∞-sheaves on Mflds and some examples

Consider categories of sheaves as a model for "generalized geometric spaces". Mfld or Inf Mfld are the basic geometric objects – sheaves valued in some "higher category" ∞-Grpd are called stacks (∞-stacks) on this site / category.

Def. C^∞-algebra is a commutative algebra A such that for any n-tuple of elements $\in A$ $(a_1, \dots, a_n) \in A$ and any C^∞-function $\varphi : \mathbb{R}^n \to \mathbb{R}$ there is an element $\varphi(a_1, \dots, a_n) \in A$ such that this assignment is functorial in the choice of φ.

$$\psi : \mathbb{R}^m \to \mathbb{R} \quad , \varphi_1, \dots, \varphi_n : \mathbb{R}^{k_i} \to \mathbb{R}$$

$$\psi\left(\varphi_1(a_1, \dots a_{k_1}), \varphi_2(a_{k_1+1} \dots), \dots, a_{k_1+\dots+k_n})\right)$$

Ex: $C^\infty(M)$ is naturally a C^∞-algebra

$$(f_1, \dots, f_n) : M \to \mathbb{R}^n \xrightarrow[\varphi]{} \mathbb{R}$$

$$\varphi(f_1, \dots f_n) \in C^\infty(M)$$

with cohomological

Def. A dg C^∞-algebra is $CDGA^{\leq 0}$ A concentrated in non-pos. degrees s.t. $\pi_0 A$ $(= H_0(A))$ has a C^∞-alg-ra str-re.

Def. (Inf manifolds)

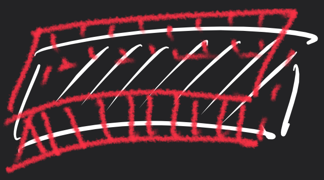
$$\begin{tikzcd} C^\infty\text{-Alg} \arrow[r, "\text{Spec}"'] & (\text{Topological spaces} + \text{Sheaf of } C^\infty\text{-algebras}) \\ A \arrow[u, phantom, "\in"] \arrow[r, mapsto] & \underline{\text{Hom}(A; \mathbb{R})} \end{tikzcd}$$

An inf manifold is an object dual under Spec to dg C^∞ Alg which satisfies the following properties:

1) $$A \rightsquigarrow A_{red} = A / \underbrace{\sqrt[\text{nil}]{A}}_{\sqrt[\text{nil}]{\pi_0 A} + A^{<0}} = C^\infty(\overset{\text{smooth manifold}}{M})$$

2) $H^*(A)$ is bounded

3) A is cohomologically complete.



Def: An ∞-stack on the category of dMfld, InfdMfld is a functor $\underline{\text{dMfld}^{op}, \text{Inf dMfld}^{op} \to \underbrace{\infty\text{-Grpd}}}$

∞-prestacks

∞-category of homotopy types

sSet / Kan complexes

An open cover $\{U_i \to \underline{U}\}$ + the smooth functions on U_i are localized from U.

open embeddings on the underlying top spaces

~~$\operatorname{Spec} \mathbb{R} \to \operatorname{Spec} \mathbb{R}[\varepsilon]/\varepsilon^2$~~

not an open cover

$\bullet$ $\odot$

$\mathbb{R}[\varepsilon]/\varepsilon^2 \xrightarrow[ev_0]{} \mathbb{R}$

An ∞-stack is an ∞-prestack which satisfies descent w.r.t. this notion of open covers.

$\mathcal{U} = \{U_i \to U\}$ - open covers

$\check{C}^n\mathcal{U} = \coprod_{(i_1 \dots i_n)} U_{i_1} \times_U \dots \times_U U_{i_n}$

simplicial object

$F : \text{dMfld}^{op} \to \infty\text{-Grpd}$

$F(\check{C}^\bullet \mathcal{U})$ – cosimplicial object in ∞-Grpd

$F(U) \xrightarrow{\sim} \lim F(\check{C}^\bullet \mathcal{U})$

$F : \mathrm{Mfld}^{op} \longrightarrow \mathrm{Set} \hookrightarrow \infty\text{-Grpd}$

$U_i \cap U_j$

$\shortparallel$

$$F(U) \xrightarrow{\simeq} \mathrm{Eq}\Big(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)\Big)$$

$F : \mathrm{Mfld}^{op} \longrightarrow 1\text{-Grpd}$

$$F(U) \xrightarrow{\simeq} \lim\Big(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k} F(U_i \times_U U_j \times_U U_k)\Big)$$

