Derived manifolds, derived intersections, etc

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Setup

Definition

Two closed submanimolds: $N \hookrightarrow L$ and $M \hookrightarrow L$ of the smooth manifold L, are said to intersect transversely, if for all points $p \in M \cap N$ two subspaces: T_pM and T_pN span the T_pL.

generalization of definition

Maps $f : N \to L$, $g : M \to L$ are called transverse if for all points $x \in M$ and $y \in M$ such that $f(x) = g(y)$, letting $p = f(x)$,

$$f_*(T_xM) + f_*(T_yN) = T_pL$$

Subspace $\{(x, y)|f(x) = g(y)\} \subseteq M \times N$ is a closed submanifold, and is the categorial fiber product $M \times_L N$. We can consider particular case when $N = \{p\} \in L$. Then f is transverse to g if there exists neighborhood U of $f^{-1}(p)$ such that $f|_U$ is a closed submersion. This recovers the fact that the fibers of any submersion are a closed submanifold.

What if transversality is not the case?

In the absence of transversality, we can make situation arbitrary bad. For example, if $f : M \to \mathbb{R}$ smooth function. $f^{-1}(\{p\})$ is a closed subset of M. And any closed subset of M can be treated as fiber over $\{0\}$ of some smooth function $f : M \to \mathbb{R}$.

Construction

for any open subset $U = M \setminus C$, where C is closed subset of manifold M. Since M is paracompact we can choose locally finite open cover $\{U_i\}_{i \in I}$ such that each U_i contained in coordinate chart. For each i, let $\varphi_i : M \to [0, 1]$ be a smooth bump function with $\mathrm{supp}(\varphi_i) \subset U_i$. Then define function $g(x) = \sum_{i \in I} \varphi_i(x)^2$. The sum is locally finite, so g is smooth. We can obtain C as zero set $g^{-1}(0)$ of function g.

In particular we can obtain Cantor set that way.

Motivation for derived intersections of derived intersections

For some applications, it is important to be able to take derived intersections of derived intersections, and so forth. For example, suppose that $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ are fiber bundles and $H : \Gamma_M(E_1) \to \Gamma_M(E_2)$ is a non-linear differential operator of degree k. Given $\psi \in \Gamma_M(E_2)$, we may be interested in those $\varphi \in \Gamma_M(E_1)$ for which $H\varphi = \psi$.
As a first approximation, one may want to study the formal solution space as follows. Consider the k^{th} order jet bundle $\pi_1^k : J^k_M E_1 \to M$. Then H corresponds to to a map of fiber bundles $\hat{H} : J^k_M E_1 \to E_2$. We can ask for the subspace $Fsol_\psi$ of $\Gamma_M(J^k_M E_1)$ on those sections λ of $J^k_M E_1$ such that $\hat{H}(\lambda) = \psi$.(and then from there try to figure out which ones arise as prolongations, i.e. which are holonomic sections). If transversality were not an issue we could form the fibered product

$$\begin{tikzcd}
R \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & M \arrow[d, "\psi"] \\
J^k E_1 \arrow[r, "\hat{H}"'] & E_2
\end{tikzcd}$$

And then from there the fibered product

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "p"'] & R \arrow[d, "\langle \pi_1^k \circ pr_1, pr_2 \rangle"] \\
M \arrow[r, "\Delta_M"'] & M \times M
\end{tikzcd}$$

We would have $Fsol_\psi = \Gamma_M(S)$

What do we want from category of derived manifolds

Thats why we need the derived manifolds. There exists a lot of approaches to this theory, but what actually we need from it?

- In every such theory of derived manifolds for smooth maps $f : M \to L$ and $g : N \to L$ the (derived)fibered product exists as a smooth object, and agrees with the ordinary fiber product.
- Of course, for any such theory to be useful, it must behave well in that derived intersections of arbitrary maps should enjoy the same nice properties as transversal intersections.

