Algebraic, differential, and complex geometry via functors of points

Dmitri Pavlov

Department of Mathematics and Statistics, Texas Tech University
https://dmitripavlov.org/

Our initial goal is understand the proof of the following theorem.

Theorem 0.1. Suppose M is a smooth manifold, a complex manifold, or a smooth algebraic variety. Then the smooth singular infinitesimal cochain complex of M is isomorphic to the de Rham complex of M.

In addition, we want to prove this theorem in a uniform manner for algebraic varieties, smooth manifolds, and complex analytic spaces, without repeating any arguments.

The abstract machinery we use to achieve this starts with a category of cartesian spaces (Definition 1.1) such as $\mathbf{A}^m$ with polynomial maps, $\mathbf{R}^m$ with smooth maps, or $\mathbf{C}^m$ with holomorphic maps. Then we construct a category of algebras (Definition 2.5) such as commutative rings, C^∞-rings, or entire functional calculus complex algebras.

1 Cartesian spaces

Definition 1.1. A *category of cartesian spaces* Cart is a category with finite products that coincides with one of the three categories defined below. More generally, we allow Cart to be an arbitrary Fermat theory (Definition 1.14). Used in 0.1*, 1.1, 1.1*, 1.3, 1.6, 1.6*, 1.7, 1.9, 1.10, 1.11*, 1.13, 1.14, 1.15, 1.15*, 1.16, 2.0*, 2.1, 2.2, 2.3, 2.3*, 2.4, 2.4*, 2.5, 2.6, 2.7, 2.7*, 2.9, 2.9*, 3.1*, 3.2, 3.2*, 3.3, 3.4*, 3.5, 3.6, 3.8*, 3.9, 3.10, 4.1, 4.1*, 6.0*, 6.2.

Typically, the commutative monoid of isomorphism classes of objects in Cart equipped with cartesian product will be isomorphic to $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ with the operation of addition. In this case, we refer to the number corresponding to $X \in \mathsf{Cart}$ as the *dimension* of X. (This is not always true, e.g., for supercartesian spaces we get the monoid $\mathbf{N} \times \mathbf{N}$.)

1.2. *Algebraic geometry*

Definition 1.3. Fix a commutative ring R. (Important examples are $R = \mathbf{Z}$ and $R = \mathbf{R}$.) The category Cart_R has natural numbers as objects, which are denoted by $\mathbf{A}^m_R$. Morphisms $\mathbf{A}^m_R \to \mathbf{A}^n_R$ are polynomial maps, which can be concretely described as n-tuples $(f_1, \ldots, f_n)$ of polynomials $f_i(x_1, \ldots, x_m)$ in m variables with coefficients in R. Polynomial maps are composed via substitution: the composition of

$$(g_1, \ldots, g_n)\colon \mathbf{A}^m_R \to \mathbf{A}^n_R, \qquad (f_1, \ldots, f_m)\colon \mathbf{A}^l_R \to \mathbf{A}^m_R$$

is

$$(g_1, \ldots, g_n) \circ (f_1, \ldots, f_m) = (g_1(f_1, \ldots, f_m), \ldots, g_n(f_1, \ldots, f_m)).$$

Used in 3.1*, 3.3.

Remark 1.4. The example of a polynomial $x^p - x$ with coefficients in $R = \mathbf{Z}/p$, where p is a prime number, demonstrates the importance of defining polynomials not as a functions on R^m, but as formal expressions, since the polynomial $x^p - x$ vanishes on all elements of $\mathbf{Z}/p$.

1.5. *Differential geometry*

Definition 1.6. The category $\mathsf{Cart}_{\mathsf{C}^\infty}$ has natural numbers as objects, which are denoted by $\mathbf{R}^m$. Morphisms are smooth (i.e., infinitely differentiable) maps. Morphisms are composed like functions. Used in 3.4*, 3.5.

The category $\mathsf{Cart}_{\mathsf{C}^\infty}$ admits all finite products: $\mathbf{R}^m \times \mathbf{R}^n \cong \mathbf{R}^{m+n}$ and $\mathbf{R}^0$ is the terminal object.

Definition 1.7. We have a canonical product-preserving functor

$$\mathsf{Cart}_{\mathbf{R}} \to \mathsf{Cart}_{\mathsf{C}^\infty}$$

that interprets a polynomial map as a smooth function. Used in 3.6.

1.8. ***Complex geometry***

Definition 1.9. The category Cart_ω has natural numbers as objects, which are denoted by $\mathbf{C}^m$. Morphisms $\mathbf{C}^m \to \mathbf{C}^n$ are entire holomorphic maps, i.e., n-tuples of formal power series in n variables with complex coefficients that are absolutely convergent on the entire $\mathbf{C}^m$. Composition is pointwise. Used in 3.8*, 3.9.

Definition 1.10. We have a canonical product-preserving functor

$$\mathsf{Cart}_{\mathbf{C}} \to \mathsf{Cart}_\omega$$

that interprets a polynomial map as an entire holomorphic function. Used in 3.10.

1.11. ***The abstract setting: Fermat theories***

Fermat theories axiomatize Hadamard's lemma in the setting of arbitrary algebraic theories that contain the theory of commutative rings $\mathsf{Cart}_{\mathbf{Z}}$.

Definition 1.12. (Adámek–Rosický–Vitale [2011, Definition 1.1].) An *algebraic theory* is a small category with finite products. A *morphism of algebraic theories* is a functor that preserves finite products. Used in 1.11*, 1.13, 1.14, 1.16.

Example 1.13. The categories Cart_R, $\mathsf{Cart}_{\mathsf{C}^\infty}$, Cart_ω are algebraic theories. The functors $\mathsf{Cart}_{\mathbf{R}} \to \mathsf{Cart}_{\mathsf{C}^\infty}$, $\mathsf{Cart}_{\mathbf{C}} \to \mathsf{Cart}_\omega$ are morphisms of algebraic theories.

Definition 1.14. (Dubuc–Kock [1984, §1].) A *Fermat theory* is a morphism of algebraic theories

$$\iota\colon \mathsf{Cart}_{\mathbf{Z}} \to \mathsf{Cart}$$

such that for any $f\colon L \times S \to T$ there is $g\colon L \times L \times S \to T$ for which the two morphisms $L \times L \times S \to T$ specified symbolically as

$$f(l_1, s) - f(l_2, s) = (l_1 - l_2) \cdot g(l_1, l_2, s)$$

coincide. Here $L = \iota(\mathbf{A}^1_{\mathbf{Z}})$, $-\colon L \times L \to L$ and $\cdot\colon L \times L \to L$ are the ι-images of polynomials $l_1 + l_2$ and $l_1 \cdot l_2$, which denote morphisms $\mathbf{A}^2_{\mathbf{Z}} \to \mathbf{A}^1_{\mathbf{Z}}$ in $\mathsf{Cart}_{\mathbf{Z}}$. Used in 1.1, 1.15, 1.16, 5.3*.

Proposition 1.15. The categories Cart_R, $\mathsf{Cart}_{\mathsf{C}^\infty}$, Cart_ω (with obvious maps from $\mathsf{Cart}_{\mathbf{Z}}$) are Fermat theories.

Proof. Given a morphism $f\colon \mathbf{A}^1_R \times \mathbf{A}^m \to \mathbf{A}^1_R$ in Cart_R, we set $g(l_1, l_2, s) = (f(l_1, s) - f(l_2, s))/(l_1 - l_2)$ as a polynomial, where the quotient is a polynomial because the numerator is a sum of multiples of $l_1^k - l_2^k$.

Given a morphism $f\colon \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}$ in $\mathsf{Cart}_{\mathsf{C}^\infty}$, we set $g(l_1, l_2, s) = (f(l_1, s) - f(l_2, s))/(l_1 - l_2)$ for $l_1 \neq l_2$, whereas for $l_1 = l_2$ we set $g(l_1, l_2, s) = \partial_l f(l, s)$, the partial derivative with respect to l.

