be vertical. They are uniquely determined by the topological type of their projection to xy-plane and the areas of all parts of the plane obtained by projection. They have two invariants: rotation number (if we orient the knot) and Thurston-Bennequin invariant. Note: contact homology distinguishes Legendrian knots which are isotopic as smooth knots and have the same rotation number and Thurston-Bennequin invariants.*

If M is a smooth manifold and N is a submanifold, then (ST^*M, ξ) contains the conormal bundle of N $(\{(x, y)\} \mid x \in N \wedge y \in ST^*_x M \wedge y|_{T_x N} = 0\}$. Smooth isotopy of N gives a Legendrian isotopy of $L(N)$. For example, smooth knot in $\mathbf{R}^3$ turns into Legendrian submanifold of $\mathbf{R}^3 \times S^2$, then contact homology gives us an invariant that distinguished the unknot.

Definition. *Let (Y^3, ξ) be a contact 3-manifold. ξ is overtwisted if there is an embedded disk $D \in Y$ such that $\xi|_{\partial D} = TD|_{\partial D}$.*

Example: Standard contact structure on $\mathbf{R}^3$ is tight.

Theorem. *(Eliashberg.) For any closed oriented Y, overtwisted contact structures are homotopy equivalent to oriented 2-plane fields.*

Classification of tight contact structures is much more subtle. Some 3-manifolds have none.

Theorem. *(Eliashberg and Gromov.) Symplectically fillable implies tight.*

Reeb vector field. *This is a vector field R such that $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$. It depends on λ, not just ξ. A Reeb orbit is a map $\gamma: \mathbf{R}/\mathbf{T} \to Y$ such that $\gamma'(t) = R(\gamma(t))$. We mod out by reparametrization. The k-fold iterate of Y is the pullback to $\mathbf{R}/KT$, where K is a positive integer. γ is embedded iff γ is not the K-fold iterate of some γ' where $k > 1$.*

Weinstein Conjecture. *For any contact form on any closed 3-manifold there is a Reeb orbit.*

Strategy. *Define Floer homology generated by Reeb orbits whose differential counts holomorphic curves. Show Floer homology is a topological invariant. Compute invariant, show its nontriviality.*

Definition. *An almost complex structure J on $\mathbf{R}_s \times Y$ is admissible if J acts compatibly with $d\lambda$, $J(\partial_s) = R$, and J is $\mathbf{R}$-invariant.*

Look at holomorphic curves in $\mathbf{R} \times Y$. Let $\gamma: \mathbf{R}/\mathbf{T} \to Y$ be a Reeb orbit. The Reeb flow preserves λ. $L_R\lambda = di_R\lambda + i_R d\lambda = d(1) + 0 = 0$. Linearization of the Reeb flow on the contact planes along γ. $P_\gamma: \xi_{\gamma(0)} \to \xi_{\gamma(0)}$ is symplectic with respect to $d\lambda$. γ is nondegenerate if $1 \notin P_\gamma$. Assume all Reeb orbits are nondegenerate.

Let $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l$ be Reeb orbits, $g \geq 0$. Define $m_g(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, b_l)$ as the set of all J-holomorphic curves $u: \Sigma \to \mathbf{R} \times Y$ where u has positive ends at $\alpha_1, \ldots, \alpha_k$, negative ends at $\beta_1, \ldots, \beta_l$ and no other ends. Here Σ is a genus g surface with $k + l$ punctures.

If $\sum_i [\alpha_i] = \sum_j [\beta_j]$ in $H_1(Y)$, define $H_2(Y, \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l)$ to be the set of relative homology classes of 2-chains z with $\partial z = \sum_i \alpha_i - \sum_j \beta_j$. This is an affine space over $H_2(Y)$. We have $m_g(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l, z) = \{u: \Sigma \to \mathbf{R} \times Y \in m_g \mid u_*[\Sigma] = z\}$. Choose a trivialization τ of ξ over the α_i and β_j. Expected dimension of the moduli space $m_g(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l, z) = (n-3)\chi(\Sigma) + 2c_1(u^*\xi, \tau) + \sum_i \mathrm{CZ}_\tau(\alpha_i) - \sum_j \mathrm{CZ}_\tau(\beta_j)$. This is the actual dimension if J is generic and $u: \Sigma \to \mathbf{R} \times Y$ is not multiply covered, then moduli space is a manifold near u of this dimension. Multiple covers prevent this theory from being complete (work in progress by Hofer-Wysocki-Zehnder).