DMfd category

The first axiom for an ∞-category **DMfd** to be "good for doing intersection theory on manifolds" should be the following:

- There should exist a fully faithful functor $i : \mathbf{Mfd} \hookrightarrow \mathbf{DMfd}$ from the category of smooth manifolds which preserves transverse pullbacks and the terminal object.

Moreover **DMfd** should be in a suitable sense closed under taking fibered products and retracts. One way to phrase this is to ask for **DMfd** the following:

- **DMfd** should have finite limits
- **DMfd** should be idempotent complete

Finally, we should make sense of the idea that derived manifolds are completely determined by how they are built out of manifolds using fibered products and retracts.

How to describe **DMfd** precisely?
We want a universal category $\mathcal{C}$ with respect to certain property(there should exist fully faithful functor $i : \mathbf{Mfd} \to \mathcal{C}$ which preserves transverse pullbacks and the terminal object).
Let $\mathcal{D}$ denote the subcategory of the ∞-category Cat_∞ of small ∞-categories consisting of those which have finite limits and are idempotent complete, and left exact functors between them. Denote by $F : \mathcal{D} \to Cat_\infty$ the functor

$$\mathcal{C} \to \mathrm{Fun}^{\pitchfork}(\mathbf{Mfd}, \mathcal{C})$$

where $\mathrm{Fun}^{\pitchfork}(\mathbf{Mfd}, \mathcal{C})$ is the full subcategory of the functor category $\mathrm{Fun}(\mathbf{Mfd}, \mathcal{C})$ on those functors preserving transverse pullbacks and the terminal object. Denote by $\mathcal{E} \to \mathcal{D}$ the Grothendieck construction of this functor, i.e. the pullback of the universal coCartesian fibration $\mathcal{Z} \to Cat_\infty$ along F:

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] \\
\mathcal{D} \arrow[r, "F"] & Cat_\infty
\end{tikzcd}$$

i.e. $\mathcal{E}$ is the ∞-category of finitely complete idempotent complete ∞-categories equipped with a functor from the category of **Mfd** which preserves transverse pullbacks and the terminal object, and left exact functors between them respecting the functor from **Mfd**.

Universal Property

The ∞-category **DMfd** of derived manifolds is the initial object in $\mathcal{E}$. Unwinding this, this means that the ∞-category **DMfd** has finite limits, is idempotent complete, and there is a functor $i : \mathbf{Mfd} \to \mathbf{DMfd}$ from the category of smooth manifolds to the ∞-category of derived manifolds which preserves transverse pullbacks and the terminal object. Moreover, we can make the following statement:

Carchedi, Steffens

For any idempotent complete ∞-category $\mathcal{C}$ which has finite limits, composition with i induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathbf{lex}}(\mathbf{DMfd}, \mathcal{C}) \longrightarrow \mathrm{Fun}^{\pitchfork}(\mathbf{Mfd}, \mathcal{C})$$

between functors from derived manifolds to $\mathcal{C}$ which preserve finite limits, to functors from manifolds which preserve transverse pullbacks and the terminal object.

$\mathbf{C}^\infty$-algebras

Definition

Category $\mathbf{C}^\infty$ is a category with objects being the non-negative integers corresponding to real Cartesian spaces $\mathbb{R}^n$. The morphisms in $\mathbf{C}^\infty$ are given by C^∞ maps between these spaces. That is, we have the following sets of morphisms $\mathbf{C}^\infty(m, n) = C^\infty(\mathbb{R}^m, \mathbb{R}^n)$.

