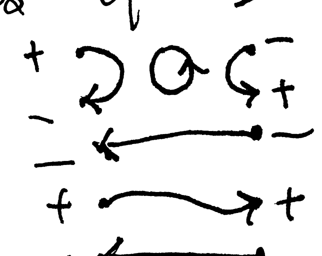
A very gentle introduction to functorial field theory

Def An oriented 1-dimensional topological field theory (TFT) is a s.m. functor $F: \mathrm{Bord}_1^{or} \to T$, where the source and target are symmetric monoidal (s.m.) categories and the functor preserves the monoidal structure up to an isomorphism. ∎

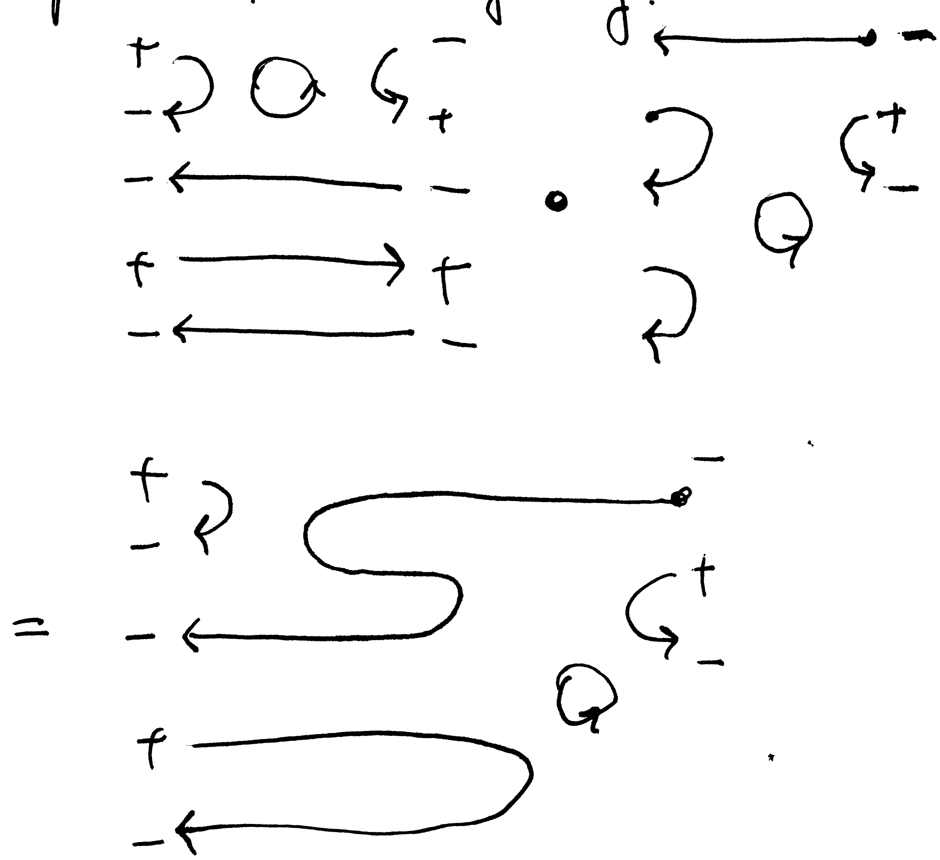
Here T is an arbitrary <u>target</u> category and Bord_1^{or} is the category of 1-dimensional oriented bordisms:

<u>objects</u> are compact oriented 0-manifolds, i.e., finite collections of points with signs: $+ \; - \; - \; + \; -$

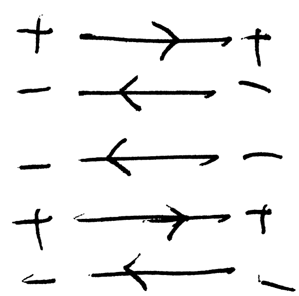
and <u>morphisms</u> are oriented diffeomorphism classes of 1-bordisms (relative boundary):



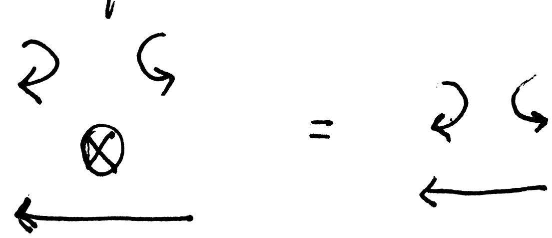
composition is gluing:



identity is the trivial bordism.



monoidal product is disjoint union (on objects and morphisms):

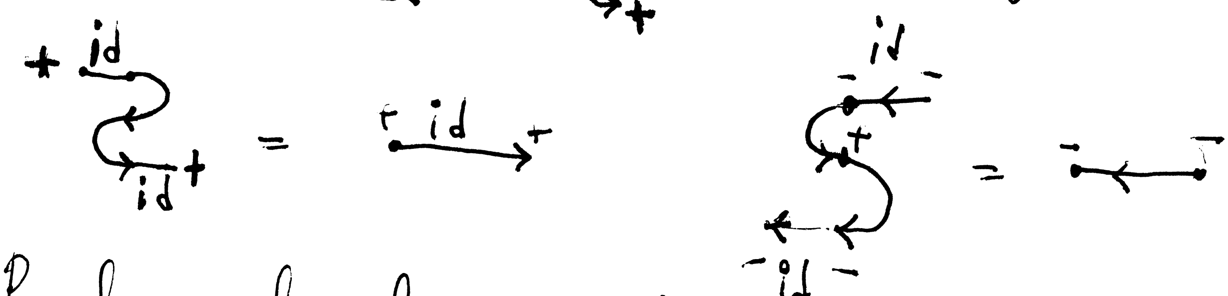


monoidal unit is the empty bordism

Generators & relations for Bord_1^{or}

<u>Prop</u> Bord_1^{or} is generated/related as a symmetric monoidal category by

$\overset{+}{\cdot}$ $\overset{-}{\cdot}$ 



Proof: classification of 1-dimensional manifolds / 1-dimensional Morse theory

<u>Remark</u> These are precisely the ~~defn~~ definition of $\cdot^-$ as the dual object of $\cdot^+$.

Prop The category of 1-dimensional oriented TFTs ~~is equi~~ with target T is equivalent to $(T^{\text{dual}})^{\sim}$ $\blacksquare$

where T^{dual} is the full subcat of <u>dualizable</u> objects in T,

and $\mathcal{C}^{\sim}$ has the same objects as $\mathcal{C}$, but only invertible maps.