CZ index in 3-dimensional case: $P_\gamma: \xi_{\gamma(0)} \to \xi_{\gamma(0)}$. Elliptic if eigenvalues are $\exp(\pm 2\pi i\theta)$, positive hyperbolic if eigenvalues are $\lambda > 0$ and λ^{-1}, negative hyperbolic if eigenvalues are $\lambda < 0$ and λ^{-1}. Elliptic case: linearized flow rotates by angle $2\pi\theta$ for some $\theta \in \mathbf{R} \setminus \mathbf{Z}$. We have $\mathrm{CZ}_\tau(\gamma) = 2\lfloor\theta\rfloor + 1$. Hyperbolic case: linearized flow rotates eigenspaces by angle πn for some $n \in \mathbf{Z}$, $\mathrm{CZ}_\tau(\gamma) = n$.

Note: if $m_g(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l)$ is nonempty then $\sum_i \int_{\alpha_i} \lambda \geq \sum_j \int_{\beta_j} \lambda$, equality only if $\{\alpha_1, \ldots, \alpha_k\} = \{\beta_1, \ldots, \beta_l\}$. Here $\int \lambda$ is "symplectic action".

Proof: if $u: \Sigma \to \mathbf{R} \times Y$ is in m_g, then $u^*d\lambda \geq 0$ on all of Σ, with equality only where $\pi \circ du = 0$. $u^*d\lambda(v_1, v_2) = d\lambda(\pi du(v_1), \pi du(v_2))$. $u^*d\lambda(v, jv) = d\lambda(\pi du(v), \pi du(jv)) = d\lambda(\pi du(v), J\pi du(v)) \geq 0$, equality iff $\pi du(v) = 0$. Apply Stokes theorem on Σ. In particular every holomorphic curve in $\mathbf{R} \times Y$ has at least one positive end. But it is possible to have no negative ends.

Compactness. *(Bargeois-Eliashberg-Hofer-Wysocki-Zehnder.) Any sequence in m_g has a subsequence which converges to a "broken" curve.*

Key point: for any $u: \Sigma \to \mathbf{R} \times Y$ in m_g we have $\int_\Sigma u^*d\lambda = \sum_i \int_{\alpha_i} \lambda - \sum_j \int_{\beta_j} \lambda$.

Cylindrical contact homology. *Chain complex generated by "good" Reeb orbits over* $\mathbf{Q}$*. Differential counts holomorphic cylinders in* $\mathbf{R} \times Y$*. Trouble with coverings. Either assume* λ *is "nice" so that there are no bad holomorphic discs or add correction term (augmentation) to deal with bad discs.*

Reference: Introduction to Symplectic Field Theory by Eliashberg-Givental-Hofer.

Cylindrical contact homology (in 3 dimensions)

Let Y be a closed oriented 3-manifold, λ be a contact form, $\lambda \wedge d\lambda > 0$, R be the corresponding Reeb vector field.

Example. $Y = S^3$, $\lambda = (x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)/2$.

Exercise. *R is tangent to the Hopf circles.*

Example. $Y = T^3 = (\mathbf{R}/2\pi\mathbf{Z})^3$. $\lambda_n = \cos(nz)dx + \sin(nz)dy$, $R_n = \cos(nz)\partial/\partial x + \sin(nz)\partial/\partial y$.

To define CCH, assume all Reeb orbits are nondegenerate. (Above examples are "Morse-Bott".) Choose J on $\mathbf{R}_s \times Y$.

$m_g(\alpha_1, \ldots, \beta_1, \ldots, z)$ is the set of all J-holomorphic curves $u\colon \Sigma \to \mathbf{R} \times Y$ such that domain Σ is a genus g surface with $k + l$ punctures, u has positive ends at α_i, negative ends at β_j. If u has a positive or negative end at the k-fold iterate of an embedded Reeb orbit γ, there are k possible asymptotic markings.

Pretend that all of these moduli spaces are manifolds of the expected dimension. (Requires abstract perturbation of Cauchy-Riemann equation: Hofer-Wysocki-Zehnder.) A Reeb orbit is "bad" if it is the k-fold iterate of γ where k is even and γ is negative hyperbolic. Otherwise it is "good". Let C_* be the free $\mathbf{Q}$-module generated by good Reeb orbits. If α is a good Reeb orbit, then $\partial\alpha = \sum_\beta k^{-1} \# m_0(\alpha, \beta, z)/\mathbf{R} \cdot \beta$, where α is the k-fold iterate of an embedded orbit.