Definition

The category $\mathbf{Alg}_{\mathbf{C}^\infty}(\mathcal{S}\mathrm{pc})$ of derived $\mathbf{C}^\infty$-algebras(of spaces) is the category of product-preserving functors from $\mathbf{C}^\infty$ to the category $\mathcal{S}\mathrm{pc}$ of spaces

Universal property 2

We can show that abstractly such an ∞-category must exist, but of course, such a definition is not very tractable. However there is the following remarkable connection between derived manifolds and $\mathbf{C}^\infty$-algebras

Carchedi, Steffens

For all idempotent complete ∞-categories $\mathcal{C}$ with finite limits, composition with the canonical functor $\mathbf{C}^\infty \to \mathbf{Mfd} \to \mathbf{DMfd}$ induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathbf{lex}}(\mathbf{DMfd}, \mathcal{C}) \longrightarrow \mathbf{Alg}_{\mathbf{C}^\infty}(\mathcal{C})$$

between the ∞-category of finite-limit preserving functor from **DMfd** to C and the ∞-category of $\mathbf{C}^\infty$-algebra objects in $\mathcal{C}$.

Let $u : \mathbf{Mfd} \to \mathbf{Top}$ be the forgetful functor to topological spaces. Then as u preserves transverse pullbacks and the terminal object, there is a unique left exact functor U making the following diagram commute

$$\begin{tikzcd}
\mathbf{Mfd} \arrow[r, "i"] \arrow[d, "u"'] & \mathbf{DMfd} \arrow[dl, "U"] \\
\mathbf{Top} &
\end{tikzcd}$$

We define U to be the underlying space functor.
Using the underlying space functor U, one can define a natural Grothendieck topology $J_{\mathbf{DMfd}}$ on **DMfd** by declaring a collection of maps $(f_\alpha : U_\alpha \to \mathcal{M})_\alpha$ to be a cover of a derived manifold $\mathcal{M}$ if and only if $(U(f_\alpha) : U(U_\alpha) \to U(\mathcal{M}))_\alpha$ is an open cover of topological spaces. Moreover, this Grothendieck topology is subcanonical.

Describing **DMfd** precisely

Since $i : \mathbf{Mfd} \to \mathbf{DMfd}$ preserves transverse pullbacks and the terminal object, the composite $\mathbf{C}^\infty \to \mathbf{DMfd}$ preserves finite products. So, for $\mathcal{M}$ any derived manifold, the functor

$$\mathbb{M}\mathrm{ap}_{\mathbf{DMfd}}(\mathcal{M}, i(-)) : \mathbf{C}^\infty \to \mathcal{S}\mathrm{pc}$$

also preserves finite products, and hence defines a $\mathbf{C}^\infty$-algebra in spaces. Therefore there is a canonically induced functor

$$\mathcal{O}_{\mathbf{DMfd}} : \mathbf{DMfd}^{op} \to \mathbf{Alg}_{\mathbf{C}^\infty}(\mathcal{S}\mathrm{pc})$$

The functor $\mathcal{O}_{\mathbf{DMfd}}$ is fully faithful, and the essential image is precisely the (homotopically) finitely presented algebras. In particular,

$$\mathbf{DMfd} \simeq (\mathbf{Alg}_{\mathbf{C}^\infty}(\mathcal{S}\mathrm{pc})^{\mathbf{fp}})^{op}$$

Finitely presented algebras

Definition

A $\mathbf{C}^\infty$-algebra $X \in \mathbf{Alg}_{\mathbf{C}^\infty}(\mathcal{S}pc)$ is finitely presented if the functor $\mathbf{Alg}_{\mathbf{C}^\infty} \to Spc$ corepresented by X preserves small filtered colimits; that is, if X is a compact object. Denote the full subcategory on the finitely presented algebras by $\mathbf{Alg}_{\mathbf{C}^\infty}(\mathcal{S}pc)^{\mathbf{fp}}$

Smooth manifolds are derived manifolds

Consider a smooth manifold M. By Whitney's embedding theorem, we may regard it as a closed submanifold of $\mathbb{R}^N$ for some large enough N. By the tubular neighbourhood theorem, there exists an open subset $M \subseteq U \subseteq \mathbb{R}^N$ which retracts onto M. Every open subset of $\mathbb{R}^N$ can be presented as a closed subset in $\mathbb{R}^{N+1} = \mathbb{R}^N_x \times R_t$ given by the fibre of a single smooth function. Indeed, let φ_U be the characteristic smooth function of an open subset U, then U is diffeomorphic to the following global fibre.

