Yang - Mills Lagrangian

- LAGRANGIANS ARE RESTRICTED BY THE FOLLOWING

↳ EX. OF SYMMETRIES

↳ QFTs SHOULD BE RENORMALIZABLE

↳ QFTs SHOULD BE FREE OF GAUGE ANOMALIES

↓

- Yang - Mills
- Klein-Gordon & Higgs
- Dirac
- Yukawa (itself not a complete dynamic Lagrangian)

Symmetries:

- Lorentz → local Lorentz transformations of spacetime manifold, acting on each tangent space
- Gauge
- Conformal
- Supersymmetry

Renormalizability

$$\underset{\substack{\uparrow \\ \text{gauge inv.} \\ \text{Lorentz inv.}}}{\mathcal{L}(\phi_1, \dots, \phi_n)} \rightsquigarrow \underset{\substack{\uparrow \\ \text{also gauge \& Lorentz inv.}}}{\mathcal{L}^k(\phi_1, \dots, \phi_n)}$$

Anomalous symmetries

Symmetries of the classical field theory, like gauge symmetries, do not necessarily hold in the QFT.

Unitarity - Hilbert sp. of states has a positive definite Hermitian scalar product.

$$\mathcal{L} = \mathrm{Re}(\bar{\Psi} D_A \Psi) + \langle d_A \Phi , d_A \Phi \rangle_E - V(\Phi) - 2 g_y \mathrm{Re}(\bar{\Psi}_L \Phi \psi_R)$$
$$- \tfrac{1}{2} \langle F_M^A , F_M^A \rangle_{Ad(P)}$$

- -

Def.

CONNECTION 1-FORM

$\pi : P \to M$, $A \in \Omega^1(P, \mathfrak{g})$ (GAUGE FIELD)

1. $r_g^* A = \mathrm{Ad}_{g^{-1}} \circ A \quad \forall g \in G$
2. $A(\hat{X}) = X$

LIN. MAP

$$A_p : T_p P \to \mathfrak{g}$$

Def. (GLOBAL) GAUGE TRANSFORMATION

Diff. $f : P \to P$ which preserves fibres of P and is G equivariant.

1. $\pi \circ f = \pi$
2. $f(p \cdot g) = f(p) \cdot g \qquad \forall p \in P , \; g \in G$

$\mathcal{G}(P)$ – set of all gauge transformations

$\pi_U : P_U \to U$, $U \underset{\text{open}}{\subset} M$ LOCAL GAUGE TRANSF.

Def.

$$C^\infty(P,G)^G := \{ \underset{\text{smooth}}{\sigma : P \to G} \mid \sigma(p \cdot g) = g^{-1} \sigma(p) g \} \quad \overset{\text{group}}{(\sigma \cdot \sigma')(p) = \sigma'(p) \cdot \sigma(p)}$$

Prop.

$$\mathcal{G}(P) \xrightarrow{\text{isom.}} C^\infty(P,G)^G$$
$$f \longmapsto \sigma_f$$
$$f(p) = p \cdot \sigma_f(p) \quad \forall p \in P$$

Prop. G - abelian $\Rightarrow$

$$C^\infty(M,G) \xrightarrow{\text{isom.}} C^\infty(P,G)^G$$
$$\tau \longmapsto \sigma_\tau$$
$$\sigma_\tau = \tau \circ \pi .$$

Def.

PHYSICAL GAUGE TRANSFORMATION.

$\pi : P \to M.$ $\qquad \tau : U \to G \;,\; U \underset{\text{open}}{\subset} M .$

RIGID WHEN τ IS A CONSTANT MAP

$C^\infty(U,G) \cong G$.

Prop.

$s : U \to P$ – a local section. Then s defines a grp. isomorphism:

$$C^\infty(P_U, G)^G \longrightarrow C^\infty(U,G)$$
$$\sigma \longmapsto \tau_\sigma = \sigma \circ s$$

$$C^\infty(U,G) \longrightarrow C^\infty(P_U,G)^G$$
$$\tau \longmapsto \sigma_\tau \text{, where}$$
$$\sigma_\tau(s(x)\cdot g) = g^{-1}\tau(x)g \quad \forall x \in U,\ g \in G .$$

Fix $s : U \to P$ $\Rightarrow$ We can identify local bundle automorphisms on the principal G-bundle $P_U \to U$ with physical gauge transformations.

* * *

$$\begin{tikzcd} G \arrow[r] & P \arrow[d, "\pi_P"] \\ & M \end{tikzcd} \quad , \quad \rho : G \longrightarrow GL(\overset{\dim k}{V})$$

Lemma

$$(P \times V) \times G \longrightarrow P \times V$$
$$(p,v,g) \longmapsto (p,v)\cdot g = (p\cdot g,\ \rho(g)^{-1}v)$$

defines a free principal right action of G on $P \times V$.

In particular, $E = {}^{(P\times V)}/_{G}$ is a smooth mfd s.t. $P \times V \to E$ is a submersion.

Thm.

E has the structure of a k-vector bundle over M, with a projection $\pi_E : E \to M$

$$[p,v] \longmapsto \pi_P(p)$$

and fibres $E_x = (P_x \times V)/G$ isomorphic to V.

$\lambda[p,v] + \mu[p,w] = [p, \lambda v + \mu w]$ $\forall p \in P$, $v,w \in V$, $\mu, \lambda \in \mathbb{K}$.

↑ vector sp. str. on E_x.

Def.

The vector bundle $E = P \times_\rho V = (P \times V)/G$ is called the *vector bundle associated* to the principal bundle P and rep. ρ.

$$\begin{tikzcd} V \arrow[r] & P \times_\rho V \arrow[d, "\pi_E"] \\ & M \end{tikzcd}$$

Local sections:

$s: U \to P$ ⇒ There is a 1-1 relation between smooth sections $\tau: U \to E$ and smooth maps $f: U \to V$, $\tau(x) = [s(x), f(x)]$ $\forall x \in U$.

Def.

$E = P \times_\rho V$. If $\rho_*: \mathfrak{g} \to \mathrm{End}(V)$ is non-triv., then the sections of E are called *charged*.

Prop.