"Theorem". *Suppose there is no contractible Reeb orbit γ bounding a disk D such that* $-1 + 2c_1(\xi|_D, \tau) + \mathrm{CZ}_\tau(\gamma) = 1$*. Then* $\partial^2 = 0$*.*

"Proof". *Let α, γ be generators,* $z \in H_2(Y, \alpha, \gamma)$, $\mathrm{CZ}_\tau(\alpha) - \mathrm{CZ}_\tau(\gamma) + 2c_1(\xi|_z, \tau) = 2$*. Look at* $m_0(\alpha, \gamma, z)$*. Compactify and look at the boundary. We have* $\#\partial(m_0(\alpha, \beta, z)/\mathbf{R}) = a\langle \partial^2\alpha, \gamma\rangle$*. For* $\Gamma \in H_1(Y)$*, let* CH_*^Γ *be the part corresponding to Reeb orbits in homology class* Γ*.*

"Theorem". *Let λ_1, λ_2 be two different contact forms and J_1, J_2 two complex structures. Suppose that both forms are nice and correspond to some contact structure. Then* $\mathrm{CH}_*^\Gamma(\lambda_1, J_1) = \mathrm{CH}_*^\Gamma(\lambda_2, J_2)$*.*

Definition. *A contact form λ on Y is called nice, if all Reeb orbits are nondegenerate and there are no contractible Reeb orbits γ bounding a disk D with* $-1 + 2c_1(\xi|_D, \tau) + \mathrm{CZ}_\tau(\gamma) \in \{1, 0, -1\}$*.*

"Proof". *Usual argument with continuation maps and chain homotopies. Niceness assumption implies that boundaries of relevant moduli spaces of cylinders consist of broken curves involving only cylinders.*

Corollary. *Let Y be a closed oriented manifold with a contact structure ξ. If there is a nice contact form for ξ, then* $\mathrm{CH}_*^\Gamma(Y, \xi)$ *is well defined.*

Corollary. *If ξ has a nice contact form and* $\mathrm{CH}_*^\Gamma(Y, \xi) \neq 0$*, then Weinstein conjecture holds for any contact form for ξ.*

Examples. T^3*. After perturbation each circle of Reeb orbits splits into two Reeb orbits, one elliptic and one positive hyperbolic. Therefore for each z with* $\tan(z) \in \mathbf{Q} \cup \{\infty\}$ *we have generators* e_z^k, h_z^k *for* $k \geq 1$ *integer. Claim:* $\partial = 0$*. Idea: There are two index 1 cylinders from* e_z *to* h_z*, opposite sign. Like* $H_*^M(S^1)$*. No other cylinders because of symplectic action. Conclusion: Let* $T = (a, b, c) \in H_1(T^3)$*. If* $c \neq 0$*, then* $\mathrm{CH}_*^\Gamma = 0$*. If* $c = 0$ *and* $(a, b) \neq (0, 0)$*, then* $\mathrm{CH}_*^\Gamma = \oplus_n H_*(S^1, \mathbf{Q})$*. Corollary: Weinstein conjecture holds for these contact structures. The contact structures determined by* λ_n *for different n are not isomorphic.*

Let (Y, ξ) be a contact manifold ($\dim Y = 3$, can be generalized to other dimensions). Choose (1) contact form λ with $\xi = \ker \lambda$ and nondegenerate Reeb orbits; (2) almost complex structure on $\mathbf{R}_s \times Y$ such that $J(\xi) = \xi$, J is compatible with $d\lambda$, $J(\partial_s) = R$, J is $\mathbf{R}$-invariant; (3) abstract perturbations to make moduli spaces of holomorphic curves transverse. Then we have contact homology algebra A over $\mathbf{Q}$. The generators are good Reeb orbits. Relations: if α and β are good Reeb orbits then $\alpha\beta = (-1)^{|\alpha|\cdot|\beta|}\beta\alpha$, where $|\alpha| = \mathrm{CZ}_\tau(\alpha) - 1 \pmod 2$. If α is a good Reeb orbit, then

$$\partial\alpha = \sum_{k\geq 0}\sum_{\beta} (\text{combinatorial factor}) \sum_{z\in H_2(Y,\alpha,\beta_i)} k - 1 + 2c_1(\xi|_z,\tau) + \underset{\tau}{\mathrm{CZ}}(\alpha) - \sum_i \underset{\tau}{\mathrm{CZ}}(\beta_i) = 1,$$

where β_i are good Reeb orbits. Extend ∂ to A by the Leibniz rule. Theorem: $\partial^2 = 0$ implies that $\mathrm{HC}_*(Y,\xi)$ depends only on Y and ξ and not on the other choice. Also a symplectic cobordism X from Y_+ to Y_- induces a DGA morphism $A_+ \to A_-$.

Example. *(S^3,ξ), ξ is the standard contact structure. Reeb orbits are Hopf circles. Perturbation gives two Reeb orbits (plus very long Reeb orbits). A_* is generated by a_k and b_k for $k \geq 1$. Also $\partial = 0$ and $|a_k| = |b_k| = 0 \pmod 2$. Since $H_1(S^3) = H_2(S^3) = 0$, A has a $\mathbf{Z}$-grading. Conclusion: $\mathrm{HC}_*(S^3,\xi) = \mathbf{Q}[z_2, z_4, \ldots]$. $\deg(z_{2k} = 2k$.*

"Theorem." If ξ is an overtwisted contact structure on Y, then $\mathrm{HC}_*(Y,\xi) = 0$. "Proof." (Mei-Lin Yau, Eliashberg.) Can find λ and J such that there is a Reeb orbit γ such that γ bounds a unique index 1 holomorphic disc in $\mathbf{R}\times Y$ and γ is shorter than all other Reeb orbits. Now $\partial\gamma = 1$. If $\partial\alpha = 0$ then $\partial(\gamma\alpha) = (\partial\gamma)\alpha \pm \gamma(\partial\alpha) = 1\alpha \pm \gamma\cdot 0 = \alpha$.

