1. Which sets of data given below define categories? Warning: some sets of data are incomplete, e.g., do not specify composition. You must reconstruct all missing data (this is a creative process and can have more than one answer). Some items below have negative answers. Some are purposefully nonsensical. If some set of data cannot be made into a category, you have to provide a proof.

- Topological spaces and proper maps. (A map $f\colon X \to Y$ is *proper* if it is continuous and for any compact subset $T \subset Y$ the subset $f^{-1}(T) \subset X$ is also compact.)
- Sets and relations. (More precisely, given two sets X and Y, morphisms $X \to Y$ are *relations* from X to Y, i.e., subsets of $X \times Y$. Two relations $R \subset X \times Y$ and $S \subset Y \times Z$ are composed as follows: $R \circ S = \{(x, z) \mid \exists y \in Y\colon (x, y) \in R \wedge (y, z) \in Z\}$.
- Sets and surjective functions.
- Sets and partially defined functions. (A partially defined function $X \to Y$ is a function $A \to Y$, where $A \subset X$. If $x \in X$, we say that f is *defined* on x (or: $f(x)$ is defined) if $x \in A$. Partially defined functions are composed as follows: if $f\colon X \to Y$ and $g\colon Y \to Z$ are partially defined function, then the composition gf is defined on $x \in X$ if f is defined on x and g is defined on $f(x)$, in which case $(gf)(x)\colon= g(f(x))$.)
- Fix a topological space X and define a category as follows. Objects are continuous functions with codomain X, i.e., $f : Y \to X$ (Y is arbitrary). (Here *morphisms* in Top play the role of *objects* in the category that we are constructing.) Morphisms from an object $f : Y \to X$ to an object $f' : Y' \to X$ are continuous maps $g : Y \to Y'$ such that $f'g = f$.
- The category Open. Objects are open subsets of $\mathbf{R}^n$, where n is arbitrary (not fixed). Morphisms $U \to V$ are infinitely differentiable maps $U \to V$.
- The category Open_*. Objects are pairs (U, x), where $U \subset \mathbf{R}^n$ is an open subset and $x \in U$. Morphisms $(U, x) \to (V, y)$ are infinitely differentiable maps $U \to V$ that map x to y.
- $\mathsf{Mat}_{\mathbf{R}}$: objects are natural numbers $n \geq 0$ and morphisms $m \to n$ are matrices of size $n \times m$. Composition is multiplication of matrices.
- $\mathsf{B}\mathbf{R}$: there is only one object $*$. Morphisms $* \to *$ are real numbers. Composition of morphisms is given by multiplication of real numbers.
- There is only one object $*$. Morphisms $* \to *$ are compactly supported continuous functions $\mathbf{R} \to \mathbf{R}$. Composition of morphisms is given by multiplication of functions.
- Poset: objects are partially ordered sets (i.e., a set X with a relation R that is reflexive ($x \leq x$), transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$), and antisymmetric ($x \leq y$ and $y \leq x$ implies $x = y$). Morphisms are functions that preserve the order: if $x \leq y$, then $f(x) \leq f(y)$.

2. Which sets of data below define functors? (Same warning as above.)

- $\mathsf{Open}_* \to \mathsf{Vect}_{\mathbf{R}}$. Send $U \subset \mathbf{R}^m$ to $\mathbf{R}^m$. Send $f\colon (U, x) \to (V, y)$ to the linear map $\mathbf{R}^m \to \mathbf{R}^n$ given by the Jacobian matrix of f at x, i.e., the entry in ith row and jth column is the value of the ith partial derivative of the jth coordinate of f at point x. In symbols: $a_{i,j} = \frac{\partial f_j}{\partial x_i}(x)$. (The jth coordinate of f is the composition $U \to V \subset \mathbf{R}^n \to \mathbf{R}$, where $\mathbf{R}^n \to \mathbf{R}$ is the projection to the jth component.)
- $\mathsf{Mat}_{\mathbf{R}} \to \mathsf{B}\mathbf{R}$: send any object $n \geq 0$ of $\mathsf{Mat}_{\mathbf{R}}$ to the only object of $\mathsf{B}\mathbf{R}$. Send a matrix of size $m \times n$ to its determinant (a morphism in $\mathsf{B}\mathbf{R}$) if $m = n$. Otherwise send it to zero.
- $\mathsf{OpenSet}\colon \mathsf{Top} \to \mathsf{Poset}$: send any topological space X to the poset $\mathsf{OpenSet}(X)$ whose elements are open subsets of X and the ordering is given by inclusion. Send any continuous map $f\colon X \to Y$ of topological spaces to the map of posets $g\colon \mathsf{OpenSet}(X) \to \mathsf{OpenSet}(Y)$ defined as follows: $g(U) = \bigcup_{V \subset f(U)} V$, where V runs over open subsets of Y.

