Next talk: March 25

Website: dmitripavlov.org/homotopy
Schedule, sources

Higher/derived differential geometry

Motivation: H/D diff. geometry is the geometry of physics

Four steps toward H/D diff geometry:

Step 0: Duality between geometry and algebra (of "simple" spaces) } Motivation: Enable Step 1 [and Step 2]

Step 1: Enhance "simple" spaces to affine C^∞-schemes } Motivation: Enable Step 2. Enable infinitesimals.

Physics: infinitesimals.

Step 2: Enhance affine schemes to derived affine schemes. Motivation: Limits, pullbacks, intersections exist and have good geometric properties

Physics: phase space (on-shell).

Step 3: Enhance derived affine schemes to derived stacks (co-sheaves). Motivation: Colimits, pushouts, quotients exist and have good geometric properties.

Physics: gauge fields.

Step 0 Duality between geometry and algebra

$\mathcal{O}$: $X \mapsto$ alg of functions $X \to k$

$$\begin{tikzcd} \text{Space} \arrow[r, shift left, "\mathcal{O}"] \arrow[r, phantom, "\perp"] & \mathrm{CAlg}^{op} \arrow[l, shift left, "\mathrm{Spec}"] \end{tikzcd}$$

Spec: $A \mapsto \mathrm{Spec}\, A$

Spectrum points: $A \to k$

geometry — algebra

an equivalence in many cases

$X_1 \xrightarrow{f} X_2$

$\mathcal{O}(X_1) \xleftarrow{\mathcal{O}(f)} \mathcal{O}(X_2)$

$\mathcal{O}(f_1 f_2) = \mathcal{O}(f_2) \circ \mathcal{O}(f_1)$, $(-)\circ f$

Proposition: (Pursell, 1957) $C^\infty(-,\mathbb{R})$ is a fully faithful functor

$\mathrm{CAlg}_{\mathbb{R},man} \subset \mathrm{CAlg}_{\mathbb{R}}$ essential image

Examples

a) $$\begin{tikzcd} \text{smooth manifolds} \arrow[r, shift left, "C^\infty(-,\mathbb{R})"] \arrow[r, phantom, "\sim \ \perp"] & \mathrm{CAlg}^{op}_{\mathbb{R},man} \arrow[l, shift left] \end{tikzcd}$$

$\mathrm{hom}(A,\mathbb{R})$ w/ smooth structure $\leftarrow\!\shortmid A$

b) $$\begin{tikzcd} \text{Stein manifold} \arrow[r, shift left, "\mathcal{O}(-,\mathbb{C})"] \arrow[r, phantom, "\sim"] & \mathrm{CAlg}^{op}_{\mathbb{C},man} \arrow[l, shift left, "\text{points Hom}(A,\mathbb{C})"] \end{tikzcd}$$

c) $$\begin{tikzcd} \boxed{\text{complex affine algebraic varieties}} \arrow[r, shift left, "\mathcal{O}(-,\mathbb{C})"] \arrow[r, phantom, "\perp"] & \text{reduced complex comm algebras}^{op} \arrow[l, shift left, "\mathrm{Spec}"] \end{tikzcd}$$

$\mathrm{Nil}\, A = 0$ (reduced)

points: Hom $(A, \mathbb{C})$ — maximal ideal

$$\begin{tikzcd} \text{compact totally disconnected Hausdorff spaces} \arrow[r, shift left, "C(-,\mathbb{Z}/2)"] \arrow[r, phantom, "\sim \text{ Stone}"] & \text{Boolean algebras}^{op} \arrow[l, shift left, "\mathrm{Hom}(-,\mathbb{Z}/2)"] \end{tikzcd}$$

$$\begin{tikzcd} \text{compact extremally disconnected Hausdorff spaces} \arrow[r, shift left, "C(-,\mathbb{Z}/2)"] \arrow[r, phantom, "\sim \text{ Stone}"] & \text{complete Boolean algebras}^{op} \arrow[l, shift left, "\mathrm{Hom}(-,\mathbb{Z}/2)"] \end{tikzcd}$$

(condensed mathematics)

$$\begin{tikzcd} \text{localizable enhanced measurable spaces} \arrow[r, shift left, "{[\text{meas. sets}]/\sim}"] \arrow[r, phantom, "\sim"] & \text{measurable Boolean algebras}^{op} \arrow[l, shift left] \end{tikzcd}$$

∞ — commutative von Neumann algebras^{op}

$$\begin{tikzcd} \text{compact Hausdorff spaces} \arrow[r, shift left, "C(-,\mathbb{C})"] \arrow[r, phantom, "\sim \text{ Gelfand}"] & \text{commutative unital } C^*\text{-algebras}^{op} \arrow[l, shift left] \end{tikzcd}$$

Proposition:
H: Hilbert space
$T \in B(H)$ — $T^* = T$ self adj
$\{T_i\}$ — $TT^* = T^*T$ normal
T generates a commutative von Neumann subalgebra $A \subset B(H)$

$\mathrm{Spec}\, A \cong$ the spectrum of T

meas. space

Grothendieck: aff schemes $:= \mathrm{CRing}^{op}$

locally ringed space $\supseteq$ affine schemes $\rightleftarrows$ commutative rings

