Fermions and the Dirac Operator

And Why We Care

TA Russell
Department of Physics
Texas Tech University

April 2, 2025

Overview

1. Fermions and Bosons in Physics
2. The Pseudo-Orthogonal Group $O(s, t)$ of Indefinite Scalar Products
3. Clifford Algebras
4. More on Exterior Aglebras
5. Structure Theorems
6. Spinors
7. Majorana and Weyl spinors
8. Spin Structures and Spinor Bundles
9. The Dirac Operator

Fermions and Bosons

In physics, fermions are a certain class of particles. The classic example is an electron. Another class of particles is called bosons. The classic example here is photons. Fermions make up *ordinary matter* while bosons are typically characterized as *force carriers*.

The so called *statistics of a particle* refers to how identical particles are distributed over the accessible energy states of a system.To this day, there are only two observed types of statistics: Fermi-Dirac statistics and Bose-Einstein statistics. This is where the names fermion and boson originate from.

Fermions and Bosons

Another well-known characterization of particles is their spin: a particle can have either integer spin (e.g., $0, 1, \ldots$) or odd half-integer spin (e.g., $1/2, 3/2, \ldots$). Spin has no classical analog.

Theorem

The Spin-Statistics Theorem. *A particle is a boson if and only if it has integer spin. A particle is a fermion if and only if it has odd half-integer spin.*

Fermions and Bosons

An important consequence of this is related to how the respective wave functions of bosons and fermions behave.

Theorem

The wave function for a system of bosons is symmetric, while the wave function for a system of fermions is antisymmetric.

Example

Consider two-particle systems $|\,\Psi_B(x,y)\,\rangle, |\,\Psi_F(x,y)\,\rangle$ of bosons and fermions, respectively. Then

$$|\,\Psi_B(y,x)\,\rangle = |\,\Psi_B(x,y)\,\rangle,$$
$$|\,\Psi_F(y,x)\,\rangle = -|\,\Psi_F(x,y)\,\rangle.$$

Definition

A symmetric bilinear form is a map

$$Q : V \times V \to \mathbb{K}$$

that is symmetric in its argument. It is called non-degenerate if for each $v \neq 0 \in V$ there exists a vector $w \in V$ such that $Q(v, w) \neq 0$.

Example

1. Denote by $\mathbb{R}^{s,t}$ the vector space $\mathbb{R}^{s+t}$ with standard basis $e_1, \ldots e_{s+t}$ and the standard symmetric bilinear for η defined by

$$\eta(e_i, e_i) = 1 \quad \forall 1 \leq is$$
$$\eta(e_i, e_i) = -1 \quad \forall s + 1 \leq i \leq t$$
$$\eta(e_i, e_j) = 0 \quad \forall i \neq j.$$

The signature of this space is (s, t), and we call $\mathbb{R}^{s,t}$ the real vector space of signature (s, t).

Example

Minkowski spacetime is the real vector space $\mathbb{R}^{s,1}$ (or equivalently $\mathbb{R}^{1,t}$; both are common in the literature).

Example

2. Consider $\mathbb{C}^d$. The non-degenerate *standard* symmetric complex bilinear form q, given on the standard basis $\{e_i\}_{1\leq i\leq d}$, is

$$\begin{aligned} q(e_i, e_i) &= +1 \quad \forall 1 \leq i \leq d, \\ q(e_i, e_j) &= 0 \qquad \forall i \neq j. \end{aligned}$$

In particular, every non-degenerate symmetric bilinear form on an $\mathbb{R}$-vector space V has a well-defined signature (s,t).

The Pseudo-Orthogonal Group $O(s, t)$ of Indefinite Scalar Products

Theorem

Every non-degenerate symmetric bilinear form Q on an $\mathbb{R}$- or $\mathbb{C}$-vector space V is isomorphic to one of those found in examples 1 and 2. In particular, every non-degenerate symmetric bilinear form on an $\mathbb{R}$-vector space V has a well-defined signature.

Note: For a $\mathbb{C}$-vector space, we *cannot* have a well-defined signature; consider multiplication by i.