Given a morphism $f\colon \mathbf{C} \times \mathbf{C}^m \to \mathbf{C}$ in Cart_ω, we set $g(l_1, l_2, s) = (f(l_1, s) - f(l_2, s))/(l_1 - l_2)$ as a formal power series. The power series for g is absolutely convergent, so defines a morphism in Cart_ω. ▮

Example 1.16. We give some interesting examples of algebraic theories that are not Fermat theories:

- Objects are sets; morphisms $A \to B$ are continuous maps $S^A \to S^B$ of topological spaces, where the *Sierpiński space* $S = \{0, 1\}$ has open sets $\{\emptyset, \{1\}, \{0, 1\}\}$. This is not a Fermat theory because it does not admit a morphism of algebraic theories from $\mathsf{Cart}_{\mathbf{Z}}$. This example is interesting because it illustrates that arities can be infinite, which is essential in this case: restricting to sets of certain cardinality will not produce the same category of product-preserving functors.
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2 From cartesian spaces to spaces

We would like to construct a fully faithful embedding

$$\mathsf{Cart} \to \mathsf{Space},$$

where Space is a category, whose objects can be interpreted as spaces that are more general than cartesian spaces.

For example:

- For Cart_R, we expect Space to contain affine schemes over R (or simply affine schemes if $R = \mathbf{Z}$) as a full subcategory;
- For $\mathsf{Cart}_{\mathsf{C}^\infty}$, we expect Space to contain smooth manifolds as a full subcategory;
- For Cart_ω, we expect Space to contain Stein spaces as a full subcategory.

For a moment, we assume that such an embedding exists and explore its properties.

Our strategy is explore a space X by studying functions on X. The latter can be defined formally as morphisms $\mathsf{X} \to \iota\mathbf{A}^1$ in the category Space, where $\mathbf{A}^1 \in \mathsf{Cart}$ denotes the object of Cart of dimension 1, which we interpret as the number line. Likewise, morphisms $\mathsf{X} \to \iota\mathbf{A}^m$ encode m-tuples of functions on X. Observe that morphisms $\mathsf{X} \to \iota\mathbf{A}^{m+n}$ are in a canonical bijective correspondence with pairs consisting of morphisms $\mathsf{X} \to \iota\mathbf{A}^m$ and $\mathsf{X} \to \iota\mathbf{A}^n$. We encode this observation using the notion of a product-preserving functor.

Definition 2.1. Denote by $\mathfrak{Fun}^\times(\mathsf{Cart}, \mathsf{Set})$ the category whose objects are product-presrving functors $\mathsf{F}\colon \mathsf{Cart} \to \mathsf{Set}$ (meaning the canonical morphisms $\mathsf{F}(U \times V) \to \mathsf{F}(U) \times \mathsf{F}(V)$ and $\mathsf{F}(1) \to 1$ are isomorphisms for all $U, V \in \mathsf{Cart}$) and morphisms are natural transformations. Used in 2.2, 2.3, 2.3*, 2.4, 2.5, 2.6, 2.7, 2.7*, 3.2, 3.6, 3.10, 4.1*, 6.1, 6.3, 6.4, 6.5, 6.6.

We are now ready to formalize the above construction that explores a space X by studying maps of the form $\mathsf{X} \to \iota\mathbf{A}^m$.

Definition 2.2. Given a fully faithful embedding

$$\iota\colon \mathsf{Cart} \to \mathsf{Space}$$

that preserves finite products, we define a functor

$$\mathsf{Y}\colon \mathsf{Space} \to \mathfrak{Fun}^\times(\mathsf{Cart}, \mathsf{Set})^{\mathsf{op}}$$

by sending $\mathsf{X} \in \mathsf{Space}$ to the product-preserving functor

$$\mathsf{Y}(\mathsf{X})\colon \mathsf{Cart} \to \mathsf{Set}, \qquad U \mapsto \mathfrak{hom}_{\mathsf{Space}}(\mathsf{X}, \iota U)$$

and a morphism $f\colon \mathsf{X}_1 \to \mathsf{X}_2$ in Space to the natural transformation

$$\mathsf{Y}(\mathsf{X}_1) \leftarrow \mathsf{Y}(\mathsf{X}_2), \qquad U \mapsto \mathfrak{hom}(f, \iota U), \qquad \mathfrak{hom}(f, \iota U)\colon \mathfrak{hom}(\mathsf{X}_2, \iota U) \to \mathfrak{hom}(\mathsf{X}_1, \iota U), \qquad g \mapsto g \circ f.$$

Used in 2.2, 2.2*, 2.3, 2.3*, 2.7, 2.7*, 2.9, 5.7, 6.0*, 6.3, 6.3*, 6.4, 6.4*.

Here $\mathfrak{hom}(A, B) = \mathfrak{hom}_{\mathsf{C}}(A, B) = \mathsf{C}(A, B)$ denotes the set of morphisms from A to B in a category C. The functor $\mathsf{Y}(\mathsf{X})$ is indeed product-preserving, as long as the functor ι is: $\mathfrak{hom}(\mathsf{X}, \iota(U \times V)) \cong \mathfrak{hom}(\mathsf{X}, \iota U \times \iota V) \cong \mathfrak{hom}(\mathsf{X}, \iota U) \times \mathfrak{hom}(\mathsf{X}, \iota V)$.

Proposition 2.3. Given a fully faithful embedding of the form

$$\iota\colon \mathsf{Cart} \to \mathsf{Space},$$

the restriction of the functor

$$\mathsf{Y}\colon \mathsf{Space} \to \mathfrak{Fun}^\times(\mathsf{Cart}, \mathsf{Set})^{\mathsf{op}}$$

along the fully faithful embedding ι produces a fully faithful functor

$$\mathsf{Y}_{\mathsf{Cart}}\colon \mathsf{Cart} \to \mathsf{Fun}^{\times}(\mathsf{Cart}, \mathsf{Set})^{\mathsf{op}}$$

that does not depend on ι and can be described as follows:

$$\mathsf{Y}(\mathsf{X})(U) = \mathrm{hom}_{\mathsf{Cart}}(\mathsf{X}, U), \qquad \mathsf{Y}(f)(U) = \mathrm{hom}_{\mathsf{Cart}}(f, U).$$

Used in 2.4.

Proof. Everything except for the fully faithfulness of the restriction of Y follows immediately from the definition of a fully faithful functor: for any $U, V \in \mathsf{Cart}$, the canonical map of sets

$$\mathrm{hom}_{\mathsf{Cart}}(U, V) \to \mathrm{hom}_{\mathsf{Space}}(\iota U, \iota V)$$

is an isomorphism. The restriction of Y is fully faithful by the Yoneda lemma:

$$\begin{aligned}\mathrm{hom}_{\mathsf{Fun}^{\times}(\mathsf{Cart},\mathsf{Set})^{\mathsf{op}}}(\mathsf{Y}(\mathsf{X}_1), \mathsf{Y}(\mathsf{X}_2)) &= \mathrm{hom}_{\mathsf{Fun}^{\times}(\mathsf{Cart},\mathsf{Set})}(\mathsf{Y}(\mathsf{X}_2), \mathsf{Y}(\mathsf{X}_1)) \\ &= \mathrm{hom}_{\mathsf{Fun}(\mathsf{Cart},\mathsf{Set})}(\mathsf{Y}(\mathsf{X}_2), \mathsf{Y}(\mathsf{X}_1)) \\ &\cong \mathsf{Y}(\mathsf{X}_2)(\mathsf{X}_1) \\ &= \mathrm{hom}_{\mathsf{Cart}}(\mathsf{X}_1, \mathsf{X}_2).\end{aligned}$$

Here the three equalities hold by definition and the isomorphism holds by the Yoneda lemma. ∎

We now turn tables and make an ansatz that Y should be a fully faithful functor even without the restriction to Cart. The other ansatz that we make here is that Y is essentially surjective, i.e., any product-preserving functor corresponds to some space. This is less obvious, but will become more clear as we compute various examples. Combined together, we see that Y should be an equivalence of categories.