3. Define a functor $\mathsf{Mat}_{\mathbf{R}}^{\mathsf{op}} \to \mathsf{Mat}_{\mathbf{R}}$. Define a functor $\mathsf{Mat}_{\mathbf{R}} \to \mathsf{Vect}_{\mathbf{R}}$. Define a functor $\mathsf{B}\mathbf{R} \to \mathsf{Mat}_{\mathbf{R}}$. Define a functor $\mathsf{Mat}_{\mathbf{R}} \to \mathsf{Open}_*$. Define a functor $\mathsf{Top}^{\mathsf{op}} \to \mathsf{Poset}$.

4. Fix a category C. A *section* of a morphism $f\colon X \to Y$ in C is a morphism $g\colon Y \to X$ such that $fg = \mathrm{id}_Y$. Give an example of a category C such that all epimorphisms have sections. Give an example of a category C and an epimorphism f in C that does not have a section. Hint: it suffices to use the examples that we studied in class.

5. Fix a category C. A *bimorphism* in C is a morphism f that is simultaneously a monomorphism and an epimorphism. Is any isomorphism a bimorphism? Give an example of a category C and a bimorphism f in C that is not an isomorphism.

6. Construct two functors $D\colon \mathsf{Set} \to \mathsf{Set}$ and $I\colon \mathsf{Set}^{\mathsf{op}} \to \mathsf{Set}$ such that for any set X we have $D(X) = I(X) = 2^X$, where 2^X denotes the set of all subsets of X. (In other words, you must define D and I on morphisms and prove that composition and identity maps are respected.)

7. Construct a functor $\mathsf{L}^1\colon \mathsf{Set} \to \mathsf{Ban_1}$ such that for any set S the Banach space $\mathsf{L}^1(S)$ is the space of functions $f : S \to \mathbf{R}$ such that the sum $\sum_{s\in S} f(s)$ exists (and is finite). Construct a functor $\mathsf{L}^\infty\colon \mathsf{Set}^{\mathsf{op}} \to \mathsf{Ban_1}$ such that $\mathsf{L}^\infty(S)$ is the space of all bounded functions $S \to \mathbf{R}$.

8. Show that the class of epimorphisms in the category of Hausdorff topological spaces coincides with the class of continuous maps whose image is dense.

9. Describe concretely all monomorphisms and epimorphisms in $\mathsf{B}\mathbf{R}$. Same question for Open and Open_*. Same question for Poset.

10. Which of the following assignments of a set $F(U)$ to an open subset $U \subset X$ of a topological space X defines a sheaf on X? (As usual you must fill in all the missing data, e.g., the restriction maps are not specified explicitly if they are obvious.) Below U denotes an arbitrary open subset of X.