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & \mathbb{R}^{N+1} \arrow[d, "\varphi_U(x)t-1"] \\
\mathbb{R}^0 \arrow[r] & \mathbb{R}
\end{tikzcd}$$

Thus U is a derived manifold by the above argument and then by the tubular neighborhood theorem the manifold M is a retract of U and thus is also a derived manifold.

Another approach to derived manifolds

There is another approach to the theory of derived manifolds that does not involve theory of homotopical $\mathbf{C}^{\infty}$-algebras. Theory of derived manifolds, based upon differential graded manifolds, and is thus a theory developed using the formalism of differentialgraded geometry — which is based on supergeometry. The central objects of study are called dg-manifolds, which are also known as Q-supermanifolds. Dg-manifolds are very tractable objects on which one can do differential geometry almost as usual, even in local coordinates.

$\mathbb{Z}$-graded algebra

$\mathbb{Z}$-graded vector spaces

The category of $\mathbb{Z}$-graded vector spaces over $\mathbb{R}$ is $\mathbf{Fun}(\mathbb{Z}, \mathbf{Vect})$, hence a $\mathbb{Z}$-graded vector space $\mathbb{V}$ is a collection $(V_n)_{n\in\mathbb{Z}}$, of vector spaces. we can define usual tensor product

$$(\mathbb{V} \otimes \mathbb{W})_n = \bigoplus_{l+k=n} V_l \otimes W_k$$

Note that $\mathbf{Vect}_{\mathbb{Z}}$ is symmetric monoidal category with

$$\underline{\mathbb{H}om}(\mathbb{V}, \mathbb{W})_n = \mathbb{H}om(\mathbb{V}, \mathbb{W}[n])$$

The category of $\mathbb{Z}$-graded commutative algebras over $\mathbb{R}$ is the category of commutative monoids in the symmetric monoidal category $\mathbf{Vect}_{\mathbb{Z}}$. They are a $\mathbb{Z}$-graded algebras with the property that

$$ab = (-1)^{|a||b|} ba$$

Structure of ringed space

Ringed space

Let $\mathbb{V} = (V_0, V_1, \ldots)$ be an $\mathbb{N}$-graded vector space (regarded as a $\mathbb{Z}$-graded one) over $\mathbb{R}$ whose total dimension is finite, i.e. $\bigoplus_{n=0}^{\infty} V_n$ is finite dimensional. We define $(V_0, \mathcal{O}_{\mathbb{V}})$ to be the ringed space whose structure sheaf assigns an open set U of V_0 the $\mathbb{Z}$-graded commutative algebra

$$\mathcal{O}_{\mathbb{V}}(U) = \mathbf{C}^{\infty}(U) \otimes_{\mathbf{Sym}((\mathbb{V})^*)} \mathbf{Sym}(\mathbb{V}^*)$$

Where **Sym**is left adjoint functor to forgetful functor $\mathbf{g}_{\mathbb{Z}}\mathbf{Com}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{Z}}$(It is the usual construction of the graded symmetric algebra of a graded vector space)

(Notice that since $\mathbb{V}$ is concentrated in non-negative degrees, $\mathbb{V}^*$ is concentrated in nonpositive degrees, as is $\mathcal{O}_{\mathbb{V}}$.)

Graded manifolds

Definition of a graded domain

A graded domain is a topological space X together with a sheaf $\mathcal{O}_X$ of $\mathbb{Z}$-graded commutative $\mathbb{R}$-algebras (concentrated in non-positive degrees), such that is isomorphic as a $\mathbb{Z}$-graded commutative ringed space to $(V_0, \mathcal{O}_{\mathbb{V}})$ for $\mathbb{V} = (V_0, V_1, \dots)$ an $\mathbb{N}$-graded vector space of total finite dimension.

