A very gentle introduction to functional field theory

Def: An oriented 1-dimensional topological field theory (TFT) is a s.m. functor \( F : \text{Bord}_1 \to T \) where the source and target are symmetric monoidal (s.m.) categories and the functor preserves the monoidal structure up to an isomorphism.

Here \( T \) is an arbitrary target category and \( \text{Bord}_1^{\text{or}} \) is the category of 1-dimensional oriented bordisms: objects are compact oriented 0-manifolds, i.e., finite collections of points with signs: \( + \leftrightarrow - \) and morphisms are 1-oriented diffeomorphism classes of 1-bordisms (relative boundary):
composition is gluing:

identity is the trivial endomorphism.

monoidal product is disjoint union (on objects and morphisms).

monoidal unit is the empty endomorphism.
Generators & relations for $\text{Bord}^0$

Prop. $\text{Bord}^0$ is generated as a symmetric monoidal category by

\[
\begin{align*}
+ & \quad - \\
\circ & \quad \circ
\end{align*}
\]

\[
\begin{align*}
+ \circ \text{id} & = t \circ \text{id} \\
\text{id} & = - \circ \text{id}
\end{align*}
\]

Proof: classification of 1-dimensional manifolds

1-dimensional Morse theory

Remark. These are precisely the definition of $-$ as the dual object of $+$. 
Prop. The category of 1-dimensional oriented TFTs is equivalent with target $T$ is equivalent to $(T_{\text{dual}})^\sim$.

Here $T_{\text{dual}}$ is the full subcat of dualizable objects in $T$, and $\sim$ has the same objects as $\sim$, but only invertible maps.

Example: $T = \text{Vect}_k$:

$(T_{\text{dual}})^\sim = (\text{Vect}_k, \text{finite-dimensional}, \text{isos})$

Proof:

$F (\uparrow \leftarrow \downarrow) = F (\uparrow) \otimes F (\downarrow) \otimes \cdots$

$\otimes: F (\uparrow), F (\downarrow), F (\circ), F (\odot)$

Generators: $V, W, V \otimes W \rightarrow k, k \rightarrow W \otimes V$

Relations: $F (\begin{array}{c} \uparrow \downarrow \\ \downarrow \uparrow \end{array}) = F (\begin{array}{c} \uparrow \downarrow \\ \downarrow \uparrow \end{array}) = \text{id}_V$. 
Example Categorical treatment of traces:

\[
\text{Def } \xrightarrow{\text{thick}} \quad \text{thick morphism}
\]

\[
\text{Its Trace is } y \xrightarrow{\infty} y
\]

Problems depend on \( z, t, b \).

Eq. relation: \((z, t, b) \rightarrow (z', t', b')\)

\[
\begin{align*}
\theta & \sim \gamma \sim \delta \\
\hat{C}(x, y) & = (z, t, b) / \sim
\end{align*}
\]
Def $X$ has trace property if

$$\exists C^{th}(X,X) \implies C(I,I)$$

$\hat{C}(X,X)$ \(\text{tr}\)

Ex

1) Vect fd all

2) comp. Haus. loc. conv. TVS, thick $\iff$ nuclear, approx. prop. $\implies$ trace prop.

Id $\in$ FinRank comp. opn., $\text{tr} = \text{tr}$ cat. trace preserving

$B(H)$ does not have a finite-Szeman less Hilbert space

Then Def trace property

$\text{tr}(f,g) = \text{tr}(f \circ g)$

we trace property needed.
Def \( X \) semi-dualizable if

\[ \exists \text{ Hom} (X, Y) \]

Prop. \( X \) s.-d.

\[ C (1, X \otimes X^\vee) \cong \bigwedge (X, y) \]

Prop. \( X \) dualizable

\[ \cong (X, y) \cong C(X, y) \]
Smooth field theories

**Def**

\[ V \downarrow \uparrow \text{vector bundle} \]

\[ M \]

A connection on \( \nabla \) is a subbundle \( H \subset TV \)

such that the induced map \( H \subset TV \rightarrow TM \)

is an isomorphism.

and \( H \) is preserved by operations.

\[ V' \rightarrow V \]

\[ V' := V \times M' \]

\[ M' \rightarrow M \]

\[ f \rightarrow f \]

V' has an induced connection.

\[ V' = \mathfrak{p} \circ f^* V. \]

**Def**

\[ s: [0, 1] \rightarrow M \quad \text{a smooth path} \]

\[ \mathcal{P} \in \mathfrak{m} \]

\[ \text{Trans: } V_{s(0)} \rightarrow V_{s(1)} \]

is defined as follows:

\[ s^* V \triangleleft \text{canonical vector field} \]

\[ [0, 1] \rightarrow \]
Properties: 1) $P \leq P'$ if for every $p' \in P'$, there exists $p \in P$ such that $Tran(p) = p'$. 