- $X = \mathbf{R}$, $F(U)$ is the set of continuous functions $U \to \mathbf{R}$.
- $X = \mathbf{R}$, $F(U)$ is the set of bounded continuous functions $U \to \mathbf{R}$.
- $X = \mathbf{C}$, $F(U)$ is the set of holomorphic functions $U \to \mathbf{C}$.
- $X = \mathbf{R}$, $F(U)$ is the set of Borel measurable functions $U \to \mathbf{R}$.
- $X = \mathbf{R}$, $F(U)$ is the set of constant functions $U \to \mathbf{R}$.
- $X = \mathbf{R}$, $F(U)$ is the set of locally constant functions $U \to \mathbf{R}$.
- $X = \mathbf{R}$, $F(U)$ is the set of increasing functions $U \to \mathbf{R}$ ($x \le y$ implies $f(x) \le f(y)$).
- $X = \mathbf{R}$, $F(U)$ is the set of integrable measurable functions $U \to \mathbf{R}$ (with finite integral).
- $X = \mathbf{R}$, $F(U)$ is the set of convex functions $U \to \mathbf{R}$ in the following sense: for any $x, y \in U$ such that $x \le y$ and $[x, y] \subset U$ and for any $0 \le t \le 1$ we have $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$.

11. Recall that a *base* B of a topological space X is a set of open subsets of X such that any open subset of X can be represented as a union of sets in the base. Below we consider B to be a category whose objects are elements of B and morphisms are inclusions (i.e., the set of morphisms between any two objects is either empty or consists of a single element). A *sheaf on a topological space X with respect to a base B of X* (or simply a *sheaf on B*) is a functor $B^{\mathsf{op}} \to \mathsf{Set}$ that satisfies the gluing property, which is formulated in exactly the same way as for sheaves on X, but using elements of B everywhere instead of arbitrary open sets. Morphisms of sheaves on B are natural transformations of functors.

- Prove that any sheaf on X restricts to a sheaf of B.
- Prove that any sheaf on X can be reconstructed (up to an isomorphism) from its restriction to B.
- Prove that the category of sheaves on X is equivalent to the category of sheaves on B.

12. A *idempotent ring* is a ring R such that $x^2 = x$ for any $x \in R$. (Rings are assumed to be associative and unital, homomorphisms of rings preserve units.)

- Show that any idempotent ring is commutative: $xy = yx$ for all x and y.
- Show that the relation $x \le y := (x = xy)$ defines a partial order on R.
- Show that given a set X, equipping 2^X (the set of subsets of X) with the following operations: $0 := \emptyset$, $x + y := (x \setminus y) \cup (y \setminus x)$, $-x := X \setminus x$, $1 := X$, $xy := x \cap y$ produces an idempotent ring.
- Recall that the supremum of a subset $A \subset R$, if it exists, is the unique element $s \in R$ such that for all $a \in A$ we have $a \le s$ and if s' is another element with the same property, then $s \le s'$. Show that in the idempotent ring 2^X every subset has a supremum.
- An *atom* in an idempotent ring is an element $a \in R$ such that $a \ne 0$ and if $0 \le b \le a$ for some $b \in R$, then $b = 0$ or $b = a$. Show that in the idempotent ring 2^X every element can be represented as the supremum of a set of atoms.
- Show that the assignment $X \mapsto 2^X$ can be extended to a contravariant functor $2^{(-)}$ from the category of sets to the category whose objects are idempotent rings in which every subset has a supremum and every element is the supremum of a set of atoms, and morphisms are homomorphisms of rings that preserve suprema ($f : R \to R'$ preserves suprema if for any $S \subset R$ we have $\sup f(S) = f(\sup S)$).
- Construct a contravariant functor going in the opposite direction. (Hint: it is useful to keep the example of the idempotent ring 2^X when constructing this functor.)

- Prove that the two functors together form an equivalence of categories.

13. Prove that any continuous function defined on a closed subset of a compact Hausdorff space can be extended to a continuous function defined on the entire space. (This is meant to be proved using what we have learned about various equivalences of categories in functional analysis, not by referencing the Urysohn lemma or the Tietze extension theorem.)

14. Recall that any Hilbert space H admits an isomorphism $H \cong H^*$, which equips H with a weak-* topology (alias weak topology). Suppose we are given an operator $P\colon H \to H$ that becomes continuous if we equip its source with the weak topology and its target with the norm topology. Prove that H decomposes as a direct sum of eigenspaces of P.

