The World of Differential Categories

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Thank you Dmitri for the invitation

Quick Hello!

- Full Name: Jean-Simon Pacaud Lemay, please feel free to call me **JS**

- I'm from Québec, Canada

- I'm a senior lecturer/associate professor at **Macquarie University** (Sydney, Australia)

- I'm a category theorist, and I study:

  • **Differential Categories**
  • Tangent Categories
  • Differential Geometry, Algebraic Geometry, Differential Algebras
  • Traced Monoidal Categories
  • Restriction Categories
  • Other stuff...

Top of Mt.Fuji, August 15 2022

If you find differential categories interesting and would like to chat/work together or even visit our category theory group at Macquarie: feel free to come to talk to me and reach out by email!

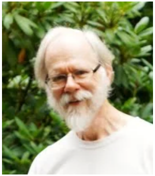
What is the Theory of Differential Categories About?

- The theory of differential categories uses category theory to provide the foundations of differentiation and has been able to formalize numerous aspects of differential calculus.

- Originally, Blute, Cockett, and Seely

R.Blute

R.Cockett



R.A.G.Seely

introduced differential categories in:

R. Blute, R. Cockett, R.A.G. Seely, **Differential Categories**, (2006)

to provide the categorical semantics of Differential Linear Logic.

- Differential categories are successful because they capture both the classical limit definition of differentiation and the more algebraic synthetic definition of differentiation. This has led to the categorical formalization of various aspects of differentiation, which is why differential categories have been become quite popular in both mathematics and computer science.

The Differential Category World: The Four Tomes

Differential Categories
Blute, Cockett, Seely - 2006

Cartesian Differential Categories
Blute, Cockett, Seely - 2009

Differential Restriction Categories
Cockett, Cruttwell, Gallagher - 2011

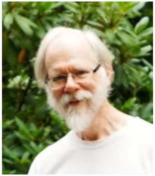
Tangent Categories
Rosický - 1984
Cockett, Cruttwell - 2014

The Differential Category World: The Four Tomes

- **Differential Categories** (2006):
  Algebraic Foundations of Differentiation

R.Blute

R.Cockett



R.A.G.Seely

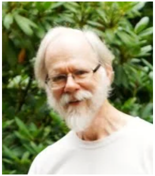
Blute, R., Cockett, R., Seely, R.A.G. **Differential Categories** (2006)

The Differential Category World: The Four Tomes

- **Differential Categories** (2006): Algebraic Foundations of Differentiation
- **Cartesian Differential Categories** (2009):
  Foundations of Differential Calculus over Euclidean Spaces $\mathbb{R}^n$

R.Blute

R.Cockett



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- **Differential Restriction Categories** (2011):
  Foundations of Differential Calculus over open subsets $U \subseteq \mathbb{R}^n$

R.Cockett

G.Cruttwell

J.Gallagher

Cockett, R., Cruttwell, G., and Gallagher, J. **Differential Restriction Categories**. (2011)

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- **Differential Categories** (2006): Algebraic Foundations of Differentiation
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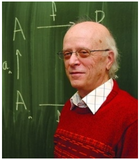
  Foundations of Differential Calculus over Euclidean Spaces $\mathbb{R}^n$
- **Differential Restriction Categories** (2011):
  Foundations of Differential Calculus over open subsets $U \subseteq \mathbb{R}^n$
- **Tangent Categories** (1984 & 2014):
  Foundations of Differential Calculus over Smooth Manifolds