Corollary. *If Y is symplectically fillable, then ξ is tight.*

Proof. *Suppose X^4 is a symplectic cobordism from (Y^3,ξ) to ϕ. Make choices to define A for Y. Then X induces a DGA morphism $\Phi\colon A \to \mathbf{Q}$. Suppose ξ is overtwisted. Then there is an α such that $\partial\alpha = 1$ and $0 = \Phi(\partial\alpha) = \Phi(1) = 1$. Symplectically fillable implies $1 \neq 0$ in HC_*. Overtwisted implies $1 = 0$ in HC_*. Possible: tight and $1 = 0$. Examples: tight but not fillable.*

Morse-Bott theory. *Model case: X is a closed smooth manifold. A smooth function $f\colon X \to \mathbf{R}$ is Morse-Bott if $\mathrm{Crit}(f)$ is a union of closed submanifolds of X, and for each $p \in \mathrm{Crit}(f,p)$ the map $H\colon T_pX \otimes T_pX \to \mathbf{R}$ is nondegenerate on the orthogonal complement of T_pS, where S is the critical submanifold containing p.*

Example. *Height function on a torus lying on its side.*

Example. *If $F \to E \to B \to \mathbf{R}$ and $E \to \mathbf{R}$ with $E \to \mathbf{R}$ Morse-Bott, then $\mathrm{Crit}(\pi^* f) = \pi^{-1}\mathrm{Crit}(f)$, where $\pi = E \to B$.*

The index of a critical submanifold can be regarded as an interval $[i_-(s), i_+(s)]$, where $i_-(s)$ is the number of negative eigenvalues of Hessian and $i_+(s) = i_-(s) + \dim S$. Perturb f to $f + \sum_s \epsilon_s \pi^* f_s$ where $f_s\colon s \to \mathbf{R}$ is Morse, $\pi\colon N \to S$ is a tubular neighborhood, ϵ_s is a small function which is positive near s and 0 elsewhere. $\mathrm{Crit}(\hat{f}) = \cup_s \mathrm{Crit}_j(f_s)$. Near s: $d\hat{f} = df + \epsilon_s\pi^* df_s$. Pick a generic metric g on X. What are the gradient flow lines of $\hat{f}$, in terms of f? Answer: "Cascades". Bourgeois (contact homology). Frauenfelder (Morse theory). Claim: flow lines of $\hat{f}$ correspond bijectively to cascades.

Exercise. *Expected dimensions agree.*

Exercise. *Find an example where cascades with $k > 1$ contribute.*

Example: torus lying on its side.

Floer homology. *Let (X^{2n},ω) be a closed symplectic manifold (assume monotone) and $\phi\colon (X,\omega) \to (X,\omega)$ be a Hamiltonian symplectomorphism. Assume fixed point come in nondegenerate manifolds, i.e., $\mathrm{Fix}(\phi)$ is a union of closed submanifolds of X and for any $p \in \mathrm{Fix}(\phi)$ we have $\ker(1 - d\phi_p\colon T_pX \to T_pX) = T_pS$ where S is the corresponding manifold of fixed points. Choose Morse functions $f_s\colon s \to \mathbf{R}$. Define $\mathrm{CF}_*^{MB}(\phi) = \oplus C_{*-\mathrm{ind}_-(s)}(f_s)$. Differential counts cascades.*

"Theorem." Can extend differential of Floer homology, continuation maps, etc. to this setting. Corollary. $\mathrm{HF}_*(\mathrm{id}_X) = H_{*-n}(X)$. Proof. $k = 0$ is reduced to Morse homology. $k > 0$ cascades: holomorphic spheres in X ruled out by monotonicity.

Example. *Cylindrical contact homology of* $(T^3, \lambda_n = \cos(nz)dx + \sin(nz)dy)$. *Claim:*

$$\mathrm{CH}_*(T^3, \lambda_n, (a, b, 0)) = \bigoplus_n H_*(S^1).$$

Proof: Just need to show there are now cascades with $k \geq 1$, *i.e., no holomorphic cylinders between Reeb orbits in different critical submanifolds. But all orbits* γ *with* $[\gamma] = (a, b, 0)$ *have the same action. A simple calculation completes the proof. Can also show that differential in contact homology algebra vanishes. (Lots of holomorphic curves with 2 positive ends.)*