Ex: BG – principal G-bundles on smooth manifolds

$M \longmapsto$ groupoid of principal G-bundles over M.

objects are
$$\begin{tikzcd} P \arrow[d, "G"] \\ M \end{tikzcd}$$
morphisms are
$$\begin{tikzcd} P' \arrow[rr, "\simeq"] \arrow[dr, "G"'] & & P \arrow[dl, "G"] \\ & M & \end{tikzcd}$$

$\downarrow$

∞-prestack $\quad$ ∞-Grpd $\rightsquigarrow N(\mathrm{Bun}_G(M))$ – ∞-Groupoid

$\downarrow$

∞-stack? $\quad$ <u>G-bundles satisfy descent.</u>

$\wr\wr$

description of G-bundles in terms of the local data.

$\{U_i \to U\}$ $\qquad$ $BG(U) \xrightarrow{\simeq} \lim\big(\prod BG(U_i) \rightrightarrows \cdots\big)$

+ assume that you are dealing with a "good cover"

all elements are contractible
+ all intersections are either empty
or locally contractible.

this is the category of G-bundles on U

this category for each U_i is $\simeq$ to G-bundles on a point

Calculation of the homotopy limit reduces to the calculation of Čech cohomology of this open cover.

Ex: Higher $U(1)$-bundles & $(n-1)$-Gerbes.

$\parallel$

principal G-bundles where G is allowed to be some "higher Lie group"

$G : \mathrm{Mfld}^{op} \longrightarrow \mathrm{Set} \qquad M \mapsto G(M) = C^\infty(M; G)$

$\times$ $\hat{}$ Inf $\qquad$?

ordinary Lie group $\qquad \infty$-Grpd

group object in $(\mathrm{Mfld}^{op} \longrightarrow \infty\text{-Grpd})$
+ stack condition

Abelian group object in ∞-Grpd apply B to it as many times as you want $\qquad B^n\,(U(1) : \mathrm{Mfld}^{op} \to \infty\text{-Grpd})$

$$B^n(U(1))(M) = B^n(U(1)(M)).$$

Higher $U(1)$-bundles are just maps $\quad M \longrightarrow B^n U(1)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\uparrow$ manifold $\quad \underline{B^{n-1}U(1)}$ the structure group

Infinitesimal theory of $B^n U(1) \Longleftrightarrow \qquad M \to BU(1)$

Deligne cohomology.

In later talks: Atiyah algebroid

$\mathrm{At}_{n-1}(M)$
$\ll$

$x : M \longrightarrow B^n U(1) \qquad T(M / B^n U(1)) = \mathrm{hofib}(TM \to T_x B^n U(1))$

// as a complex

$TM \oplus C^\infty(M)[n-1]$

› Binary brackets on this are given by the action of vector fields

› The higher $(n+1)$-bracket is given by the class of H viewed as an $(n+1)$-closed diff. form on M.

$$[X_1, \dots, X_{n+1}] = \underset{\underset{\Omega^{n+1}_{cl}(M)}{\rotatebox{90}{$\in$}}}{H}(X_1, \dots, X_{n+1}) \qquad \underline{[x_{top}]} \in H^{n+1}(M; \mathbb{Z})$$

$$0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$$

$$\underline{B^n U(1) \to B^{n+1}\mathbb{R} \to B^{n+1}\mathbb{Z}} \text{ - fiber sequence}$$

$T(M/B^n U(1))$

$$\begin{tikzcd}
M \arrow[r, "x"] \arrow[d, "x_{top}"'] & B^n U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "0"] \\
B^{n+1}\mathbb{Z} \arrow[r, equal] & B^{n+1}\mathbb{Z} \arrow[r] & B^{n+1}\mathbb{R}
\end{tikzcd}$$

$T(M/-)$ is a right adjoint $\Rightarrow$ it preserves pullbacks

$T(M/*) = TM$ $\qquad$ $T(M/B^{n+1}\mathbb{R}) \simeq$

$\qquad\qquad\qquad\qquad \simeq TM \oplus C^\infty(M)[n]$

$T(M/B^{n+1}\mathbb{Z}) = TM$

$$\underline{T(M/B^n U(1)) = TM \oplus C^\infty(M)[n-1]}$$

$$H : TM \to TM \oplus C^\infty(M)[n]$$