J.Rosický

R.Cockett

G.Cruttwell

J. Rosický **Abstract tangent functors** (1984)

R. Cockett, G. Cruttwell **Differential structure, tangent structure, and SDG** (2014)

The Differential Category World: The Four Tomes

Differential Categories

Blute, Cockett, Seely - 2006

Cartesian Differential Categories

Blute, Cockett, Seely - 2009

Differential Restriction Categories

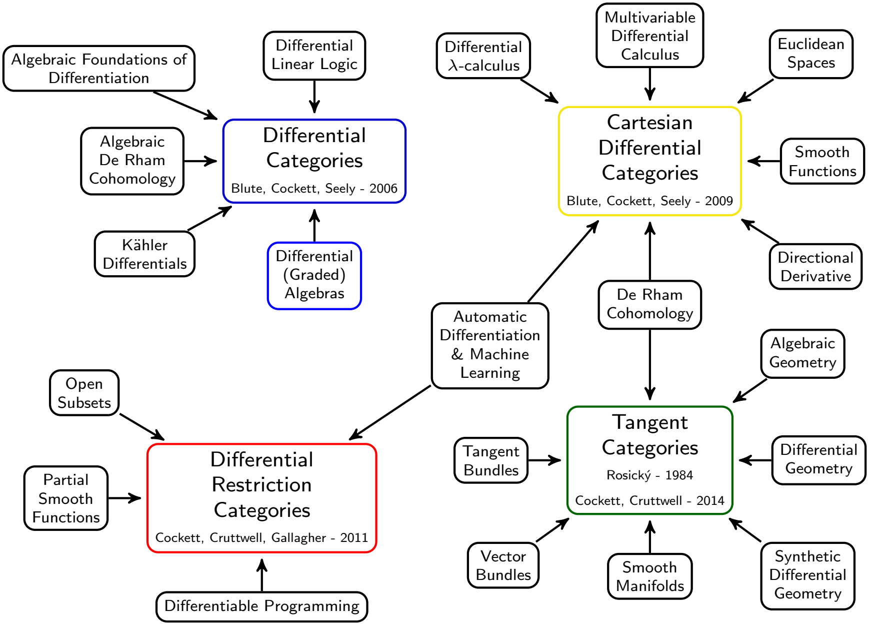
Cockett, Cruttwell, Gallagher - 2011

Tangent Categories

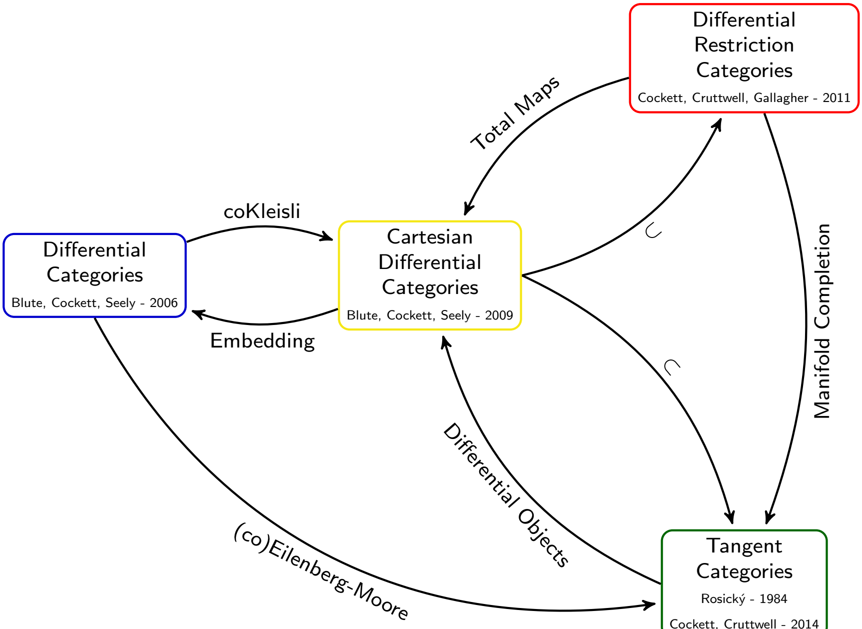
Rosický - 1984

Cockett, Cruttwell - 2014

The Differential Category World: A Taster



The Differential Category World: It's all connected!



Today's Story

TODAY'S STORY: A tour of the world of differential categories.

The plan is:

- Introduction to Differential Categories, setting up for next week's talk by Chiara Sava
- A brief look at Cartesian differential categories and tangent categories

- **Differential Categories**: Algebraic Foundations of Differentiation

Some introductory references for today

Blute, R., Cockett, R., Seely, R.A.G. **Differential Categories** (2006)

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. (2019)

Blute, R., Lucyshyn-Wright, R.B.B. and O'Neill, K. **Derivations in codifferential categories**. (2016)

Lemay, J.-S.P. **Differential algebras in codifferential categories**. (2019)

Terminology: Differential Categories vs. Codifferential Categories

Differential categories were originally introduced from the point of view of Linear Logic. So they are about:

- Comonads, comonoids, coalgebras, etc.

However, if we want to talk about differentiation in algebra, we actually need the dual notion of **co**differential categories:

- Monads, monoids, algebras, etc.

So notion of differentiation from algebra fit more naturally in a codifferential category.

But I don't like the term codifferential category... it scares people away!

So I've going on a crusade to propose the following terminology change:

- To call differential categories instead coalgebraic/geometric differential categories.
- To call codifferential categories instead algebraic differential categories, or just differential categories. So I am going to do this today.

Hopefully you'll agree with this convention after we see the definition...

Differential Categories - Definition

An (algebraic) differential category (née codifferential category) is:

- A k-linear symmetric monoidal category,
- With a differential modality which is:
  - An algebra modality
  - Equipped with a deriving transformation.

Differential Categories - Definition

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k-linear symmetric monoidal Categories - Definition

For a fixed commutative semiring[1] k, a k-linear symmetric monoidal category is a symmetric monoidal category (we are going to work in the strict setting for simplicity):

$$\mathbb{X} \qquad \otimes \qquad I \qquad \sigma : A \otimes B \xrightarrow{\cong} B \otimes A$$

which is enriched over k-modules:

- Every homset $\mathbb{X}(A, B)$ is a k-module, we can add maps together $f + g$, have zero maps 0, can scalar multiply maps $r \cdot f$ (where $r \in k$), and composition preserves the k-module structure:

$$f \circ (r \cdot g + s \cdot h) \circ k = r \cdot (f \circ g \circ k) + s \cdot (f \circ h \circ k)$$

- The monoidal product $\otimes$ also preserves the k-linear structure:

$$f \otimes (r \cdot g + s \cdot h) \otimes k = r \cdot (f \otimes g \otimes k) + s \cdot (f \otimes h \otimes k)$$

We need addition to talk about the Leibniz rule and zero to talk about the constant rule. Note that this definition does not assume (bi)products or negatives.

Example

Let $\mathbb{K}$ be a field and and let $\text{VEC}_{\mathbb{K}}$ to be the category of all $\mathbb{K}$-vector spaces and $\mathbb{K}$-linear maps between them. $\text{VEC}_{\mathbb{K}}$ is an $\mathbb{K}$-linear monoidal category with the usual monoidal and $\mathbb{K}$-linear structure.

[1] Recall that a semiring is a ring without negatives.

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Algebra Modality - Definition

An **algebra modality** on a symmetric monoidal category $\mathbb{X}$ is

$$SSA \xrightarrow{\mu} SA \qquad\qquad A \xrightarrow{\eta} SA$$

$$\begin{array}{ccc} SA & \xrightarrow{\eta} & SSA \\ {\scriptstyle S(\eta)}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu} \\ SSA & \xrightarrow[\mu]{} & SA \end{array} \qquad\qquad \begin{array}{ccc} SSSA & \xrightarrow{\mu} & SSA \\ {\scriptstyle S(\mu)}\downarrow & & \downarrow{\scriptstyle \mu} \\ SSA & \xrightarrow[\mu]{} & SA \end{array}$$

equipped with two natural transformations:

$$SA \otimes SA \xrightarrow{\mathsf{m}} SA \qquad\qquad I \xrightarrow{\mathsf{u}} SA$$