15. Recall the category of enhanced measurable spaces. More precisely, we constructed two different categories: PreEMS and its quotient $\mathsf{StrictEMS}$ modulo the equivalence relation of equality almost everywhere. Consider the category $\mathsf{CAlg}^*_{\mathbf{C}}$ of commutative *complex *-algebras*, defined as follows. Objects are complex algebras A equipped with a complex-antilinear operation $*\colon A \to A$ (the involution) such that $(ab)^* = b^*a^*$, $1^* = 1$, and $a^{**} = a$. Morphisms are morphisms of complex algebras $f\colon A \to B$ such that $f(a^*) = f(a)^*$.

- Extend the following construction to a functor $\mathsf{L}^\infty\colon \mathsf{PreEMS}^{\mathsf{op}} \to \mathsf{CAlg}^*_{\mathbf{C}}$. Send an enhanced measurable space (X, Σ, N) to the complex *-algebra of *bounded* morphisms $(X, \Sigma, N) \to (\mathbf{C}, \Sigma_{\mathbf{C}}, \{\emptyset\})$, where $\Sigma_{\mathbf{C}}$ denotes the Borel σ-algebra of $\mathbf{C}$. All operations are pointwise, with f^* being the pointwise complex conjugate of f. Here a morphism is *bounded* if it factors (when restricted to a conegligible subset of X) through some bounded subset of $\mathbf{C}$.
- Show that if C is a category with an equivalence relation R on its sets of morphisms, then precomposing a functor $\mathsf{C}/R \to \mathsf{D}$ with the quotient functor $\mathsf{C} \to \mathsf{C}/R$ establishes a bijection between functors $\mathsf{C}/R \to \mathsf{D}$ and functors $\mathsf{C} \to \mathsf{D}$ that send equivalent morphisms in C to equal morphisms in D. Apply this observation to construct a functor $\mathsf{L}^\infty\colon \mathsf{StrictEMS}^{\mathsf{op}} \to \mathsf{CAlg}^*_{\mathbf{C}}$
- [***] (Very optional.) Prove that the functor L^∞ is *not* faithful: there is an object (X, Σ, N) in the quotient $\mathsf{StrictEMS}$ and two different morphisms $f, g\colon (X, \Sigma, N) \to (\mathbf{C}, \Sigma_{\mathbf{C}}, \{\emptyset\})$ such that $\mathsf{L}^\infty(f) = \mathsf{L}^\infty(g)$.
- Consider again the category PreEMS. Show that the following relation of *weak equality almost everywhere* gives rise to another quotient category of PreEMS, denoted simply by EMS: $f \approx g\colon (X, \Sigma, N) \to (X', \Sigma', N')$ if for any $m \in \Sigma'$ the symmetric difference $f^*m \oplus g^*m$ is a negligible subset of X.
- Construct a functor $\mathsf{L}^\infty\colon \mathsf{EMS}^{\mathsf{op}} \to \mathsf{CAlg}^*_{\mathbf{C}}$ and prove that it is faithful.
- [***] (Very optional.) A functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ is *full* if for any objects $X, Y \in \mathsf{C}$ the map of sets $\mathsf{C}(X, Y) \to \mathsf{D}(X, Y)$ is surjective. Show that the functor L^∞ is full.

16. Recall the category Open of open subsets of $\mathbf{R}^n$ (for any $n \geq 0$) and infinitely differentiable maps. (Attain extra bonus points by working with arbitrary second countable Hausdorff smooth manifolds instead.)

