Classification of irreducible representations of Heisenberg groups and algebras

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Definitions of Heisenberg groups and algebras

Definition. A *symplectic vector space* is a pair (V, ω), where V is a finite-dimensional real vector space and ω is a nondegenerate real skew-symmetric bilinear form.

The dimension of a symplectic vector space is always even. We denote it by $2n$.

Definition. A *Heisenberg group* for a symplectic vector space (V, ω) is the Lie group with the underlying manifold $V \times \mathbf{R}$ and the multiplication $(u, s)(v, t) = (u + v, s + t + \omega(u, v)/2)$ where $u, v \in V$ and $s, t \in \mathbf{R}$.

The map $t \mapsto (0, t)$ is a Lie group homomorphism from $\mathbf{R}$ to the Heisenberg group. Its image coincides with the center of the Heisenberg group. The dimension of the Heisenberg group equals $2n + 1$.

Definition. A *Heisenberg algebra* for a symplectic vector space (V, ω) is the Lie algebra with the underlying vector space $V \oplus \mathbf{R}$ and the commutator $[(u, s), (v, t)] = (0, \omega(u, v))$ where $u, v \in V$ and $s, t \in \mathbf{R}$.

The map $t \mapsto (0, t)$ is a Lie algebra homomorphism from $\mathbf{R}$ to the Heisenberg algebra. Its image coincides with the center of the Heisenberg algebra. The dimension of the Heisenberg algebra equals $2n + 1$.

Theorem. The map $V \oplus \mathbf{R} \to V \times \mathbf{R}$ that sends (v, t) to (v, t) for every $v \in V$ and $t \in \mathbf{R}$ is the exponential map from a Heisenberg algebra to the corresponding Heisenberg group.

Definitions of representations of Heisenberg groups and algebras

Heisenberg groups do not have irreducible finite-dimensional representations of dimension greater than 1. This follows from the same result for Lie algebras, which we explain later. Hence we need to go to the infinite-dimensional case. Finite-dimensional representations are unitarizable. In the infinite-dimensional case this is not always true and has to be assumed. We restrict ourselves to the group of unitary operators on a Hilbert space. We define a representation of a Heisenberg group as a continuous homomorphism from the Heisenberg group to the group of the unitary operators. To talk about continuity we need to put a topology on the group of unitary operators. One must be very careful about this topology. For example, if we choose the norm topology, then from Stone's theorem and the corresponding result for Lie algebras, which we explain later, it follows that there are no irreducible unitary representations of a Heisenberg group on an infinite-dimensional Hilbert space.

Definition. A *representation of a Heisenberg group* is a continuous homomorphism from the Heisenberg group to the topological group of the unitary operators on a Hilbert space equipped with the strong operator topology. Here the strong operator topology is the weakest topology on the set of all bounded operators such that all evaluation maps at points are continuous.

The strong operator topology is weaker than the norm topology. Hence we can hope to find some irreducible infinite-dimensional unitary representations of a Heisenberg group. In fact, there is a family of such representations called Shrödinger representations, as we explain later.

Now we want to define representations of Heisenberg algebras. Existence of a faithful trace in the finite-dimensional case immediately implies that there are no irreducible finite-dimensional Heisenberg algebra representations of dimension greater than 1. Hence we have to go to the infinite-dimensional case. It is an easy exercise in functional analysis [Reed and Simon, §VIII.5, Example 2] that there are no irreducible representations of a Heisenberg algebra in the Lie algebra of bounded operators on an infinite-dimensional Hilbert space. Hence we must allow unbounded operators. Stone's theorem, which we explain below, tells us that there is a bijective correspondence between representations of $\mathbf{R}$ in the group of the unitary operators with the strong topology and skew-adjoint operators. Hence it makes sense to look at the Lie algebra of the skew-adjoint operators defined on a dense subspace. However, it is unclear how one can define a commutator of two such operators, in particular, what should be the domain of the commutator. Here is one possible solution to this problem.

Definition. A *representation of a Heisenberg algebra* is a Lie algebra homomorphism from the Heisenberg algebra to the Lie algebra of skew-symmetric endomorphisms of a dense subspace D of a Hilbert space H.

It does not hurt to point out that continuity of such homomorphism follows from linearity. The commutator of two skew-symmetric endomorphisms of a dense subspace D is again an skew-symmetric endomorphism of D in an obvious way, hence the target is indeed a Lie algebra.

We remark that the condition of skew-adjointness was relaxed to the condition of skew-symmetry. Any Lie algebra has the zero operator and if we require it to be skew-adjoint or even closed, then $D = H$ and we do not want this to happen.