such that for every object A, $(SA, \mathsf{m}, \mathsf{u})$ is a commutative monoid:

$$\begin{array}{ccc} SA & \xrightarrow{\mathsf{u}\otimes 1} & SA \otimes SA \\ {\scriptstyle 1\otimes\mathsf{u}}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mathsf{m}} \\ SA \otimes SA & \xrightarrow[\mathsf{m}]{} & SA \end{array} \qquad \begin{array}{ccc} SA \otimes SA \otimes SA & \xrightarrow{\mathsf{m}\otimes 1} & SA \otimes SA \\ {\scriptstyle 1\otimes\mathsf{m}}\downarrow & & \downarrow{\scriptstyle \mathsf{m}} \\ SA \otimes SA & \xrightarrow[\mathsf{m}]{} & SA \end{array} \qquad \begin{array}{ccc} SA \otimes SA & \xrightarrow{\sigma} & SA \otimes SA \\ & {\scriptstyle \mathsf{m}}\searrow & \downarrow{\scriptstyle \mathsf{m}} \\ & & SA \end{array}$$

and μ is a monoid morphism:

$$\begin{array}{ccc} SSA \otimes SSA & \xrightarrow{\mu\otimes\mu} & SA \otimes SA \\ {\scriptstyle \mathsf{m}}\downarrow & & \downarrow{\scriptstyle \mathsf{m}} \\ SSA & \xrightarrow[\mu]{} & SA \end{array} \qquad\qquad \begin{array}{ccc} K & \xrightarrow{\mathsf{u}} & SSA \\ & {\scriptstyle \mathsf{u}}\searrow & \downarrow{\scriptstyle \mu} \\ & & SA \end{array}$$

Algebra Modality - Rough Idea

- S(A) $\equiv$ set of differentiable/smooth functions $A \to I$ (whatever that means).
- $\mu \equiv$ function composition
- $\eta \equiv$ identity function/linear function
- m $\equiv$ function multiplication
- u $\equiv$ multiplication unit/constant function.

Algebra Modality - Example Polynomials

Example

A commutative monoid in $\text{VEC}_{\mathbb{K}}$ is precisely a commutative $\mathbb{K}$-algebra.

$$\text{Sym}(V) := \mathbb{K} \oplus V \oplus (V \otimes_{sym} V) \oplus \ldots = \bigoplus_{n \in \mathbb{N}} V^{\otimes^n_{sym}}$$

where $\otimes_{sym}$ is the symmetrized tensor power of V.

If $X = \{x_1, x_2, \ldots\}$ is a basis of V, then $\text{Sym}(V) \cong \mathbb{K}[X]$.
In particular for $\mathbb{K}^n$, $\text{Sym}(\mathbb{K}^n) \cong \mathbb{K}[x_1, \ldots, x_n]$.

Then the algebra modality structure can be described in terms of polynomials as

$$\begin{aligned} \eta : V &\to \mathbb{K}[X] \\ x_i &\mapsto x_i \end{aligned} \qquad \begin{aligned} \mu : \text{Sym}\left(\mathbb{K}[X]\right) &\to \mathbb{K}[X] \\ P\left(p_1(\vec{x_1}), \ldots, p_n(\vec{x_n})\right) &\mapsto P\left(p_1(\vec{x_1}), \ldots, p_n(\vec{x_n})\right) \end{aligned}$$

$$\begin{aligned} \text{u} : \mathbb{K} &\to \mathbb{K}[X] \\ 1 &\mapsto 1 \end{aligned} \qquad \begin{aligned} \text{m} : \mathbb{K}[X] \otimes \mathbb{K}[X] &\to \mathbb{K}[X] \\ p(\vec{x}) \otimes q(\vec{y}) &\mapsto p(\vec{x})q(\vec{y}) \end{aligned}$$

which we extend by linearity. Therefore, μ and η correspond to polynomial composition, while m and u correspond to polynomial multiplication.

Differential Categories - Definition

An (algebraic) differential category (née codifferential category) is:

- A k-linear symmetric monoidal category,
- With a differential modality which is:
  - An algebra modality
  - Equipped with a **deriving transformation**.

Deriving Transformation - Definiton

A **deriving transformation** for an algebra modality on an k-linear symmetric monoidal category is a natural transformation:

$$SA \xrightarrow{\quad \mathsf{d} \quad} SA \otimes A$$

whose axioms are based on the basic identities from differential calculus.

IDEA: $f(x) \mapsto f'(x) \otimes \mathsf{d}x$

- **[D.1]**: Constant rule: $c' = 0$
- **[D.2]**: Product rule: $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
- **[D.3]**: Linear rule: $x' = 1$
- **[D.4]**: Chain rule: $(f \circ g)'(x) = f'(g(x))g'(x)$
- **[D.5]**: Interchange rule: $\dfrac{d^2 f(x,y)}{dxdy} = \dfrac{d^2 f(x,y)}{dydx}$

Example

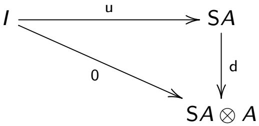
Let V be a $\mathbb{K}$-vector space with basis $X = \{x_1, x_2, \dots\}$.