Definition 2.4. Given a category of cartesian spaces Cart, the corresponding *category of spaces* Space is defined as

$$\mathsf{Fun}^{\times}(\mathsf{Cart}, \mathsf{Set})^{\mathsf{op}}.$$

The canonical embedding

$$\iota\colon \mathsf{Cart} \to \mathsf{Space}$$

is given by the functor $\mathsf{Y}_{\mathsf{Cart}}$ from Proposition 2.3. Used in 2.0*, 2.2, 2.3, 2.3*, 2.4, 2.5, 2.7, 2.7*, 2.8, 2.8*, 2.9, 2.9*, 3.3, 4.1, 5.0*, 5.1*, 5.2, 5.9, 5.11, 5.12*, 5.14, 5.14*, 5.15*, 5.18, 6.0*, 6.1, 6.2, 6.2*, 6.3, 6.4, 6.4*, 6.5, 6.6.

We also introduce a concise notation for the category of product-preserving functors $\mathsf{Cart} \to \mathsf{Set}$. The choice of terminology is motivated by the above considerations and the next section, in which specific choices of Cart yields certain categories of algebras in the usual sense.

Definition 2.5. Given a category of cartesian spaces Cart, the corresponding *category of algebras* Alg is defined as

$$\mathsf{Fun}^{\times}(\mathsf{Cart}, \mathsf{Set}),$$

i.e.,

$$\mathsf{Alg} = \mathsf{Space}^{\mathsf{op}}, \qquad \mathsf{Space} = \mathsf{Alg}^{\mathsf{op}}.$$

Used in 0.1*, 2.5, 2.6, 2.7, 2.7*, 2.8, 2.8*, 2.9, 3.2, 3.5, 3.9, 4.1, 5.1*, 5.2, 5.3, 5.9, 5.9*, 5.11, 5.12*, 5.13, 5.14, 5.15, 5.18*.

Remark 2.6. Any category of cartesian spaces Cart is equipped with a product-preserving functor $\mathsf{Cart}_{\mathbf{Z}} \to \mathsf{Cart}$. Restricting along this functor induces a functor

$$\mathsf{Alg} = \mathsf{Fun}^{\times}(\mathsf{Cart}, \mathsf{Set}) \to \mathsf{Fun}^{\times}(\mathsf{Cart}_{\mathbf{Z}}, \mathsf{Set}) \simeq \mathsf{CRing}.$$

(The rightmost equivalence of categories will be established below.) Thus, any category of algebras admits a forgetful functor to the category of commutative rings, justifying the terminology.

To confirm that we have indeed achieved our goal of making the functor Y an equivalence of categories, we establish the following result.

Proposition 2.7. Given $\iota\colon \mathsf{Cart} \to \mathsf{Space}$ as constructed in Definition 2.4, the functor

$$\mathsf{Y}\colon \mathsf{Space} \to \mathfrak{Fun}^{\times}(\mathsf{Cart}, \mathsf{Set})^{\mathsf{op}} = \mathsf{Alg}^{\mathsf{op}}$$

as constructed in Definition 2.2 is an equivalence of categories that is isomorphic to the identity functor $\mathrm{id}\colon \mathsf{Space} \to \mathsf{Alg}^{\mathsf{op}} = \mathsf{Space}$. Used in 2.8*, 6.6*.

Proof. The statement is nontrivial: even though we defined $\mathsf{Space} = \mathsf{Alg}^{\mathsf{op}}$, there is no reason why some specific functor $\mathsf{Y}\colon \mathsf{Space} \to \mathsf{Alg}^{\mathsf{op}}$ should be isomorphic to the identity functor. However, in our case this follows from the Yoneda lemma again: for any $\mathsf{F} \in \mathsf{Space} = \mathfrak{Fun}^{\times}(\mathsf{Cart}, \mathsf{Set})^{\mathsf{op}}$, the object $\mathsf{Y}(\mathsf{F}) \in \mathfrak{Fun}^{\times}(\mathsf{Cart}, \mathsf{Set})^{\mathsf{op}}$ is naturally isomorphic to F because for any $U \in \mathsf{Cart}$ we have

$$\mathsf{Y}(\mathsf{F})(U) = \mathfrak{hom}_{\mathsf{Space}}(\mathsf{F}, U) = \mathfrak{hom}_{\mathfrak{Fun}^{\times}(\mathsf{Cart},\mathsf{Set})^{\mathsf{op}}}(\mathsf{F}, U) = \mathfrak{hom}_{\mathfrak{Fun}^{\times}(\mathsf{Cart},\mathsf{Set})}(U, \mathsf{F}) \cong \mathsf{F}(U). \blacksquare$$

We now spell out more explicitly how to pass between a space and its algebra of functions.

Definition 2.8. The adjoint equivalence of categories

$$\mathfrak{Spec}\colon \mathsf{Alg}^{\mathsf{op}} \to \mathsf{Space}, \qquad \mathcal{O}\colon \mathsf{Space} \to \mathsf{Alg}^{\mathsf{op}}$$

is given by identity functors. Used in 2.8*, 2.9, 2.9*, 5.1*.

Thus, given an algebra $\mathcal{A} \in \mathsf{Alg}^{\mathsf{op}}$, the corresponding space is $\mathsf{A} = \mathfrak{Spec}(\mathcal{A}) \in \mathsf{Space}$. Given a space $\mathsf{X} \in \mathsf{Space}$, the corresponding algebra of functions is $\mathcal{A} = \mathcal{O}(\mathsf{X}) \in \mathsf{Alg}$. This choice of fonts will be used below to distinguish between spaces and algebras.

We remark that Proposition 2.7 proves that $\mathcal{O}$ is naturally isomorphic to Y. The new notation $\mathcal{O}$ is justified by the different perspective on this functor: we now interpret $\mathcal{O}(\mathsf{X})$ as the algebra of functions on the space X. Of course, in retrospect, this is exactly what Y does: $\mathsf{Y}(\mathsf{X})(\mathbf{A}^1)$ is precisely the set of maps $\mathsf{X} \to \mathbf{A}^1$, i.e., the underlying set of $\mathcal{O}(\mathsf{X})$.

We formalize this observation in the following result.

Proposition 2.9. The adjoint equivalence $\mathcal{O} \dashv \mathfrak{Spec}$ of Definition 2.8 is isomorphic to the following adjoint equivalence. The right adjoint functor is

$$\mathfrak{Spec}\colon \mathsf{Alg}^{\mathsf{op}} \to \mathsf{Space}, \qquad \mathcal{A} \mapsto (U \mapsto \mathfrak{hom}_{\mathsf{Alg}^{\mathsf{op}}}(\mathcal{A}, \mathsf{F}(U))),$$

where $\mathsf{F}(U)$ denotes the algebra of functions on a cartesian space U:

$$\mathsf{F}(U) \in \mathsf{Alg}^{\mathsf{op}}, \qquad \mathsf{F}(U)\colon \mathsf{Cart} \to \mathsf{Set}, \qquad \mathsf{F}(U)(V) = \mathsf{Cart}(U, V).$$

The left adjoint functor is

$$\mathcal{O}\colon \mathsf{Space} \to \mathsf{Alg}^{\mathsf{op}}, \qquad \mathsf{A} \mapsto (U \mapsto \mathfrak{hom}_{\mathsf{Space}}(\mathsf{A}, \mathsf{G}(U))),$$

where $\mathsf{G}(U)$ denotes $U \in \mathsf{Cart}$ interpreted as an object of Space:

$$\mathsf{G}(U) \in \mathsf{Space}, \qquad \mathsf{G}(U)\colon \mathsf{Cart} \to \mathsf{Set}, \qquad \mathsf{G}(U)(V) = \mathsf{Cart}(U, V).$$

Proof. Both $\mathsf{F}(U)\colon \mathsf{Cart} \to \mathsf{Set}$ and $\mathsf{G}(U)\colon \mathsf{Cart} \to \mathsf{Set}$ are the corepresentable functors of U. The isomorphism of adjunctions now follows from the Yoneda lemma. $\blacksquare$

In our case, spaces are encoded by their algebra of functions, so the above definition of $\mathfrak{Spec}$ and $\mathcal{O}$ may seem quite meaningless. Below, we will see how one can replace Space with an equivalent category, so that the above correspondence becomes nontrivial. For instance, in the case of Cart_ω we can replace Space with the equivalent category of pro-objects in the category of globally finitely presented Stein spaces.