Since we have already mentioned skew-symmetry and skew-adjointness, let us briefly recall their properties.

Definition. An operator T is *skew-symmetric* if its adjoint operator T^* exists and $T \subset -T^*$. Here we write $A \subset B$ whenever B extends A.

If T is skew-symmetric, then its domain is a dense subspace, because the adjoint operator is defined only for such operators. Also T^{**} exists and is equal to the closure of T. We have the following four classes of operators:

skew-symmetric	$T \subset T^{**} \subset -T^*$
closed skew-symmetric	$T = T^{**} \subset -T^*$
essentially skew-adjoint	$T \subset T^{**} = -T^*$
skew-adjoint	$T = T^{**} = -T^*$

Connection between representations of Heisenberg groups and algebras

In the finite-dimensional case the standard functor from the category of finite-dimensional Lie groups to the category of finite-dimensional Lie algebras maps every finite-dimensional representation of a Lie group to a finite-dimensional representation of the corresponding Lie algebra. In this section we establish an analog of this construction for the infinite-dimensional case.

Theorem. (Stone, 1932.) The map $A \mapsto (t \mapsto \exp(tA))$ establishes a bijective correspondence between skew-adjoint operators A on a Hilbert space and representations of $\mathbf{R}$ in the group of unitary operators with the strong topology.

Theorem. Suppose we have a representation of a Heisenberg group. Fix an element of the corresponding Heisenberg algebra and take the Lie algebra homomorphism from $\mathbf{R}$ to the Heisenberg algebra that sends 1 to this element. Compose it with the exponential map and then with the representation. We obtain a continuous homomorphism from $\mathbf{R}$ to the topological group of unitary operators with the strong topology. By Stone's theorem we get an skew-adjoint operator. Hence we have a mapping F from the Heisenberg algebra to skew-adjoint operators. To obtain a Lie algebra homomorphism, denote by D the intersection of the domains of all operators in the image of F and restrict everything to D. We claim that D is a dense subspace and the restriction of F is a representation of the Heisenberg algebra.

In the finite-dimensional case there is a functor that goes the other way round: Every morphism of Lie algebras is mapped to a morphism of the corresponding simply connected Lie groups. Moreover, this functor is an equivalence of the categories of Lie algebras and simply connected Lie groups. In the infinite-dimensional case this is not true. Later we give an example of a representation of a Heisenberg algebra that does not correspond to any representation of the corresponding Heisenberg group.

Elementary properties of representations

Definition. A representation of a Heisenberg group on a Hilbert space H is called *irreducible* if H is nontrivial and any closed subspace of H that is invariant under the action of the group coincides with H or with the zero subspace.

Proposition. If F is an irreducible representation of a Heisenberg group, then for all real t we have $F(0,t) = \exp(ht)I$ for some unique imaginary number h, where I is the identity operator.

Proof. The image of $(0,t)$ commutes with the image of F for all t. By the infinite-dimensional Schur's lemma we have $F(0,t) = \lambda(t)I$ for some continuous homomorphism $\lambda\colon \mathbf{R} \to \mathbf{U}$, where $\mathbf{U}$ is the group of complex numbers of norm 1. Obviously, $\lambda(t) = \exp(ht)$ for some unique imaginary number h.

Definition. The number h defined in the previous proposition for an arbitrary irreducible representation of a Heisenberg group is called the *parameter* of the representation. If $h = 0$, then the representation is called *trivial*.

Proposition. The map $u \mapsto ((v,t) \mapsto \exp(u(v)))$ establishes a bijective correspondence between $\mathrm{Hom}(V, \mathbf{I})$ and the set of all isomorphism classes of trivial irreducible representations of the Heisenberg group for a symplectic vector space (V, ω). Here $\mathbf{I} := \{z \in \mathbf{C} \mid \Re z = 0\}$ denotes the set of all imaginary numbers.

Proof. A trivial representation factors through the Lie group homomorphism $(v,t) \mapsto v$ from the Heisenberg group to the vector space V with the additive Lie group structure.

Definition. A representation of a Heisenberg algebra on a Hilbert space H with a dense subspace D is called *irreducible* if H is nontrivial and any closed subspace G of H such that $G \cap D$ is invariant under the action of the algebra coincides with H or with the zero subspace.

Proposition. If F is an irreducible representation of a Heisenberg algebra, then for all real t we have $F(0,t) = htI$ for some unique imaginary number h, where I is the identity operator.