Zariski Spectrum

points = prime ideals = $\mathrm{Hom}(A, \text{int. domain } k)$

$\mathrm{Spec}\, k \to X$ point

Step 1 Enhancing simple spaces to affine schemes

$C^\infty M$ $\qquad x \in A \quad p \in \mathbb{R}[x] \quad p(x) \in A$ $\qquad$ C^∞-ring: $p \in C^\infty(\mathbb{R})$, $p(x) \in A$

a) $\mathrm{CAlg}_{\mathbb{R}} \hookrightarrow C^\infty\mathrm{Ring}$ (Lawvere, Dubuc)

opposite category = "affine schemes"

b) $\mathrm{CAlg}_{\mathbb{C}} \hookrightarrow \mathrm{EFCAlg}$ entire functional calculus algebra

algebraic geometry: c) $\mathrm{RedCAlg}_{\mathbb{C}} \hookrightarrow \mathrm{CAlg}_{\mathbb{C}}$ $\quad x^n = 0 \Rightarrow x = 0$

Enables infinitesimals $\simeq \{a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R},\ x^2 = 0\}$

Example a) $\mathrm{Spec}\, \mathbb{R}[x]/(x^2) = I$ = 1st order infinitesimal neighborhood of $0 \in \mathbb{R}$

Remark: $\mathrm{CAlg}_{\mathbb{R}} \hookrightarrow C^\infty\mathrm{Ring}$ does not preserve coproducts

$\otimes_{\mathbb{R}} \qquad \otimes_{C^\infty} \qquad C^\infty M \otimes_{C^\infty} C^\infty N \cong C^\infty(M \times N) \qquad C^\infty\mathrm{Ring}^{op}(I, M) \cong \mathrm{Der}(C^\infty M, \mathbb{R}) \cong TM$

Benefits: a) infinitesimals

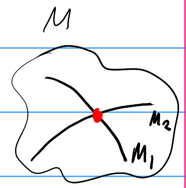
b) limits & colimits exist, and some of them have better geom. properties.

(not all limits & colimits have good geometric properties

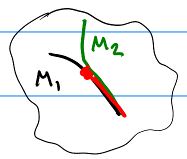
b') finite products are well-behaved.

Sources: A. Kock: Synthetic Geometry of Manifolds.

Step 2 Enhancing affine schemes to derived affine schemes



codim $M_1 \cap M_2$ = codim M_1 + codim M_2 (*)



$M_1 \cap M_2$ can be any closed subset

(*) can fail

Nontransversal pullbacks do not exist or have bad properties.

Solution: add pullbacks (& other homotopy limits) formally (but ensure existing good limits are preserved)

Finite products already exist $\Rightarrow$ need to add (homotopy) limits over Δ

$\{0, \ldots, n\}$ nondecreasing maps

$\mathrm{Fun}(\Delta, \mathrm{AffSch})$

Affine schemes $\hookrightarrow$ cosimplicial affine schemes

$\wr\wr$ $\qquad\qquad$ $\wr\wr$ $\mathrm{Fun}(\Delta^{op}, \mathrm{CAlg})$

commutative algebras $\hookrightarrow$ simplicial commutative algebras (w/ simplicial eq.)

ensures existing transversal pullbacks are preserved

normalized chains $\wr\wr$ Dold-Kan correspondence

differential graded commutative algebras (with quasi-isomorphisms)

$S: C^\infty M \to \mathbb{R}$

$\{dS = 0\}$

derived affine schemes

$\simeq$ the opposite relative category ($\simeq$ ∞-category) of

- differential graded C^∞-rings
- d. g. EFC Alg
- d. g. commutative $\mathrm{Alg}_{\mathbb{C}}$

Step 3 Enhance (derived) affine schemes to (derived) stacks

(derived) affine schemes $\hookrightarrow$ sheaves / ∞-sheaves on (derived) affine schemes

Nonfree quotients (or colimits) do not exist or have bad properties

Solution: add (homotopy) colimits formally (but ensure existing good ones are preserved)

$\underline{\mathcal{C}}$: (Derived) affine schemes $\xhookrightarrow{\text{Yoneda embedding}}$ $Sh_\infty(\mathcal{C}) = Fun(\mathcal{C}^{op}, sSet)$ localized at covering sieves

(simplicial presheaves on $\mathcal{C}$ — ∞-presheaves)

$X \mapsto (S \mapsto AffSch(S,X))$ $\qquad U, V \subset_{open} X,\ U \cup V = X \qquad U \sqcup_{U \cap V} V \xrightarrow{\sim} X$

$\dim(X/G) = \dim X - \dim G$; $*/G \cong *$; $\dim */G = 0 \neq \dim * - \dim G$

Example $G \curvearrowright *$; homotopy quotient: $\underbrace{BG}$; $Map(\underset{\text{manifold}}{M}, BG) \simeq \{$groupoid of principal G-bundles over $M\}$

delooping/classifying stack of G

$\dim(\text{geom. stack } S) = \sum_k (-1)^k \dim S_k = \sum_k (-1)^k \dim \pi_k S$

Example Derived critical locus: $\mathcal{F} \xrightarrow{S} \mathbb{R}$ action functional

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F} \arrow[d, "0"] \\ \mathcal{F} \arrow[r, "dS"] & T^*\mathcal{F} \end{tikzcd}$$

$\dim BG = \sum_k (-1)^k \dim \pi_k BG$

$= -\dim G.$

Goal What is $B_\nabla G$, the $\boxed{\text{derived}}$ stack of principal ∞-bundles w/ structure Lie ∞-group G and a connection?

Physics: gauge field

Source: Bunk – Müller – Nuiten – Szabo.