The deriving transformation can be described in terms of polynomials as follows:

$$\mathsf{d} : \mathbb{K}[X] \to \mathbb{K}[X] \otimes V$$

$$\mathsf{p}(x_1, \dots, x_n) \mapsto \sum_{i=1}^{n} \frac{\partial \mathsf{p}}{\partial x_i}(x_1, \dots, x_n) \otimes x_i$$

D.1 - Constant Rule

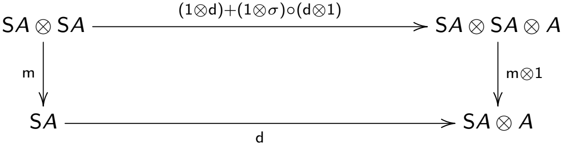


Example

For a constant polynomial $\mathrm{p}(x_1, \ldots, x_n) = r$:

$$\sum_{i=1}^{n} \frac{\partial \mathrm{p}}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i = 0$$

D.2 - Product Rule

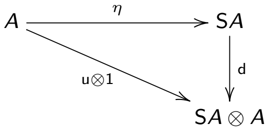


Example

For polynomials $p(x_1, \ldots, x_n)$ and $q(x_1, \ldots, x_n)$:

$$\begin{aligned}&\sum_{i=1}^{n} \frac{\partial pq}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i \\ &= \sum_{i=1}^{n} p(x_1, \ldots, x_n)\frac{\partial q}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i + \sum_{i=1}^{n} \frac{\partial p}{\partial x_i}(x_1, \ldots, x_n) q(x_1, \ldots, x_n) \otimes x_i\end{aligned}$$

D.3 - Linear Rule



Example

For a monomial of degree 1, $\mathrm{p}(x_1, \ldots, x_n) = x_j$:

$$\sum_{i=1}^{n} \frac{\partial x_j}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i = 1 \otimes x_j$$

D.4 - Chain Rule

$$
\begin{array}{ccccc}
SSA & & \xrightarrow{\quad\mu\quad} & & SA \\
\downarrow{\scriptstyle d} & & & & \downarrow{\scriptstyle d} \\
SSA \otimes SA & \xrightarrow[\mu \otimes d]{} & SA \otimes SA \otimes A & \xrightarrow[m \otimes 1]{} & SA \otimes A
\end{array}
$$

Example

For polynomials $p(x_1, \ldots, x_n)$ and $q(x)$:

$$\sum_{i=1}^{n} \frac{\partial q(p(x_1, \ldots, x_n))}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i = \sum_{i=1}^{n} \frac{\partial q}{\partial x_i}(p(x_1, \ldots, x_n)) \frac{\partial q}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i$$

D.5 - Interchange Rule

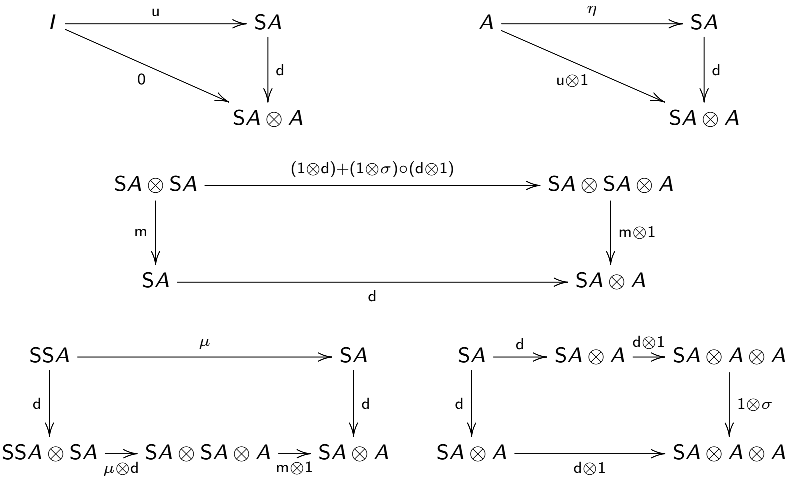
$$
\begin{array}{ccccc}
SA & \xrightarrow{\;\mathrm{d}\;} & SA \otimes A & \xrightarrow{\;\mathrm{d}\otimes 1\;} & SA \otimes A \otimes A \\
\downarrow{\scriptstyle \mathrm{d}} & & & & \downarrow{\scriptstyle 1\otimes\sigma} \\
SA \otimes A & & \xrightarrow[\;\mathrm{d}\otimes 1\;]{} & & SA \otimes A \otimes A
\end{array}
$$

Example

For a polynomial $p(x_1, \ldots, x_n)$:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\frac{\partial p}{\partial x_i}}{\partial x_j}(x_1, \ldots, x_n) \otimes x_j \otimes x_i = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\frac{\partial p}{\partial x_i}}{\partial x_j}(x_1, \ldots, x_n) \otimes x_i \otimes x_j$$

Deriving Transformation - Definiton



Example: Polynomials

Example

$\mathsf{VEC}_{\mathbb{K}}$ is a differential category, with differential modality Sym and deriving transformation given by polynomial differentiation:

$$\mathsf{d} : \mathbb{K}[X] \to \mathbb{K}[X] \otimes V$$
$$\mathsf{p}(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} \frac{\partial \mathsf{p}}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i$$

- This example generalizes to modules over any commutative semiring.
- In fact, the free commutative monoid monad (if it exists) on an k-linear symmetric monoidal category is always a differential modality:

Lemay, J.-S. P. **Coderelictions for Free Exponential Modalities**. (2021)

Blute, R., Lucyshyn-Wright, R.B.B. and O'NeilL, K. **Derivations in codifferential categories**. (2016)

- This gives us lots of examples, such as the category of sets and relations, where the differential modality is given by finite bags, or the opposite category of modules, where the differential modality is given by the cofree cocommutative coalgebra.

Example: Smooth Functions

Example

Recall that a $\mathcal{C}^\infty$**-ring** is commutative $\mathbb{R}$-algebra A such that for each smooth map $f : \mathbb{R}^n \to \mathbb{R}$ there is a function $\Phi_f : A^n \to A$ and such that the Φ_f satisfy certain coherences between them.

Ex. For a smooth manifold M, $\mathcal{C}^\infty(M) = \{f : M \to \mathbb{R} | \ f \text{ smooth}\}$ is a $\mathcal{C}^\infty$-ring.