$\langle \cdot, \cdot \rangle_V$ - G-inv. scalar product on V. Then the bundle metric $\langle \cdot, \cdot \rangle_E$ is given by

$$\langle [p,v], [p,w] \rangle_{E_x} = \langle v, w \rangle_V, \text{ for arb. } p \in P_x.$$

Ex.

$$\begin{tikzcd} G \arrow[r] & P \arrow[d] \\ & M \end{tikzcd}$$, $\mathrm{Ad}: G \to GL(\mathfrak{g})$.

Then the assoc. v. bundle $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g}$.

ADJOINT BUNDLE

$$\begin{tikzcd} \mathfrak{g} \arrow[r] & \mathrm{Ad}(P) \arrow[d] \\ & M \end{tikzcd}$$.

Thm.

$$\mathcal{G}(P)\times E \longrightarrow E$$

$$(f,[p,v]) \mapsto f\cdot[p,v] = [f(p),v] = [p\cdot\tau_f(p),v]$$

Thm. (Action of Physical G. Transf. on E)

$s: U\to P$, $\Phi: U\to E$ – local sections. We write

$$\Phi(x) = [s(x), \phi(x)] \quad \forall x\in U.$$

smooth map $U\to V$

Suppose f is a local bundle automorphism of P over U and $\tau_f: U\to G$ the assoc. physical gauge transformation. Then

$$(f\cdot\Phi)(x) = [s(x), \rho(\tau_f(x))\phi(x)].$$

* * *

Def.

LOCAL CONNECTION DET. BY S

$s: U\to P$. $A_s\in\Omega^1(U,\mathfrak{g})$, $A_s = A\circ Ds = s^*A$

$\{\partial_\mu\}$ – local coordinate basis vector fields on U

We set $A_\mu = A_s(\partial_\mu)$.

$\{e_a\}$ – basis of the Lie algebra

$$A_\mu = \sum_{a=1}^{\dim\mathfrak{g}} A^a_\mu e_a$$

$\in C^\infty(U,\mathbb{R})$

Corr. real valued 1-forms $A^a_s\in\Omega^1(U)$ are called LOCAL GAUGE BOSON FIELDS.

Principal bundles can have many local gauges.

$s_i: U_i\longrightarrow P$

$s_j: U_j\longrightarrow P$ $\quad U_i\cap U_j\neq\emptyset$.

PHYSICAL GAUGE TRANSF.

$$s_i(x) = s_j(x)\cdot g_{ij}(x) \quad \forall x\in U_i\cap U_j \qquad g_{ij}: U_{ij}\longrightarrow G$$

$A_i = A_{s_i} \in \Omega^1(U_i, \mathfrak{g})$ $\qquad$ $A_j = A_{s_j} \in \Omega^1(U_j, \mathfrak{g})$

$\mu_G \in \Omega^1(G, \mathfrak{g})$, $\mu_G(v) = D_g L_{g^{-1}}(\overset{T_gG}{v})$

$\mu_{ji} = g_{ji}^* \mu_G \in \Omega^1(U_i \cap U_j, \mathfrak{g})$

Thm.

$$A_i = \mathrm{Ad}_{g_{ji}^{-1}} \circ A_j + \mu_{ji} \quad \text{on} \quad U_i \cap U_j .$$

Thm. (Global statement)

$P \to M$, $A \in \Omega^1(P, \mathfrak{g})$, $f \in \mathcal{G}(P)$. Then f^*A is a conn 1-form on P and

$$f^*A = \mathrm{Ad}_{\tau_f^{-1}} \circ A + \tau_f^* \mu_G .$$

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$$TP = V \oplus \underset{\substack{\uparrow \\ \text{horizontal subbundle}}}{H} , \quad \pi_H : TP \to H$$

$$\ker \underset{\substack{\uparrow \\ \text{conn 1-form}}}{A} = H$$

Def.

$$\overset{\text{CURVATURE 2-FORM}}{F} \in \Omega^2(P, \mathfrak{g}) , \quad F(X,Y) = dA(\pi_H(X), \pi_H(Y)) \quad \forall X, Y \in T_pP$$

Prop.

1. $r_g^* F = \mathrm{Ad}_{g^{-1}} \circ F \quad \forall g \in G$.
2. $\tilde{X} \underset{\substack{\uparrow \\ \text{insertion of the vector field } \tilde{X}.}}{\lrcorner} F = 0 \quad \forall X \in \mathfrak{g}$

Thm $\qquad F = dA + \frac{1}{2}[A, A]$.

Def.

$s : U \to P \qquad A \in \Omega^1(P, \mathfrak{g}) \rightsquigarrow A_s \in \Omega^1(U, \mathfrak{g})$

$F_s \in \Omega^2(U, \mathfrak{g})$

$$F_s = F \circ (Ds, Ds) = s^* F$$

$$\{\partial_\mu\} \Rightarrow F_{\mu\nu} = F_s(\partial_\mu, \partial_\nu).$$

$$\{e_a\} \Rightarrow F_{\mu\nu} = \sum_{a=1}^{\dim \mathfrak{g}} F^a_{\mu\nu} e_a$$

Prop.

$F_s = dA_s + \frac{1}{2}[A_s, A_s]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

If G is abelian, then $[A_\mu, A_\nu] = 0$.

Thm.

$F_i = Ad_{g_{ji}^{-1}} \circ F_j$ on $U_i \cap U_j$.

Cor.

G-abelian $\Rightarrow$ F_s is independent of s $\Rightarrow$ determines a well defined, global, closed 2-form $F_M \in \Omega^2(M, \mathfrak{g})$.

Hodge star.

(M, g) - n-dim oriented pseudo-Riemannian mfd.

$\uparrow$ usually has Lorentzian signature

g + orientation = canonical volume form on M

$e_1, \dots, e_n$ of T_pM

$$dvol_g(e_1, \dots, e_n) = +1.$$

$$dvol_g = \sqrt{|g|}\, dx^1 \wedge \dots \wedge dx^n$$

$\uparrow$

$|\det(g_{\mu\nu})|$

$\nwarrow$ $g(\partial_\mu, \partial_\nu)$.

Def.