- Construct a functor $\mathsf{C}^\infty\colon \mathsf{Open}^{\mathsf{op}} \to \mathsf{CAlg}_{\mathbf{R}}$ to the category of real commutative algebras. Send an object $U \in \mathsf{Open}$ to its algebra of smooth functions $U \to \mathbf{R}$.
- Prove that the functor C^∞ is faithful.
- Consider the map of sets $U \to \mathsf{Hom}(\mathsf{C}^\infty(U), \mathbf{R})$ that sends a point $u \in U$ to the homomorphism $\mathsf{C}^\infty(U) \to R$ $(f \mapsto f(u))$. Prove that this maps of sets is bijective. Hint: consider evaluating homomorphisms on coordinate functions $x_i \in \mathsf{C}^\infty(U)$. Hint: (re)familiarize yourself with Hadamard's lemma from elementary analysis.
- Prove that a subset $W \subset \mathsf{Hom}(\mathsf{C}^\infty(U), \mathbf{R}) \cong U$ is closed in U if and only if there is $f \in \mathsf{C}^\infty(U)$ such that $W = \{h\colon \mathsf{C}^\infty(U) \to \mathbf{R} \mid h(f) = 0\}$. Hint: (re)familiarize yourself with the smooth Tietze extension theorem, also known as the Whitney extension theorem.
- [**] (Optional.) Use the previous items to show that the functor C^∞ is full.

17. Suppose R is a commutative ring. A *radical ideal* if R is an ideal I of R such that $x^n \in I$ for some $n \geq 0$ implies $x \in I$. Consider the poset $\mathsf{Spec} R$ of radical ideals of R ordered by inclusion.

- Prove that the poset $\mathsf{Spec} R$ admits suprema and infima for all subsets.
- Prove that in the poset $\mathsf{Spec} R$ we have $x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x \wedge y_i)$ for any arbitrary set I. Here $\wedge$ denotes infimum and $\bigvee$ denotes the supremum of a family.

- Promote the above construction to a functor $\mathsf{Spec}\colon \mathsf{CRing} \to \mathsf{Frame}$. Here Frame has as objects posets that admit arbitrary suprema, finite infima, and $x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \wedge y_i)$ holds. Morphisms in Frame are order-preserving maps that preserve arbitrary suprema and finite infima.
- Construct a functor $\mathsf{Top} \to \mathsf{Frame}$. Prove that this functor is not faithful.
- Prove that it is faithful when restricted to Hausdorff spaces. Prove that $\mathsf{Spec}(\mathbf{Z})$ is not in the image of the restricted functor.

18. Review the definition of a Banach space. Consider the category Ban of Banach spaces and *contractive maps.* These are linear maps f such that $\|f(x)\| \le \|x\|$ for all x. Consider the category Ball of *compact unit balls.* It has *compact unit balls* as objects, defined as pairs (V, B) consisting of a Hausdorff locally convex topological vector space V and a compact Hausdorff topological subspace $B \subset V$ such that B is *balanced* (i.e., $0 \in B$ and for any $x \in B$ and number t such that $|t| \le 1$ we have $tx \in B$), and B is *convex* (i.e., for any $x, y \in B$ and real numbers $r \ge 0$ and $s \ge 0$ such that $r + s \le 1$ we have $rx + sy \in B$). Morphisms $(V, B) \to (V', B')$ are continuous linear maps $V \to V'$ that send B to B'.

- Construct a functor $\mathsf{Ban}^{\mathsf{op}} \to \mathsf{Ball}$ that sends a Banach space X to the unit ball $(X^*, X^*_{\le 1})$, where X^* denotes the space of linear functionals on X equipped with the weak-* topology and $X^*_{\le 1}$ denotes its subspace of functionals of norm at most 1. (Look up the Banach–Alaoglu theorem.)
- Construct a functor $\mathsf{Ball} \to \mathsf{Ban}^{\mathsf{op}}$ that sends a unit ball (V, B) to the Banach space of continuous linear functionals on V, with the norm given by the supremum over B.
- Show that monomorphisms in Ban are precisely injective maps.
- Show that epimorphisms in Ball are precisely those morphisms $(V, B) \to (V', B')$ for which the maps $V \to V'$ and $B \to B'$ are surjective.
- Assuming the functors defined above form an equivalence of categories, prove that given an inclusion $A \to B$ of Banach spaces, any linear functional on A can be extended to a linear functional on B that has the same norm. You may use the fact that monomorphisms are precisely epimorphisms in the opposite category.