For every $\mathbb{R}$-vector space V, there is a free $\mathcal{C}^\infty$-ring over V, $\mathsf{S}^\infty(V)$. This induces a differential modality on $\mathsf{VEC}_\mathbb{R}$. In particular, $\mathsf{S}^\infty(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n)$, and the deriving transformation is given by the usual differentiating of smooth functions:

$$\mathsf{d} : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$$

$$f \longmapsto \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \otimes x_i$$

Cruttwell, G.S.H., Lemay, J.-S. P. and Lucyshyn-Wright, R.B.B. **Integral and differential structure on the free C^∞-ring modality**. (2019)

Other Examples

Example

Other examples can be found in:

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. (2019)

which in particular includes:

- The free Rota-Baxter monad is a differential modality
- The exterior algebra monad on finite dimensional $\mathbb{Z}_2$-vector spaces
- Fun fact: the free differential algebra monad is **NOT** a differential modality!

Example

Every categorical model of Differential Linear Logic gives a (coalgebraic) differential category.

Fiore, M. **Differential structure in models of multiplicative biadditive intuitionistic linear logic** (2007)

Important examples include:

- Fininiteness Spaces, Köthe spaces, etc.
  - Ehrhard, T. **An introduction to differential linear logic: proof-nets, models and antiderivatives**. (2018)
- Convenient vector spaces
  - Blute, R., Ehrhard, T. and Tasson, C. **A convenient differential category** (2012)

Things we can do in differential categories

- Derivations and Kähler differentials
  - Blute, R., Lucyshyn-Wright, R.B.B. and O'NeilL, K. **Derivations in codifferential categories**. (2016)
- Hochschild complex, de Rham complex, and (co)homology
  - O'Neill, K. **Smoothness in codifferential categories** (PhD Thesis) (2017)
- Differential algebras
  - Lemay, J.-S.P. **Differential algebras in codifferential categories**. (2019)
- Antiderivatives, integration, and Taylor Series
  - Ehrhard, T. **An introduction to differential linear logic: proof-nets, models and antiderivatives**. (2018)
  - Cockett, R., Lemay, J.-S.P. **Integral Categories and Calculus Categories**. (2018)
  - Lemay, J.-S.P. **Convenient Antiderivates for Differential Linear Categories**. (2020)
  - Lemay, J.-S.P. **An Ultrametric for Cartesian Differential Categories for Taylor Series Convergence**. (2024)
- Exponential Functions and Laplace Transforms:
  - Lemay, J.-S.P. **Exponential Functions for Cartesian Differential Categories**. (2018)
  - Kerjean, M., Lemay, J.-S.P. **Laplace Distributors and Laplace Transformations for Differential Categories**. (2024)
- Reverse differentiation:
  - Cruttwell, G., Gallagher, P., Lemay, J-S. P., Pronk, D. **Monoidal Reverse Differential Categories**. (2023)

Things we can do in differential categories

- **Derivations and Kähler differentials** (← Talk next week!)
  - Blute, R., Lucyshyn-Wright, R.B.B. and O'NeilL, K. **Derivations in codifferential categories**. (2016)
- Hochschild complex, **de Rham complex** (← Talk next week!), and (co)homology
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- **Differential GRADED algebras** (← Talk next week!)
  - Lemay, J.-S.P. **Differential algebras in codifferential categories**. (2019)
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Well we have a monad...

Since a differential modality is a monad, we can ask what can we say about its Kleisli category and its Eilenberg-Moore category?

- Kleisli category = Smooth maps
- Eilenberg-Moore category = Smooth manifolds

Differential Categories - Smooth Maps

Recall that for a monad S, its Kleisli category Kl(S) is the category with the same objects as the base category but where a map from X to Y in Kl(S) is a map:

$$X \to \mathsf{S}(Y)$$

Every differential category has a notion of a *smooth map* given by the **opposite** category of the Kleisli category. So a **smooth map** from A to B is a map:

$$B \to \mathsf{S}(A)$$

Example

Let's consider our $\mathcal{C}^\infty$-ring differential modality example, where $\mathsf{S}^\infty(\mathbb{R}^n) := \mathcal{C}^\infty(\mathbb{R}^n)$.

$$f : \mathbb{R}^n \to \mathbb{R} \qquad f \text{ is a smooth function}$$

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$$\frac{f : \mathbb{R}^n \to \mathbb{R} \qquad f \text{ is a smooth function}}{f \in \mathcal{C}^\infty(\mathbb{R}^n)}$$

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Example

Let's consider our $\mathcal{C}^\infty$-ring differential modality example, where $\mathsf{S}^\infty(\mathbb{R}^n) := \mathcal{C}^\infty(\mathbb{R}^n)$.

$$\frac{\dfrac{f : \mathbb{R}^n \to \mathbb{R} \qquad f \text{ is a smooth function}}{f \in \mathcal{C}^\infty(\mathbb{R}^n)}}{q_f : \mathbb{R} \to \mathcal{C}^\infty(\mathbb{R}^n) \qquad q_f \text{ linear map in } \mathsf{VEC}_\mathbb{R},\ q_f(1) = f}$$

Differential Categories - Smooth Maps

Recall that for a monad S, its Kleisli category KI(S) is the category with the same objects as the base category but where a map from X to Y in KI(S) is a map:

$$X \to \mathsf{S}(Y)$$

Every differential category has a notion of a *smooth map* given by the **opposite** category of the Kleisli category. So a **smooth map** from A to B is a map:

$$B \to \mathsf{S}(A)$$

Example

Let's consider our $\mathcal{C}^\infty$-ring differential modality example, where $\mathsf{S}^\infty(\mathbb{R}^n) := \mathcal{C}^\infty(\mathbb{R}^n)$.

$$\begin{array}{c} f : \mathbb{R}^n \to \mathbb{R} \qquad f \text{ is a smooth function} \\ \hline\hline f \in \mathcal{C}^\infty(\mathbb{R}^n) \\ \hline\hline q_f : \mathbb{R} \to \mathcal{C}^\infty(\mathbb{R}^n) \qquad q_f \text{ linear map in } \mathsf{VEC}_\mathbb{R},\ q_f(1) = f \\ \hline\hline \mathbb{R} \to \mathbb{R}^n \qquad \text{in } \mathsf{KI}(\mathsf{S}^\infty) \end{array}$$