3 Examples of spaces

3.1. *Algebraic geometry*

Recall the category Cart_R from Definition 1.3.

Proposition 3.2. Given a commutative ring R, the category $\mathsf{Alg} = \mathsf{Fun}^\times(\mathsf{Cart}_R, \mathsf{Set})$ is equivalent to the category CAlg_R of commutative algebras over R. Used in 3.3, 3.6, 3.10.

Proof. We describe mutually inverse functors going in both directions. Given a commutative algebra $\mathcal{A}$ over R, we construct a product-preserving functor

$$\mathcal{F}\colon \mathsf{Cart}_R \to \mathsf{Set}$$

by sending a cartesian space $\mathbf{A}^m_R \in \mathsf{Cart}_R$ to the set $\mathsf{U}(\mathcal{A})^m$, where $\mathsf{U}(\mathcal{A})$ denotes the underlying set of the algebra $\mathcal{A}$. We send a morphism $\mathbf{A}^m_R \to \mathbf{A}^n_R$ given by an n-tuple $(f_1, \ldots, f_n)$ of polynomials $f_i(x_1, \ldots, x_m)$ in m variables with coefficients in R to the map of sets

$$\mathsf{U}(\mathcal{A})^m \to \mathsf{U}(\mathcal{A})^n, \qquad (a_1, \ldots, a_m) \mapsto (f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m)),$$

where in the right side we evaluate the polynomials f_i on elements of $\mathcal{A}$ using the R-algebra structure of $\mathcal{A}$.

Given a product-preserving functor $\mathcal{F}\colon \mathsf{Cart}_R \to \mathsf{Set}$, we construct an R-algebra $\mathcal{A}$ as follows. The underlying set $S = \mathsf{U}(\mathcal{A})$ of $\mathcal{A}$ is the set $\mathcal{F}(\mathbf{A}^1)$. The various operations on $\mathcal{A}$ of arity 0 (i.e., constants), 1, and 2 are given by evaluating $\mathcal{F}$ on the indicated morphisms in Cart_R:

arity	operation	morphism	polynomial
0	$0\colon S^0 \to S$	$\mathbf{A}^0 \to \mathbf{A}^1$	0
0	$1\colon S^0 \to S$	$\mathbf{A}^0 \to \mathbf{A}^1$	1
1	$-\colon S^1 \to S$	$\mathbf{A}^1 \to \mathbf{A}^1$	$-x_1$
2	$+\colon S^2 \to S$	$\mathbf{A}^2 \to \mathbf{A}^1$	$x_1 + x_2$
2	$\cdot\colon S^2 \to S$	$\mathbf{A}^2 \to \mathbf{A}^1$	$x_1 x_2$

These operations saitsfy the axioms of a commutative algebra because $\mathcal{F}$ is a functor. For example, the associativity of multiplication follows from the fact that $\mathcal{F}$ preserves the commutativity of the following diagram:

$$\begin{tikzcd}
\mathbf{A}^3 \arrow[r, "(x_1+x_2,x_3)"] \arrow[d, "(x_1,x_2+x_3)"'] & \mathbf{A}^2 \arrow[d, "(y_1+y_2)"] \\
\mathbf{A}^2 \arrow[r, "(y_1+y_2)"] & \mathbf{A}^1,
\end{tikzcd}$$

where we use different variables x, y for different stages of the composition. ▮

The following definition represents a fundamental breakthrough in mathematics.

Definition 3.3. The category of spaces Space constructed from $\mathsf{Cart}_{\mathbf{Z}}$ (Definition 1.3) as described in Definition 2.4 and which is equivalent to the opposite category of commutative rings by Proposition 3.2, will be referred to as the *category of affine schemes*. Likewise, for Cart_R, where R is a commutative ring, we get the category of affine schemes over R.

Later we will see how to construct from affine schemes more general spaces like schemes and algebraic spaces.

3.4. *Differential geometry*

Recall the category $\mathsf{Cart}_{\mathsf{C}^\infty}$ from Definition 1.6.

Definition 3.5. The category $\mathsf{C}^\infty\mathsf{Ring}$ of C^∞-rings is the category Alg constructed from the category of cartesian spaces $\mathsf{Cart}_{\mathsf{C}^\infty}$ (Definition 1.6) using Definition 2.5.

C^∞-rings were introduced by Lawvere in 1967. The theory was then developed by Dubuc. Later, an expository account by Moerdijk and Reyes [1991] appaeared.

To get a more concrete picture of C^∞-rings, we first observe that C^∞-rings are, in particular, commutative real algebras.

Definition 3.6. The forgetful functor

$$\mathsf{C}^\infty\mathsf{Ring} \to \mathsf{CAlg}_{\mathbf{R}}$$

is given by the functor

$$\mathsf{Fun}^\times(\mathsf{Cart}_{\mathsf{C}^\infty}, \mathsf{Set}) \to \mathsf{Fun}^\times(\mathsf{Cart}_{\mathbf{R}}, Set)$$

induced by the precomposition with the inclusion

$$\mathsf{Cart}_{\mathbf{R}} \to \mathsf{Cart}_{\mathsf{C}^\infty}$$

as in Definition 1.7.

Proposition 3.7. Suppose $\mathcal{A} \in \mathsf{C}^\infty\mathsf{Ring}$ and I is an ideal of the underlying commutative real algebra of $\mathcal{A}$. Then there is a unique C^∞-ring structure on the quotient real algebra $\mathcal{A}/I$ such that the quotient map $\mathcal{A} \to \mathcal{A}/I$ is a morphism of C^∞-rings.

Proof. See, for example, Proposition 1.2 in Moerdijk and Reyes [1991]. ▮

3.8. ***Complex geometry***

Recall the category Cart_ω from Definition 1.9.

Definition 3.9. The category $\mathsf{EFCRing}$ of EFC-rings (complex algebras with entire functional calculus) is the category Alg constructed from the category of cartesian spaces Cart_ω (Definition 1.9) using Definition 2.5.

To get a more concrete picture of C^∞-rings, we first observe that C^∞-rings are, in particular, commutative real algebras.

Definition 3.10. The forgetful functor

$$\mathsf{EFCRing} \to \mathsf{CAlg}_{\mathbf{C}}$$

is given by the functor

$$\mathsf{Fun}^\times(\mathsf{Cart}_\omega, \mathsf{Set}) \to \mathsf{Fun}^\times(\mathsf{Cart}_{\mathbf{C}}, Set)$$

induced by the precomposition with the inclusion

$$\mathsf{Cart}_{\mathbf{C}} \to \mathsf{Cart}_\omega$$

as in Definition 1.10. Used in 2.9*.

Recall that a *Stein manifold* is a complex manifold that admits a proper holomorphic immersion into some $\mathbf{C}^n$. More generally, a *Stein space* is a complex analytic space (i.e., a locally ringed space that is locally isomorphic to the vanishing locus of some ideal of holomorphic functions on $\mathbf{C}^n$) whose reduction is a Stein manifold. A Stein spaces is *globally finitely presented* if it admits a closed embedding in $\mathbf{C}^n$ whose defining ideal is globally finitely generated.

Proposition 3.11. The category of globally finitely presented Stein spaces is contravariantly equivalent to the category of finitely presented EFC-algebras. The equivalence sends a Stein space to its EFC-algebra of global sections.

Proof. See Proposition 1.13 in Pridham [2020]. ▮

Proposition 3.12. The category of Stein spaces of finite embedding dimension is contravariantly equivalent to the category of finitely generated EFC-algebras (alias commutative holomorphically finitely generated algebras). The equivalence sends a Stein space to its EFC-algebra of global sections.