The Pseudo-Orthogonal Group $O(s, t)$ of Indefinite Scalar Products

Definition

Let V be a $\mathbb{K}$-vector space with a non-degenerate symmetric bilinear form Q. Then the psuedo-orthogonal group of (V, Q) is defined as the automorphism group of Q. That is,

$$O(V, Q) = \{f \in GL(V) \mid Q(fv, fw) = Q(v, w) \quad \forall v, w \in V\}.$$

Example

In particular, the groups $O(1, t)$, $O(s, 1)$ are called Lorentz groups, where:

$$O(s, t) = \{A \in GL(V) \mid \eta(Av, Aw) = \eta(v, w) \quad \forall, v, w \in \mathbb{R}^{s+t}\}.$$

Some properties of pseudo-orthogonal groups:

- ▷ The group $O(s,t)$ is a lineaer Lie group.
- ▷ If both $s, t \neq 0$, then $O(s,t)$ is not compact.
- ▷ Let $\mathfrak{o}(s,t)$ be the associated Lie algebra of $O(s,t)$. Then

$$\mathfrak{o}(s,t) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{o}(s+t) \otimes_{\mathbb{R}} \mathbb{C}.$$

The Pseudo-Orthogonal Group $O(s,t)$ of Indefinite Scalar Products

Example

- $SO(s,t) = \{A \in O(s,t) \mid \det A = 1\}$ is called a proper or special pseudo-orthogonal group.
- $O^+(s,t) = \{A \in O(s,t) \mid A \text{ has time-orientability } +1\}$ is called a orthochronous pseudo-orthogonal group.
- $SO^+(s,t) = SO(s,t) \cap O^+(s,t)$ is called a proper orthochronous pseudo-orthogonal group.

One last note before moving on: The subgroup $SO^+(s,t)$ is the connected component of the identity in $O(s,t)$.

Definition

Let V be a $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) equipped with a symmetric bilinear form Q. A Clifford algebra of (V, Q) is a pair $(Cl(V, Q), \gamma)$ where

- $CL(V, Q)$ is an associative $\mathbb{K}$-algebra with unit element 1.
- $\gamma : V \to CL(V, Q)$ is a linear map with

$$\{\gamma(v), \gamma(w)\} = -2Q(v, w) \cdot 1 \quad \forall v, w \in V.$$

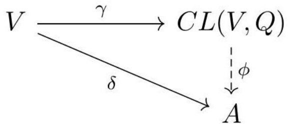
Definition

(Con't)

- If A is another associative $\mathbb{K}$-algebra with unit element 1 and $\delta : V \to A$ is a $\mathbb{K}$-linear map with

$$\{\delta(v), \delta(w)\} = -2Q(v, w) \cdot 1 \quad \forall v, w \in V$$

then there exists a unique algebra homomorphism $\phi : CL(VQ) \to A$ such that the the diagram below commumtes.



Note: Such structures actually define a category, which I will denote $\underline{\mathbb{K}\mathtt{Alg}(Q)}$. Taking the obvious choice of objects and morphisms, we can instead define a Clifford algebra as the initial object in $\underline{\mathbb{K}\mathtt{Alg}(Q)}$.

Morally speaking we can view the linear map γ as a linear square root of the symmetric bilinear form $-Q$. It suffices to demand that

$$\gamma(v)^2 = -Q(v, v) \cdot 1.$$

We claim that Clifford algebras exist and in fact are unique up to isomorphism. Uniqueness is trivial. Existence on the other hand requires some elbow grease.

Theorem

For every finite-dimensional $\mathbb{K}$-vector space V with a symmetric bilinear form Q, there exists a Clifford algebra $CL((V, Q), \gamma)$.

Proof.

(Sketch) Consider the tensor algebra of a $\mathbb{K}$-vector space V, which we denote $T(V)$. Let $J(Q)$ denote the two-sided ideal in $T(V)$ generated by

$$\{v \otimes v + Q(v, v) \cdot 1 \mid v \in V\}.$$

Define:

$$CL(V, Q) := T(V)/J(Q).$$

$\square$

Clifford Algebras

Some basic corollaries are as follows:

Theorem

The image of the vector space V under γ generates $CL(V, Q)$ multiplicatively.