Differential Categories - Smooth Maps

Recall that for a monad S, its Kleisli category $\mathsf{Kl}(\mathsf{S})$ is the category with the same objects as the base category but where a map from X to Y in $\mathsf{Kl}(\mathsf{S})$ is a map:

$$X \to \mathsf{S}(Y)$$

Every differential category has a notion of a *smooth map* given by the **opposite** category of the Kleisli category. So a **smooth map** from A to B is a map:

$$B \to \mathsf{S}(A)$$

Example

Let's consider our $\mathcal{C}^\infty$-ring differential modality example, where $\mathsf{S}^\infty(\mathbb{R}^n) := \mathcal{C}^\infty(\mathbb{R}^n)$.

$$\dfrac{\dfrac{\dfrac{f : \mathbb{R}^n \to \mathbb{R} \qquad f \text{ is a smooth function}}{f \in \mathcal{C}^\infty(\mathbb{R}^n)}}{q_f : \mathbb{R} \to \mathcal{C}^\infty(\mathbb{R}^n) \qquad q_f \text{ linear map in } \mathsf{VEC}_\mathbb{R},\ q_f(1) = f}}{\dfrac{\mathbb{R} \to \mathbb{R}^n \qquad \text{in } \mathsf{Kl}(\mathsf{S}^\infty)}{\mathbb{R}^n \to \mathbb{R} \qquad \text{in } \mathsf{Kl}(\mathsf{S}^\infty)^{op}}}$$

Differential Categories - Smooth Maps

Amongst the smooth maps we have:

- The constant maps:

$$B \longrightarrow I \xrightarrow{\mathsf{u}} \mathsf{S}(A)$$

- The linear maps:

$$B \longrightarrow A \xrightarrow{\eta} \mathsf{S}(A)$$

- The product of smooth maps:

$$B \otimes C \xrightarrow{f \otimes g} \mathsf{S}(A) \otimes \mathsf{S}(A) \xrightarrow{\mathsf{m}} \mathsf{S}(A)$$

- The composition of smooth maps:

$$C \xrightarrow{g} \mathsf{S}(B) \xrightarrow{\mathsf{S}(f)} \mathsf{SS}(A) \xrightarrow{\mu} \mathsf{S}(A)$$

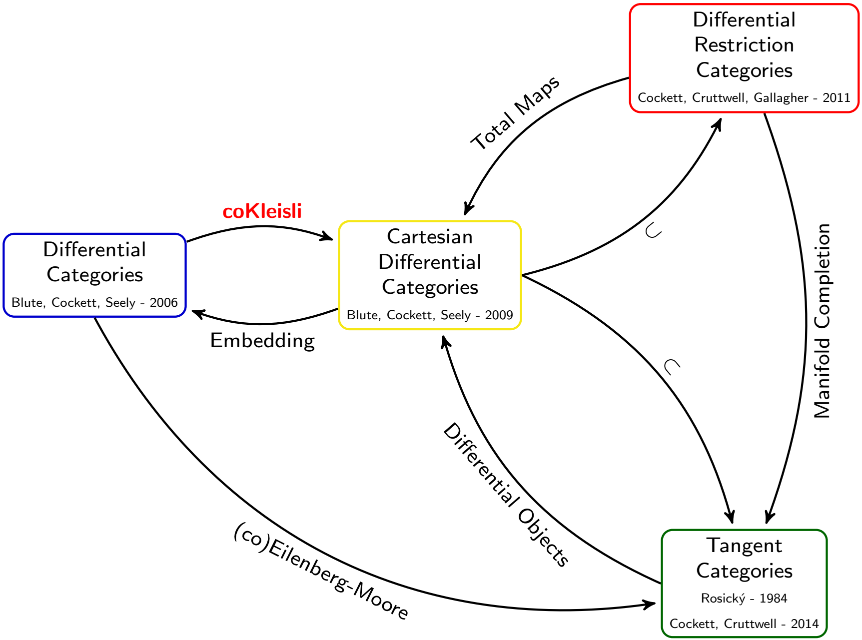
The differential of a smooth map $f : B \to \mathsf{S}(A)$ is then:

$$B \xrightarrow{f} \mathsf{S}(A) \xrightarrow{\mathsf{d}} \mathsf{S}(A) \otimes A$$

So the deriving transformation axioms describe differentiation of constants, identity maps, composition, etc. in the *Kleisli category*!

We can make this precise by looking at: **Cartesian Differential Categories!**

The Differential Category World: It's all connected!



Cartesian Differential Categories

- Categorical foundations of differential calculus over Euclidean spaces
- Categorical semantics of differential λ-calculus

Some introductory references:

Blute, R., Cockett, R., Seely, R.A.G. **Cartesian Differential Categories** (2009)

Garner, R, and Lemay, J-S P. **Cartesian differential categories as skew enriched categories**.

Manzonetto, G. **What is a Categorical Model of the Differential and the Resource λ-Calculi?**. (2012)

Cartesian Differential Categories - Brief Definition

A **Cartesian differential category** is a category with finite products $\times$ such that:

- Each homset is a k-linear module, but only precomposition preserves the k-module structure (think polynomials);
- With a **differential combinator** D which sends every map $f : X \to Y$ to its derivative:

  $$\mathsf{D}[f] : X \times X \to Y$$

  such that seven axioms hold which capture key identities of the total derivative such as the chain rule:

  $$\mathsf{D}[g \circ f](x, y) = \mathsf{D}[g]\,(f(x), \mathsf{D}[f](x, y))$$

Differential Combinator - Main Example

Example

Let SMOOTH be the category of real smooth functions, that is, the category whose objects are the Euclidean vector spaces $\mathbb{R}^n$ and whose maps are smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$, which is actually an m-tuple of smooth functions:

$$F = \langle f_1, \dots, f_m \rangle \qquad f_i : \mathbb{R}^n \to \mathbb{R}$$

SMOOTH is a CDC where the differential combinator is defined as the total derivative of a smooth function, which is given by the sum of partial derivatives. So for a smooth function