Definition of a graded manifold

A graded manifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ is a second countable Hausdorff space M with a sheaf of $\mathbb{Z}$-graded commutative $\mathbb{R}$-algebras $\mathcal{O}_{\mathcal{M}}$, which is locally isomorphic to a graded domain. We denote the $\mathbb{Z}$-graded commutative $\mathbb{R}$-algebra $\Gamma_M(\mathcal{O}_{\mathcal{M}})$ by $\mathbf{C}^\infty(\mathcal{M})$. Smooth maps of graded manifolds are morphisms of ringed spaces.

Let $\mathcal{I}_0$ be the ideal sheaf locally generated by elements of $\mathcal{O}_{\mathcal{M}}$ of negative degree. Then $|\mathcal{M}| := (M, \mathcal{O}_{\mathcal{M}}/\mathcal{I}_0)$ is a smooth manifold, called the core of M. Notice that there is a canonical smooth map $\mathcal{M} \to |\mathcal{M}|$.

Remark

In fact, $\mathcal{M} \to |\mathcal{M}|$ is the unit of an adjunction exhibiting the category **Mfd** as a reflective subcategory of graded manifolds, with the core-functor $|-|$ as a left adjoint. In particular, any morphism $f : \mathcal{M} \to \mathcal{N}$ of graded manifolds fits into a commutative square

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "f"] \arrow[d, "\pi_{\mathcal{M}}"'] & \mathcal{N} \arrow[d, "\pi_{\mathcal{N}}"] \\
|M| \arrow[r, "|f|"'] & |N|
\end{tikzcd}$$

Vector bundles

Notice that, in particular, for an $\mathbb{N}$-graded vector space $\mathbb{V}$, $(V_0, \mathcal{O}_{\mathbb{V}})$ is a graded manifold. Hence, there is a canonical functor $\mathbf{Vect}_{\mathbb{N}} \to \mathbf{Mfd}_{\mathbb{N}}$. Morphisms in the image of this functor are called linear. From here, it is straightforward to (geometrically) define vector bundles over a graded manifold M, and their category is equivalent to that of locally free sheaves of $\mathcal{O}_{\mathcal{M}}$-modules concentrated in non-positive degrees.

Differential graded manifolds

Given a vector bundle $\mathcal{V} \to M$ over a graded manifold, we can associate a graded $\mathcal{O}_{\mathcal{M}}$-module of sections:

$$\underline{\Gamma}_{\mathcal{M}}(\mathcal{V})_n = \Gamma_{\mathcal{M}}(\mathcal{V}[n])$$

The $\mathbb{Z}$-graded module of sections of a vector bundle form a locally free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules, and conversely, any locally free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules arises in this way.

Definition of vector field of degree n

A vector field of degree n on a graded manifold $\mathcal{M}$ is a derivation $D : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}[n]$. Regarding n as a grading, this is locally free sheaf of graded $\mathcal{O}_{\mathcal{M}}$-modules, which we denote by $\mathcal{T}\mathcal{M}$—the tangent sheaf. Of course, this is the module of sections of the tangent bundle $T\mathcal{M} \to M$.
A vector field of degree $+1$ whose graded commutator $[D, D] = 2D^2 = 0$ is called a cohomological vector field.

Differential graded manifolds

Definition of differential graded manifold

A differential graded manifold or **dg**-manifold is a pair $(\mathcal{M}, D)$ of a graded manifold $\mathcal{M}$ and a cohomological vector field D. A morphism $\varphi : (\mathcal{M}, D) \to (\mathcal{M}', D')$ of **dg**-manifolds is a morphism $\varphi : \mathcal{M} \to \mathcal{M}'$ of graded manifolds with the property that D' is φ-related to D, that is, for any f in $C^\infty(\mathcal{M}')$,

$$\varphi^* D' f = D \varphi^* f$$

We denote the resulting category as $\mathbf{dg} - \mathbf{Mfd}_{\leq 0}$. (It is more precisely the category of **dg**-manifolds which are concentrated in non-positive degrees.)