Theorem

If $Q = 0$ then there exists an algebra isomorphism from $(CL(V, 0), \cdot) \simeq (\Lambda^* V, \wedge)$

where γ is given by the standard embedding of V into $\Lambda^* V$.

Theorem

Suppose that $\dim_{\mathbb{K}} V = n$ with orthonormal basis $\{e_i\}$ for (V, Q). Then the elements $\{\gamma(e_{i1}) \cdot \cdots \cdot \gamma(e_{ik})$ span $Cl(V, Q)$ as a vector space. In particular, this implies

$$\dim_{\mathbb{K}} Cl(V, Q) \leq 2^n.$$

More on Exterior Algebras

Theorem

There exists a canonical isomorphism of vector spaces

$$\Lambda^* V \to Cl(V, Q)$$

given by

$$e_{i1} \wedge \ldots \wedge e_{ik} \mapsto \gamma(e_{i1}) \cdots \gamma(e_{ik}).$$

In particular, we have

$$\dim_{\mathbb{K}} Cl(V, Q) = 2^n.$$

Theorem

Let $(Cl(V, Q), \gamma)$ be a Clifford algebra. Then the linear map $\gamma : V \to Cl(V, Q)$ is injective.

Recall that the usual tensor algebra can be decomposed as the sum of its even and odd parts; we can piggyback off of this and realize Clifford algebras in the same way. That is to say, we have

$$Cl(V, Q) = Cl^0(V, Q) \oplus Cl^1(V, Q).$$

where the first term is the even part and the second is the odd part. With this comes the structure of a $\mathbb{Z}/2$-graded associative algebra. In particular, we can view the even part as being a subalgebra of $Cl(V, Q)$.

Structure Theorems

Theorem

There exists an isomorphism of complex associative algebras

$$\mathbb{C}l(s+t) \simeq Cl(s,t) \otimes_{\mathbb{R}} \mathbb{C}.$$

That is, complex representations of $Cl(s,t)$ are equivalent to complex representations of $\mathbb{C}l(s,t)$.

Structure Theorems

These last two slides lead to some structure theorems, as one would hope for. These characterize the Clifford algebras depending on $n = s + t$ (for the complex case) and $n = s + t$ and $\rho = s - t$ (for the real case). More details can be found in Hamilton (Tables 6.1, 6.2, 6.3, and 6.4). In particular, the dimension of V depends on n.

Spinors

Definition

The vector space of Dirac spinors is $\triangle_n := \mathbb{C}^N$. The corresponding Dirac spinor representation of the complex Clifford algebra is:

$$\rho : \mathbb{C}l(n) \to \mathrm{End}\triangle_n.$$

Definition

The bilinear map

$$\mathbb{R}^{w,t} \times \triangle_n \to \triangle_n$$
$$(X, \psi) \mapsto X \cdot \psi = \rho(\gamma(X))\psi$$

is called the mathematical Clifford multiplication of a spinor with a vector. The physical Clifford multiplication is simply the mathematical one but multiplied by $-i$.

Spinors

Theorem

Consider the restriction of the Dirac spinor representation to the even subalgebra $\mathbb{C}l^0(n)$. If n is odd, the induced representation is irreducible; if n is even, the induced representation can be decomposed as:

$$\mathbb{C}l^0(n) \simeq End(\triangle_n^+) \oplus End(\triangle_n^-).$$

Spinors

Definition

The group of invertible elements in $Cl(s,t)$ is defined by

$$Cl^\times(s,t) = \{x \in Cl(s,t) \mid \exists y \in Cl(s,t) : xy = yx = 1\}.$$

Theorem

$Cl^\times(s,t)$ is an open subset of $Cl(s,t)$ and is therefore a Lie group.