$F : \mathbb{R}^n \to \mathbb{R}^m$, its derivative $\mathsf{D}[F] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is then defined as:

$$\mathsf{D}[F](\vec{x}, \vec{y}) := \mathbf{J}(F)(\vec{x}) \cdot \vec{y} = \left\langle \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i}(\vec{x}) y_i, \dots, \sum_{i=1}^{n} \frac{\partial f_m}{\partial x_i}(\vec{x}) y_i \right\rangle$$

In particular for smooth function $f : \mathbb{R}^n \to \mathbb{R}$:

$$\mathsf{D}[f](\vec{x}, \vec{y}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}) y_i$$

Cartesian Differential Categories - Other Main Examples

Example

Any category with finite biproduct $\oplus$ is a CDC, where for a map $f : A \to B$:

$$\mathsf{D}[f] := \; A \oplus A \xrightarrow{\quad \pi_1 \quad} A \xrightarrow{\quad f \quad} B$$

For example, $\mathsf{VEC}_{\mathbb{K}}$ is a CDC where $\mathsf{D}[f](x, y) = f(y)$.

Example

For any commutative semiring k, let Poly_k be the Lawvere theory of polynomials, that is, the category whose objects are $n \in \mathbb{N}$ and where a map $P : n \to m$ is a tuple of polynomials:

$$P = \langle p_1(\vec{x}), \ldots, p_m(\vec{x}) \rangle \qquad p_i(\vec{x}) \in R[x_1, \ldots, x_n]$$

POLY_k is a CDC where for a map $P : n \to m$ with $P = \langle p_1(\vec{x}), \ldots, p_m(\vec{x}) \rangle$, $\mathsf{D}[P] : n \times n \to m$ is:

$$\mathsf{D}[P] := \left\langle \sum_{i=1}^{n} \frac{\partial p_1(\vec{x})}{\partial x_i} y_i, \ldots, \sum_{i=1}^{n} \frac{\partial p_m(\vec{x})}{\partial x_i} y_i \right\rangle$$

where $\sum_{i=1}^{n} \frac{\partial p_i(\vec{x})}{\partial x_i} y_i \in R[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Note that $\mathsf{POLY}_{\mathbb{R}}$ is a sub-CDC of SMOOTH.

Cartesian Differential Categories - Other Examples

Example

- Abelian functor calculus

Bauer, K., Johnson, B., Osborne, C., Riehl, E. and Tebbe, A. **Directional derivatives and higher order chain rules for abelian functor calculus.** (2018)

- Models of the differential λ-calculus

Bucciarelli, A., Ehrhard, T. and Manzonetto. G. **Categorical models for simply typed resource calculi.** (2010)

Manzonetto, G. **What is a Categorical Model of the Differential and the Resource λ-Calculi?**. (2012)

J.R.B. Cockett, R. and Gallagher, J. **Categorial models of the differential λ-calculus** (2019)

- Cofree Cartesian differential categories

Cockett, J.R.B. and Seely, R.A.G. **The Faa di bruno construction**. (2011)

Garner, R, and Lemay, J-S P. **Cartesian differential categories as skew enriched categories**.

Lemay, J-S P. **A Tangent Category Alternative to the Faa di Bruno Construction**.

Lemay, J-S P. **Properties and Characterisations of Cofree Cartesian Differential Categories**.

Another important source of CDC comes from differential categories!

The coKleisli Category of a Differential Category is a CDC

Recall that earlier we defined the differential of a smooth map $f : B \to \mathsf{S}(A)$ as

$$B \xrightarrow{f} \mathsf{S}(A) \xrightarrow{\mathsf{d}} \mathsf{S}(A) \otimes A$$

But this is not a Kleisli map!

The differential combinator $\mathsf{D}[f] : B \to \mathsf{S}(A \times A)$ is defined as follows:

$$B \xrightarrow{f} \mathsf{S}(A) \xrightarrow{\mathsf{d}} \mathsf{S}(A) \otimes A \xrightarrow{1 \otimes \eta} \mathsf{S}(A) \otimes \mathsf{S}(A)$$

$$\xrightarrow{\mathsf{S}(\iota_0) \otimes \mathsf{S}(\iota_1)} \mathsf{S}(A \times A) \otimes \mathsf{S}(A \times A) \xrightarrow{\mathsf{m}} \mathsf{S}(A \times A)$$

Theorem

For a differential category with finite (bi)products, the opposite category of its Kleisli category is a Cartesian differential category.

Example

- POLY_k is a sub-CDC of the coKleisli category $\mathsf{Kl}(\mathsf{Sym})^{op}$
- SMOOTH is a sub-CDC of the coKleisli category $\mathsf{Kl}(\mathsf{S}^\infty)^{op}$.

What can we do with Cartesian differential categories?

- Study and solve differential equations, and also study exponential functions, trigonometric functions, hyperbolic functions, etc.

  Cockett, R., Cruttwel, G., Lemay, J-S. P., **Differential equations in a tangent category I: Complete vector fields, flows, and exponentials**.

  Lemay, J-S.P., **Exponential Functions for Cartesian Differential Categories**.

- Linearization, Jacobians and gradients:

  Cockett, R., Lemay, J-S.P., **Linearizing Combinators**.

  Lemay, J-S.P., **Jacobians and Gradients for Cartesian Differential Categories**.

- Foundations for automatic differentiation and machine learning algorithms via **reverse differentiation**.