What cohomological vector field does

sheaf of cdga

A cohomological vector field D on $\mathcal{M}$ makes $(\mathcal{O}_{\mathcal{M}}, D)$ into a sheaf of commutative differential graded algebras over $\mathbb{R}$. Indeed, D becomes a differential turning $(\mathcal{O}^{\bullet}_{\mathcal{M}}, D)$ into a cochain complex, since D has degree 1 and $D^2 = 0$, and since it is a derivation, the algebra structure is compatible with this differential. The above condition is the same as asking f^* to be a map of cdgas. The following functor is fully faithful.

$$\mathbf{C}^{\infty} : \mathbf{dg} - \mathbf{Mfd}_{\leq 0} \longrightarrow \mathbf{dg}{-}\mathbb{R}\mathbf{Alg}^{op}_{\leq 0}$$

differential graded $\mathbf{C}^\infty$-algebras

Definition

differential graded $\mathbf{C}^\infty$-algebra is a commutative differential graded $\mathbb{R}$-algebra $(\mathcal{A}^\bullet, d)$, together with the additional structure of a lift of the induced commutative $\mathbb{R}$-algebra structure on $\mathcal{A}_0$ to the structure of a $\mathbf{C}^\infty$-algebra. A morphism $f : (\mathcal{A}, d) \to (\mathcal{A}', d')$ between two such algebras is a morphism of differential graded $\mathbb{R}$-algebras such that the morphism $f^0 : \mathcal{A}^0 \to \mathcal{A}'^0$ is a morphism of $\mathbf{C}^\infty$-algebras.

Theorem

There exists a cofibrantly generated, almost simplicial, model category structure on the category of (non-positively graded) dg-$\mathbf{C}^\infty$-algebras, unique with the property that $f : (\mathcal{A}, d) \to (\mathcal{A}', d')$ is a weak equivalence (respectively fibration) if and only if the induced map of underlying cochain complexes is, with respect to the projective model structure on cochain complexes $\mathbf{Ch}(\mathbb{R})_{\leq 0}$

Equivalence of models

Definition

Denote by $\underline{\mathbf{dgC^{\infty}Alg}}_{\leq 0}$ the 1-category of non-positively graded $\mathbf{dg}-\mathbf{C}^{\infty}$-algebras and denote by $\mathbf{dgC^{\infty}Alg}_{\leq 0}$ the ∞-category associated to the above model category.

Theorem

Category $\mathbf{Alg}_{\mathbf{C}^{\infty}}(Spc)$ is equivalent to the category $\mathbf{dgC^{\infty}Alg}_{\leq 0}$

As a corollary we obtain the following fact:

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here is a canonical equivalence:

$$\mathbf{DMfd} \simeq (\mathbf{dgC^{\infty}Alg}^{\mathbf{fp}}_{\leq 0})^{op}$$

between the opposite of the ∞-category of homotopically finitely presented $\mathbf{dg}-\mathbf{C}^{\infty}$-algebras, and the ∞-category of derived manifolds.