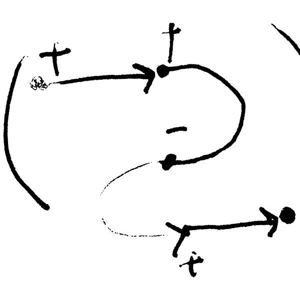
Example: $T = \mathrm{Vect}_k$:

$$(T^{\text{dual}})^{\sim} = \left(\mathrm{Vect}_k^{\text{finite-dimensional}}, \text{isos}\right)$$

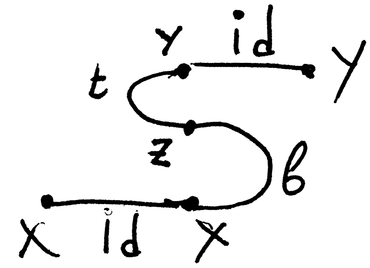
Proof $F(\overset{+}{\cdot}\ \overset{-}{\cdot}\ \overset{+}{\cdot}\ \overset{+}{\cdot}) = F(\overset{+}{\cdot}) \otimes F(\overset{-}{\cdot}) \otimes \cdots$

~~Def~~: $F(\overset{+}{\cdot})$, $F(\overset{-}{\cdot})$, $F($  $)$, $F($  $)$,

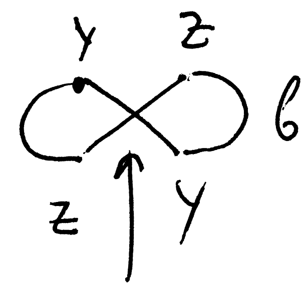
Generators: $F(\overset{+}{\cdot}) = V$, $F(\overset{-}{\cdot}) = W$, $V \otimes W \to k$, $k \to W \otimes V$

Relations: $F($  $) = F(\overset{+}{\cdot} \longrightarrow \overset{+}{\cdot}) = \mathrm{id}_V$.

Example Categorical treatment of traces:

Def $X \xrightarrow{\bullet} Y$ thick = 

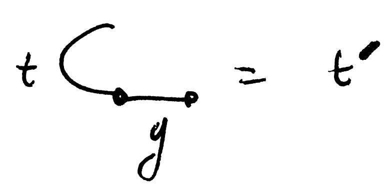
Def $X \xrightarrow{\bullet} X$ thick morphism

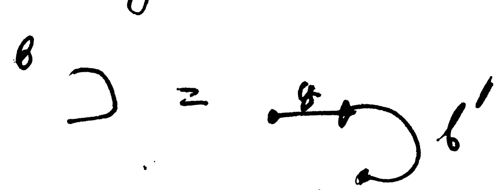
its trace is 

$s_{Y,Z}$ switching

Problem tr depends on Z, t, b.

Eq. relation: $(Z, t, b) \to (Z', t', b')$

$Z \xrightarrow{g} Z'$ 



Def $\hat{C}(X, Y) = (Z, t, b)/\sim$.

$\downarrow$

$C^{tk}(X, Y)$

Def X has <u>trace property</u> if

$$\begin{tikzcd}
& \hat{C}(X,X) \arrow[dl] \arrow[dr, "tr"'] & \\
C^{tk}(X,X) \arrow[rr, dashed, "\exists"] & & C(I,I)
\end{tikzcd}$$

Ex 1) $Vect_{fd}$ <u>all</u>

2) compl. Haus. loc. conv. TVS.

thick $\Longleftrightarrow$ nuclear

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \\
X' \arrow[r] & Y' \arrow[u]
\end{tikzcd}$$

Banach

$Y' \hat{\otimes} \mathrm{Hom}(X', \mathbb{C})$

$\cong$

$\mathrm{Hom}(X',$

approx. prop. $\Rightarrow$ trace prop.

$id \in \overline{\text{FinRank}}$ comp. oper.

$tr_{nuc} = tr_{cat}$.

$B(H)$ <u>does not</u> have a.p.; every Hilbert space does.

Enflo-Szankowski

Then **Def** trace pairing

$tr(f,g) = tr(f \circ g)$

<u>no</u> trace property needed.

Def X semi-dualizable if
$\exists$ Hom $(X, 1)$

Prop X s.-d.
$\mathcal{C}(1, Y \otimes X^\vee) \xrightarrow{\cong} \hat{\mathcal{C}}(X, Y)$

Prop X dualizable
$\hat{\mathcal{C}}(X, Y) \xrightarrow{\cong} \mathcal{C}(X, Y)$

Smooth field theories

Def
$$\begin{array}{c} V \\ \downarrow p \\ M \end{array}$$ vector bundle

A <u>connection</u> on $\begin{array}{c} V \\ \downarrow p \\ M \end{array}$ is a vector subbundle $H \subset TV$

such that the induced map $H \subset TV \xrightarrow{Tp} TM$

is an isomorphism.

and H is preserved by operations $+, \cdot$.

Def
$$\begin{tikzcd} V' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & V \arrow[d] \\ M' \arrow[r, "f"'] & M \end{tikzcd}$$

$$V' := V \times_M M'$$

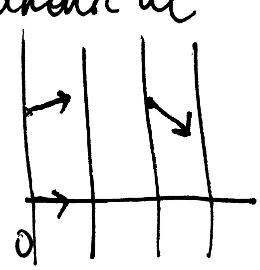
V' has an induced connection.

$$V' = \cancel{p}\, f^* V.$$

Def $s : [0,1] \to M$ a smooth path

$\mathrm{Tran}^P_s : V_{s(0)} \to V_{s(1)}$ is

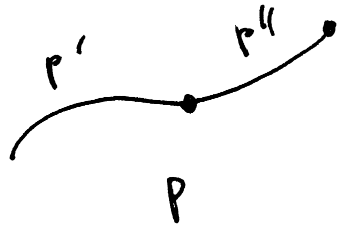
defined as follows: $\begin{array}{c} s^* V \\ \downarrow \\ [0,1] \end{array}$ canonical vector field



Properties: 1) $P \cong P'$ diffeomorphism

$$\mathrm{Tran}_P = \mathrm{Tran}_{P'}.$$

2) $P = [0,0]$: $\mathrm{Tran}_{[0,0]} = \mathrm{id}.$

3) 

$\mathrm{Tran}_p = \mathrm{Tran}_{p''} \circ \mathrm{Tran}_{p'}.$

This looks like a field theory!

Equip 1-dimensional bordisms with a geometric structure given by a smooth map to M: $\mathrm{Bord}_1^{or}(M)$

Problem: composition is undefined!

Solutions: 1) allow piecewise-~~linear~~ smooth paths

2) Consider thin homotopy eq classes

3) consider homotopy classes ~~only~~ (doesn't work for us because parallel transport is not invariant)

4) Segal spaces: <u>split paths</u> instead of <u>gluing</u>

Details:

1) Sitting instants:

$[0,1] \xrightarrow{s} M$ has <u>sitting instants</u>

if $s|_{[0,\varepsilon)}$ and $s|_{(1-\varepsilon,1]}$

are constant.