SCALAR PRODUCT OF FORMS

$\mathbb{K} = \mathbb{C}, \mathbb{R}$

$$\langle \cdot , \cdot \rangle : \Omega^k(M,\mathbb{K}) \times \Omega^k(M,\mathbb{K}) \longrightarrow C^\infty(M,\mathbb{K})$$

$$\langle \omega, \eta \rangle = \sum_{\mu_1 < \dots < \mu_k} \omega_{\mu_1 \dots \mu_k} \eta^{\mu_1 \dots \mu_k}$$

$$= \frac{1}{k!} \omega_{\mu_1 \dots \mu_k} \eta^{\mu_1 \dots \mu_k}$$

$\omega_{\mu_1 \dots \mu_k} = \omega(\partial_{\mu_1}, \dots, \partial_{\mu_k})$ in a local chart (U, ϕ).

Def.

Hodge star operator

$$\star : \Omega^k(M,\mathbb{K}) \longrightarrow \Omega^{n-k}(M,\mathbb{K})$$

uniquely defined

lin. map.

$$\langle \omega, \eta \rangle dvol_g = \omega \wedge \star \eta \qquad \forall \omega, \eta \in \Omega^k(M,\mathbb{K})$$

$e_1, \dots, e_n$ -on. oriented basis of tangent vect

and $\alpha^i(e_j) = \delta^i_j \Rightarrow dvol_g = \alpha^1 \wedge \dots \wedge \alpha^n$

$\star dvol_g = (-1)^t \cdot 1$

$\star 1 = dvol_g$

Def.

L^2-scalar product of forms

$$\langle \cdot , \cdot \rangle_{L^2} : \Omega^k_0(M,\mathbb{K}) \times \Omega^k_0(M,\mathbb{K}) \longrightarrow \mathbb{K}$$

compact support

$$\langle \omega, \eta \rangle_{L^2} := \int_M \langle \omega, \eta \rangle dvol_g$$

Def.

$E \to M$ with $\langle \cdot, \cdot \rangle_E$. We then get induced bundle metrics on the vector bundle $\Lambda^k T^*M \otimes E$.

twisted k-form

$$\langle \cdot , \cdot \rangle_E : \Omega^k(M,E) \times \Omega^k(M,E) \longrightarrow C^\infty(M)$$

Choose local frame $e_1, \dots, e_r$ for E over $U \subset M$ and expand k-forms F, G twisted with E as

$$F = \sum^{r} F_i \otimes e_i$$

$$\Omega^k(U, \mathbb{K})$$

$$G = \sum^{r} G_i \otimes e_i$$

$$\langle F, G \rangle_E = \sum_{i,j=1}^{r} \langle F_i, G_i \rangle \langle e_i, e_j \rangle_E$$ ← independent of the choice of $\{e_i\}$.

$$\ast : \Omega^k(M, E) \longrightarrow \Omega^{n-k}(M, E)$$

$$\ast F = \sum_{i=1}^{r} (\ast F_i) \otimes e_i$$

$$\langle \cdot, \cdot \rangle_{E, L^2} : \Omega^k_0(M, E) \times \Omega^k(M, E) \longrightarrow \mathbb{K}$$

$$\langle \omega, \eta \rangle_{E, L^2} = \int_M \langle \omega, \eta \rangle_E \, dvol_g$$

The codifferential

(M, g) – oriented, semi-Riemannian mfd, $\dim = n$ and signature (s, t)

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

Def.

$$d^\ast : \Omega^{k+1}(M) \longrightarrow \Omega^k(M)$$

$$d^\ast = (-1)^{t + nk + 1} \ast d \ast$$

Thm.

$$\langle d\omega, \eta \rangle_{L^2} = \langle \omega, d^\ast \eta \rangle_{L^2}$$

$$\omega \in \Omega^k(M)$$

$$\eta \in \Omega^{k+1}(M)$$

Def.

$\gamma^*: I \longrightarrow P$ is called a horizontal lift of $\gamma: I \longrightarrow M$ if:

1. $\pi \circ \gamma^* = \gamma$
2. the velocity vectors $\dot{\gamma}^*(t)$ are horizontal

elements of $H_{\gamma^*(t)}$

Thm.

$\gamma: [a,b] \longrightarrow M$ with $\gamma(a) = x$. Let p be a point in P_x. Then there exists a unique horizontal lift γ_p^* of γ with $\gamma_p^*(a) = p$.

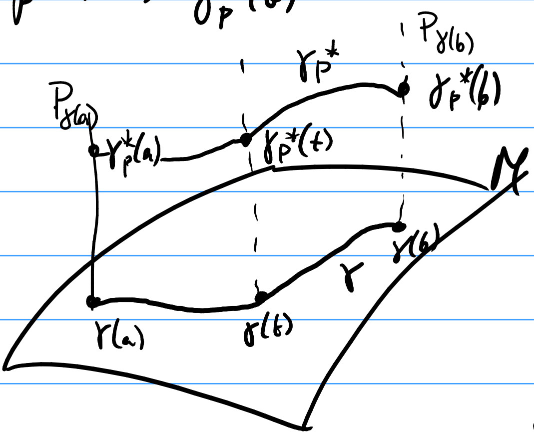
Def.

$\gamma: [a,b] \longrightarrow M$

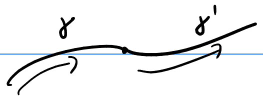
parallel transp. in principal bundle

$$\Pi_\gamma^A : P_{\gamma(a)} \longrightarrow P_{\gamma(b)}$$

$$p \longmapsto \gamma_p^*(b)$$



$$\begin{tikzcd} P \arrow[d, "\pi"] \\ M \end{tikzcd}$$



Thm (Properties of parallel transport)

1. Π_γ^A is a smooth map between fibers and it does not depend on the parametrization of γ.
2. $\Pi_{\gamma * \gamma'}^A = \Pi_{\gamma'}^A \circ \Pi_\gamma^A$
3. $\gamma^{-1} \Rightarrow \Pi_{\gamma^{-1}}^A = (\Pi_\gamma^A)^{-1}$ $\Rightarrow$ p.tr. is a diffeomorphism of fibers

4. G-equivariance: $\Pi_\gamma^A \circ r_G = r_G \circ \Pi_\gamma^A \quad \forall_{g\in G}$

Parallel transport in assoc. vect. bundle

$P \to M \qquad \rho : G \to GL(V)$

Thm.

For a curve $\gamma:[0,1]\to M$, the map

$$\Pi_\gamma^{E,A} : E_{\gamma(0)} \to E_{\gamma(1)}$$

$$[p,v] \longmapsto [\Pi_\gamma^A(p), v]$$

is a well-defined lin. isomorphism.