Proof. The image of $(0,t)$ commutes with the image of F for all t. By the infinite-dimensional Schur's lemma we have $F(0,t) = \lambda(t)I$ for some continuous homomorphism $\lambda\colon \mathbf{R} \to \mathbf{C}$. Since $\lambda(t)I$ is skew-symmetric, we have $\lambda(t) = ht$ for some unique imaginary number h.

Definition. The number h defined in the previous proposition for an arbitrary irreducible representation of a Heisenberg algebra is called the *parameter* of the representation. If $h = 0$, then the representation is called *trivial*.

Proposition. The map $u \mapsto ((v,t) \mapsto u(v))$ establishes a bijective correspondence between $\mathrm{Hom}(V, \mathbf{I})$ and the set of all isomorphisms classes of trivial irreducible representations of the Heisenberg algebra for a symplectic vector space (V, ω).

Proof. A trivial representation factors through the Lie algebra homomorphism $(v,t) \mapsto v$ from the Heisenberg algebra to the vector space V regarded as an abelian Lie algebra.

Schrödinger representations of Heisenberg groups and algebras

In this section we define a series of nontrivial irreducible representations of Heisenberg groups and algebras. First we need to look deeper into the structure of a symplectic vector space.

Definition. A *polarization* of a symplectic vector space (V, ω) is a pair (W, X) of subspaces of V such that $V = W \oplus X$, the form ω vanishes on W and X and defines a nondegenerate pairing between W and X, which we denote by $w \cdot x$. A *Lagrangian subspace* is a vector subspace of V that appears as a part of a (unique) polarization of (V, ω).

It follows that $\omega(w + x, w' + x') = w \cdot x' - w' \cdot x$ for arbitrary $w, w' \in W$ and $x, x' \in X$.

Definition. A *Schrödinger representation* of a Heisenberg group with a nonzero parameter $h \in \mathbf{I}$ corresponding to a polarization (W, X) of a symplectic vector space (V, ω) is the representation R_h of the Heisenberg group for the symplectic vector space (V, ω) on the Hilbert space $L^2(W, \mathbf{C})$ such that for arbitrary $w, z \in W$, $x \in X$, $t \in \mathbf{R}$, and $f \in L^2(W, \mathbf{C})$ we have $R_h(w + x, t)(f)(z) = \exp(h(t + z \cdot x + w \cdot x/2)) f(z + w)$.

Theorem. Any Schrödinger representation of a Heisenberg group is irreducible.

Definition. A *Schrödinger representation* of a Heisenberg algebra with a nonzero parameter $h \in \mathbf{I}$ corresponding to a polarization (W, X) of a symplectic vector space (V, ω) is the representation S_h of the Heisenberg algebra for the symplectic vector space (V, ω) on the Hilbert space $L^2(W, \mathbf{C})$ with the dense subspace $S(W, \mathbf{C})$ of Schwartz functions such that for arbitrary $w, z \in W$, $x \in X$, $t \in \mathbf{R}$, and $f \in S(W, \mathbf{C})$ we have $S_h(w, x, t)(f)(z) = h(t + z \cdot x) f(z) + \partial_w f(z)$. Here ∂_w is the derivation corresponding to the constant vector field on W with value w.

Theorem. Any Schrödinger representation of a Heisenberg algebra is irreducible.

Uniqueness of irreducible representations with a given parameter

Theorem. (von Neumann, 1931.) Every irreducible representation of a Heisenberg group with a nonzero parameter h is unitarily equivalent to the Schrödinger representation with the parameter h.

Hypothesis. (Stone, 1930.) Every irreducible representation of a Heisenberg algebra with a nonzero parameter h is unitarily equivalent to the Schrödinger representation with the parameter h.

Stone's hypothesis turned out to be wrong. In fact, Stone gave a sketch of a supposed proof of this hypothesis in his paper. When von Neumann tried to understand this sketch, he realized that one needs to put some integrability condition on representations of the Heisenberg algebra. In the end he obtained his theorem, which he published with a complete proof in 1931.

In the next section we look at the simplest nontrivial case $n = 1$ and $|h| = 1$. We give a counterexample to Stone's hypothesis in this case. Later we discuss a correction to Stone's hypothesis.

Examples

The easiest case is $n = 0$. It is obvious that the one-dimensional Schrödinger representations $t \to \exp(ht)I$ and $t \to htI$ are the only nontrivial irreducible representations.