Paths with sitting instants can be composed!

<u>Problem:</u> paths with sitting instants don't occur in physics.

2) Thin homotopy classes.

$[0,1] \xrightarrow[s,\ s']{} M$ are <u>thin homotopic</u> if

$$\exists\ [0,1] \times [0,1] \xrightarrow{h} M$$

$h_0 = s$, $h_1 = s'$, $\underbrace{\operatorname{rank} h \leq 1}_{\text{"thin"}}$,

Instead of smooth maps $B \to M$

we take their thin homotopy equiv. classes.

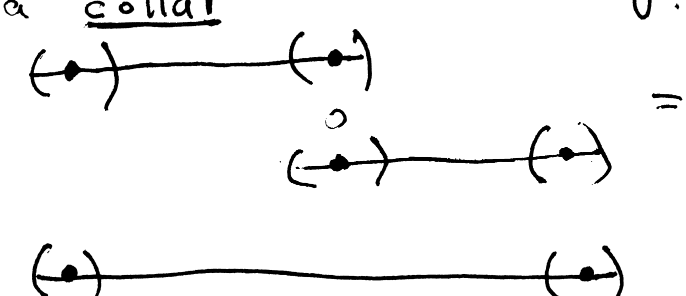
Every equivalence class contains a map with sitting instants.

Composition is well-defined!

<u>Problem</u>: geometric structures such as Riemannian metrics, symplectic forms, spin structures cannot be defined on <u>equivalence classes</u>, only on <u>actual bordisms</u>.

3) Collars: equip objects, i.e., 0-dimensional manifolds, with a 1-dimensional <u>germ</u> (neighborhood) called a <u>collar</u>

<u>Example</u>:



This resolves <u>all</u> of the above objections!

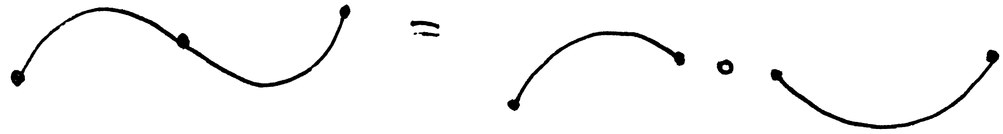
__Problem__ While this approach provides an elegant framework for many applications, it is hard to work with in __practice__ because the space of __objects__ is infinite-dimensional.

(Existing model-categorical tools allow one to work with infinite-dimensional spaces of morphisms, but not objects.)

4) Segal categories of bordisms.

Instead of __gluing__ 2 bordisms into 1, __split__ 1 bordism into 2!

Example:



__Problem:__ This doesn't seem to form a category! Composition is not always defined.

__Solution:__ Use the framework of model categories to interpret the above construction as a "resolution" for an actual category. Secretly this amounts to allowing piecewise-smooth maps, ~~[illegible]~~ but __without__ associated technical difficulties, i.e., we only work with smooth, not piecewise-smooth, maps.

Smooth categories

Physically meaningful field theories sends smooth families (i.e., bundles) of bordisms to smooth families of linear maps between vector spaces (i.e., smooth linear maps of smooth vector bundles).

*)

Idea (*) Any mathematical structure can be made "smooth" by considering sheaves with values in this structure.

First approximation For any fixed smooth manifold S consider S-indexed families: $\mathrm{Bord}_1^{or}(X)_S := \begin{cases}\mathrm{Ob}: \\ \mathrm{Mor}\end{cases}$

$$\begin{array}{ccc} B & \xrightarrow{C^\infty} & X \\ \downarrow \leftarrow \text{bundle} & & \\ S & & \end{array}$$

Vect_S = vector bundles over S

$$FT_1^{or}(X)_S := \mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^{or}(X)_S, \mathrm{Vect}_S).$$

We want to look at "compatible families" for all possible S.

A smooth map $S \to S'$ (<u>think</u> open subset) induces $f^*: \mathrm{Bord}_1^{or}(X)_{S'} \to \mathrm{Bord}_1^{or}(X)_S$

$$f^*: \mathrm{Vect}_{S'} \to \mathrm{Vect}_S$$

$$f^*: FT_1^{or}(X)_{S'} \to FT_1^{or}(X)_S.$$

<u>Roughly</u>: $\forall S$: $F_S \in FT_1^{or}(X)_S$.

such that $\forall S \xrightarrow[f]{} S'$: $f^* F_{S'} \xrightarrow[\alpha_f]{\cong} F_S$

rather isomorphism.

$$S \xrightarrow{f} S' \xrightarrow{f'} S''$$

$$\begin{tikzcd}
f^* f'^* F_{S''} \arrow[rr, "\alpha_{f' \circ f}"] \arrow[dr, "f^*\alpha_{f'}"', "\cong"] & \# & F_S \\
& f^* F_{S'} \arrow[ur, "\cong \alpha_f"'] &
\end{tikzcd}$$

How do we axiomatize / define such objects?

How can we compute with them?

Recall (*):

<u>Definition:</u> A <u>presheaf</u> (on smooth manifolds) with values in T is a functor $\mathrm{Man}^{op} \longrightarrow T$.

<u>Definition</u> A <u>morphism</u> of presheaves is a natural transformation of functors $\mathrm{Man}^{op} \longrightarrow T$.

<u>What is T?</u>

Functor of points

Last time:

A smooth field theory F is:

1) $F_S \in FT_1^{or}(X)_S$

2) an isomorphism $\forall S' \xrightarrow{f} S$:

$$f^* F_S \xrightarrow{\cong} F_{S'}$$

3) coherence condition for $S'' \to S' \to S$.

Today F is a global section of the presheaf $FT_1^{or}(X)$.

Definition A presheaf on Man is a functor $F : Man^{op} \to Set$.

A morphism of presheaves is a natural transformation of functors.

Ideology A presheaf F on Man is a "smooth space"
elements of $F(S)$ are smooth maps $S \to F$
$F(f)$: $F(S) \to F(S')$ is precomposition $S' \to S \to F$.

Example Any smooth manifold M is a generalized smooth space:

$$y(M) : \mathrm{Man}^{op} \to \mathrm{Set}$$
$$S \mapsto C^\infty(S, M).$$
$$S' \xrightarrow{f} S \mapsto C^\infty(f, M): C^\infty(S, M) \to C^\infty(S', M)$$

Def $y : \mathrm{Man} \longrightarrow \mathrm{Fun}(\mathrm{Man}^{op}, \mathrm{Set})$ is the Yoneda embedding.

It sends a smooth manifold to its associated generalized space.