Spinors

Some notation:

$$
\begin{aligned}
S^{s,t}_{+} &:= \{v \in \mathbb{R}^{s,t} \mid \eta(v,v) = +1\} \\
S^{s,t}_{-} &:= \{v \in \mathbb{R}^{s,t} \mid \eta(v,v) = -1\} \\
S^{s,t}_{\pm} &:= S^{s,t}_{+} \cup S^{s,t}_{-}.
\end{aligned}
$$

Spinors

Theorem

The following subsets of $CL(s,t)$ form subgroups of $Cl^{\times}(s,t)$:

$$Pin(s,t) = \{v_1 \cdots v_r \mid v_i \in S^{s,t}_{\pm}\}, r \geq 0$$
$$Spin(s,t) = Pin(s,t) \cap Cl^0(s,t)$$
$$Spin^+(s,t) = \{v_1 \cdots v_{2p} w_1 \cdots w_{2q} \mid v_i \in S^{s,t}_{+}, w_j \in S^{s,t}_{-}, p, q \geq 0\}.$$

In the case $(s,t) = (n,0)$, we denote these instead as $Pin(n)$ etc. These subgroups inherit the subset topology from $Cl(s,t)$. These three subsets are respectively called the pin, spin, and orthochronous spin groups.

There is a well-defined map

$$R : Pin(s,t) \times \mathbb{R}^{s,t} \to \mathbb{R}^{s,t}$$
$$(u,x) \mapsto (-1)^{deg(u)} uxu^{-1}.$$

This also yields a map

$$\lambda : Pin(s,t) \to O(s,t)$$
$$u \mapsto R_u := R(u,-)$$

which corresponds to reflections in the hyperplane $v^{\perp} \subset \mathbb{R}^{s,t}$. Also note that λ is a continuous homomorphism.

Theorem

(Cartan-Dieudonnè Theorem) Every element of $O(s,t)$ can be written as a composition of at most $2(s+t)$ reflections in $v_i^{\perp}$ with vectors $v_i \in S_{\pm}^{s,t}$.

Spinors

Theorem

- λ *is open and surjective with kernel* $\{\pm 1\}$.
- *The preimages of* λ *of the subgroups* $SO(s,t), SO^+(s,t)$ *are respectively equal to* $Spin(s,t), Spin^+(s,t)$ *and thus are open subgroups of* $Pin(s,t)$.
- *The restrictions of* λ

$$\begin{aligned} Spin(s,t) &\to SO(s,t) \\ Spin^+(s,t) &\to SO^+(s,t) \end{aligned}$$

  are surjective with kernel $\{\pm 1\}$.

Theorem

We can define unique Lie group structures on $Pin, Spin, Spin^+$ *so that* λ *is a smooth double covering of the Lie groups.*

For $n \geq 3$ the λ maps

$$\begin{aligned} Spin(n) &\to SO(n) \\ Spin^+(n,1) &\to SO^+(n,1) \\ Spin^+(1,n) &\to SO^+(1,n) \end{aligned}$$

are the universal coverings. This is related to the fact that the fundamental groups of $SO(n), SO^+(1,n), SO^+(n,1)$ are $\mathbb{Z}/2$.

Spinors

Now that we have a bunch of Lie groups, we of course have corresponding Lie algebras with the typical commutator $[x, y] = xy - yx$.

Definition

$$M(s,t) := span\{e_i e_j \in Cl(s,t) \mid 1 \leq i < j \leq s+t\}$$

M is a vector space, and in particular a Lie subalgebra of $\mathfrak{cl}^\times(s,t)$ with dimension

$$\frac{1}{2}(s+t)(s+t-1).$$

In fact:

$$\mathfrak{spin}^+(s,t) = M(s,t).$$

Spinors

We end this section with a nice result

Theorem

The differential of the map

$$\lambda : Spin^+(s,t) \to SO^+(s,t)$$

is given by

$$\lambda_* : \mathfrak{spin}^+(s,t) \to \mathfrak{so}^+(s,t)$$
$$\lambda_*(z)x = [z,x] = zx - xz$$

for all $s \in \mathbb{R}^{s,t}$. *For any* $z \in \mathfrak{spin}^+(s,t)$, *we can recover* z *from its image* $\lambda_*(z)$ *via*

$$z = \frac{1}{2}\sum_{k<l} \eta(\lambda_*(z)e_k, e_l)\eta_k\eta_l e_k e_l.$$

Majorana and Weyl spinors

Recall for a Dirac spinor with even n:

$$\mathbb{C}l^0(n) \simeq \mathrm{End}(\triangle_n^+) \oplus \mathrm{End}(\triangle_n^-).$$