19. Given a topological space X, consider the category Cov_X of open covers of X. Objects are open covers of X, defined as maps of sets $f\colon I \to \mathsf{Open}(X)$ such that $\bigcup_{i\in I} f(i) = X$, where $\mathsf{Open}(X)$ denotes the collection of open subsets of X. Morphisms $(f\colon I \to \mathsf{Open}(X)) \to (g\colon J \to \mathsf{Open}(X))$ are maps of sets $p\colon I \to J$ such that for every $i \in I$ we have $f(i) \subset g(p(i))$.

- Extend the following construction to a functor $\mathsf{Cov}_X \to \mathsf{Set}$. Send an open cover $f\colon I \to \mathsf{Open}(X)$ to the set $\pi_0(f)$, defined as the quotient of I by the equivalence relation $\sim$, where $i \sim i'$ if $f(i) \cap f(i') \ne \emptyset$.
- If X is a locally connected topological space (look it up), explain how the functor π_0 relates to the traditional set of connected components.
- [**] (Optional.) Suppose $X = \{0,1\}^{\mathbf{N}}$ (an infinite product of two-point topological spaces; also known as the Cantor space). Is there an open cover f of X such that the canonical map from the set of connected components of X to $\pi_0(X)(f)$ is an isomorphism?
- Look up the definition of a groupoid and the category Grpd of groupoids and functors. Extend the following construction to a functor $\mathsf{Cov}_X \to \mathsf{Grpd}$. Send an open cover $f\colon I \to \mathsf{Open}(X)$ to the groupoid $\pi_{\le 1}(f)$. Objects are elements of I. Morphisms $i \to i'$ are equivalence classes of chains of elements of I: $i = i_0, i_1, \ldots, i_k = i'$ such that $f(i_j) \cap f(i_{j+1}) \ne \emptyset$. Two chains are equivalent if they can be connected by a sequence of elementary transformations or their inverses. An elementary transformation takes two consecutive elements i_j, i_{j+1} and replaces them with a single i'_j such that $f(i_j) \cap f(i_{j+1}) \cap f(i'_j) \ne \emptyset$.
- Compute $\pi_{\le 1}(S^1)(f)$, where S^1 is a circle and f is a sufficiently fine cover, for example, three overlapping intervals.

20. Look up the definition of a *covering space* in topology.

- Prove that for a topological space X, covering spaces $A \to X$ over X form a category. Morphisms $(A \to X) \to (B \to X)$ are continuous maps $A \to B$ that commute with the map to X.
- Prove that for a topological space X, the following construction defines a category $\pi_{\le 1}(X)$. Objects are points in X. Morphisms $x \to x'$ are equivalences classes of continuous maps $p\colon [0,1] \to X$ such that $p(0) = x$, $p(1) = x'$. Two paths $p, q\colon [0,1] \to X$ are equivalent if they are homotopic. Composition is given by concatenating paths (define exactly how).

- Given a covering space $p\colon A \to X$, prove that the following construction defines a functor $\pi_{\leq 1}(X) \to \mathsf{Set}$. Send a point $x \in X$ to the fiber $A_x = \{a \in A \mid p(a) = x\}$. Send a path $p\colon [0,1] \to X$ to the map of sets $A_p\colon A_{p(0)} \to A_{p(1)}$ defined as follows. Given $a \in A_{p(0)}$, denote by $q\colon [0,1] \to A$ the unique lift of p such that $q(0) = a$. (Look up the lifting property for covering spaces.) Then A_p sends a to the point $q(1)$.