Φ - a section of E, $x\in M$, $X\in T_xM$

$\gamma:(-\varepsilon,\varepsilon)\to M$ with $\gamma(0)=x$

$\dot\gamma(0)=X$.

For each $t\in(-\varepsilon,\varepsilon)$, parallel transport the vect. $\Phi(\gamma(t))\in E_{\gamma(t)}$ back to E_x along γ.

$$D(\Phi,\gamma,x,A) = \frac{d}{dt}\Big|_{t=0} \Big(\Pi_{\underbrace{\gamma_t}_{\text{the restriction of } \gamma \text{ to } [0,t]}}^{E,A}\Big)^{-1}(\Phi(\gamma(t)))\in E_x$$

Thm. $s: U\to P$

s - local gauge, $A_s = s^*A$, $\phi: U\to V$, $\Phi=[s,\phi]$.

section of assoc. v.b.

Then the vector $D\in E_x$ is given by

$$D(\Phi,x,\gamma,A) = [s(x),\ d\phi(X) + \rho_*(A_s(X))\phi(x)]$$

Def.

assoc. v.b.

Φ – section of E, $X \in \mathfrak{X}(M)$. Then the covariant derivative $\nabla^A_X \Phi$ of the section of E defined by

$$\nabla^A_X \Phi(x) = D(\Phi, \gamma, x, A)$$

$$\nabla^A : \Gamma(E) \longrightarrow \Omega^1(M, E)$$

In local gauge $\nabla^A_X \Phi = [s, \nabla^A_X \phi]$, where

$$\nabla^A_X \phi = d\phi(X) + \rho_*(A_s(X))\phi$$

$$\nabla^A_X \phi(x) = d\phi(X_x) + \rho_*(A_s(X_x))\phi(x) \in V.$$

Prop.

$$\nabla^A_{fX} \Phi = f \nabla^A_X \Phi \quad \text{and}$$

$f, \lambda \in C^\infty(M, \mathbb{R})$

$$\nabla^A_X (\lambda \Phi) = (L_X \lambda)\Phi + \lambda \nabla^A_X \Phi$$

If $\{\partial_\mu\}$ local basis on U, we get

$$\nabla^A_\mu \phi = \nabla^A_{\partial_\mu} \phi = \partial_\mu \phi + \underbrace{A_\mu \phi}_{\rho_*(A_\mu)\phi}$$

minimal coupling

„charged" whenever

$\rho_* : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is non-trivial

Remark: $\nabla^A_X \Phi = [s, \nabla^A_X \phi] = d\phi(X) + \rho_*(A_s(X))\phi$

In physics

Need to show independence from the choice of s.

$$\nabla^A : \Gamma(E) \longrightarrow \Omega^1(M,E)$$

$$d : C^\infty(M) \longrightarrow \Omega^1(M) \Rightarrow d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

We want $d_A : \Omega^k(M,E) \longrightarrow \Omega^{k+1}(M,E)$

Def.

$$\underset{\substack{\uparrow \\ \text{well-defined}}}{\wedge} : \Omega^k(M) \times \Omega^l(M,E) \longrightarrow \Omega^{k+l}(M,E)$$

In standard case, we get a product of two scalars. Here, we a product of a scalar and a vector in E.

Pick $e_1,\dots,e_r$ - local basis of E over $U\subset M$.

$\omega \in \Omega^k(M,E)$

$$\omega = \sum_{i=1}^r \underset{\substack{\uparrow \\ \Omega^k(U)}}{\omega_i} \otimes e_i$$

Def. ext. covariant derivative (covariant differential)

$$d_\nabla : \Omega^k(M,E) \longrightarrow \Omega^{k+1}(M,E)$$

covariant derivative on $E \to M$

$$\underset{\substack{\uparrow \\ \text{independent of the choice of } \{e_i\}.}}{d_\nabla \omega} = \sum_{i=1}^r \left(d\omega_i \otimes e_i + (-1)^k \omega_i \wedge \nabla e_i\right)$$

Prop.

$\Omega^k(M,E)$

- $d_\nabla(\omega + \omega') = d_\nabla \omega + d_\nabla \omega'$
- $d_\nabla(\sigma \otimes e) = \underset{\substack{\uparrow \\ \Omega^k(M)}}{d\sigma} \otimes e + (-1) \sigma \wedge \nabla \underset{\substack{\uparrow \\ \Gamma(E)}}{e}$

- $d_\nabla \big|_{\Gamma(E)} = \nabla. \qquad (\Gamma(E) = \Omega^0(M,E))$

$$d_\nabla(\sigma\wedge\omega) = d\underset{\substack{\uparrow\\ \Omega^k(M)}}{\sigma}\wedge\omega + (-1)^k\sigma\wedge d_\nabla \underset{\substack{\uparrow\\ \Omega^l(M,E)}}{\omega}$$

! $d_\nabla \circ d_\nabla \neq 0$ Related to the curvature F^∇.

$P\to M$, $\rho: G\to \mathrm{End}(V)$, $E = \underset{(P\times V)/G}{P\times_\rho V}$, $A\in\Omega^1(P,\mathfrak{g})$.

Def.

$$\wedge : \Omega^k(M,\mathfrak{g})\times\Omega^l(M,V) \longrightarrow \Omega^{k+l}(M,V)$$

$$(\alpha,\omega)\longmapsto \alpha\wedge\omega$$

Expand $\omega = \sum_{i=1}^n \omega_i\otimes \underset{\substack{\uparrow\\ \text{basis of } V}}{v_i}$ $\Rightarrow$ $\alpha\wedge\omega = \sum_{i=1}^n (\rho_*(\alpha)v_i)\wedge\omega_i$

$s: U\to P$ $\overset{W}{\cancel{\alpha}}\in\Omega^l(M,E)$ defines a form $\overset{W_s}{\cancel{\alpha_s}}\in\Omega^l(M,V)$

Thm.

With respect to $s: U\to P$ we can write

$$(d_A\omega)_s = d\omega_s + A_s\wedge\omega_s \qquad \omega\in\Omega^k(M,E)$$

For twisted forms:

$E \to M$ with $\langle \cdot , \cdot \rangle_E$ and compatible covariant derivative ∇.