2) $P = \{0, 0\}$: $Tran(0,0) = id$. 

3) $p \to p''$ $Tran(p) = Tran(p) \circ Tran(p)$. 

This looks like a field theory!

Equip 1-dimensional manifolds with a geometric structure given by a smooth map to $M$: $\text{Bord}_1(M)$. 

Problem: composition is undefined! 

Solutions: 1) allow piecewise-smooth paths. 

2) consider thin homotopy classes. 

3) consider homotopy classes locally (doesn't work for us because parallel transport is not invariant) 

Segal spaces: split paths instead of gluing.
1) Sitting instants:

\[ [0, 1] \xrightarrow{s} M \] has sitting instants

if \( s \big|_{[0, \varepsilon)} \) and \( s \big|_{(-1-\varepsilon, 1]} \)

are constant.

Paths with sitting instants
can be composed!

Problem: paths with sitting
instants don't occur in physics.

2) Thin homotopy classes.

\[ [0, 1] \xrightarrow{s, s'} M \] are thin homotopic

if

\( \exists [0, 1] \times [0, 1] \xrightarrow{h} M \)

\( h_0 = s, \ h_1 = s' \),

\( \text{rank } h \leq 1 \),

"thin"

Instead of smooth maps \( B \to M \)
we take their thin homotopy equiv. classes.

Every equivalence class contains a map with sitting instants. Composition is well-defined!

Problem: geometric structures such as Riemannian metrics, symplectic forms, spin structures cannot be defined on equivalence classes, only on actual bordisms.

3) Collars: equip objects, i.e., 0-dimensional manifolds with a 1-dimensional germ (neighborhood called a collar.

Example: \((\text{-}) \xrightarrow{\alpha} (\ast)\) = \((\ast) \xrightarrow{\beta} (\ast)\) = \((\ast) \xrightarrow{\gamma} (\ast)\)

This resolves all of the above objections!
Problem While this approach provides an elegant framework for many applications, it is hard to work with in practice because the space of objects is infinite-dimensional.

(Existing model-categorical books allow one to work with infinite-dimensional spaces of morphisms, but not objects.)

4) Segal categories of bordisms. Instead of gluing 2 bordisms into 1, split 1 bordism into 2!

Example:

\[
\text{\begin{tikzpicture}
\fill (-0.5,0) circle (1.5pt);
\fill (0.5,0) circle (1.5pt);
\draw (-0.5,0) .. controls (0,-0.5) .. (0.5,0);
\end{tikzpicture}} = \begin{tikzpicture}
\fill (-0.5,0) circle (1.5pt);
\fill (0.5,0) circle (1.5pt);
\fill (0,-0.5) circle (1.5pt);
\fill (0,0.5) circle (1.5pt);
\draw (-0.5,0) .. controls (0,-0.5) .. (0.5,0);
\draw (-0.5,0) .. controls (0,0.5) .. (0.5,0);
\end{tikzpicture}
\]
Problem: This doesn't seem to form a category.
Composition is not always defined.

Solution: Use the framework of model categories to interpret the above construction as a "resolution" for an actual category. Secretly this amounts to allowing piecewise-smooth maps, but without associated technical difficulties, i.e., we only work with smooth, not piecewise-smooth, maps.
Smooth categories

Physically meaningful field theories

send smooth families (i.e., bundles) of bordisms to smooth families of linear maps between vector spaces (i.e., smooth linear maps of smooth vector bundles).

Idea (*) Any mathematical structure can be made "smooth" by considering sheaves with values in this structure.

First approximation For any fixed smooth manifold $S$ consider $S$-indexed families: $\text{Bord}_1^\text{or}(X)_S := \{\text{Ob}: B \rightarrow X, \text{Mor}: S \rightarrow \text{bundle}\}$

\[\text{Vect}_S = \text{vector bundles over } S\]

\[\text{FT}_1^\text{or}(X)_S := \text{Fun}(\text{Bord}_1^\text{or}(X)_S, \text{Vect}_S)\]
We want to look at "compatible families" for all possible $S$.

A smooth map $S \to S'$ (think open subset) induces $f^*: \text{Bord}_1^\text{or}(X)_S \to \text{Bord}_1^\text{or}(X)_{S'}$.

$f^*: \text{Vect}_S \to \text{Vect}_{S'}$

$f^*: \text{FT}_1^{\text{or}}(X)_S \to \text{FT}_1^{\text{or}}(X)_{S'}$.