21. Suppose A is a locally compact Hausdorff abelian group (typical cases include $A = \mathbf{R}$, $A = \mathbf{Z}$, and $A = \mathrm{U}(1)$). Recall (Example 7.18) the Pontryagin dual group $\mathsf{PD}(A) = \mathsf{Hom}(A, \mathrm{U}(1))$. Recall also the two functors of the form $\mathsf{LocCompHausAb} \to \mathsf{Ban_1}$, namely, $\mathsf{L^1} \circ \mathsf{HaarMeas}$ and $\mathsf{C_0} \circ \mathsf{PD}$. Here $\mathsf{L^1}(\mathsf{HaarMeas}(A))$ (henceforth $\mathsf{L^1}(A)$) is the Banach space of finite measures on the measurable space $\mathsf{HaarMeas}(A)$ and $\mathsf{C_0}(\mathsf{PD}(A))$ is the Banach space of $\mathbf{C}$-valued continuous functions on the topological group $\mathsf{PD}(A)$ that vanish at infinity. The *Fourier transform* on A is defined as a bounded map $F_A\colon \mathsf{L^1}(A) \to \mathsf{C_0}(\mathsf{PD}(A))$ that sends $\mu \in \mathsf{L^1}(A)$ to the function $\chi \mapsto \int \chi \cdot \mu$, where $\chi \in \mathsf{PD}(A)$, i.e., $\chi\colon A \to \mathrm{U}(1)$, and $\chi \cdot \mu$ denotes the product of a measurable function and a finite measure. Prove that the collection of maps F_A defined above is a natural transformation from $\mathsf{L^1} \circ \mathsf{HaarMeas}$ to $\mathsf{C_0} \circ \mathsf{PD}$.

22. Prove that the category $\mathsf{Ban_1}$ admits arbitrary small coproducts. Prove that the category Ban admits finite coproducts, but does not admit infinite coproducts.

23. Suppose V and W are two Banach spaces in the category $\mathsf{Ban_1}$ of Banach spaces and contractive linear maps. A *contractive bilinear map* is a bilinear map $b\colon V, W \to A$ such that $\|b(v,w)\| \leq 1$ for all $v \in V$ and $w \in W$ such that $\|v\| \leq 1$ and $\|w\| \leq 1$. Prove the existence of a Banach space $V \hat{\otimes}_{\mathbf{C}} W$ with the following universal property: contractive bilinear maps $V, W \to A$ are in natural bijection with contractive linear maps $V \hat{\otimes}_{\mathbf{C}} W \to A$. Prove that the span of elements of the form $v \otimes_{\mathbf{C}} w$ is dense in $V \hat{\otimes}_{\mathbf{C}} W$. Does it always coincide with $V \hat{\otimes}_{\mathbf{C}} W$?

24. Suppose S is a set and $P \subset S \times S$ is a set of pairs of elements in S. Prove that the quotient set $S/{\sim}$, where $\sim$ is the equivalence relation generated by P, is isomorphic to the coequalizer of $P \underset{p_1}{\overset{p_0}{\rightrightarrows}} S$, where $p_0, p_1\colon P \to S$ are the projection maps.

25. Denote by $\mathsf{U}\colon \mathsf{CAlg}_k \to \mathsf{Vect}_k$ the forgetful functor from the category of commutative k-algebras to the category of vector spaces over k. Fix a vector space $V \in \mathsf{Vect}_k$. Consider the functor $\mathsf{F}\colon \mathsf{CAlg}_k \to \mathsf{Set}$ such that $\mathsf{F}(A)$ is the set of linear maps $V \to \mathsf{U}(A)$ and for a homomorphism $f\colon A \to A'$ the map $F(f)$ sends $g\colon V \to \mathsf{U}(A)$ to $f \circ g\colon V \to \mathsf{U}(A')$. Prove that F is corepresentable. Identify its corepresenting object as a familiar object from your algebra course.

26. Fix some vector spaces $V, W \in \mathsf{Vect}_k$. Consider the functor $\mathsf{F}\colon \mathsf{Vect}_k^{\mathsf{op}} \to \mathsf{Set}$ such that $\mathsf{F}(A)$ is the set of linear maps $A \otimes_k V \to W$ and $F(f)$ for a linear map $f\colon A \to A'$ sends a map $A' \otimes_k V \to W$ to its composition with $A \otimes_k V \to A' \otimes_k V$. Prove that F is representable. Identify its representing object as a familiar object from your algebra course. Same question when Vect_k is replaced by $\mathsf{Ban_1}$ and $\otimes_k$ is replaced by $\hat{\otimes}_{\mathbf{C}}$.