Lemma (Yoneda) $\exists$ canonical isomorphism between $\{y(M) \longrightarrow \mathcal{F}\}$ and $\mathcal{F}(M)$.

Proof Given $y(M) \xrightarrow{\alpha} \mathcal{F}$, $\alpha(id_M) \in \mathcal{F}(M)$.

Given $\beta \in \mathcal{F}(M)$, define $y(M) \longrightarrow \mathcal{F}$
$$S \xrightarrow{f} M \mapsto f^*\beta.$$

<u>Examples</u>: 1) smooth manifolds.

2) mapping spaces.

$$C^\infty(M, N) \in \mathrm{Fun}(\mathrm{Man}^{op}, \mathrm{Set})$$

$$S \to C^\infty(M, N) \quad \leftrightarrow \quad S \times M \xrightarrow{C^\infty} N$$

$$C^\infty(M,N)(S) := C^\infty(S \times M, N)$$

3) differential forms:

$$\Omega^k(M) := \text{diff } k\text{-forms on } M$$

$$\Omega^k(f) := \text{pullback}$$

abstract de Rham complex:

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} - - - -$$

4) Can now make sense of

$$\Omega^1 C^\infty(M, N)!$$

Last time:

Functors $\mathrm{Man}^{op} \to \mathrm{Set}$ are "generalized smooth spaces"

Today: 1) Gluing property
2) Stacks
3) Principal G-bundles w/ ∇ as a stack.
4) The de Rham complex of $B_\nabla G$

1) Gluing property:

$F: \mathrm{Man}^{op} \to \mathrm{Set}$

$$F(M) \cong \{y(M) \to F\}$$

Suppose $M = U \cup V$, U and V open in M.

$$\begin{tikzcd} & F(M) \arrow[dl] \arrow[dr] & \\ F(U) \arrow[dr] & & F(V) \arrow[dl] \\ & F(U \cap V) & \end{tikzcd}$$

should be cartesian (pullback) square!

In other words: a map $y(M) \to F$ should be the same data as maps $y(U) \xrightarrow{\alpha} F$, $y(V) \xrightarrow{\beta} F$ such that $\alpha|_{U \cap V} = \beta|_{U \cap V}$

Definition (Only works for Man)

A sheaf on Man is

a presheaf F on Man

such that:

1) $\forall M$ $\forall U, V$ open in M : $M = U \cup V$

$$\begin{tikzcd}
 & \mathcal{F}(M) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\wedge" near start] & \\
\mathcal{F}(U) \arrow[dr] & & \mathcal{F}(V) \arrow[dl] \\
 & \mathcal{F}(U \cap V) &
\end{tikzcd}$$

is a pullback square

2) $\forall \{M_i\}_{i \in I}$: $\mathcal{F}\left(\coprod_{i \in I} M_i\right) \xrightarrow{\cong} \prod_{i \in I} \mathcal{F}(M_i)$

All examples so far are sheaves.

Nonexample: Fix a Lie group G.

$\mathcal{F}(M) :=$ the set of isomorphism classes of principal G-bundles with connection over M.

Not a sheaf: $M = S^1$, $U = \subset$, $V = \supset$, $U \cap V = \,=$

$\mathcal{F}(U) = \mathcal{F}(V) = \mathcal{F}(U \cap V) = \{*\}$

$\uparrow$ the trivial principal G-bundle

$\mathcal{F}(S^1) = \left\{ \begin{matrix} \text{conjugacy classes} \\ \text{of elements of } G \end{matrix} \right\}$.

Second attempt: replace Set by Grp. (groupoids).

$\mathcal{F}: \mathrm{Man}^{op} \to \mathrm{Grp}$

$\mathcal{F}(M) :=$ the groupoid of principal G-bundles with connection and connection-preserving isomorphisms:

Claim: $\mathcal{F}$ is a sheaf!

Example: M, U, V as above

$$\mathcal{F}(U) = \mathcal{F}(V) = * /\!/ G$$

$$\mathcal{F}(U \cap V) = * /\!/ G^2$$

$$\begin{tikzcd} & \mathcal{F}(U) \arrow[r] & \mathcal{F}(U \cap V) \\ \mathcal{F}(M) \arrow[ur] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \mathcal{F}(V) \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} * /\!/ G \arrow[r] & * /\!/ G^2 \\ & * /\!/ G \arrow[u] \end{tikzcd}$$

$$* /\!/ G \times * /\!/ G$$

$$* /\!/ G^2$$

$$\begin{tikzcd} G_1 \arrow[dr, "p_1"'] & \times & G_2 \arrow[dl, "p_2"] \\ & H & \end{tikzcd}$$

Ob: $g_1 \in G_1$; $g_2 \in G_2$

$p_1(g_1) \xrightarrow[\alpha]{\cong} p_2(g_2)$

Mor: $(g_1, g_2, \alpha) \to (g_1', g_2', \alpha')$

are $g_1 \xrightarrow[q_1]{} g_1'$, $g_2 \xrightarrow[q_2]{} g_2'$

such that

$$\begin{tikzcd} p_1 g_1 \arrow[r, "\alpha"] \arrow[d, "p_1(q_1)"] \arrow[dr, phantom, "\#"] & p_2 g_2 \arrow[d, "p_2(q_2)"] \\ p_1 g_1' \arrow[r, "\alpha'"'] & p_2 g_2' \end{tikzcd}$$

Ob: $g_1, g_2 = *$ $\quad \alpha = (h_1, h_2)$, $h_1, h_2 \in G$

Mor: $\alpha \to \alpha'$: $q_1, q_2 \in G$

$$\begin{tikzcd} * \arrow[r, "{h_1, h_2}"] \arrow[d, "{q_1, q_1}"'] & * \arrow[d, "{q_2, q_2}"] \\ * \arrow[r, "{h_1', h_2'}"'] & * \end{tikzcd}$$

$h_2 = 1$; $q_1 = q_2$

$$\begin{tikzcd} \cdot \arrow[r, "{h_1, h_2}"] \arrow[d, "{1, 1}"'] & \cdot \arrow[d, "{h_2^{-1}, h_2^{-1}}"] \\ \cdot \arrow[r, "{h_1 h_2^{-1}, 1}"'] & \cdot \end{tikzcd}$$

Chern - Weil theory

Man := the category of smooth manifolds and smooth maps

$\mathrm{PreSh}(T) := \mathrm{Fun}(\mathrm{Man}^{op}, T)$

$\mathrm{Sh}(T) :=$ ~~Fun(~~ full subcategory of $\mathrm{PreSh}(T)$ consisting of presheaves with the gluing property.