Associated exterior covariant derivative:

$$d_\nabla : \Omega^k(M,E) \longrightarrow \Omega^{k+1}(M,E)$$

Def.

covariant codifferential

$$d^*_\nabla : \Omega^{k+1}(M,E) \longrightarrow \Omega^k(M,E)$$

$$d^*_\nabla = (-1)^{k+nk+1} * d_\nabla *$$

Thm.

$$\langle d_\nabla \omega , \eta \rangle_{L^2,E} = \langle \omega , d^*_\nabla \eta \rangle_{E,L^2}$$

- (M,g) - n-dim oriented pseudo-Riemannian
- $P \to M$, G - compact $\dim G = r$.
- Ad-inv. positive def. scalar product on $\mathfrak{g}$, $\langle \cdot , \cdot \rangle_{\mathfrak{g}}$.
- $\langle \cdot , \cdot \rangle_{\mathfrak{g}}$ -on. $T_1, \dots, T_r$ for $\mathfrak{g}$.

Ad-inv. $\langle \cdot , \cdot \rangle_{\mathfrak{g}}$ determines a bundle on
$\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g}$, $\langle \cdot , \cdot \rangle_{\mathrm{Ad}(P)}$.

A - connection 1-form , $F^A \in \Omega^2(P, \mathfrak{g})$ defines
$F^A_M \in \Omega^2(M, \mathrm{Ad}(P))$.

Def.

$\omega \in \Omega^k(P, \mathfrak{g})$

1. ω is __horizontal__ if for all $p \in P$
   $\omega_p(X_1, \dots, X_k) = 0$ whenever at least one X_j is vertical

2. __of type Ad__ if
   $r_g^* \omega = Ad_{g^{-1}} \circ \omega$ for all $g \in G$

1. & 2. $\Rightarrow$ $\Omega^k_{hor}(P, \mathfrak{g})^{Ad}$

Prop.

1. $A, A' \in \Omega^1(P, \mathfrak{g}) \Rightarrow A - A' \in \Omega^1_{hor}(P, \mathfrak{g})^{Ad}$
   Moreover, if $\omega \in \Omega^1_{hor}(P, \mathfrak{g})^{Ad}$, then $A + \omega$ is a connection on P.
2. The curvature F^A is an element of $\Omega^2_{hor}(P, \mathfrak{g})^{Ad}$

$\Downarrow$

the set of all connections

$\mathcal{A}(P)$ is an affine space over $\Omega^1_{hor}(P, \mathfrak{g})^{Ad}$.

Thm.

$$\Omega^k_{hor}(P, \mathfrak{g})^{Ad} \underset{\text{isom.}}{\simeq} \Omega^k(M, Ad(P))$$

Corollary:

1. $A - A' \in \Omega^1(M, Ad(P))$ $\quad$ $\Omega^1(P)$ $\quad$ $\Omega^1(M)$
2. $F^A = F^A_M \in \Omega^2(M, Ad(P))$

vacuum field

↓ ↓

„Gauge bosons, the differences $A_\mu - A_\mu^0$, are fields that transform in the adjoint representation of G under gauge transformations"

Def.

$$\overset{\text{Yang-Mills Lagrangian}}{\mathcal{L}_{YM}}[A] = -\frac{1}{2}\langle F^A_M, F^A_M\rangle_{\mathrm{Ad}(P)}$$

Thm.

$\mathcal{L}_{YM}$ is gauge invariant, i.e.

$$\mathcal{L}_{YM}[f^*A] = \mathcal{L}_{YM}[A].$$

for all $f \in \mathcal{G}(P)$.

$s : U \to P$ - local gauge.

$$F^A_s = s^* F^A \in \Omega^2(U, \mathfrak{g})$$

$$F^A_{\mu\nu} = F^A_s(\partial_\mu, \partial_\nu)$$

$$F^A_s = \overset{\Omega^2(U)}{F^{Aa}_s} \otimes T_a$$

$$F^A_{\mu\nu} = \overset{C^\infty(U)}{F^{Aa}_{\mu\nu}} T_a$$

$$\mathcal{L}_{YM} = -\frac{1}{2}\langle F^A_s, F^A_s\rangle_{\mathfrak{g}} = -\frac{1}{4}\langle F^A_{\mu\nu}, F^{A\mu\nu}\rangle_{\mathfrak{g}} =$$

$$= -\frac{1}{4} F^{Aa}_{\mu\nu} F^{A\mu\nu}_a$$

$$F^A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$ and

$$F^{Aa}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \underbrace{f_{bca}}_{\text{structure constants}} A^b_\mu A^c_\nu$$

$$[T_a, T_b] = \sum_{c=1}^{r} f_{abc} T_c$$

$$f_{bca} = f_{abc}$$

$$\mathcal{L}_{YM} = -\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right)$$

$$\overbrace{-\frac{1}{2}\, f_{abc}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) A^{b\mu} A^{c\nu}}^{\text{cubic}}$$

$$\overbrace{-\frac{1}{4}\, f_{abc} f_{ade}\, A^b_\mu A^c_\nu A^{d\mu} A^{e\nu}}^{\text{quartic}}.$$

Def.

$\mathcal{A}(P)$ – affine over $\Omega^1_{hor}(P,\mathfrak{g}) \cong \Omega^1(M, \mathrm{Ad}(P))$

$\alpha \longmapsto \Lambda(\alpha)$

$S_{YM} : \mathcal{A}(P) \longrightarrow \mathbb{R}$

$$S_{YM}[A] = -\frac{1}{2}\langle F^A_M, F^A_M\rangle_{\mathrm{Ad}(P), L^2} =$$

Yang-Mills action

$$= -\frac{1}{2}\int_M \langle F^A_M, F^A_M\rangle_{\mathrm{Ad}(P)}\, d\mathrm{vol}_g$$

closed (compact w/o boundary)

$$\left.\frac{d}{dt}\right|_{t=0} S_{YM}[A + t\alpha] = 0 \quad ., \ \alpha \in \Omega^1_{hor}(P,\mathfrak{g})^{\mathrm{Ad}}$$

critical points of S_{YM}

Thm.

A is a critical point of S_{YM} $\Longleftrightarrow$ A satisfies Y-M equation $d^*_A F^A_M = 0$, i.e. $d_A * F^A_M = 0$.

Ex.