Fully faithfulness

Denote by $\mathbf{DgMfd}_{\leq 0}$ the ∞-category obtained from $\mathbf{dgMfd}_{\leq 0}$ by formally inverting the weak equivalences. Consider the canonical functor of 1-categories:

$$\mathbf{C}^{\infty} : \mathbf{dgMfd}_{\leq 0} \longrightarrow (\underline{\mathbf{dgC}^{\infty}\mathbf{Alg}}_{\leq 0})^{op}$$

$$\mathcal{M} \longrightarrow \mathbf{C}^{\infty}(\mathcal{M})$$

It can be proved that this functor sends weak equivalences to quasi-isomorphisms, hence there is an induced functor between ∞-categories

$$\mathbf{C}^{\infty} : \mathbf{DgMfd}_{\leq 0} \to \mathbf{dgC}^{\infty}\mathbf{Alg}^{op}_{\leq 0}$$

This induced functor will be fully faithful

Equivalence of models

We can construct equivalence of categories $\mathbf{DgMfd}_{\leq 0} \simeq \mathbf{DMfd}$

Example of computation of derived intersection

We will compute derived intersection of point with itself in over real line $\mathbb{R}$.
The ambient space is the real line $\mathbb{R}$. The ring of smooth functions on it is $A = C^{\infty}(\mathbb{R})$
The point $0 \in \mathbb{R}$ is given by the ideal of functions vanishing at that point:

$$I = \{f \in C^{\infty}(\mathbb{R}) | f(0) = 0\}.$$

The ring of functions on the point 0 itself is the quotient $A/I \cong \mathbb{R}$.
We need to compute derived tensor product $\mathbb{R} \otimes^{\mathbb{L}}_{C^{\infty}(\mathbb{R})} \mathbb{R}$
Take the left module $\mathbb{R}$. By Hadamard's lemma, any smooth function that vanishes at zero can be written as $x * g(x)$, where x is the coordinate function. That is, the ideal I is generated by the function x. This allows us to write resolution for $\mathbb{R}$:

$$0 \longrightarrow C^{\infty}(\mathbb{R}) \xrightarrow{*x} C^{\infty}(\mathbb{R}) \xrightarrow{ev_0} \mathbb{R} \longrightarrow 0$$

ev_0 is evaluation at zero, $f \rightarrow f(0)$.
$*x$ is multiplication by the coordinate function x.

Thus our resolution $P_\bullet$ consists of two terms in degrees -1 and 0 (using cohomological grading):

$$P_\bullet = C^\infty(\mathbb{R}) \xrightarrow{*x} C^\infty(\mathbb{R})$$

Now we take the resolution $P_\bullet$ and tensor it over $C^\infty(\mathbb{R})$ with the second module $\mathbb{R}$:

$$P_\bullet \otimes C^\infty(\mathbb{R}) = C^\infty(\mathbb{R}) \otimes_{C^\infty(\mathbb{R})} \mathbb{R} \xrightarrow{*x\otimes id} C^\infty(\mathbb{R}) \otimes_{C^\infty(\mathbb{R})} \mathbb{R} = \mathbb{R} \xrightarrow{0} \mathbb{R}$$

Thus, the derived tensor product $\mathbb{R} \otimes^{\mathbb{L}}_{C^\infty(\mathbb{R})} \mathbb{R}$ is the graded algebra:

$$\mathbb{R}[-1] \oplus \mathbb{R}$$

Connection with Lie algebroids

Definition

A Lie ∞-algebroid $(\mathfrak{g}, A)$ consists of

- a commutative, associative, unital (super) algebra A
- an $\mathbb{N}$-graded A-module $\mathfrak{g}^*$ such that $V_0 = A$
- on $\Lambda^\infty_A \mathfrak{g}^*$ regarded as a graded commutative algebra over the ground field, a graded degree $+1$ derivation
$$d : \Lambda^\infty_A \mathfrak{g}^* \to \Lambda^\infty_A \mathfrak{g}^*$$
such that $d^2 = 0$

$\mathrm{CE}_A(\mathfrak{g}) := (\Lambda^\bullet_A \mathfrak{g}^*, d)$ is called Chevalley-Eilenberg **CdgA**. If we demand all A-modules to be not just of finite rank but also projective, then this reproduces the notion of differential graded manifold, but with opposite homological degrees. This means in practice Lie algebroids and differential graduated manifolds are different, Nevertheless they are encoded by same algebraic structure.