$\mathrm{Sh} : \mathrm{PreSh}(T) \longrightarrow \mathrm{Sh}(T)$ is the <u>sheafification</u> functor.

$$\begin{tikzcd}
\mathrm{PreSh}(\mathrm{Set}) \arrow[r, phantom, "\supset"] \arrow[r, bend right, "\mathrm{Sh}"'] & \mathrm{Sh}(\mathrm{Set}) \\
\mathrm{PreSh}(\mathrm{Grp}) \arrow[u, phantom, "\cap"] \arrow[r, phantom, "\supset"] \arrow[r, bend right, "\mathrm{Sh}"'] & \mathrm{Sh}(\mathrm{Grp}) \arrow[u, phantom, "\cap"]
\end{tikzcd}$$

$\Omega^k \in \mathrm{Sh}(\mathrm{Set})$: $\quad \Omega^k(M) =$ diff. k-forms on M

$\mathrm{Sh}(\mathrm{Ab})$. $\quad \Omega^k(f) = f^*$ pullback of forms

$\Omega^0 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k \xrightarrow{d}$ $\qquad \mathbb{Z} \to \mathrm{Sh}(\mathrm{Set})$.

is a chain complex in $\mathrm{Sh}(\mathrm{Ab})$.

$B_\nabla G \in \mathrm{Sh}(\mathrm{Grp})$: $\quad B_\nabla G(M) := \begin{cases} \text{Ob: principal } G\text{-b. } P \text{ w/ } \nabla \text{ on } M \\ \text{Mor: connection-pres. isos.} \end{cases}$

$E_\nabla G \in \mathrm{Sh}(\mathrm{Grp})$: $E_\nabla G(M) :=$ Ob: $P \underset{M}{\rightleftarrows} s$

Mor: $\begin{tikzcd} P \arrow[r] \arrow[dr] & Q \arrow[d] \\ & M \end{tikzcd}$ $\quad \begin{tikzcd} P \arrow[r] & Q \\ M \arrow[u, "s"] \arrow[ur, "s"'] & \end{tikzcd}$

1) We have an equivalence:

$$E_\nabla G \underset{(-)+M.C.}{\overset{s^*\theta}{\rightleftarrows}} \Omega^1 \mathfrak{g}$$

trivial bundle. , i.e. a principal G-bundle w/ connection

and a section is the same thing as a connection 1-form.

The map $E_\nabla G$ is a connection on $E_\nabla G \to B_\nabla G$, the universal connection

2) We have a canonical action

$G \curvearrowright E_\nabla G$: $g\cdot(P, s) = (P, g\cdot s)$.

$G \curvearrowright \Omega^1\mathfrak{g}$: $g\cdot\alpha = g^*\theta + Ad_{g^{-1}}(\alpha)$.

3) Recall: $G \curvearrowright F$, $F \in PreSh(Set)$: $G \in PreSh(Grp)$

$F/\!/G$: $(F/\!/G)(S) = F(S) /\!/ G(S)$, $S \in Man$

$G \curvearrowright A \in Set$: $A/\!/G := \begin{cases} Ob = A \\ Mor(a_1, a_2) = \{g \in G \mid g a_1 = a_2\} \end{cases}$

$\pi_0(A/\!/G) =$ iso classes of $A/\!/G$

$= A/G$, the traditional quotient.

Example: $F = y(M)$, $G \curvearrowright M$: free action, i.e., no fixed points

$M/\!/G \longrightarrow M/G$, $M/G \in PreSh(Set)$.

$S \in Man$: $C^\infty(S, M) /\!/ C^\infty(S, G) \longrightarrow C^\infty(S, M/G)$

Remark: If $G \curvearrowright M$ is proper, then $M/G \in PreSh(Set)$ is a manifold, i.e., representable

Example $G \curvearrowright *$: $*//G \not\cong */G \cong *$.

Indeed: $\pi_1(*//G) = G$.

$\pi_1(*) = *$.

Example : $E_\nabla G // G \xrightarrow[\text{forget } s.]{\sim} B_\nabla G$

is an equivalence because locally every principal G-bundle has a section.

~~Def~~ ~~E~~

Summary $\begin{array}{c} E_\nabla G \\ \downarrow \\ B_\nabla G \end{array}$ is a ~~la~~ universal principal G-bundle with connection.

Corollary $\Omega^*(B_\nabla G) = eq\left(\Omega^* E_\nabla G \rightrightarrows \Omega^*(G \times E_\nabla G)\right)$

= basic forms ~~on~~ on $E_\nabla G$

1) G-invariant

2) $\iota_g(-) = 0$.

Thm (Freed - Hopkins) $\Omega^* E_\nabla G = (\underbrace{\mathrm{Kos}\ g^*}_{\text{Weil algebra}}, d)$

$\mathrm{Kos}\ V = \mathrm{Sym}(V_1 \xrightarrow{id} V_2)$.

Thus, $\Omega^* B_\nabla G = ((\mathrm{Sym}\ g^*)^G, d = 0)$.

Recall. the Borel quotient of $G \curvearrowright X$ is $X_{hG} := (EG \times_G X)$.

<u>Def</u> The stacky Borel quotient of $G \curvearrowright X$ is $(E_\nabla G \times X) /\!\!/ G. = \mathbb{X}_{G,\nabla}$

Thm (Freed–Hopkins) $G \curvearrowright X$:

a) $\Omega^*(E_\nabla G \times X) = \Omega^*(X, \mathrm{Kos}\ g^*)$

$d = d_X + d_K$.

b) $\Omega^*(\mathbb{X}_{G,\nabla}) =$ basic subcomplex of <u>Weil model</u> for equivariant cohomology.

Chern-Simons theory via Bundle 2-gerbes

Recall:

1) Cart: the cartesian site
$\mathrm{Ob} = \{0, 1, 2, \dots\}$
$\mathrm{Mor}(m, n) = C^\infty(\mathbb{R}^m, \mathbb{R}^n)$.