$G = U(1) \Rightarrow F_s$ - independent of the choice of $s \Rightarrow$

define $F_M \in \Omega^2(M, \underset{i\mathbb{R}}{\overset{\wr\wr}{u(1)}})$

Bianchi id. (satisfied by all conn.'s)

Maxwell equations $\begin{cases} dF_M = 0 \\ d * F_M = 0 \end{cases}$

for a source-free electromagn. field.

Def.

Y-M moduli space $(\text{Y-M connections})/\mathcal{G}(P)$.

Ex.

M - Riemannian $\Rightarrow ** = 1$ on 2 forms on M

We consider $F^A_M \in \Omega^2(M, Ad(P))$ s.t.

$$*F^A_M = F^A_M$$

or

$$*F^A_M = -F^A_M$$

self-dual connections (instantons)

Bianchi id. is satisfied for all $A \Rightarrow$ instantons satisfy Y-M equations.

Instanton moduli space for $G = SU(2)$ and $G = SO(3) \to$

$\to$ Donaldson theory.

Massive bosons

$\mathcal{L}_{YM}$ describes massless bosons.

Arguments from physics $\Rightarrow$ gauge bosons of mass m are described by adding a term of the form

$$\frac{1}{2} m^2 A^\nu_a A^a_\nu$$

In local gauge!

$$\mathcal{L}'_{YM} = \mathcal{L}_{YM} + \sqcup$$

$\frac{1}{2} m^2 \langle A_M, A_M \rangle_{Ad(P)}$

A does not define an element $A_M \in \Omega^1(M, Ad(P))$

$\Rightarrow$ $\mathcal{L}_{YM}'$ is not independent of the choice of l.g.

Solution: Higgs mechanism!

Klein-Gordon & Higgs Lagrangians

Def. cplx. scalar field

$\phi : M \to \mathbb{C}$ oriented pseudo-Riemannian

multiplet of —— " ——

$\phi : M \to \mathbb{C}^r$

Std. Hermitian product $\langle v, w \rangle = v^+ w$

$d\phi \in \Omega^1(M, \mathbb{C}^r)$ with induced Hermitian sc. pr.

Def. free Klein-Gordon Lagrangian

$$\mathcal{L}_{KG} = \underbrace{\langle d\phi, d\phi \rangle}_{\text{kinetic term}} - \underbrace{m^2 \langle \phi, \phi \rangle}_{\text{(KG) mass term}}$$

$\langle d\phi, d\phi \rangle = \langle \partial^\mu \phi, \partial_\mu \phi \rangle$ ← in local coordinates

Def. potential

$V : \mathbb{R} \to \mathbb{R}$ Higgs Lagrangian

$\mathcal{L}_H = \langle d\phi, d\phi \rangle - V(\phi)$ ← in SM: a quadratic polynomial in $\phi^+\phi$. (order 4)

- (M, g) - n-dim or. pseudo-Riemannian mfd
- $P \to M$, G - compact $\dim G = r$
- $\rho : G \to GL(W) \rightsquigarrow E = P \times_\rho W$

cplx. assoc. complex vect. bundle

- G-inv. Hermitian product $\langle \cdot, \cdot \rangle_W$ with assoc. bundle metric on E.

·

Def.

$\dim W = 1 \Rightarrow$ the section of E is a compl. scalar field

$\dim W > 1 \Rightarrow$ the section of E is a multipl. — " — .

multiplet space

$d_A : \Gamma(E) \longrightarrow \Omega^1(M,E) \quad + \quad \langle \cdot , \cdot \rangle_E$ on $\Omega^1(M,E)$

Def. $\Phi = [s, \phi]$

$$\mathcal{L}_{KG}[\Phi, A] = \langle d_A \Phi , d_A \Phi \rangle_E - m^2 \langle \Phi , \Phi \rangle_E$$

in local coords.: $\langle \nabla^{A\mu} \Phi , \nabla^A_\mu \Phi \rangle_E$

Klein-Gordon Lagrangian for a multiplet of complex scalar fields Φ of mass m coupled to a gauge field A

$$\mathcal{L}_{KG}[\Phi, A] = \underbrace{(\partial_\mu \phi)^\dagger (\partial_\mu \phi) - m^2 \phi^\dagger \phi}_{\text{free field}} + (\partial^\mu \phi)^\dagger (A_\mu \phi) - (\phi^\dagger A_\mu)(\partial^\mu \phi) - \phi^\dagger \underset{\int(A_\mu)}{A^\mu A_\mu} \phi$$

interactions

$\Phi = [s, \phi]$, $\nabla^A_\mu \phi = \partial_\mu \phi + A_\mu \phi$

Gauge invariant.

$\mathcal{L}_{KG}[f^{-1}\Phi, f^* A] = \mathcal{L}_{KG}[\Phi, A]$

Gauge fields are not dynamic !

$$\mathcal{L}_{KGYM} = \mathcal{L}_{KG} + \mathcal{L}_{YM} = \langle d_A \Phi , d_A \Phi \rangle_E - m^2 \langle \Phi , \Phi \rangle_E - \frac{1}{2} \langle \overset{\in \Omega^2(M, Ad(P))}{F^A_M} , F^A_M \rangle_{Ad(P)}$$

$\downarrow$ $\mathrm{tr}(\mathrm{ad}_{v_1}, \mathrm{ad}_{v_2})$

G - Semisimple

More generally:

$$\mathcal{L}_H + \mathcal{L}_{YM} = \langle d_A \Phi , d_A \Phi \rangle_E - V(\Phi) - \frac{1}{2} \langle F^A_M , F^A_M \rangle_{Ad(P)}$$

Dirac Lagrangian

Fermions – spinor fields on spacetime

- (M,g) – n-dim, oriented, time-oriented pseudo-Riemannian spin mfd of sign. (s,t).