Roughly: $\forall S: F_S \in \text{FT}_1^{\text{or}}(X)_S$ such that $\forall S \to S': F_S \xrightarrow{\cong} F_{S'}$ rather isomorphism.

$s \xrightarrow{f} s' \xrightarrow{f'} s''$

$f^* f'^* F_{s''} \cong \alpha_{f' \circ f} F_s$

$f^* \alpha_f \cong \# f^* F_{s'} \cong \alpha_f$

How do we axiomatize/define such objects?

How can we compute with them?
Recall (\(\star\)):

Definition: A presheaf (on smooth manifolds) is a functor \(\text{Man}^\text{op} \to T\).

Definition: A morphism of presheaves is a natural transformation of functors \(\text{Man}^\text{op} \to T\).

What is \(T\)?
Functor of points

Last time:

A smooth field theory $F$ is:

1) $F_S \in FT^*_1(X)_S$

2) an isomorphism $\forall S' \in S:
   \quad f^* F_{S'} \cong F_S$

3) coherence condition for $S' \to S' \to S$.

Today $F$ is a global section of the presheaf $FT^*_1(X)$.

Definition A presheaf on Man is a functor $F: \text{Man}^{\text{op}} \to \text{Set}$.

A morphism of presheaves is a natural transformation of functors.

Idea: A presheaf $F$ on Man is a "smooth space" elements of $F(S)$ are smooth maps $S \to F$

$F(f): F(S) \to F(S')$ is precomposition $S' \to S \to F$. 
Example. Any smooth manifold $M$ is a generalized smooth space:

$y(M) : \text{Man}^{op} \to \text{Set}$

$S \mapsto C^\infty(S, M)$,

$f \colon S' \to S \mapsto C^\infty(f, M) : C^\infty(S', M) \to C^\infty(S, M)$

Def. $y : \text{Man} \to \text{Fun} (\text{Man}^{op}, \text{Set})$

is the Yoneda embedding.

It sends a smooth manifold to its associated generalized space.

Lemma (Yoneda) There canonical isomorphism between

$\{ y(M) \to F \}$ and $F(M)$.

Proof. Given $y(M) \to F$, $\alpha(\text{id}_M) \in F(M)$.

Given $\beta \in F(M)$, define $y(M) \to F$

$f \colon S' \to S \mapsto f^* \beta$. 


Examples: 1) smooth manifolds.
2) mapping spaces.

\[ C^\infty (M, N) \in \text{Fun}(\text{Man}^{\text{op}}, \text{Set}) \]
\[ S \to C^\infty (M, N) \leftrightarrow S \times M \xrightarrow{\partial} N \]
\[ C^\infty (M, N)(S) := C^\infty (S \times M, N) \]

3) differential forms:

\[ \Omega^k (M) := \text{diff k-forms on } M \]
\[ \Omega^k (f) := \text{pullback} \]

abstract de Rham complex:

\[ \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \]

4) can now make sense of

\[ \Omega^* C^\infty (M, N) \]
Last time:

Functors $\text{Man}^{op} \to \text{Set}$ are "generalized smooth spaces."

Today: 1) Gluing property

2) Stacks

3) Principal $G$-bundles with $G$ as a stack.

4) The de Rham complex of $BG$.

1) Gluing property:

$F: \text{Man}^{op} \to \text{Set}$

$F(M) \cong \{ y(M) \to F \}$

Suppose $M = U \cup V$, $U$ and $V$ open on $M$.

$F(M)$ should be the data of $y(U) \to F$, $y(V) \to F$, and $y(U \cap V) \to F$.

A (pullback) square:

In other words: a map $y(M) \to F$ should be the same data as maps $y(U) \to F$, $y(V) \to F$ such that $\Delta^{y(U \cap V)} = \rho^{y(U \cap V)}$. 
Definition (Only works for Man)

A sheaf on Man is a presheaf $F$ on Man such that:

1) $\forall M \forall U, V : M = U \cup V$ open in $M$

\[
\begin{array}{ccc}
F(M) & \overset{\top}{\sim} & F(U) \\
\downarrow & & \downarrow \\
F(U \cup V) & \overset{\sim}{\sim} & F(V)
\end{array}
\]

is a pullback square

2) $\forall \{M_i\}_{i \in I} : F\left( \bigsqcup_{i \in I} M_i \right) \overset{\sim}{\sim} \prod_{i \in I} F(M_i)$

All examples so far are sheaves.
Nonexample: Fix a Lie group $G$.

$\mathcal{F}(M)$: the set of isomorphism classes of principal $G$-bundles with connection over $M$.