2) $\underbrace{\mathrm{Sh}}(\mathrm{Cart}, \mathrm{Set}) \subset \mathrm{PreSh}(\mathrm{Cart}, \mathrm{Set})$
sheaves: ~~satisfy~~ satisfy gluing $\qquad = \mathrm{Fun}(\mathrm{Cart}^{op}, \mathrm{Set})$

"generalized smooth manifolds"

$\mathrm{Cart} \hookrightarrow \mathrm{Man} \hookrightarrow \mathrm{Sh}(\mathrm{Cart}, \mathrm{Set})$

Examples: a) $M, N \in \mathrm{Man} \Rightarrow \underline{C^\infty}(M, N) \in$

b) $\Omega^n, \Omega^n_{closed} \in \mathrm{Sh}$

3) $\mathrm{Sh}(\mathrm{Cart}, \mathrm{Grpd}) \subset \mathrm{PreSh}(\mathrm{Cart}, \mathrm{Grpd})$
$E_\nabla G \cong \Omega^1 \mathfrak{g}$
$\qquad = \mathrm{Fun}(\mathrm{Cart}^{op}, \mathrm{Grpd})$

"Smooth (C^∞) stacks"

Examples: a) Vect, Vect_∇

b) $\mathbb{B}G$ $\quad \mathbb{B}_\nabla G$

c) Bord $\quad \mathrm{Bord}_\nabla \dots$

Ex: $E_\nabla G \to B_\nabla G$ univ prin G-bun w/ ∇

Theorem (Freed - Hopkins)

$$\Omega^*(E_\nabla G) = (\mathrm{Kos}\ \mathfrak{g}^*, d)$$
$$= \mathrm{Sym}\left(\mathfrak{g}^*_1 \xrightarrow[d=id]{} \mathfrak{g}^*_2\right).$$

$$\Omega^*(\mathbb{B}_\nabla G) = \left((\mathrm{Sym}\ \mathfrak{g}^*_2)^G, 0 \right)$$
$$\parallel$$
$$\Omega^*_{cl}(\mathbb{B}_\nabla G)$$

Today 1) introduce $\mathbb{B}^n A$, $\mathbb{B}^n_\nabla A$

A abelian Lie group

($n=1$: recover $\mathbb{B}A$, $\mathbb{B}_\nabla A$.)

2) compute $\mathbb{B}_\nabla G \to \mathbb{B}^3_\nabla U(1)$

Geometric models for cohomology

$H^n(-,\mathbb{R})$ $\quad\Omega^n_{cl}$

$H^n(-,\mathbb{Z})$ $\quad(*)?\ \xrightarrow[\mathrm{cc}(n)]{}$ bundle gerbes w/con

K^n $\quad\mathrm{Vect}_\nabla$

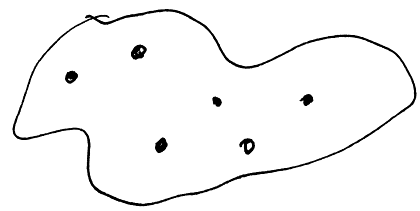
TMF $\quad$? (open problem)

MO $\cancel{\text{...}} = S^0$ $\quad$ Geometric families of bordisms

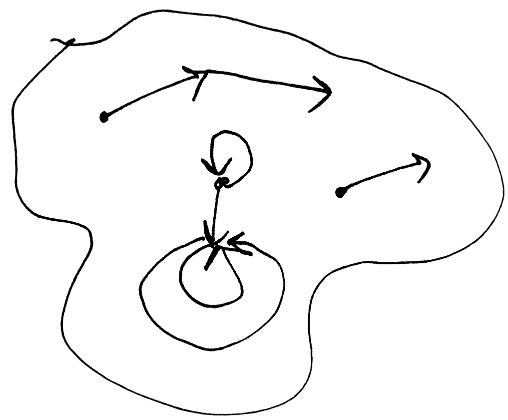
Names: Cheeger-Simons char, Deligne cohomology, ordinary diff cohom, bundle $(n-1)$-gerbes (w/∇), circle n-bundles (w/∇)

A few words about higher groupoids

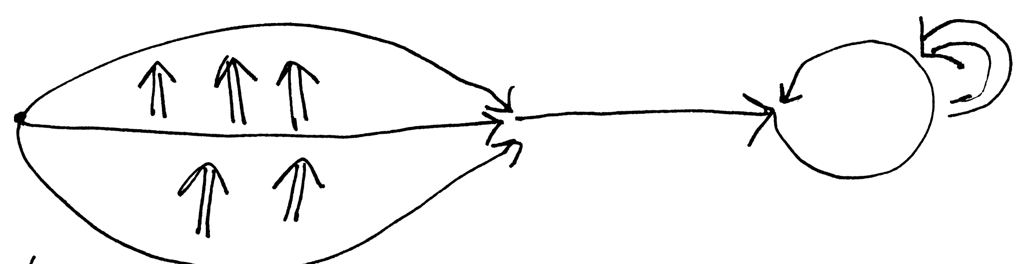
Set



Groupoid



2-groupoid



'Def' An n-category/groupoid is a) Set $n=0$

b) a category/groupoid "enriched" in $(n-1)$-categories/groupoids

$\mathrm{Ob} \in \mathrm{Set}$

$\forall X, Y \in \mathrm{Ob}: \quad \mathrm{Mor}(X,Y) \in (n-1)$-category/groupoid

$$\mathrm{Mor}(X_{n-1}, X_n) \times \mathrm{Mor}(X_{n-2}, X_{n-1}) \cdots\cdots \times \mathrm{Mor}(X_0, X_1) \to \mathrm{Mor}(X_0, X_n)$$

homotopy associative

For groupoids need inverses.

Examples: 1) Alg, Bimod, Intertwiners

2) Man, Corr $X \leftarrow C \to Y$, Isos

Models for higher groupoids

n	n-groupoids
$n=0$	Set
$n=1$	Groupoids
$n=2$	Bigroupoids
$n\geq 2$	Simplicial ~~sets~~ sets X ~~with~~ that are n-truncated: $\pi_k(X,x)=0$ $\forall x\in X_0$ $\forall k>n$.

k-simplices are k-morphisms.

<u>In practice</u>: many n-groupoids for $n>1$ are <u>abelian</u>: k-morphisms form an abelian group for $0\leq k\leq n$.

Instead of sSet have sAb

Dold-Kan correspondence

$$\underset{\text{weak equiv}}{\mathrm{sAb}} \underset{\Gamma}{\overset{N}{\rightleftarrows}} \underset{\text{quasi-iso}}{\mathrm{Ch}_{\geq 0}}$$

$X_0 \leftarrow X_1 \leftarrow \cdots$

induced by $\Delta \to \mathrm{Ch}_{\geq 0}$
$m \mapsto N\mathbb{Z}[\Delta^m]$.