The left-hand side is called a left-handed Weyl spinor, and similarly the right-hand side is called a right-handed Weyl spinor.
If a spinor representation $\triangle$ admits a real structure σ, we call it a Majorana spinor.
There are also symplectic versions of Majorana spinors.
If σ commutes with the projection map of a Dirac spinor onto (for instance) its left-handed Weyl spinor component, we call $\triangle$ a Majorana-Weyl spinor. There are a variety of different ways to view Majorana spinors (e.g., Majorana but not Weyl, Majorana and Weyl but not Majorana-Weyl, etc.); see Table 6.6.

Majorana and Weyl spinors

By themselves, Weyl spinors are not sufficient to describe massive particles, such as electrons. But nothing is lost; considering both left-handed and right-handed components simultaneously, we can handle massive particles.

With the possible exception of neutrinos, none of the particles in the standard model of particle physics are represented by Majorana spinors. However, such spinors do appear in supersymmetric theories. But, who knows if supersymmetry is a real thing. So then, are Majorana particles worth studying (physically speaking)?

Yes! In the context of condensed matter, there are quasiparticles that have been observed to behave as Majorana particles.

Majorana and Weyl spinors

Take a Dirac spinor Ψ. We can then write it as

$$\Psi = (u_+, u_-)$$

where u_+ and u_- are left- and right-handed Weyl spinors. What about Majorana spinors? Given this above decomposition, we can get ourselves a Majorana spinor by taking:

$$\Psi_M = (u_+, -i\sigma^2(u_+)^*).$$

Definition

Let M be a smooth manifold. A pseudo-Riemannian metric g of signature (s, t) is a section $g \in \Gamma(T^*M \otimes T^*M)$ that defines at each point $x \in M$ a non-degenerate, symmetric bilinear form

$$g_x : T_xM \times T_xM \to \mathbb{R}$$

of signature (s, t).

Here, Γ is in reference to what is called a chirality operator; for sake of brevity, this is not discussed much here. Morally speaking, Γ refers to the handedness of Weyl spinors. In $Cl(1,3)$ there are so called $\gamma = i\Gamma$ matrices that generate the representation $Cl(1,3)$ via certain anti-commutation relationships. In particular, there are four γ^i, $1 = 0, 1, 2, 3$ we start with, and we can construct $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$.

Recall that the frame bundle has the structure of a principal $O(s,t)$-bundle.

Definition

- (M,g) is called orientable if the frame bundle can be reduced to a principal $SO(s,t)$-bundle under the embedding $SO(s,t) \subset O(s,t)$.
- (M,g) is called time-orientable if the frame bundle can be reduced to a principal $O^+(s,t)$-bundle under the embedding $O^+(s,t) \subset O(s,t)$.
- (M,g) is called orientable and time-orientable if the frame bundle can be reduced to a principal $SO^+(s,t)$-bundle under the embedding $SO^+(s,t) \subset O(s,t)$.

An orientation of M is just an orientation of TM. Time-orientation is a bit trickier. TM admits maximally g-positive definite vector subbundles $W \to M$, of which any two are homotopic. A time-orientation then is a choice of orientation on W.

Spin Structures and Spinor Bundles

Suppose (M, g) is oriented and time-oriented. Denote the $SO^+(s,t)$-frame bundle by

$$\pi_{SO} : SO^+(M) \to M$$

and recall the double covering $\lambda : Spin^+(s,t) \to SO^+(s,t)$.