Not a sheaf: $M = S'$, $U = C$, $V = C$, $U \cap V = -$

$\mathcal{F}(U) = \mathcal{F}(V) = \mathcal{F}(U \cap V) = \{x^2\}
\uparrow
\text{the trivial principal } G\text{-bundle}

$\mathcal{F}(S') = \{\text{conjugacy classes}\}$ of elements of $G$.

Second attempt: replace set by $G\text{grp.}$ (groupoids).

$F: \text{Man}^{\text{op}} \to G\text{grp}$

$F(M)$: the groupoid of principal $G$-bundles with connection and connection-preserving isomorphisms.

Claim: $F$ is a sheaf!
Example: \( M, \gamma, V \) as above

\[
\begin{align*}
F(u) &= F(v) = \ast \sslash G \\
F(u \cap v) &= \ast \sslash G^2 \\
F(u) \rightarrow F(u \cap v)
\end{align*}
\]

\[
\ast \sslash G \rightarrow \ast \sslash G^2
\]

\[
G_1 \times G_2 \\
\ast \sslash G
\]

\[
\ast \sslash G^2
\]

\[
\ast \sslash H
\]

Ob: \( g_1 \in G_1, g_2 \in G_2 \)

\[
p_1(g_1) \xrightarrow{\alpha} p_2(g_2)
\]

Mon: \( (g_1, g_2, \alpha) \rightarrow (g'_1, g'_2, \alpha') \)

are \( g_1 \rightarrow g'_1, g_2 \rightarrow g'_2 \)

such that

\[
\begin{align*}
p_1(g_1) &\rightarrow p_2(g_2) \\
p_1(g'_1) &\rightarrow p_2(g'_2)
\end{align*}
\]

\[
\alpha \circ \alpha'
\]

Ob: \( \gamma_1, \gamma_2 = \ast \quad \alpha = (h_1, h_2) \)

Mon: \( q_1, q_2 \in G \)

\[
\begin{array}{c}
h_1, h_2 \rightarrow x \\
q_1, q_2 \rightarrow q_1, q_2
\end{array}
\]

\[
\begin{array}{c}
1, 1 \\
h_1, h_2^{-1}
\end{array}
\]

\[
\begin{array}{c}
h_1, h_2 \\
h_1^{-1}, h_2^{-1}, 1
\end{array}
\]
Chern–Weil theory

$\text{Man} := \text{the category of smooth manifolds and smooth maps}$

$\text{PresSh}(T) := \text{Fun}(\text{Man}^{op}, T)$

$\text{Sh}(T) := \text{Fun}^k$, full subcategory of $\text{PresSh}(T)$ consisting of presheaves with the gluing property.

$\text{Sh}: \text{PresSh}(T) \longrightarrow \text{Sh}(T)$ is the sheafification functor.

$\text{PresSh}(\text{Set}) \xrightarrow{\text{Sh}} \text{Sh}(\text{Set})$

$\text{PresSh}(\text{Grp}) \xrightarrow{\text{Sh}} \text{Sh}(\text{Grp})$

$\Omega^k \in \text{Sh}(\text{Set}): \quad \Omega^k(M) = \text{diff. k-forms on } M$

$\text{Sh}(\text{Ab})$. \quad \Omega^k(f) = f^* \text{ pullback of form}$

$\Omega^k \xrightarrow{d} \ldots \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^0 \xrightarrow{\otimes Z \rightarrow \text{Sh}(\text{Set})}$

is a chain complex in $\text{Sh}(\text{Ab})$.

$B_V G \in \text{Sh}(\text{Grp}): \quad B_V G(M) := \left\{ \begin{array}{ll}
\text{Ob: principal } G\text{-}P \text{-bundle } \pi:M \longrightarrow \text{M} \\
\text{Mor: connection } - \text{pres.} \\
\end{array} \right\}$

$E_V G \in \text{Sh}(\text{Grp}): \quad E_V G(M) := \left\{ \begin{array}{ll}
\text{Ob: } \frac{P \times S_k}{\text{M}} \text{-bundle } \pi:P \longrightarrow Q \\
\text{Mor: } \frac{\pi}{\text{M}} \\
\end{array} \right\}$
We have an equivalence:

\[ E \downarrow G \xrightarrow{s^* \Theta} \pi_1^{-1} g \]

\( \text{i.e. a principal trivial bundle.} \)

\( G \)-bundle w/ connection and a section is the same thing as a connection 1-form.

The map \( E \downarrow G \) is a connection on \( E \downarrow G \to B \downarrow G \), the universal connection.

2) We have a canonical action

\( G \triangleleft \pi_1^{-1} \) \( G \): \( g \cdot (p, s) = (p, g \cdot s) \).

\( G \triangleleft \pi_1^{-1} \) \( g \cdot \alpha = g \cdot \Theta + \text{Ad}_{\alpha\Gamma}^{-1} (\alpha) \).