Explanation: $X_0 \xleftarrow{d} X_1 \xleftarrow{d} X_2 \xleftarrow{d} \cdots$ $\quad d^2 = 0$

encodes an ∞-groupoid (k-morphisms are an abelian group)

Objects: X_0

1-Morphisms: $A, B \in X_0$ $\quad f : A \to B$ $\quad f \in X_1 : df = A - B.$

2-Morphisms:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow h"] & B \end{tikzcd} \qquad h \in X_2 : \quad dh = f - g.$$

$$\underset{0}{\overset{}{d(dh)}} = df - dg = (A-B) - (A-B)$$

k-morphisms: $\alpha \Rightarrow \beta$ $\quad p \in X_k : \quad dp = \alpha - \beta.$

Remark: the same element $f \in X_p$ $\; p > 0$

encodes isos between <u>different</u> pairs of objects A, B: $\quad A - B = df.$

In particular: the space of <u>all</u> k-morphisms is $X_0 \times X_1 \times \cdots \times X_k.$

Illustration: Sing X (singular simp. set.) $\qquad \mathbb{Z}[\mathrm{Sing}\, X]$ (singular chains)

Example: $G = A$ abelian Lie group $A = U(1)$

$\cancel{(\mathbb{B}U(1))(X)}$ $\mathbb{B}_\nabla U(1)(X)$ = groupoid $\left\{ \begin{array}{l} \text{objects: } \Omega^1(X) \\ \text{morphisms } \omega_0 \to \omega_1: \\ f \in C^\infty(X, U(1)) \\ df = \omega_0 - \omega_1 \end{array} \right\}$

cart

cart

$\mathbb{B}U(1)(X) =$ groupoid $\{*\}$ object

$Mor(*,*) = C^\infty(X, U(1))$.

In the language of chain complexes:

$$\left(\mathbb{B}U(1)\right)(X) = \overset{0}{0} \leftarrow \overset{1}{C^\infty(X, U(1))}$$

$$\left(\mathbb{B}_\nabla U(1)\right)(X) = \Omega^1(X) \xleftarrow{d\log} C^\infty(X, U(1))$$

Definition: $$\left(\mathbb{B}^n U(1)\right)(X) = \overset{0}{0} \leftarrow \cdots\cdots \overset{n-1}{0} \leftarrow \overset{n}{C^\infty(X, U(1))}$$

$$\left(\mathbb{B}^n_\nabla U(1)\right)(X) = \cancel{0 \leftarrow \cdots 0 \leftarrow C^\infty X \leftarrow C^\infty(X, U(1))}$$

$$\overset{0}{\Omega^n X} \leftarrow \overset{1}{\Omega^{n-1} X} \leftarrow \cdots \overset{n-1}{\Omega^1 X} \leftarrow \overset{n}{C^\infty(X, U(1))}$$

$$\Omega^n X \leftarrow \cdots\cdots \quad \Omega^1 X \leftarrow \Omega^0 X \leftarrow C^\infty(X,$$

bundle $(n-1)$-gerbes
circle n-bundles
Cheeger-Simons characters
Deligne cohomology
ordinary differential cohomology

Recall

$$\begin{tikzcd}
\mathrm{PSh}(\mathrm{Man},\mathrm{Set}) \arrow[r, hook, "\text{f.f.}"] \arrow[d, shift right, "\#"'] & \mathrm{PSh}(\mathrm{Man},\mathrm{Grp}) \arrow[r, hook] & \mathrm{PSh}(\mathrm{Man},\mathrm{sSet}) & \mathrm{PSh}(\mathrm{Man},\mathrm{Ch}) \arrow[l, "\text{faithful, not full}"'] \\
\mathrm{Sh}(\mathrm{Man},\mathrm{Set}) \arrow[r, hook, "\text{f.f.}"] \arrow[u, hook, shift right, "\text{f.f.}"'] & \mathrm{Sh}(\mathrm{Man},\mathrm{Grp}) \arrow[r, hook] \arrow[u, hook, "\text{f.f.}"'] & \mathrm{Sh}(\mathrm{Man},\mathrm{sSet}) \arrow[u, hook, "\text{f.f.}"'] & \mathrm{Sh}(\mathrm{Man},\mathrm{Ch}_{\geq 0}) \arrow[l, "\text{faithful, not full}"'] \arrow[u, hook, "\text{f.f.}"']
\end{tikzcd}$$

$\Omega^n, \Omega^n_{cl}, \underline{C^\infty}(M,N) \in \mathrm{Sh}(\mathrm{Man},\mathrm{Set})$

$\mathbb{B}G, \mathbb{B}_\nabla G, \mathrm{Vect}_\nabla \in \mathrm{Sh}(\mathrm{Man},\mathrm{sSet})$

$\mathbb{B}^n A, \mathbb{B}^n_\nabla A \in \mathrm{Sh}(\mathrm{Man},\mathrm{Ch}_{\geq 0})$

f.f. = fully faithful

$\mathrm{PSh}(\mathrm{Man}, \mathcal{D}) = \mathrm{Fun}(\mathrm{Man}^{op}, \mathcal{D})$

$\mathrm{Sh}(\mathrm{Man}, \mathcal{D})$ = full subcategory satisfy descent.

$\mathrm{Sh}(\mathrm{Man}, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Sh}(\mathrm{Cart}, \mathcal{D})$ (false for presheaves)

~~Sometimes repl.~~

Def A : abelian Lie group

$\mathfrak{a}$: Lie algebra of A

$L \xrightarrow{\ker} \mathfrak{a} \xrightarrow{\exp} A$

$(2\pi i \mathbb{Z} \longrightarrow i\mathbb{R} \xrightarrow{\exp} U(1)$

$2\pi i \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times)$

Def (Deligne - Tate) $\mathbb{Z}(p) := (2\pi i)^p \cdot \mathbb{Z}$

$B \subset \mathbb{R}$ subring subgroup

$B(p) := (2\pi i)^p \cdot B \subset \mathbb{C}$

$B(p) \otimes_B B(q) \to B(p+q)$

Remark Del: $\Omega^n \leftarrow \cdots \leftarrow \Omega^1 \leftarrow U(1)$

$\wr$

$$\underbrace{\Omega^n/\Omega^n_{ex}}_{\text{iso classes of bundle gerbes with connection}} \quad 0 \cdots 0 \leftarrow U(1)_\delta$$

$\mathrm{Del} \longrightarrow \Omega^n/\Omega^n_{cl}\,[0]$ computes the iso class

$U(1)_\delta \longrightarrow \mathrm{Del}$ embeds flat bundle gerbes

Remark Exact triangle / cofiber sequence

$$\begin{array}{lcccccl}
\text{degree } 0 & & n-1 & & n & & n+1 \\
0 \leftarrow & \cdots & 0 & \leftarrow & \mathbb{Z} & \leftarrow & 0 \qquad B^n\mathbb{Z} \\
 & & & & & & \qquad\ \downarrow \\
\Omega^n \leftarrow & \cdots & \Omega^1 & \leftarrow & \Omega^0 & \leftarrow & 0 \qquad B^n_\nabla \mathbb{R} \\
 & & & & & & \qquad\ \downarrow \\
\Omega^n \leftarrow & \cdots & \Omega^1 & \leftarrow & U(1) & & \qquad B^n_\nabla U(1) \\
 & \cdots & & \leftarrow & \Omega^0 & \leftarrow & \mathbb{Z} \qquad\ \downarrow \\
0 \leftarrow & \cdots & 0 & \leftarrow & 0 & \leftarrow & \mathbb{Z} \qquad B^{n+1}\mathbb{Z}
\end{array}$$

Embedding of trivial bundle gerbes

$$\begin{tikzcd}
\Omega^n & \arrow[l] \cdots & \Omega^0 \arrow[l] \arrow[d, hook] & 0 \arrow[l] \\
\Omega^n \arrow[u, phantom] & \arrow[l] \cdots & \Omega^0 \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}$$

$\mathbb{E}^n_\nabla\, U(1) = \quad \Omega^n$

Important observation:

the ∞-groupoid of morphisms dom $= B$

B some bundle $(n-1)$-gerbe

is <u>precisely</u> the ∞-groupoid of

bundle $(n-2)$-gerbes!