- a spin structure $Spin^+(M)$ together with cplx. spinor bundle $S \to M$.
- a Dirac form $\langle \cdot , \cdot \rangle$ on the Dirac spinor spc. $\Delta = \Delta_n$ with associated Dirac bundle metric $\langle \cdot , \cdot \rangle_S$.
  $\langle \Psi, \Phi \rangle_S = \bar{\Psi} \Phi$

$\Delta_n = \mathbb{C}^N$ ← Dirac spinors

N - given by values

n	$\mathbb{C}l(n)$	N
Even	$End(\mathbb{C}l(\mathbb{C}^N))$	$2^{n/2}$
Odd	$End(\mathbb{C}^N) \oplus End(\mathbb{C}^N)$	$2^{(n-1)/2}$

$\rho : \mathbb{C}l(n) \longrightarrow End(\Delta_n)$ ← Dirac spinor representation of the cplx. Clifford algebra

$\mathbb{C}l(n) \cong Cl(s,t) \otimes_{\mathbb{R}} \mathbb{C}$ $n = s+t$

Def.

$Cl(s,t) = Cl^0 \oplus Cl^1$

$S_+^{s,t} = \{ v \in \mathbb{R}^{s,t} \mid \eta(v,v) = +1 \}$

$S_\pm^{s,t} = S_+^{s,t} \cup S_-^{s,t}$

$Pin(s,t) = \{ v_1 \cdots v_r \mid v_i \in S_\pm^{s,t}, r \geq 0 \}$

$Spin(s,t) = Pin(s,t) \cap \underline{Cl^0(s,t)} = \{ v_1 v_2 \cdots v_{2r} \mid v_i \in S_\pm^{s,t}, r \geq 0 \}$

$Spin^+(s,t) = \{ v_1 \cdots v_{2p} w_1 \cdots w_{2q} \mid v_i \in S_+^{s,t}, w_j \in S_-^{s,t}, p, q \geq 0 \}$

↑ orthochronous spin group

subgroups of $Cl^\times(s,t)$

$\lambda : Pin(s,t) \longrightarrow O(s,t)$ ← composition of reflections $\mathbb{R}^{s,t} \to \mathbb{R}^{s,t}$ $(v \in S_\pm^{s,t} \; R(v,\cdot))$

$u \longmapsto R_u = R(u,\cdot)$

λ restricts to $\lambda : Spin(s,t) \longrightarrow SO(s,t)$

$\lambda : Spin^+(s,t) \longrightarrow SO^+(s,t)$

$\ker \lambda = \mathbb{Z}_2$, double coverings

Def. (Dirac form)

Fix $\delta = \pm 1$. $\langle \cdot , \cdot \rangle : \Delta_n \times \Delta_n \longrightarrow \mathbb{C}$ ← non-deg. $\mathbb{R}$-bilinear form

1. $\langle X \cdot \Psi, \Phi \rangle = \delta \langle \Psi, X \cdot \Phi \rangle$, $\forall X \in \mathbb{R}^{s,t}$ $\forall \Psi, \Phi \in \Delta_n$
2. $\langle \Psi, \Phi \rangle = \langle \Phi, \Psi \rangle^*$
3. $\langle \Psi, c\Phi \rangle = c \langle \Psi, \Phi \rangle = \langle c^* \Psi, \Phi \rangle$

(we don't assume that it is positive-def.)

Invariant under the action of $Spin^+(s,t)$.

Physical gamma matrices: (For $Cl(1,3)$)

$$\Gamma_0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \qquad \Gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

Weyl repr. of the Clifford algebra.

(M,g) - pseudo-Riemannian of sign. $(\underbrace{+,\dots,+}_{s},\underbrace{-,\dots,-}_{t})$

Frame bundle = principal $O(s,t)$ - bundle

Def. $SO(s,t)$ - bundle under emb $SO(s,t) \subset O(s,t)$

(M,g) is called orientable and time-orientable $SO^+_{s,t}$ ~~so~~ $O^+(s,t)$
if the frame bundle can be reduced to a principal $SO^+(s,t)$ - bundle under emb. $SO^+(s,t) \subset O(s,t)$.

Def.
$\pi_{SO} : SO^+(M) \longrightarrow M$ $SO^+(s,t)$-frame bundl.
A spin structure on M is a $Spin^+(s,t)$-principal bundle $\pi_{Spin}: Spin^+(M) \longrightarrow M$ with a double covering $\Lambda : Spin^+(M) \longrightarrow SO^+(M)$ s.t.

right actions

$$\begin{tikzcd}
Spin^+(M) \times Spin^+(s,t) \arrow[r] \arrow[dd, "\Lambda \times \lambda"] \arrow[ddr, phantom, "2"] & Spin^+(M) \arrow[dd, "\Lambda"] \arrow[dr, "\pi_{Spin}"] & \\
 & & M \\
SO^+(M) \times SO^+(s,t) \arrow[r] & SO^+(M) \arrow[ur, "\pi_{SO}"'] &
\end{tikzcd}$$

A spin structure is a λ-equivariant bundle morph $\Lambda : Spin^+(M) \longrightarrow SO^+(M)$, i.e., λ-reduction of $SO^+(M)$.

Thm.

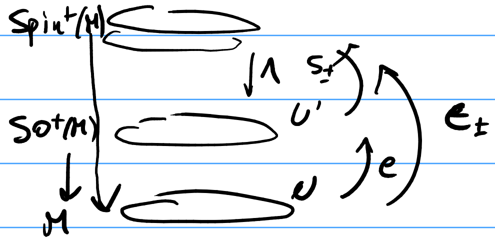
1. The frame budl $SO^+(M)$ admits a spin str. iff the second Stiefel-Whitney class of M vanishes, $w_2(M) = 0$. (We then say M is spin)
2. If $SO^+(M)$ admits a spin str., then there is a (non-can.) bijection between the set of isom. classes of spin str. on M and the cohomology group, $H^1(M;\mathbb{Z}_2)$.

$$\begin{tikzcd}
 & B\mathbb{Z}_2 \arrow[d] \\
M \arrow[ur] \arrow[r] \arrow[dr, "TM + or."'] \arrow[dr, phantom, "2_{\cong}", shift left=3] & BSpin(n) \arrow[d, "B\lambda"] \\
 & BSO(n)
\end{tikzcd}$$

Def.

We call a local section $e=(e_1,\dots,e_n)$ of $SO^+(M)$ a vielbein.

Lemma: Suppose we have a spin str. on M. Then for every e on a contractible open set $U\subset M$ there exists precisely two local sections $\epsilon_\pm$ of $Spin^+(M)$ over U.