3) Recall: \( G \triangleleft F \), \( F \in \text{HSh} (\text{Grp}) \): \( G \in \text{PrSh} (\text{Set}) \).

\( F / / G \): \( (F / / G)(S) : = \text{F}(S) \uparrow G(S) \).

Define \( G \triangleleft A \in \text{Set} \): \( \text{A/} G := \left\{ \begin{array}{l} \text{Ob} = A \\ \forall \alpha \in G \exists \gamma \in G | \gamma \cdot \alpha = \gamma \cdot \alpha_2 \\ \forall \gamma \in G \end{array} \right\} \).

\( \text{J}_0 (A/ G) = \text{is classes of A/} G \)

\( = A / G \), the traditional quotient.

**Example:** \( F = \text{y}(M) \), \( G \triangleleft M \): free action, 1 \(-2\)-no fixed points

\( M / / G \to M / G, M / G \in \text{PreSh} (\text{Set}) \).

\( S \triangleleft M \): \( \text{C}^0 (S, M) \uparrow \text{C}^0 (S, G) \to \text{C}^0 (S, M / G) \)

**Remark:** If \( G \triangleleft M \) is proper, then \( M / G \in \text{PreSh} (\text{Set}) \) is a manifold, i.e., representable.
Example \[ E \xrightarrow{\nabla} G \xrightarrow{\parallel} G \xrightarrow{\nabla} G \]

Indeed: \[ \pi_1(*/G) = G, \]
\[ \pi_1(*) = * . \]

Example: \[ E \nabla G \xrightarrow{\parallel} G \xrightarrow{\nabla} B \nabla G \]

is an equivalence because locally every principal \( G \)-bundle has a section.

### Summary

\[ E \nabla G \]

\[ \downarrow \]

\[ B \nabla G \]

is a universal principal \( G \)-bundle with connection.

### Auxiliary

\[ \Omega^1_{*}(B \nabla G) = \pi_1(\Omega^1_E G) \Rightarrow \Omega^1(G \times E \times G) \]

= basic forms \( \Omega^1 \) on \( E \nabla G \)

1. \( G \)-invariant
2. \( \Omega^1(-) = 0 \).

The (Freed - Hopkins): \[ \Omega^{*}_G E \nabla G \approx (K_{03}, g^*, d) \]

\[ K_{03}\ G = \ Sym(\frac{\nabla id}{2}). \]

Wald algebra

Thus, \( \Omega^*_G B \nabla G \approx (\ Sym \ g^*, G^{*} \ G, d = 0) \).
Recall, the Borel quotient is $X \overset{\sim}{=} (EG \times X) / _\alpha G_x$. The stacky Borel quotient of $G \times X$ is $(E \nabla (G \times X)) / G_x = X \overset{\sim}{=} X \times G / G_x$.

The (Freed–Hopkins) $G \times X$:

1) $\Omega^* (E \nabla (G \times X)) = \Omega^*(X, \text{kos } g^*)$

2) $\Omega^* (X \times G, \nabla) = \text{base subcomplex of Weil model for equivariant cohomology}$. 
Cheeger--Simons theory via

Bundle 2-gerbes

Recall:
1) \(\text{Cart}: \) the cartesian site
\[\mathcal{O}_0 = \{0, 1, 2, \ldots, \}\]
\[\text{Hom}(m, n) = C^\infty(\mathbb{R}^m, \mathbb{R}^n).\]

2) \(\text{Sh}(\text{Cart}, \text{Set}) \subset \text{Presh}(\text{Cart}, \text{Set})\)
as sheaves:

3) \(\text{Fun}(\text{Cart}^{op}, \text{Set})\)

"generalized smooth manifolds"

\[\text{Cart} \rightarrow \text{Man} \rightarrow \text{Sh}(\text{Cart}, \text{Set})\]

Examples:

a) \(M, N \in \text{Man} \Rightarrow C^\infty(M, N)\)

b) \(\Omega^h, \Omega^h_c \in \text{Sh}\)

closed \(\in \text{Sh}\)

3) \(\text{Sh}(\text{Cart}, \text{Grpd}) \subset \text{Presh}(\text{Cart}, \text{Grpd})\)

"Smooth \(C^\infty\) stacks"

Examples:

a) Vect, Vect

b) \(\text{B}_G, \text{B}_G^\triangleright G\)

c) \(\text{B}_G^\triangleright \text{B}_G\)