In other words: morphisms have <u>geometry</u> (like curvature!).

~~Curvature~~:

Curvature

$$
\begin{array}{ccccccccc}
\Omega^n & \leftarrow & \cdots & & \leftarrow & \Omega^0 & \leftarrow & \mathbb{Z} & \leftarrow 0 \\
\downarrow & & \downarrow & & & \downarrow & & \downarrow & \\
\Omega^{n+1}_{\text{closed}} & \leftarrow & \Omega^n & \leftarrow & \cdots & \Omega^1 & \leftarrow & \Omega^0 & \leftarrow \mathbb{R}_\delta \\
\uparrow & & & & & & & & \\
\Omega^{n+1}_{\text{closed}} & & 0 & - & - & - & - & - & 0
\end{array}
$$

underlying topological object / discard connection

$$
\begin{array}{ccccccc}
\Omega^n & \leftarrow & \cdots & \cdots & \Omega^0 & \leftarrow & \mathbb{Z} \\
& & & & \downarrow & & \\
0 & - & - & 0 & 0 & \leftarrow & \mathbb{Z}_{n+1}
\end{array}
$$

Dixmier-Douady class

embedding of <u>flat</u> bundle gerbes

$$
\begin{array}{ccccccc}
& & & & & & U(1)_\delta \cong \\
\Omega^n_{\text{closed}} & \longleftarrow & \Omega^{n-1} & \cdots & \Omega^0 & \leftarrow & U(1) \\
\downarrow & & & & & & \\
\Omega^n & \leftarrow & \cdots \Omega^{n-1} & - & - & - & -
\end{array}
$$

The differential hexagon

bundle (n-1)-gerbes w/ band $U(1)$

n - gerbes w/ band $\mathbb{Z}$

trivial b.g.

$$\Omega^n/\Omega^n_{ex} \oplus \mathbb{R}_\delta[n]$$

$$\cong$$

$$\Omega^n \leftarrow \cdots \Omega^0$$

$$\Big\downarrow \text{inclusion of trivial}$$

flat b.g.

$$\Omega^n_{cl} \leftarrow \Omega^{n-1} \leftarrow \cdots U(1) \xrightarrow{\text{inclusion of flat}} \Omega^n \leftarrow \cdots \Omega^1 \leftarrow \overset{n}{\underline{U}(1)} \xrightarrow[d]{\text{curvature}} \Omega^{n+1} \leftarrow 0 \cdots$$

$$\cong \qquad\qquad\qquad\qquad\qquad \cong$$

$$\text{---}\; U(1)_\delta \qquad\qquad \text{---}\; \Omega^1 \leftarrow \Omega^0 \leftarrow \overset{n+1}{\mathbb{Z}}$$

$$\Big\downarrow \text{topological class}$$

$$0 \;\text{---}\; 0 \leftarrow \overset{n+1}{\mathbb{Z}}$$

__Def__ $B(p)^{\infty}_{\emptyset}$ $\quad\quad \Omega^n$

0

$$B(p) \longrightarrow \Omega^0 \longrightarrow \Omega^1 \longrightarrow \cdots \longrightarrow \Omega^{p-1}$$

$$B(p) \otimes B(q) \longrightarrow B(p+q)$$

$x \in \mathbb{Z}$

$$x \cup y = \begin{cases} \cancel{x \wedge y} \quad x \cdot y & \deg x = 0 \\ x \wedge dy & \deg x > 0, \underbrace{\deg y = q}_{\text{last}} \\ 0, & \text{---} \end{cases}$$

Punchline

action higher CS-thy

connection $(2k+1)$-form

$$S_{CS} : \widehat{H}^{2k+1}(\Sigma) \to U(1)$$

$$c \longmapsto \int_\Sigma c \cup c$$

$\omega \wedge d\omega$

$(4k+3)$ - dimensional

$$\mathbb{C}\mathbb{C}^* \underset{\mathcal{O}}{\overset{\mathrm{Spec}}{\rightleftarrows}} \mathrm{CH}$$

$A \to \mathcal{O}(\mathrm{Spec}\, A)$

injective : sufficiently many char

$X \to \mathrm{Spec}(\mathcal{O} X)$

injective : Hausdorff

surjective : any char is eval

compact

$f \,\square\, i_V \,\sqcup\, W$

Open

$C \to C'$ $\quad$ $A \to A'$

$$\begin{array}{ccc} CA & \xrightarrow{\sim} & CA' \\ \downarrow & & \downarrow \\ C'A & \xrightarrow{\sim} & C'A' \end{array}$$

Electromagnetism

$E : \mathbb{R}_t \to \Omega^1 \mathbb{R}^3$ electric field

~~M~~ $B : \mathbb{R}_t \to \Omega^2 \mathbb{R}^3$ magnetic field / flux

$J_E : \mathbb{R}_t \to \Omega^2 \mathbb{R}^3$ electric current

$\rho_E : \mathbb{R}_t \to \Omega^3 \mathbb{R}^3$ electric charge density

Relativistic: $M^4 := \mathbb{R}_t \times \mathbb{R}^3 \quad (c = 1)$

$$F := B - dt \wedge E \quad \in \Omega^2 M$$

$$j_E := \rho_E - dt \wedge J_E \quad \in \Omega^3 M$$

Maxwell's equations: $dF = 0$

$$d \star F = j_E$$

With magnetic current $j_B \in \Omega^3 M$:

$j = j_B, j_E$ compact support on any space like slice N $\qquad dF = j_B$

$[j|_N] \in H^3_{c.s.}(N, \mathbb{R})$ $\qquad d \star F = j_E$

$\Big\} \int_{\text{space-like}} j$ = total charge

$$[j|_N] \in \ker H^3_{c.s.}\mathbb{R} \to H^3 \mathbb{R}$$

Dirac charge quantization: $[j|_N] \in H^3_{c.s.}(N, \mathbb{Z})$.

comes from ↑