Proof. See Theorem 3.23 in Pirkovskii [2015]. ▮

4 Properties of spaces

Proposition 4.1. Suppose Space is a category of spaces constructed in Definition 2.4 from a category of cartesian spaces Cart, and $\mathsf{Alg} = \mathsf{Space}^{\mathsf{op}}$ is the corresponding category of algebras. Then the category Alg (hence also Space) admits all small limits and small colimits, and is a locally presentable category. Furthermore, small limits and sifted colimits in Alg are computed objectwise.

Proof. All cited facts are proved by Adámek–Rosický–Vitale [2011]. Every algebraic category, such as $\mathtt{Fun}^{\times}(\mathsf{Cart}, \mathsf{Set})$, is complete [2011, Corollary 1.22], cocomplete [2011, Theorem 4.5], locally finitely presentable [2011, Example 6.22(1)], the forgetful functor $\mathtt{Fun}^{\times}(\mathsf{Cart}, \mathsf{Set}) \to \mathtt{Fun}(\mathsf{Cart}, \mathsf{Set})$ preserves limits [2011, Proposition 1.21] and sifted colimits [2011, Proposition 2.5], i.e., is an algebraic functor. ∎

5 Open and closed subspaces

Our goal in this section is to formalize the notions of open and closed subspaces in the category Space.

5.1. *Closed subspaces*

What does it mean for a morphism $\mathsf{F} \to \mathsf{X}$ to define a closed subspace in the category Space? Equivalently, what does it mean for the corresponding morphism $\mathcal{X} \to \mathcal{F}$ in the category Alg to correspond to a closed subspace?

Recall that the Tietze extension theorem proves that any continuous function on a closed subspace of a normal topological space can be extended to the whole space. There are analogs of this result in the smooth and holomorphic cases, all formulated in a similar manner for closed subspaces. Therefore, it is quite reasonable to require that $\mathcal{X} \to \mathcal{F}$ is surjective on underlying sets, or, in more categorical terms, is a regular epimorphism.

Additionally, the algebra $\mathcal{X}$ must be reduced: if $x^n = 0$ for some $x \in \mathcal{X}$ and $n \geq 0$, then $x = 0$, i.e., $\mathcal{X}$ has no nonzero nilpotent elements. For instance, the surjective homomorphism of C^∞-rings $\mathsf{C}^\infty(\mathbf{R}) \to \mathbf{R}[x]/(x^n)$ that computes the derivatives of order less than n at point 0 geometrically corresponds to an inclusion $\mathtt{Spec}(\mathbf{R}[x]/(x^n)) \to \mathbf{R}$, whose domain will be interpreted later as a subspace of $\mathbf{R}$ given by

$$\mathtt{Spec}(\mathbf{R}[x]/(x^n)) = \{x \in \mathbf{R} \mid x^n = 0\},$$

where the right side must be interpreted using the internal logic of Space, as described below. If $n = 1$, then $\mathtt{Spec}(\mathbf{R}[x]/(x^n)) = \mathtt{Spec}(\mathbf{R}) = \{0\}$, which is a closed subspace of $\mathbf{R}$. The algebra $\mathbf{R}$ has no nonzero nilpotent elements. If $n > 1$, then $\mathtt{Spec}(\mathbf{R}[x]/(x^n))$ is the $(n-1)$st order infinitesimal neighborhood of $\{0\}$, whose closure contains the whole formal neighborhood of $\{0\}$ given by $\mathtt{Spec}(\mathbf{R}[[x]])$, where the algebra $\mathbf{R}[[x]]$ has no nonzero nilpotent elements, unlike $\mathbf{R}[x]/(x^n)$.

Definition 5.2. A *closed subspace* of a space X is a morphism of spaces $\mathsf{F} \to \mathsf{X}$ such that the corresponding morphism of algebras $\mathcal{X} \to \mathcal{F}$ is surjective and $\mathcal{F}$ is reduced. Used in 5.3, 5.3*, 5.11, 5.17.

Proposition 5.3. Given $\mathcal{X} \in \mathsf{Alg}$, the correspondence

$$I \mapsto (\mathcal{X} \to \mathcal{X}/I), \qquad (\varphi\colon \mathcal{X} \to \mathcal{F}) \mapsto (\ker \varphi)$$

between ideals in the underlying commutative ring of $\mathcal{X}$ and isomorphism classes (in the coslice category $\mathcal{X}/\mathsf{Alg}$) of surjective homomorphisms $\mathcal{X} \to \mathcal{F}$. Under this correspondence, reduced quotients $\mathcal{F}$ correspond to *radical ideals* I, i.e., $I = \sqrt{I}$, where $\sqrt{I} = \mathtt{rad}(I) = \{a \in \mathcal{X} \mid \exists n \geq 0\colon a^n \in I\}$. This correspondence is contravariant: an inclusion $\mathsf{F}_1 \subset \mathsf{F}_2$ of closed subspaces (as subobjects of a space X) corresponds to an inclusion $I_1 \supset I_2$ of radical ideals, and vice versa. Used in 5.3*, 5.9, 5.9*, 5.11, 5.13*, 5.15, 5.15*, 5.17.

Proof. The first part is valid for any Fermat theory (Dubuc–Kock [1984, Proposition 1.2]) and is proved by invoking Hadamard's axiom. The second part is an elementary result about commutative rings. ∎

Proposition 5.3 can be interpreted as saying that a closed subspace F of a space X is uniquely characterized by the ideal of functions on X that vanish on F.

We now examine the poset of closed subspaces of a space X, which turns out to be isomorphic to the poset of closed subspaces of a certain topological space, the Zariski spectrum of the corresponding algebra $\mathcal{X}$.

First, we review some material about such posets, treated abstractly (they are known as locales), as well as their relation to the traditional topological spaces. For more information, see the book by Picado and Pultr [2012].

Below, we axiomatize the properties of *open* subsets of a topological space. In this section, we take them to be the formal complements of closed subspaces introduced above. In the next section, we will see how to work with open subspaces directly.

Definition 5.4. The category Frm of frames is defined as follows. A *frame* is a poset F that admits suprema of arbitrary subsets, infima of finite subsets, and the map $a \mapsto a \wedge b$ preserves suprema for any fixed $b \in F$. A *homomorphism of frames* $f\colon F \to F'$ is a map of posets that preserves arbitrary suprema and finite infima. Used in 5.5, 5.6, 5.6*, 5.9, 5.9*.

Definition 5.5. The category Loc of *locales* is defined as $\mathsf{Frm}^{\mathsf{op}}$. Used in 5.3*, 5.6, 5.6*, 5.7, 5.8, 5.8*, 5.9.

Definition 5.6. The functor $\Omega\colon \mathsf{Top} \to \mathsf{Loc}$ sends a topological space X to the poset $\Omega(X)$ of open subsets of X and a continuous map $f\colon X \to X'$ to the homomorphism of frames $\Omega(f)\colon \Omega(X') \to \Omega(X)$.

Thus, a locale can be thought of as an abstract topological space, in which elements of the poset play the role of open subsets, but there need not be any underlying set of points.

Any open $a \in L$ in some $L \in \mathsf{Loc}$ induces a locale $L_a \in \mathsf{Loc}$, which as a poset is precisely $L_a = \{b \in L \mid b \leq a\}$. There is a canonical map $\iota_a\colon L_a \to L$ in Loc, given by the homomorphism of frames $b \mapsto b \wedge a$. Thus, $\iota_a\colon L_a \to L$ can be thought of as an inclusion of the open subspace corresponding to the element $a \in L$.

The functor Ω has a right adjoint $\mathsf{Sp}\colon \mathsf{Loc} \to \mathsf{Top}$, which can be described as follows.