Example:
\(E \rightarrow \text{B}_G, \text{universal principal } G\text{-bundle}\)
Theorem (Freed-Hopkins)
$$\Sigma^\infty_+ (E \gg H) = (Kos \{ g^* \}, d)$$
$$= \text{Sym} \left( \left( g^* \rightarrow g^* \right) \right) \frac{1}{2}$$
$$\Sigma^\infty_+ (B \gg H) = \left( \left( \text{Sym} \left( g^* \right) \right), 0 \right)$$

Today
1) introduce $\int B^A, \int B^A$
A abelian Lie group
$n = 1$: recover $\int B^A, \int B^A$.
2) compute $\int \int B^G \rightarrow B^U(1)$

---

Geometric models for cohomology

$H^n(-, \mathbb{R})$
$H^n(-, \mathbb{Z})$
$K^n$

$\Omega^\infty_+ \left( \bullet \right) \rightarrow \text{bundle gerbes w/com}$

$\text{Veect}_\n$

$\text{TMF}$

$\text{NO}: \text{Geometric families of bordisms}$

Names: Cheeger-Simons classes, Deligne cohomology, ordinary $\mathbb{F}$-cohom. bundle $(n-1)$-gerbes, circle $n$-bundles $(n, \n)$
A few words about higher groupoids

Set

Groupoid

2-groupoid

Def: An n-groupoid is a) Set n=0
b) a category "enriched" in (n-1)-groupoids

\forall x, y \in \text{Ob} \in \text{Set}

\text{Mor}(X, Y) \in (n-1)-\text{groupoid}

\text{Mor}(X_0, X_1) \times \text{Mor}(X_{n-2}, X_{n-1}) \cdots \times \text{Mor}(x_0, x_1)

\Rightarrow \text{Mor}(x_0, x_n)

For groupoids, homotopy is associative.

Examples:
1) Alg
2) \text{Mor}
Bimod
\text{Corr} x \leftarrow C \rightarrow y

Intertwiners
Isos
Models for higher groupoids

- $n$-groupoids
- Set
- Groupoids
- Bigrroupoids

For $n > 2$:

- Simplicial sets $X$ with
  - $k$-simplices that are $n$-truncated:
  - Are $k$-morphisms. $T^k(X, x) = 0$ \(\forall x \in X_0\)
  - \(\forall k > n\).

In practice, many $n$-groupoids for $n > 1$ are abelian: $k$-morphisms form an abelian group for $0 \leq k \leq n$.

Instead of $sSet$ have $sAb$

Dold-Kan correspondence

\[
\begin{array}{rl}
\Delta & \rightarrow \text{Ch}_\geq 0 \\
m & \mapsto \mathbb{N}Z[\Delta^m] \\
sAb & \xrightarrow{\mathbb{N}} \text{Ch}_\geq 0 \\
\end{array}
\]

weak equiv
gauged
Explanation: \[ X_0 \xrightarrow{d} X_1 \xrightarrow{d} X_2 \xrightarrow{d} \quad d^2 = 0 \]

encodes an \( \infty \)-groupoid (\( k \)-morphisms are \( \infty \)-groupoids)

Objects: \( X_0 \)

1. Morphisms: \( A, B \in X_0, f: A \to B, f \in X_1, df = A-B \).

2. Morphisms:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow h & & \downarrow g \\
B & \xrightarrow{g} & A
\end{array}
\]

\( h \in X_2, dh = f - g \).

\( d(dh) = df - dg = (A-B) - (A-B) \)

\( k \)-morphisms:

\( p \in X_k, dp = \alpha - \beta \).

Remark: the same element \( f \in X_p, p > 0 \)

encodes \( \infty \)-spaces between different pairs of objects \( A, B \): \( A-B = df \).

In particular: the space of all \( k \)-morphisms is \( X_0 \times X_1 \times \cdots \times X_k \).

Illustration: \( \text{Sing } X \quad \mathbb{Z}[\text{Sing } X] \)

\( \text{Singular simp. set.} \quad \text{Singular chains} \)
Example: \( G = \text{a abelian Lie group} \)
\( A = U(1) \)

\( (B \mathcal{G}) \cong \text{groupoid} \)
\( \text{objects: } \Omega^1(X) \)
\( \text{morphisms: } f \in \Omega^0(x, U(1)) \)
\( df = \omega_0 - \omega_1 \)

\[ B \mathcal{U}(1)(X) = \text{groupoid} \{ \star \} \text{ Object} \]
\[ Mor(\star, \star) = \Omega^0(X, U(1)) \]

In the language of chain complexes:

\[ (B \mathcal{U}(1))(X) = \]
\[ 0 \leftarrow \Omega^1(X) \xleftarrow{\frac{d\log}{d\log}} \Omega^0(X, U(1)) \]

Definition: \( (B^n \mathcal{U}(1))(X) = \)
\[ 0 \leftarrow \cdots \leftarrow 0 \leftarrow \Omega^0(X, U(1)) \]

