∞-Sheaves on Manifolds and Some Examples

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Abstract

This talk is a brief introduction to the theory of ∞-stacks on manifolds and some examples of such stacks. In particular, we show how with minimal machinery set up we can recover a beautiful connection between differential cohomology and deformation theory of higher $U(1)$-gerbes.

Consider categories of sheaves as a model for "generalized geometric spaces". Manifolds (Mfld) or Infinitesimal Manifolds (InfMfld) are the basic geometric objects—sheaves valued in some "higher category", like ∞-groupoids (∞-Grpd), are called stacks (∞-stacks) on this site/category and are treated as generalized spaces.

Definition 1 (***C***$^{\infty}$-algebra). A C^{∞}-algebra is a commutative algebra A such that for any n-tuple of elements $(a_1,\ldots,a_n)\in A$ and any C^{∞}-function $\varphi:\mathbb{R}^n\to\mathbb{R}$, there is an element $\varphi(a_1,\ldots,a_n)\in A$ such that this assignment is functorial in the choice of φ. For instance, given $\psi:\mathbb{R}^m\to\mathbb{R}$ and $\varphi_1,\ldots,\varphi_m:\mathbb{R}^{k_i}\to\mathbb{R}$, evaluating the composition respects the functorial action:

$$\psi(\varphi_1(a_{1,1},\ldots,a_{1,k_1}),\ldots,\varphi_m(a_{m,1},\ldots,a_{m,k_m}))=(\psi\circ(\varphi_1\times\cdots\times\varphi_m))(a_{1,1},\ldots,a_{1,k_1},\ldots,a_{m,1},\ldots,a_{m,k_m})$$

Example: $C^{\infty}(M)$ is naturally a C^{∞}-algebra. A tuple $(f_1,\ldots,f_n)$ corresponds to a map $M\to\mathbb{R}^n$, which can be composed with φ to yield $\varphi(f_1,\ldots,f_n)\in C^{\infty}(M)$.

Definition 2 (dg ***C***$^{\infty}$-algebra). A differential graded (dg) C^{∞}-algebra is a commutative differential graded algebra ($CDGA^{\leq 0}$) A concentrated in non-positive homological degrees such that π_0A (which is naturally $H_0(A)$) has a C^{∞}-algebra structure.

Definition 3 (Infinitesimal Manifold). We have an adjunction diagram:

$$\mathrm{Spec}(-):\mathrm{dg}C^{\infty}\text{-Alg}\rightleftarrows(\text{Topological spaces}+\text{Sheaf of }\mathrm{dg}C^{\infty}\text{-algebras}):\Gamma(-;\mathcal{O})$$

An infinitesimal manifold (inf-manifold) is an object dual under Spec to a dg-C^{∞}-algebra A possessing special properties. The dg C^{∞}-algebra A must satisfy the following properties:

1. A has a reduction to a smooth manifold: $A\rightsquigarrow A_{\text{red}}:=H^0(A)/\,\mathrm{Nilrad}=C^{\infty}(M)$ for some smooth manifold M.
2. $H^*(A)$ is bounded.
3. A is cohomologically complete (equivalent to its Postnikov completion).

Definition 4 (∞-Stack). An ∞-prestack on the categories of manifolds and inf-manifolds is a functor

$$\mathsf{Mfld}^{op},\,\mathsf{InfMfld}^{op}\longrightarrow\infty\text{-}\mathsf{Grpd}$$

where ∞-Grpd represents the ∞-category of homotopy types (e.g., simplicial sets or Kan complexes).

To proceed to stacks, we first need a notion of open covers. An open cover $\{U_i \to U\}$ utilizes open embeddings onto the underlying topological spaces, supplemented with the condition that smooth functions on U_i are localized from U. (Note that the formal neighborhood $\operatorname{Spec}\mathbb{R} \to \operatorname{Spec}\mathbb{R}[\varepsilon]/\varepsilon^2$ does *not* constitute an open cover).

An ∞-stack is precisely an ∞-prestack that satisfies descent with respect to this notion of open covers. Given an open cover $\mathcal{U} = \{U_i \to U\}$, we construct the Čech nerve $\check{C}^\bullet\mathcal{U}$, the simplicial object consisting of intersections $\check{C}^n\mathcal{U} = \coprod_{i_1 \dots i_n} U_{i_1} \times_U \cdots \times_U U_{i_n}$. For a stack F, $F(\check{C}^\bullet\mathcal{U})$ is a cosimplicial object in ∞-Grpd. The descent condition forces the canonical equivalence:

$$F(U) \xrightarrow{\sim} \lim F(\check{C}^\bullet\mathcal{U})$$

We can unpack this for ordinary low-categorical stacks:

- For a 0-stack $F : \mathsf{Mfld}^{op} \to \mathsf{Set} \hookrightarrow \infty\text{-}\mathsf{Grpd}$:

$$F(U) \overset{\sim}{\to} \mathrm{Eq}\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)\right)$$

- For a 1-stack $F : \mathsf{Mfld}^{op} \to 1\text{-}\mathsf{Grpd}$:

$$F(U) \overset{\sim}{\to} \lim\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j) \Rrightarrow \prod_{i,j,k} F(U_i \times_U U_j \times_U U_k)\right)$$