Definition 5.7. The functor $\mathsf{Sp}\colon \mathsf{Loc} \to \mathsf{Top}$ send a locale L to the topological space $\mathsf{Sp}(L)$ defined as follows. The underlying set of $\mathsf{Sp}(L)$ is the set $\mathsf{hom}_{\mathsf{Loc}}(\mathbf{1}, L)$. Every $a \in L$ induced a subset $U_a = \{\iota_a \circ u \mid u\colon \mathbf{1} \to L_a\}$, and these subsets are precisely the open subsets. Used in 5.6*, 5.7, 5.8, 5.8*, 5.10.

Definition 5.8. A *spatial locale* is a locale in the essential image of the functor Ω. A *sober space* is a topological space in the essential image of the functor Sp. Used in 5.8*.

Consider the adjunction $\Omega \dashv \mathsf{Sp}$ defined above. A locale L is spatial if and only if the unit $L \to \mathsf{Sp}(\Omega(L))$ is an isomorphism. A topological space S is sober if and only if the counit $\Omega(\mathsf{Sp}(S)) \to S$ is an isomorphism. Thus, $\Omega \dashv \mathsf{Sp}$ restricts to an equivalence of categories between sober topological spaces and spatial locales.

A topological space S is sober if and only if the closure map on subsets of S with its domain restricted to singleton subsets and its codomain restricted to irreducible closed subsets (nonempty closed subsets that cannot be represented as a union of two proper closed subsets) is a bijection. In particular, all Hausdorff spaces are sober and all sober spaces are T0. On the other hand, there are T1-spaces that are not sober and there are sober spaces that are not T1.

Proposition 5.9. The poset of radical ideals of any $\mathcal{X} \in \mathsf{Alg}$ is a frame: it admits suprema of arbitrary subsets, infima of finite subsets, and the map $I \mapsto I \wedge J$ preserves suprema for any fixed radical ideal J. This construction can be promoted to a functor $\mathsf{Zar}\colon \mathsf{Alg} \to \mathsf{Frm}$, equivalently, $\mathsf{Zar}^{\mathsf{op}}\colon \mathsf{Space} \to \mathsf{Loc}$. Used in 5.9, 5.10.

Proof. The poset of radical ideals of $\mathcal{X}$ only depends on the underlying commutative ring of $\mathcal{X}$, so we assume $\mathcal{X}$ is a commutative ring. The supremum of a family $\{I_k\}_{k\in K}$ of radical ideals is

$$\mathsf{rad}\left(\sum_{k\in K} I_k\right).$$

The infimum of a family $\{I_k\}_{k\in K}$ of radical ideals is their interesection $\bigcap_{k\in K} I_k$, which coincides with their product if K is finite. Finally, for the distributivity property observe that

$$I \cap \mathsf{rad}\left(\sum_{k\in K} J_k\right) \subset \mathsf{rad}\left(\sum_{k\in K} I \cap J_k\right)$$

because any $a \in I \cap \mathtt{rad}\left(\sum_{k\in K} J_k\right)$ satisfies $a^m \in \sum_{k\in K} J_k$ for some $m \geq 0$, but then

$$a^{m+1} = a \cdot a^m \in \sum_{k\in K} IJ_k = \sum_{k\in K} I \cap J_k.$$

The functor $\mathsf{Alg} \to \mathsf{Frm}$ sends a morphism $f\colon A \to A'$ of algebras to the morphism of frames that sends a radical ideal I of A to the radical of the ideal generated by its image in A'. Finite infima (which coincide with finite products of ideals) are preserved: if $a \in \prod_{k\in K} \sqrt{f_* I_k}$ for a finite set K and ideals $I_k < A$, then some power of a belongs to $\prod_{k\in K} f_*(I_k) = f_*(\prod_{k\in K} I_k)$, hence $a \in \mathtt{rad}(f_*(\prod_{k\in K} I_k))$. Arbitrary suprema are preserved: if $a \in \mathtt{rad}(\sum_{k\in K} \sqrt{f_* I_k})$, then some power of a belongs to $\sum_{k\in K'} \sqrt{f_* I_k}$ for some finite subset $K' \subset K$, hence some other power of a belongs to $\sum_{k\in K'} f_* I_k = f_*(\sum_{k\in K'} I_k)$ ∎

Remark 5.10. The topological space $\mathsf{Sp}(\mathsf{Zar}(A))$ is the usual Zariski spectrum of a commutative ring (or algebra) A.

Remark 5.11. Below we will consider subcategories of Alg (or Space) defined by certain finiteness conditions, which result in an essentially small category, i.e., a category that is equivalent to a small category. The above description of the poset of closed subspaces may then involve different types of radical ideals, e.g., germ-determined radical ideals.

5.12. *Open subspaces*

What does it mean for a morphism $\mathsf{U} \to \mathsf{X}$ to define an open subspace in the category Space? Equivalently, what does it mean for the corresponding morphism $\mathcal{X} \to \mathcal{U}$ in the category Alg to correspond to an open subspace?

In many known examples of categories of spaces (such as topological spaces), inclusions of open subspaces are monomorphisms, but are typically not regular monomorphisms (meaning they do not arise as equalizers of pairs of continuous maps). Thus, we can expect the corresponding morphism of algebras $\mathcal{X} \to \mathcal{U}$ to be an epimorphism, but we cannot expect it to be surjective.

A typical example of a nonsurjective epimorphism of algebras is given by localization with respect to a multiplicative subset: given an ordinary commutative ring $\mathcal{A}$ and a subset $S \subset \mathcal{A}$, the morphism $\mathcal{A} \to \mathcal{A}[S^{-1}]$ is an epimorphism of algebras that is typically not surjective. Indeed, we expect that many functions on X will become invertible once restricted to U. If S is the set of all such functions, by the universal property of localizations of rings, we have a canonical homomorphism $\mathcal{X}[S^{-1}] \to \mathcal{U}$. We make an ansatz that this homomorphism is an isomorphism.

Different multiplicative subsets S of $\mathcal{X}$ can give rise to the same localization homomorphism $\mathcal{X} \to \mathcal{X}[S^{-1}]$.

Definition 5.13. A *saturated multiplicative subset* of $\mathcal{X} \in \mathsf{Alg}$ is a multiplicative subset S of $\mathcal{X}$ such that S contains all $c \in \mathcal{X}$ for which $\sqrt{(c)}$ contains $\bigcap_{a\in S} \sqrt{(a)}$. Used in 5.14, 5.15*, 5.18*.

The last condition can be intuitively explained as follows. Recall that radical ideals of $\mathcal{X}$ are in a contravariant bijective correspondence with closed subspaces of X. The radical ideal $\sqrt{(c)}$ corresponds to the zero locus of c, which is a closed subspace of X. Thus, $\sqrt{(c)}$ contains $\bigcap_{a\in S} \sqrt{(a)}$ if and only if the zero locus of c is contained in the union of zero loci of all $a \in S$. In other words, the nonvanishing locus of c contains the intersection of nonvanishing loci of all $a \in S$. But the latter should coincide with U, which means that the nonvanishing locus of c contains U, as desired.

Definition 5.14. An *open subspace* of a space X is a morphism of spaces $\mathsf{U} \to \mathsf{X}$ such that the corresponding morphism of algebras $\mathcal{X} \to \mathcal{U}$ is isomorphic (in the slice category $\mathcal{X}/\mathsf{Alg}$) to the localization with respect to a saturated multiplicative subset $S \subset \mathcal{X}$. Used in 5.16, 5.17, 6.0*.

We expect a bijective correspondence between open and closed subspaces of any $\mathsf{X} \in \mathsf{Space}$ given by passing to the complement.

Proposition 5.15. Given $\mathcal{X} \in \mathsf{Alg}$, the correspondence

$$I \mapsto S = \{a \in \mathcal{X} \mid I \subset \sqrt{(a)}\}, \qquad S \mapsto I = \bigcap_{a\in S} \sqrt{(a)}$$

between saturated multiplicative subsets S and radical ideals I in the underlying commutative ring of $\mathcal{X}$ is bijective.