Def. $Spin^+(M) \to M$ - spin structure

$\kappa : Spin^+(s,t) \longrightarrow GL(\Delta_n)$

(Dirac) spinor bundle

$$S = Spin^+(M) \times_\kappa \Delta_n$$

Locally:

$e \rightsquigarrow \epsilon_\pm \qquad \psi : U \to S$ - local section $\Longrightarrow$

$\psi_\pm : U \longrightarrow \Delta_n$, $\psi = [\epsilon_\pm, \psi_\pm]$

We have the associated covariant derivative on the spinor bundle S defined by the spin connection A_{spin}, denoted by ∇.

Local presentation: $A^\epsilon_{spin} = \epsilon^* A_{spin} \in \Omega^1(U, spin^+(s,t))$

Def.

Dirac operator

$$\underset{\cdot}{D} : \Gamma(S) \to \Gamma(S) \quad , \quad D\psi = \gamma^{ab} e_a \nabla_{e_b} \psi$$

$D\psi = [\epsilon, D\psi]$, where

Independent of the local vielbein.

$$D\psi = i\Gamma^a\left(d\psi(e_a) - \frac{1}{4}\omega_{abc}\Gamma^{bc}\psi\right)$$

$\frac{1}{2}[\Gamma^b, \Gamma^c]$

Def. Dirac Lagrangian for free spinor field

$$\mathcal{L}_D[\psi] = Re\langle \psi, D\psi\rangle_S - m\langle \psi,\psi\rangle_S = Re(\bar{\psi}D\psi) - m\bar{\psi}\psi$$

$\psi\in\Gamma(S)$

Dirac operator

$D:\Gamma(S)\longrightarrow\Gamma(S)$

For gauge fields we need:

- $P\to M$, G - compact, $\dim G = r$
- $\rho: G\to GL(V)$, $E = P\times_\rho V\longrightarrow M$.
- G-invariant sc. product on V, $\langle\cdot,\cdot\rangle_V$ with assoc. v. bundle metric $\langle\cdot,\cdot\rangle_E$ on E. Together with the Dirac form on $S\to M$, we get a Hermitian sc. pr. $\langle\cdot,\cdot\rangle_{S\otimes E}$ on $S\otimes E$. $(\langle\psi,\Phi\rangle_{S\otimes E} = \bar{\psi}\Phi)$.

$\bar{\Psi} = \langle\Psi,\cdot\rangle$

$s: U\longrightarrow P$ - local gauge, $v_1,\dots,v_s$ - on. basis for V. $\Rightarrow$

$$\psi = \begin{pmatrix}\psi_1\\ \vdots\\ \psi_s\end{pmatrix} \qquad \phi = \begin{pmatrix}\phi_1\\ \vdots\\ \phi_s\end{pmatrix}$$

ψ_i, ϕ_j - sections of S over U.

$$\langle\psi,\Phi\rangle_{S\otimes E} = \bar{\psi}\Phi = \sum_{j=1}^{s}\bar{\psi}_j\phi_j$$

Def.

$\Gamma(S\otimes E)\longrightarrow\Gamma(S\otimes E)$

$$\mathcal{L}_D[\psi,A] = Re\langle\psi, D_A\psi\rangle_{S\otimes E} - m\langle\psi,\psi\rangle_{S\otimes E} =$$

$$= Re\langle\bar{\psi}D_A\psi\rangle - m\bar{\psi}\psi.$$

Local gauge of P + on. basis for V + local vielbeine for TM with assoc. local trivialization ϵ of $\mathrm{Spin}^+(M)$

$$\Downarrow$$

$$\mathcal{L}_D = \mathrm{Re}\sum_{j=1}^{s} i\,\bar\psi_j\,\overbrace{\Gamma^p\Big(\partial_p - \underbrace{\tfrac{1}{4}\omega_{pqr}\,\Gamma^{qr}}_{}\Big)\psi_j}^{\text{kin. term}\cdot\psi_j} - \overbrace{\sum_{j=1}^{s} m\,\bar\psi_j\psi_j}^{\text{Dirac mass term}} + \underbrace{\mathrm{Re}\sum_{j=1}^{s} i\,\bar\psi_j\,\Gamma^p\underbrace{(A_p\psi)_j}_{}}_{\text{interactions}}$$

describes a coupling between the spinor field and metric g.

physical gamma matrices

map with values in $\Delta\otimes V$

ψ_j - maps with values in Δ

$\mathcal{L}_D$ is gauge invariant. $\mathcal{L}_D[f^{-1}\psi, f^*A] = \mathcal{L}_D[\psi, A]$

For dynamic gauge field:

$$\mathcal{L}_{YM} + \mathcal{L}_D = \mathrm{Re}(\bar\psi D_A\psi) - m\bar\psi\psi - \tfrac{1}{2}\langle F_M^A, F_M^A\rangle_{Ad(P)}$$

Chiral fermions $\dim M = n$ - even!

Signature of M = $(1, n-1)$ or $(n-1, 1)$

We can decompose spinors $\psi, \Phi \in \Gamma(S)$ into left-handed (positive) and right-handed (negative) componen

$$\bar\psi\Phi := \langle\psi, \Phi\rangle_S = \langle\psi_L, \Phi_R\rangle_S + \langle\psi_R, \Phi_L\rangle_S =$$

$$\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$$ Weyl spinors

$$= \bar\psi_L\Phi_R + \bar\psi_R\Phi_L$$

Prop. 7.6.7 full statement

Dirac bundle metric is null on the subbundles S_L and S_R and pairs left-handed with right-handed spinors. In particular, the decomposition $S = S_L \oplus S_R$ is not orthogonal with respect to the Dirac bundle metric.

Same holds for twisted bundles:

$$S\otimes E = (S_L\otimes E)\oplus(S_R\otimes E)$$

Prop.

On even dimensional oriented and time-oriented Lorentzian manifolds, the g. inv. Dirac Lagrangian for twisted spinors can be written as

$$\mathcal{L}_D = Re(\bar\psi_L D_A\psi_L + \bar\psi_R D_A\psi_R) - 2m\,\cancel{Re}(\bar\psi_L\psi_R)$$

It is not clear how to define mass terms and at the same time keep the Lagrangian inv.:

- non-zero masses for gauge bosons
- different masses for fermions in the same g. multiplet
- non-zero masses for twisted chiral fermions

It is known from experiments that realistic theory has to include twisted chiral fermions with non-zero mass.

Solution: Higgs field!