$P \xrightarrow{\pi} M$ $\quad P_x = G$ lie group $\quad P \curvearrowleft G$ free and transitive

H-distribution s.t. $D_p\pi : H_p \to T_{\pi(p)}M$ isomorphism

$r_{g*}H_p = H_{pg}$

Connection form $A \in \Omega^1(P, \mathfrak{g})$ s.t.

1) $r_g^* A = Ad_{g^{-1}} \cdot A$

2) $A(\tilde{X}) = X$ where $\tilde{X}$ is fundamental vector field corresponding to X

$$\tilde{X}_p = \frac{d}{dt} p\exp(tX)$$

Fact: Connection forms $\Longleftrightarrow$ Horizontal distributions

$H_p \longrightarrow A_p(\tilde{X}_p + Y_p) = X$

$Y_p \in H_p$

$H_p = \ker A_p \longleftarrow A$

$E_G \to M \quad C^\infty(U) \quad U \subset M$

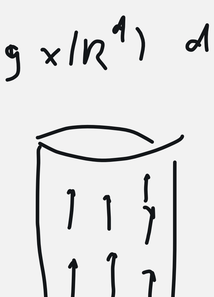
$A(E_G)(U) : \Gamma(p^{-1}(U), Tp^{-1}(U))^G$ G invariant vector fields

$C^\infty(U) \curvearrowright A(E_G)(U)$

θ-invariant vector field

$TG \simeq G \times \mathfrak{g} \quad T(U\times G) \simeq (U\times G)\times(\mathfrak{g}\times\mathbb{R}^d) \quad d = \dim M$

$\theta|_{U\times\{e\}} : U \longrightarrow \mathfrak{g}\times\mathbb{R}^d$


$U\times\{e\}$

$\underline{A(E_G)(U)} \longrightarrow \Gamma(U, \mathfrak{g}\times\mathbb{R}^d)$

locally free $C^\infty(U)$-sheaf

$\dim(\mathfrak{g}\times\mathbb{R}^d)$

Atiyah Bundle

$p^* \quad E_G \xrightarrow{p} M$

$\mu : p^*At(E_G) \xrightarrow{\simeq} TE_G$

$y \in p^{-1}(x)$

$\xi \longmapsto \xi(y)$

$Ad(E_G) \to M \quad E_G \times \mathfrak{g}$

$(p, v) \sim (pg, Ad(g^{-1})v)$

Lemma Ad sheaf g-inv vect fields on E_G s.t. lie in $\ker Dp : TE_G \to TM$

$(z, v) \in E_G \times \mathfrak{g}$

$\varphi(z, v)_{zg} = \frac{d}{dt} z\exp(tv)g$

$\varphi(zg_0, Ad(g_0^{-1})v)$

$\exp(tAd(g_0^{-1})v) = g_0^{-1}\exp(tv)g_0$

$Ad(E_G) \xrightarrow[\varphi]{} At(E_G)$

in ker Dp

$rk(Ad(E_G)) = \dim\mathfrak{g} = \underbrace{(\dim M + \dim\mathfrak{g})}_{At(E_G)} - \underbrace{\dim M}_{TM}$

$0 \to Ad(E_G) \to At(E_G) \underset{s}{\overset{d}{\rightleftarrows}} TM \to 0 \qquad d\circ s = id$

$\downarrow$

M

$p^*TM \xrightarrow{s} p^*At(E_G) \xrightarrow{\mu} TE_G$

conn $\simeq$ Sections of Atiyah sequence

Lie groups $\Leftrightarrow$ Lie algebras

Lie groupoids $\sim$ Lie algebroids

$G \underset{t}{\overset{s}{\rightrightarrows}} M$ ~ surjective submersions

$$\begin{tikzcd} G^{(2)} \arrow[r] \arrow[d] \arrow[rr, bend left, "m"] & G \arrow[d, "t"] & G \\ G \arrow[r, "s"'] & M & \end{tikzcd}$$

$u : M \to G$ unit

$i : G \to G$ inverse

Lie algebroid

$A \to M \quad [s_1, s_2] \quad a : A \to TM$

$[\sigma, f\tau] = f[\sigma, \tau] + \underbrace{(L_{a(\sigma)}f)\tau}$

$0 \to Ad \to At \xrightarrow{a} TM \to 0$

p^*f

Lie algebroid algebraic

$C(A) = \bigoplus_k C^k(A)$

$C^k(A) = \Gamma(\Lambda^k A^*)$

$\underline{(C(A), d)}$

Alg $\to$ d $\quad d\omega(\sigma_1, \dots \sigma_n) = \sum(-1)^i L_{a(\sigma_i)}\omega(\sigma_1, \dots \hat{\sigma}_i, \dots \sigma_n)$

$+ \sum(-1)^{i+j}\omega([\sigma_i, \sigma_j], \sigma_1, \dots \hat{\sigma}_i, \dots, \hat{\sigma}_j)$

Q s.t. $Q^2 = 0 = \frac{1}{2}[Q, Q]$

$a(s)f = i_s(Qf)$

$L_s = [i_s, Q] = i_sQ + Qi_s$

$i_{[s_1, s_2]} = [L_{s_1}, i_{s_2}]$

(i_{s_1}, i_{s_2}, Q)

$A \to B$

$\searrow \swarrow$

M

$[L_{s_1}, L_{s_2}] = L_{[s_1, s_2]}$

$[a(s_1), a(s_2)] = a[s_1, s_2]$

$a(s_2)f$

$\shortparallel$

$L_{s_2}f$

$[s_1, fs_2]$

$i_{[s_1, fs_2]} = [L_{s_1}, i_{fs_2}] = [L_{s_1}, fi_{s_2}](\omega)$

$= L_{s_1}(fi_{s_2}(\omega)) - fi_{s_2}(L_{s_1}\omega)$

$= L_{s_1}(f)i_{s_2}(\omega) + \underbrace{fL_{s_1}i_{s_2}(\omega) - fi_{s_2}L_{s_1}\omega}$

$a(s_1)(f)i_{s_2}(\omega) \qquad f[L_{s_1}, i_{s_2}](\omega)$

$\underbrace{i_{a(s_1)(f)s_2}\omega \qquad i_{f[s_1, s_2]}}$