Proof. Given a radical ideal I, consider the subset $S = \{a \in \mathcal{X} \mid I \subset \sqrt{(a)}\}$. Since $\sqrt{(1)} = (1) = \mathcal{A}$, we have $1 \in S$. If $a \in S$ and $a' \in S$, then $aa' \in S$ because for any $b \in I$ we have $b^m = ac$ and $b^{m'} = a'c'$ for some $m, m' \geq 0$ and $c, c' \in \mathcal{A}$, so $b^{m+m'} = aa'(cc')$, i.e., $b \in \sqrt{(aa')}$. Finally, S is saturated since $\bigcap_{a\in S} \sqrt{(a)} \supset I$ by construction, so if $\sqrt{(c)}$ contains $\bigcap_{a\in S} \sqrt{(a)}$, it also contains I.

Given a saturated multiplicative subset $S \subset \mathcal{X}$, consider the ideal $I = \bigcap_{a\in S} \sqrt{(a)}$. This ideal is radical because any intersection of radical ideals is again radical.

Suppose that S is a saturated multiplicative subset of $\mathcal{X}$, $I = \bigcap_{a\in S} \sqrt{(a)}$, and $S' = \{a \in \mathcal{X} \mid I \subset \sqrt{(a)}\}$. Then $S = S'$ by definition of a saturated multiplicative subset.

Finally, suppose I is a radical ideal of $\mathcal{X}$, $S = \{a \in \mathcal{X} \mid I \subset \sqrt{(a)}\}$, and $I' = \bigcap_{a\in S} \sqrt{(a)}$. By construction, $I \subset I'$. **To do:** Finish the proof. ▮

We conclude by examining the notion of an open cover in the category Space.

Definition 5.16. A family of inclusions of open subspaces $\mathsf{U}_i \to \mathsf{X}$ ($i \in I$) is an *open cover* of X if the supremum of U_i for all $i \in I$ in the poset of open subspaces of X equals X. Used in 5.16*, 5.18, 6.1.

The following characterization of open covers follows immediately from the formula for suprema.

Proposition 5.17. If open subspaces $\mathsf{U}_i \to \mathsf{X}$ are complements of closed subspaces $\mathsf{F}_i \to \mathsf{X}$ given by radical ideals I_i of $\mathcal{X}$, then $\{\mathsf{U}_i\}_{i\in I}$ cover X if and only if $1 \in \sum_{i\in I} I_i$, i.e., $1 \in \mathcal{X}$ is a finite sum of elements of I_i.

The following property is crucial for defining sheaves in the next section. Roughly, it says that fusing elements of an open cover together along their overlapping parts (given by intersections) yields back the original space.

Proposition 5.18. Given an open cover $\{\mathsf{U}_i \to \mathsf{X}\}_{i\in I}$ of a space X, the diagram

$$\coprod_{j,k\in I} \mathsf{U}_j \cap \mathsf{U}_k \rightrightarrows \coprod_{i\in I} \mathsf{U}_i \to \mathsf{X}$$

is a coequalizer diagram in Space. Used in 6.3*, 6.4*.

Proof. Passing to the opposite category Alg, we have to show that the corresponding diagram

$$\mathcal{X} \to \prod_{i\in I} \mathcal{U}_i \rightrightarrows \prod_{j,k\in I} \mathcal{U}_j \cap \mathcal{U}_k$$

is an equalizer diagram. Recall now that the morphisms $\mathcal{X} \to \mathcal{U}_i$ are given by localizations with respect to saturated multiplicative subsets $S_i \subset \mathcal{X}$. **To do:** Finish the proof. ▮

6 Sheaves as generalized spaces

The idea behind defining the category Space from a category of cartesian spaces Cart is that a space can be completely described by its algebra of functions.

Some very important examples of spaces cannot be described in this manner. For instance, the complex projective space of any dimension only admits constant holomorphic functions. We want to include complex projective spaces in our formalism, which forces us to further enlarge the category Space.

The following construction is formally very similar to the one used to obtain Space from Cart. This is remarkable, since the aims of the two constructions are quite different.

We would like to construct a fully faithful embedding

$$\kappa\colon \mathsf{Space} \to \mathsf{GenSpace},$$

where Space is a category constructed in Definition 2.4 from a category of cartesian spaces Cart, whereas $\mathsf{GenSpace}$ is a category of generalized spaces that we will construct below.

For example:

- For Cart_R, we expect $\mathsf{GenSpace}$ to contain schemes and algebraic spaces over R as a full subcategory;
- For $\mathsf{Cart}_{\mathsf{C}^\infty}$, we expect $\mathsf{GenSpace}$ to contain infinite-dimensional smooth manifolds as a full subcategory;
- For Cart_ω, we expect $\mathsf{GenSpace}$ to contain complex analytic spaces as a full subcategory.

For a moment, we assume that such an embedding exists and explore its properties.

Our strategy is explore a generalized space E by studying maps *into* E from objects of Space. This stands in constrast to the construction of Space from Cart, where we studied maps *from* a space X to an object of Cart. In particular, instead of (covariant) functors

$$\mathsf{Cart} \to \mathsf{Set}$$

we get (contravariant) functors

$$\mathsf{Space}^{\mathsf{op}} \to \mathsf{Set},$$

since a morphism $\mathsf{P} \to \mathsf{P}'$ in Space induces a map of sets

$$\mathrm{hom}(\mathsf{P}', \mathsf{E}) \to \mathrm{hom}(\mathsf{P}, \mathsf{E})$$

in the opposite direction.

Furthermore, just as before, the resulting functor

$$\mathrm{hom}(-, \mathsf{E})\colon \mathsf{Space}^{\mathsf{op}} \to \mathsf{Set}$$

can be expected to preserve (finite) products. Indeed, products in $\mathsf{Space}^{\mathsf{op}}$ are coproducts in Space, i.e., disjoint union of spaces, denoted by $\sqcup$. The functor $\mathrm{hom}(-, \mathsf{E})$ preserves products if the canonical map

$$\mathrm{hom}(\mathsf{P} \sqcup \mathsf{P}', \mathsf{E}) \to \mathrm{hom}(\mathsf{P}, \mathsf{E}) \times \mathrm{hom}(\mathsf{P}', \mathsf{E})$$

is an isomorphism of sets. This can be interpreted as saying that a map $\mathsf{P} \sqcup \mathsf{P}' \to \mathsf{E}$ can be identified with a pair of maps $\mathsf{P} \to \mathsf{E}$ and $\mathsf{P}' \to \mathsf{E}$, which is indeed a reasonable property to demand.

However, we can expect more. If P and P' are open subspaces of Q (not necessarily disjoint) such that $\mathsf{Q} = \mathsf{P} \cup \mathsf{P}'$, we expect that a map $\mathsf{Q} \to \mathsf{E}$ can be identified with a pair of maps $\mathsf{P} \to \mathsf{E}$ and $\mathsf{P}' \to \mathsf{E}$ whose restrictions to the intersection $\mathsf{P} \cap \mathsf{P}'$ coincide.

We encode this observation using the notion of a sheaf.

Definition 6.1. Denote by

$$\mathrm{Fun}^{\mathrm{sh}}(\mathsf{Space}^{\mathsf{op}}, \mathsf{Set})$$

the full subcategory of $\mathrm{Fun}(\mathsf{Space}^{\mathsf{op}}, \mathsf{Set})$ comprising *sheaves*, i.e., functors

$$\mathsf{F}\colon \mathsf{Space}^{\mathsf{op}} \to \mathsf{Set}$$

such that the following *gluing property* is satisfied: for any open cover $\{\mathsf{U}_i \to \mathsf{X}\}_{i \in I}$ of a space X the canonical maps

$$\mathsf{F}(\mathsf{X}) \to \prod_{i \in I} \mathsf{F}(\mathsf{U}_i) \rightrightarrows \prod_{j,k \in I} \mathsf{F}(\mathsf{U}_j \cap \mathsf{U}_k)$$

exhibit $\mathsf{F}(\mathsf{X})$ as the equalizer of the right two maps, i.e., the left map is an injective map of sets whose image is the subset of the middle term comprising those elements on which the right two maps coincide. Used in 6.2*, 6.3, 6.4, 6.5, 6.6.