Bundle \((n-1)\) - gerbes
Circle \(n\)-bundles
Cheeger - Simons characters
Deligne \& Cohomology
Ordinary differential cohomology
Recall

\[ \text{PSh}(\text{Man}, \text{Set}) \xrightarrow{\text{Sil}} \text{PSh}(\text{Man}, \text{Grop}) \xrightarrow{\text{PSh}(\text{Man}, \text{Set}) \xrightarrow{\text{faithful}} \text{PSh}(\text{Man}, \text{Set}))} \text{faithful} \]

\[ \text{Sh}(\text{Man}, \text{Set}) \xrightarrow{\text{Sil}} \text{Sh}(\text{Man}, \text{Grop}) \xrightarrow{\text{Sh}(\text{Man}, \text{Set}) \xrightarrow{\text{faithful}} \text{Sh}(\text{Man}, \text{Set}))} \text{not fact} \]

\[ \text{Sh}(\text{Man}, \text{D}) = \text{Fun}(\text{Man}^{\text{op}}, \text{D}) \]

\[ \text{Sh}(\text{Man}, \text{D}) \text{ is full subcategory} \]

\[ \text{satisfy descent} \]

\[ \text{Sh}(\text{Man}, \text{D}) \xrightarrow{\text{faithful}} \text{Sh}(\text{Cart}, \text{D}) \text{ (false for presheaves)} \]

\[ \text{Def} \quad A: \text{abelian Lie group} \]

\[ a: \text{Lie algebra of } A \]

\[ L \xrightarrow{\ker} a \xrightarrow{\exp} A \]

\[ (2\pi i \mathbb{Z} \rightarrow \mathbb{R} \xrightarrow{\exp} U(1)) \]

\[ 2\pi i \mathbb{Z} \xrightarrow{} \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \]

\[ \text{Def (Deligne - Tate)} \quad Z(p) := (2\pi i)^p \cdot \mathbb{Z} \]

\[ B(p) := (2\pi i)^p \cdot \mathbb{C} \]

\[ B(p) \otimes B(q) \rightarrow B(p+q) \]
Remark \( \text{Del} : \mathbb{R}^n \leftarrow \cdots \leftarrow \mathbb{R}^1 \leftarrow U(1) \)

\[ \mathbb{R}^n / \mathbb{R}_{ex} \mathbb{R}^n \leftarrow U(1)_5 \]

iso classes
of bundle gerbes
with connection

\( \text{Del} \rightarrow \mathbb{R}^n / \mathbb{R}_{ex} \mathbb{R}^n \) [0] computes
the iso class

\[ U(1)_5 \rightarrow \text{Del} \quad \text{embeds flat bundle gerbes} \]

Remark

\( \text{Exact triangle / cofiber sequence} \)

\[ 0 \leftarrow \mathbb{R}^n \leftarrow \cdots \leftarrow \mathbb{R}^1 \leftarrow \mathbb{R}^0 \leftarrow 0 \]

\[ B^h \mathcal{Z} \]

\[ B^h \mathcal{R} \]

\[ B^h U(1) \]

\[ 0 \leftarrow \cdots \leftarrow \mathbb{Z} \leftarrow 0 \]

\[ B^{n+1} \mathcal{Z} \]
Embedding of trivial bundle gerbes

\[ \mathcal{E}^n \leftarrow - \quad - \quad - \quad n^o \leftarrow 0 \]

\[ \mathcal{E}^n \leftarrow - \quad - \quad - \quad - \quad \leftarrow n^o \leftarrow \mathbb{Z} \]

\[ \mathcal{E}^n_{\mathbb{U}(1)} = \mathcal{O}^n \]
Important observation:
the ∞-groupoid of morphisms \( \text{dom} = B \)
\( B \) some bundle \((n-1)-\text{gerbe}\)
is precisely the \(\infty\)-groupoid of
bundle \((n-2)-\text{gerbes}\)!

In other words: morphisms have
geometry (like curvature!).