The next case is $n = 1$. Assume $|h| = 1$. Fix an irreducible representation F of the Heisenberg group with these parameters. Choose $w \in W$ and $x \in X$ such that $w \cdot x = 1$. We have group homomorphisms from $\mathbf{R}$ to the Heisenberg group: $s \to (sw, 0)$, $t \to (tx, 0)$, and $u \to (0, u)$. Composing with our representation and applying Stone's theorem we see that $F(sw, 0) = \exp(sP)$, $F(tx, 0) = \exp(tQ)$, and $F(0, u) = \exp(hu)I$ for some skew-adjoint operators P and Q. The images of these three homomorphisms generate the Heisenberg group, therefore F is determined uniquely by P and Q. Suppose we are given two arbitrary skew-adjoint operators P and Q. When do they correspond to some representation of the Heisenberg group? All relations between the elements of the Heisenberg group are generated by the three commutator relations between three one-parameter subgroups. The identity operator commutes with everything, hence two of these relations are always satisfied. The only nontrivial one is $(sw, 0)(tx, 0)(sw, 0)^{-1}(tx, 0)^{-1} = (0, st)$. Therefore the pair (P, Q) corresponds to a representation of the Heisenberg group if and only if $\exp(sP)\exp(tQ)\exp(sP)^{-1}\exp(tQ)^{-1} = \exp(hst)I$.

The corresponding example for Heisenberg algebras is almost the same. Let us point the differences. We have $F(sw, 0) = sP$, $F(tx, 0) = tQ$, and $F(0, u) = huI$ for some skew-symmetric operators P and Q. The only nontrivial relation is $[(sw, 0), (tx, 0)] = (0, st)$. The pair (P, Q) corresponds to a representation of the Heisenberg algebra if and only if $[sP, tQ] = hstI$ or, equivalently, $[P, Q] = hI$.

We already know that representations of Heisenberg groups automatically produce representations of Heisenberg algebras: If P and Q are skew-adjoint operators and $\exp(sP)\exp(tQ)\exp(sP)^{-1}\exp(tQ)^{-1} = \exp(hst)I$, then $[\hat{P}, \hat{Q}] = hI$, where $\hat{P}$ and $\hat{Q}$ are P and Q restricted to the dense subspace D defined earlier. The converse of this statement is false.

Here is an example of two operators P and Q, which are not only skew-symmetric but also essentially skew-adjoint (hence they have unique skew-adjoint extensions $\bar{P}$ and $\bar{Q}$, which are their closures) such that the equality $\exp(s\bar{P})\exp(t\bar{Q})\exp(s\bar{P})^{-1}\exp(t\bar{Q})^{-1} = \exp(hst)I$ does not hold in general. Suppose M is the Riemann surface of the square root without the origin, $H = L^2(M, \mathbf{C})$ and D is a dense subspace of H consisting of all compactly supported smooth functions. Define two essentially skew-adjoint endomorphisms of D as follows: $P = \partial_x$ and $Q = hx + \partial_y$. Here ∂ means partial derivative and hx means multiplication by a function. Obviously $[P, Q] = hI$. However, the corresponding relation for the exponents does not hold. This is due to the fact that $\exp(t\partial_x)$ moves a function in the horizontal direction by t units and $\exp(t\partial_y)$ does the same for the vertical direction. Hence the commutator $\exp(sP)\exp(tQ)\exp(sP)^{-1}\exp(tQ)^{-1}$ moves the function to another sheet of the Riemann surface and multiplies it by some other function. However, the other side $\exp(hst)I$ is just multiplication by constant, it cannot move the function to another sheet. Hence the equality does not hold. The idea of this examples is due to Nelson. Reed and Simon give the details in §VIII.5. Fuglede in his paper gives a similar example of this kind with weaker conditions on P and Q.

Correction to Stone's hypothesis

Theorem. (Rellich, 1946; Dixmier, 1958; Kilpi, 1962.) Every irreducible representation F of a Heisenberg algebra with a nonzero parameter h such that for every $w \in W$ and for every $x \in X$ with $w \cdot x = 1$ the operator $F(w,0)^2 + F(x,0)^2$, which is a symmetric endomorphism of D, is essentially self-adjoint, is unitarily equivalent to the Schrödinger representation with the parameter h.

References

In this survey we touched a large area of mathematics. We had to omit many interesting results. For example, Schmüdgen in his two papers discusses a large class of counterexamples to Stone's hypothesis and modifications that make it correct. Putnam's book is a good survey of the area. Two surveys by Summers and Rosenberg are also useful. The book by Jørgensen and Moore is on the same topic but with an emphasis on Banach spaces and smooth functions instead of Hilbert spaces and analytic functions. Chapter 11 of the book by Barut and Raczka contains an exposition of Gårding's general representation theory of Lie algebras by unbounded operators.

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