  Cockett, R., Cruttwell, G., Gallagher, J., Lemay, J.-S. P., MacAdam, B., Plotkin, G., & Pronk, D. (2020). **Reverse derivative categories.**

  Wilson, P., & Zanasi, F. Reverse **Derivative Ascent: A Categorical Approach to Learning Boolean Circuits.**

  Cruttwell, G., Gallagher, J., & Pronk, D. **Categorical semantics of a simple differential programming language.**

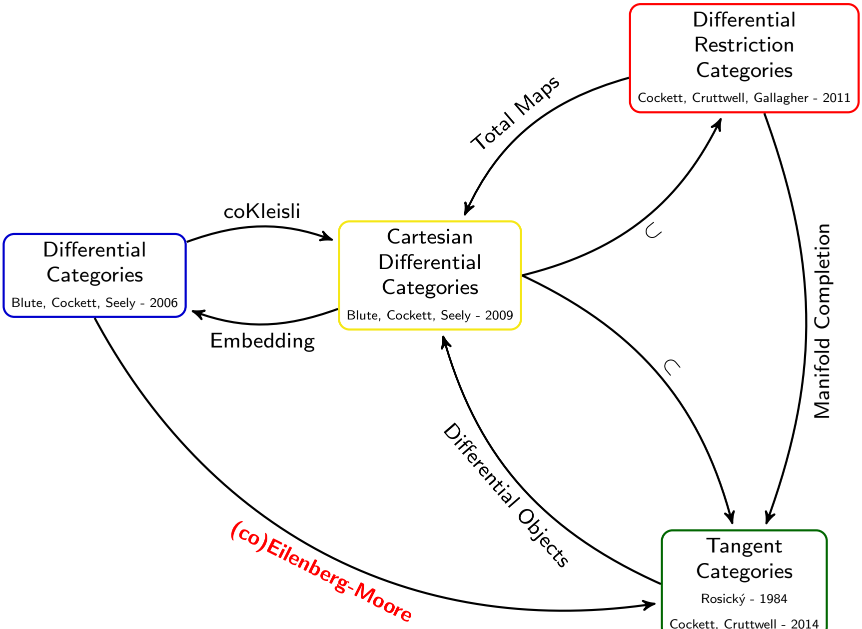
  Cruttwell, G., Gavranovic, B., Ghani, N., Wilson, P., & Zanasi, F. **Categorical Foundations of Gradient-Based Learning.**

Well we have a monad...

Since a differential modality is a monad, we can ask what can we say about its Kleisli category and its Eilenberg-Moore category?

- Kleisli category $=$ Smooth maps
- Eilenberg-Moore category $=$ Smooth manifolds

The Differential Category World: It's all connected!



A quick word on Tangent Categories

Tangent Categories:

- Formalize differential calculus on smooth manifold and their tangent bundles
- Formalize notions from differential geometry, algebraic geometry, synthetic differential geometry, etc.

J. Rosický **Abstract tangent functors** (1984)

R. Cockett, G. Cruttwell **Differential structure, tangent structure, and SDG** (2014)

R. Garner **An embedding theorem for tangent categories** (2018)

Tangent Categories - Brief Definition

Briefly a tangent category is a category $\mathbb{X}$ equipped with an endofunctor:

$$\mathsf{T} : \mathbb{X} \to \mathbb{X}$$

where for an object A we think of $\mathsf{T}(A)$ as the tangent bundle of A

Equipped some natural transformations that capture the essential properties of the tangent bundle of smooth manifolds, such as the natural projection, vector bundle, local triviality, etc.

Lots of important concepts from differential geometry can be formalized in tangent categories:

- Tangent Spaces
- Euclidean Spaces
- Vector Fields
- Lie Bracket
- Vector Bundles
- Connections
- Differential/Sector Forms and Cohomology
- Differential Equations

Tangent Categories - Main Examples

Example

The category of finite dimensional smooth manifolds, SMAN is a tangent category where the tangent bundle functor maps a smooth manifold M to its usual tangent bundle $\mathsf{T}(M)$. This relates tangent categories to differential geometry.

Example

The category of commutative rings, CRING, is a tangent category with tangent functor which maps a commutative ring R to its ring of dual numbers $\mathsf{T}(R) = R[\epsilon]$. This relates tangent categories to algebra.

Example

The category of affine schemes, $\mathsf{AFF} = \mathsf{CRING}^{op}$, is a tangent category with tangent functor which maps a commutative ring R to the free commutative R-algebra over its Kahler module $\mathsf{T}(R) = \mathsf{Sym}_R(\Omega_R)$. This relates tangent categories to algebraic geometry.

Tangent categories have also found connections to synthetic differential categories, operad theory, group theory, etc.

Tangent Categories – Relation to Differential Categories

Proposition

Every Cartesian differential category is a tangent category:

$$T(A) = A \times A \qquad\qquad T(f)(x, y) = \langle f(x), D[f](x, y) \rangle$$

Conversely, the subcategory of **differential objects** *of a tangent category is a CDC.*

Theorem (Cockett, Lemay, Lucyshyn-Wright)

The (opposite category of the) Eilenberg-Moore category of a differential category is a tangent category.

R. Cockett, R., Lemay, J-S. P., Lucyshyn-Wright, R. **Tangent Categories from the Coalgebras of Differential Categories.**

Which means that in a way, we can think of S-algebras as "abstract smooth manifolds" or "abstract affine schemes". Chiara will talk more about S-algebras next week.

Preview of Chiara's Talk Next Week

- Chiara will talk about how to formalize the notion of **differential graded algebras** in a differential category. She'll also talk derivations, Kahler modules, de Rham cohomology, and S-algebras.

That's all folks! Hope you enjoyed!

If you find differential/tangent categories interesting and have ideas, I hope you will start working with them! I am always happy to chat about differential categories, so feel free to come talk to me or reach out by email. I'm very interested in new ideas for what to do with differential category/tangent categories.

The Differential Category World: It's all connected!

Hope you enjoyed it!
Thanks for listening!
Merci!