Example: Principal G-bundles

Consider the stack BG parameterizing principal G-bundles on smooth manifolds. We build an ∞-prestack via the assignment:

$$M \longmapsto \{\text{groupoid of principal } G\text{-bundles over } M\}$$

which is sent to ∞-Grpd uniformly via the nerve $N(\mathrm{Bun}_G(M))$. The objects are locally trivial bundles $P \downarrow M$ and morphisms are G-bundle isomorphisms. G-bundles provably satisfy descent, guaranteeing that:

$$BG(U) \xrightarrow{\sim} \lim\left(\prod_i BG(U_i) \rightrightarrows \dots\right)$$

This describes the moduli space of G-bundles cleanly in terms of local intersection data. Assuming we isolate our geometry to a "good cover" $\mathcal{U} = \{U_i\}$—meaning all localized intersections align to contractible spaces—the category for each U_i collapses: it is categorically equivalent to G-bundles on a point. Evaluating the respective homotopy limit then simplifies strictly to the calculation of the Čech cohomology of the open cover.

Example: Higher $U(1)$-bundles and $(n-1)$-gerbes

These generalize principal bundles to the case of more general "structure group". A standard Lie group G is equivalent to a stack

$$G : \mathsf{Mfld}^{op} \to \mathsf{Grp} \subset \mathsf{Set} \subset \infty\text{-}\mathsf{Grpd}$$

given by $M \mapsto G(M) = C^\infty(M; G)$.

A higher Lie group is just a group object within the ∞-category of stacks. Because $U(1)$ operates as an abelian group object intrinsically inside ∞-Grpd, we can iterate the delooping functor B as many times as we want:

$$B^nU(1) : \mathsf{Mfld}^{op} \longrightarrow \infty\text{-}\mathsf{Grpd} \quad \text{characterized by} \quad B^nU(1)(M) = B^n(U(1)(M))$$

The corresponding Higher $U(1)$-bundles reflect are classified by maps from the manifold into the classifying stack: $M \to B^nU(1)$. The associated infinitesimal deformation geometry naturally reconstructs classical Deligne cohomology as we shall sketch below.

The Atiyah Algebroid example and differential cohomology

One of the most striking features of the Bunk–Nuiten–Mueller framework is how it naturally encodes the Atiyah algebroid and connects it to differential/Deligne cohomology. For any classifying map $x : M \to B^nU(1)$, the Atiyah algebroid is formalized intrinsically as the relative tangent complex. Seen primarily as a complex, it acts as the canonical homotopy fiber:

$$At_{n-1}(x) := T(M/B^nU(1)) = \mathrm{hofib}(TM \to T_xB^nU(1))$$

This precisely evaluates to $TM \oplus C^\infty(M)[n-1]$ as we shall see below.

- The binary brackets along this structure is the standard action of vector fields.
- The higher $(n+1)$-bracket receives non-trivial curvature shifts from the integration class, given definitively through the de Rham differential class H characterizing an $(n+1)$-closed differential form on M:

  $$[X_1, \dots, X_{n+1}] = H(X_1, \dots, X_{n+1}) \in \Omega^{n+1}_{cl}(M)$$

  This directly corresponds to the class $[x_{top}] \in H^{n+1}(M; \mathbb{Z})$ classifying the higher $U(1)$-bundle, i.e. it is precisely the closed form representative of the integral class.

To see this consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$$

Delooping $n+1$ times yields the fiber sequence $B^nU(1) \to B^{n+1}\mathbb{R} \to B^{n+1}\mathbb{Z}$. We thus get the following diagram:

$$\begin{tikzcd}
M \arrow[r, "x"] \arrow[d, "x_{top}"] & B^nU(1) \arrow[r] \arrow[d] & * \arrow[d, "0"] \\
B^{n+1}\mathbb{Z} \arrow[r] & B^{n+1}\mathbb{Z} \arrow[r] & B^{n+1}\mathbb{R}
\end{tikzcd}$$

Taking relative tangent evaluations utilizing $T(M/-)$ is a right adjoint and thus preserves limits. Consequently, we have:

$$\begin{aligned}
T(M/*) &= TM \\
T(M/B^{n+1}\mathbb{Z}) &= TM \\
T(M/B^{n+1}\mathbb{R}) &\simeq TM \oplus C^\infty(M)[n]
\end{aligned}$$

As a result we get the relative tangent complex:

$$T(M/B^nU(1)) = TM \oplus C^\infty(M)[n-1]$$

where the non-trivial higher bracket is given by the de Rham class

$$H \in \mathrm{Hom}(TM, TM \oplus C^\infty(M)[n]) \leftrightsquigarrow \Omega^{n+1}_{cl}(M)$$

This equivalence holds because $\mathrm{Hom}_{L_\infty\text{-algd}}(TM, TM \oplus C^\infty(M)[n]) \simeq CE^n_{cl}(TM)$.

Thus we see that in the Bunk–Nuiten–Mueller framework, the Atiyah algebroid of a higher $U(1)$-gerbe naturally encodes the differential cohomology class corresponding to the gerbe.