**Curvature:**
\[
\mathbb{R}^h \leftarrow \cdots \cdots \leftarrow \mathbb{R}^0 \leftarrow \mathbb{Z} \leftarrow 0
\]

**Underlying topological object**

\[
\mathbb{R}^n \leftarrow \cdots \cdots \leftarrow \mathbb{R}^0 \leftarrow \mathbb{Z} \xrightarrow{\text{Dixmier}} \mathbb{Z}^{n+1} \xrightarrow{\text{Duskin}} \mathbb{Z}^{n+1}
\]

**Embedding of flat bundle gerbes**

\[
\mathbb{R}^{\text{closed}} \leftarrow \mathbb{R}^{n-1} \cdots \cdots \mathbb{R}^{0} \leftarrow \mathbb{Z} \xrightarrow{ \mathbb{Z} \otimes \mathbb{U}(1) } 0
\]
The differential hexagon
bundle $(\mathcal{H}^n)$-gerbes w/ bundle $\mathcal{U}(1)$
$n$-gerbes w/ bundle $\mathbb{Z}$

trivial e.g.

$$\mathbb{Z}^n/\mathbb{Z} \oplus \mathbb{R}_e [n]$$

$$\mathbb{Z}^n \leftarrow \cdots \leftarrow \mathbb{R}^0$$

inclusion of trivial

Flat e.g.

$$\mathbb{Z}^n \leftarrow \mathbb{Z}^{n-1} \leftarrow \cdots \leftarrow \mathbb{R}$$

inclusion of flat

curvature

$$\mathbb{R}^n \leftarrow \cdots \leftarrow \mathbb{R}^0 \leftarrow \mathbb{Z}^{n+1}$$

topological class

$$\mathbb{Z} \leftarrow \cdots \leftarrow \mathbb{R}^0 \leftarrow \mathbb{Z}^{n+1}$$

$$\mathbb{Z} \leftarrow \cdots \leftarrow \mathbb{R}^0 \leftarrow \mathbb{Z}^{n+1}$$

$$\mathbb{Z} \leftarrow \cdots \leftarrow \mathbb{R}^0 \leftarrow \mathbb{Z}^{n+1}$$
\[ P \Rightarrow \exists \, x : A \xrightarrow{B} x \Rightarrow C \]

\[ B(f) : \psi \Rightarrow x \Rightarrow \exists \, y : B(p+x) \Rightarrow (b | \phi) \]

\[ \begin{array}{l}
\end{array} \]
Ponchline

action \quad \text{higher} \quad \text{CS-tay} \quad \text{connection} \quad (2k+1)-\text{form}

\[ S_{cs} : \wedge^{2k+1} (\Sigma) \to U(1) \]

\[ \otimes \quad \sum \quad \omega \wedge dw \]

\((4k+3)\) -- dimensionless

\[ \mathbb{Z} \xrightarrow{\text{Spec}} C^H \]

\[ A \to O(\text{Spec} A) \quad \text{injective: sufficiently many char} \]

\[ X \to \text{Spec} (\mathcal{O} X) \quad \text{injective: Hausdorff} \quad \text{surjective: any char is eval compact} \]

\[ \begin{array}{ccc}
\mathbb{D} & \circlearrowleft & \mathbb{W} \\
\mathbb{C} & \to & \mathbb{C}'
\end{array} \\
\begin{array}{ccc}
\mathcal{A} & \to & \mathcal{A}' \\
\mathcal{A} & \to & \mathcal{A}'
\end{array} \\
\begin{array}{ccc}
\mathcal{C}' & \to & \mathcal{C}'A1
\end{array} \]
Electromagnetism

\[ E : \mathbb{R}^+_t \to \mathbb{R}^1 \times \mathbb{R}^3 \text{ electric field} \]

\[ B : \mathbb{R}^+_t \to \mathbb{R}^2 \times \mathbb{R}^3 \text{ magnetic field/flux} \]

\[ J_E : \mathbb{R}^+_t \to \mathbb{R}^2 \times \mathbb{R}^3 \text{ electric current} \]

\[ \rho_E : \mathbb{R}^+_t \to \mathbb{R}^2 \times \mathbb{R}^3 \text{ electric charge density} \]

Relativistic:

\[ M^4 := \mathbb{R}^+_t \times \mathbb{R}^3 \quad (c = 1) \]

\[ \mathcal{F} := B - dt \wedge E \in \mathcal{M}^2 \]

\[ \mathcal{F}' := \rho_E - dt \wedge J_E \in \mathcal{M}^3 \]

Maxwell's equations:

\[ d \mathcal{F} = 0 \]

\[ d \times \mathcal{F} = \mathcal{J}_E \]

With magnetic current \( \mathcal{J}_B \in \mathcal{M}^3 \):

\[ \mathcal{J} = \mathcal{J}_B, \text{ integrable support on any space-like} \]

\[ \mathcal{J}(N) \in H^3_{c.s.}(N, \mathbb{R}) \text{ slice } N \]

\[ d \times \mathcal{F} = \mathcal{J}_E \]

\[ \mathcal{J}_B \in \ker H^3_{c.s.} \to H^3_{c.s.} \]

[\mathcal{J}_B] \in H^3(N, \mathbb{Z}) \text{ Dirac charge quantization} \]

\[ \text{comes from } \uparrow \]