Remark 6.2. The same definition can be used to define sheaves valued in any category that admits small limits, not just Set. The equalizers and products must be taken in this new category. In particular, below we assume that the embedding

$$\kappa^{\mathrm{op}} \colon \mathsf{Space}^{\mathsf{op}} \to \mathsf{GenSpace}^{\mathsf{op}}$$

is a sheaf (valued in the opposite category of generalized spaces), in the same manner as the functor

$$\iota \colon \mathsf{Cart} \to \mathsf{Space}$$

was previously assumed to preserve finite products.

In particular, the gluing property applied to the cover of $\mathsf{U}_1 \sqcup \mathsf{U}_2$ by its two summands implies that any sheaf F preserves binary products. Likewise, if $I = \emptyset$ and $\mathsf{X} = \emptyset$ is the initial object in Space, we see that $\mathsf{F}(\emptyset)$ is singleton set, i.e., F preserve terminal objects, hence also finite products.

We are now ready to formalize the above construction that explores a generalized space $\mathbf{E}$ by studying maps of the form $\mathsf{P} \to \mathbf{E}$ for arbitrary spaces $\mathsf{P} \in \mathsf{Space}$.

Definition 6.3. Given a fully faithful embedding of the form

$$\kappa \colon \mathsf{Space} \to \mathsf{GenSpace}$$

such that the functor

$$\kappa^{\mathrm{op}} \colon \mathsf{Space}^{\mathsf{op}} \to \mathsf{GenSpace}^{\mathsf{op}}$$

is a sheaf, we define a functor

$$\mathbf{Y} \colon \mathsf{GenSpace} \to \mathtt{Fun}^{sh}(\mathsf{Space}^{\mathsf{op}}, \mathsf{Set})$$

by sending $\mathbf{E} \in \mathsf{GenSpace}$ to the sheaf

$$\mathbf{Y}(\mathbf{E}) \colon \mathsf{Space}^{\mathsf{op}} \to \mathsf{Set}, \qquad \mathsf{P} \mapsto \mathtt{hom}_{\mathsf{GenSpace}}(\kappa\mathsf{P}, \mathbf{E})$$

and a morphism $f \colon \mathbf{E}_1 \to \mathbf{E}_2$ in $\mathsf{GenSpace}$ to the natural transformation

$$\mathbf{Y}(\mathbf{E}_1) \to \mathbf{Y}(\mathbf{E}_2), \qquad \mathsf{P} \mapsto \mathtt{hom}(\kappa\mathsf{P}, f), \qquad \mathtt{hom}(\kappa\mathsf{P}, f) \colon \mathtt{hom}(\kappa\mathsf{P}, \mathbf{E}_1) \to \mathtt{hom}(\kappa\mathsf{P}, \mathbf{E}_2), \qquad g \mapsto f \circ g.$$

Used in 6.6.

By Proposition 5.18, the functor $\mathbf{Y}(\mathbf{E})$ is indeed a sheaf, as long as κ^{op} is a sheaf: the functor $\mathtt{hom}(\kappa(-), \mathbf{E})$ preserves whatever limits are preserved by κ^{op}.

Proposition 6.4. Given a fully faithful embedding of the form

$$\kappa \colon \mathsf{Space} \to \mathsf{GenSpace}$$

such that the functor

$$\kappa^{\mathrm{op}} \colon \mathsf{Space}^{\mathsf{op}} \to \mathsf{GenSpace}^{\mathsf{op}}$$

is a sheaf, the restriction of the functor

$$\mathbf{Y} \colon \mathsf{GenSpace} \to \mathtt{Fun}^{sh}(\mathsf{Space}^{\mathsf{op}}, \mathsf{Set})$$

along the fully faithful embedding κ produces a fully faithful functor

$$\mathbf{Y}_{\mathsf{Space}}\colon \mathsf{Space} \to \mathfrak{Fun}^{\mathfrak{sh}}(\mathsf{Space}^{\mathsf{op}}, \mathsf{Set})$$

that does not depend on κ and can be described as follows:

$$\mathbf{Y}(\mathbf{E})(\mathsf{P}) = \mathfrak{hom}_{\mathsf{Space}}(\mathsf{P}, \mathbf{E}), \qquad \mathbf{Y}(f)(\mathsf{P}) = \mathfrak{hom}_{\mathsf{Space}}(\mathsf{P}, f).$$

Used in 6.5.

Proof. Everything except for the fully faithfulness of the restriction of $\mathbf{Y}$ follows immediately from the definition of a fully faithful functor: for any $U, V \in \mathsf{Space}$, the canonical map of sets

$$\mathfrak{hom}_{\mathsf{Space}}(U, V) \to \mathfrak{hom}_{\mathsf{GenSpace}}(\kappa U, \kappa V)$$

is an isomorphism. The restriction of $\mathbf{Y}$ is fully faithful by the Yoneda lemma and lands in sheaves (as opposed to arbitrary functors) by Proposition 5.18. ∎

We now turn tables and make an ansatz that $\mathbf{Y}$ should be a fully faithful functor even without the restriction to Space. The other ansatz that we make here is that $\mathbf{Y}$ is essentially surjective, i.e., any sheaf corresponds to some generalized space. This is less obvious, but will become more clear as we compute various examples. Combined together, we see that $\mathbf{Y}$ should be an equivalence of categories.

Definition 6.5. Given a category of spaces Space, the corresponding *category of generalized spaces* $\mathsf{GenSpace}$ is defined as

$$\mathfrak{Fun}^{\mathfrak{sh}}(\mathsf{Space}^{\mathsf{op}}, \mathsf{Set}).$$

The canonical embedding

$$\kappa\colon \mathsf{Space} \to \mathsf{GenSpace}$$

is given by the functor $\mathbf{Y}_{\mathsf{Space}}$ from Proposition 6.4. Used in 6.0*, 6.2, 6.3, 6.4, 6.4*, 6.5, 6.6.

To confirm that we have indeed achieved our goal of making the functor $\mathbf{Y}$ an equivalence of categories, we establish the following result.

Proposition 6.6. Given $\kappa\colon \mathsf{Space} \to \mathsf{GenSpace}$ as constructed in Definition 6.5, the functor

$$\mathbf{Y}\colon \mathsf{GenSpace} \to \mathfrak{Fun}^{\mathfrak{sh}}(\mathsf{Space}^{\mathsf{op}}, \mathsf{Set})$$

as constructed in Definition 6.3 is an equivalence of categories that is isomorphic to the identity functor.

Proof. Same as the proof of Proposition 2.7. ∎

7 References

[1984] Eduardo J. Dubuc, Anders Kock. On 1-form classifiers. Communications in Algebra 12:12 (1984), 1471–1531. doi:10.1080/00927878408823064. 1.14, 5.3*.

[1991] Ieke Moerdijk, Gonzalo E. Reyes. Models for Smooth Infinitesimal Analysis. Springer, 1991. doi:10.1007/978–1–4757–4143–8. 3.5*, 3.7*.

[2011] Jiří Adámek, Jiří Rosický, Enrico M. Vitale. Algebraic theories. Cambridge Tracts in Mathematics 184 (2011), Cambridge University Press. doi:10.1017/cbo9780511760754. 1.12, 4.1*.

[2012] Jorge Picado, Aleš Pultr. Frames and Locales. Topology without points. Frontiers in Mathematics (2012), Birkhäuser. doi:10.1007/978–3–0348–0154–6. 5.3*.

[2015] Alexei Yu. Pirkovskii. Holomorphically finitely generated algebras. Journal of Noncommutative Geometry 9 (2015), 215–264. arXiv:1304.1991, doi:10.4171/JNCG/192. 3.12*.

[2020] J. P. Pridham. A differential graded model for derived analytic geometry. Advances in Mathematics 360 (2020), 106922. arXiv:1805.08538v1, doi:10.1016/j.aim.2019.106922. 3.11*.