Definition

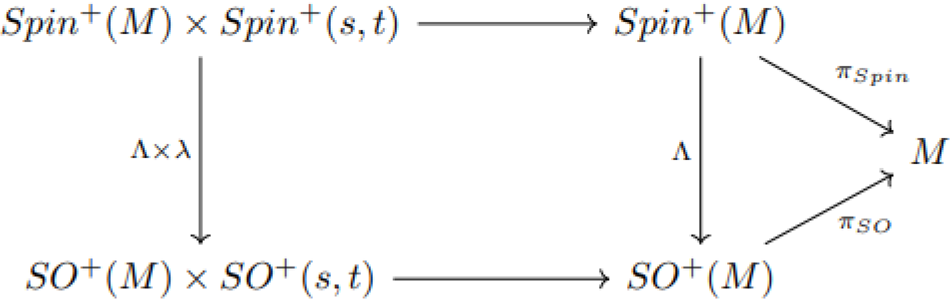
A spin structure on M is a $Spin^+(s,t)$-principal bundle

$$\pi_{Spin} : Spin^+(M) \to M$$

with a double covering

$$\Lambda : Spin^+(M) \to SO^+(M)$$

such that the diagram (next slide) commutes.


$Spin^+(M) \times Spin^+(s,t)$
$Spin^+(M)$
$\Lambda \times \lambda$
Λ
π_{Spin}
M
π_{SO}
$SO^+(M) \times SO^+(s,t)$
$SO^+(M)$

Spin Structures and Spinor Bundles

Definition

Two spin structures

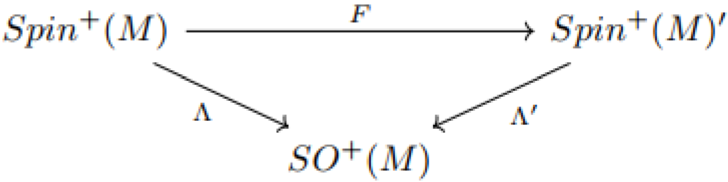
$$\Lambda : Spin^{+}(M) \to SO^{+}(M)$$
$$\Lambda' : Spin^{+}(M)' \to SO^{+}(M)$$

are called isomorphic if there exists a $Spin^{+}(s,t)$-equivariant bundle isomorphism

$$F : Spin^{+}(M) \to Spin^{+}(M)'$$

such that the diagram (next slide) commutes.

Spin Structures and Spinor Bundles



Spin Structures and Spinor Bundles

Theorem

(Existence and Uniqueness of Spin Structures)

- *The frame bundle* $SO^+(M)$ *admits a spin structure if and only if the second Stiefel-Whitney class of* M *vanishes, i.e.,* $w_2(M) = 0$.
- *If* $SO^+(M)$ *admits a spin structure, then there is a bijection between the set of isomorphism classes of spin structures on* M *and the cohomology group* $H^1(M; \mathbb{Z}/2)$.

Spin Structures and Spinor Bundles

Definition

Let $Spin^+(M) \to M$ be a spin structure on M and

$$\kappa : Spin^+(s,t) \to GL(\triangle)$$

the spinor representation. Then the (Dirac) spinor bundle is the associated complex vector bundle

$$S = Spin^+(M) \times_\kappa \triangle$$

over M. Sections of S are called spinor fields or spinors. Note that the spinor bundle may depend on the choice of spin structure.

Theorem

Let $S \to M$ be the spinor bundle associated to a spin structure.

- *There exists a well-defined bilinear Clifford multiplication*

$$
\begin{aligned}
TM \times S &\to S \\
(X, \psi) &\mapsto X \cdot \psi
\end{aligned}
$$

  on the level of bundles, restricting to a map $T_pM \times S_p$ in every point $p \in M$. This map also induces a well-defined Clifford multiplication of forms with spinors.
- *If the dimension n of M is even, then S splits as a direct sum of complex Weyl spinor bundles $S = S_+ \oplus S_-$, where $S_\pm = Spin^+(M) \times \kappa \triangle^\pm$.*

Definition

Let $\triangle = \triangle_n$ be the complex spinor representation of $Cl(s,t)$. Fix a constant $\delta = \pm 1$ and consider non-degenerate $\mathbb{R}$-bilinear forms

$$\langle -, - \rangle : \triangle \times \triangle \to \mathbb{C}$$

with the following properties:

- $\langle X \cdot \psi, \phi \rangle = \delta \langle \psi, X \cdot \phi \rangle$
- $\langle \psi, \phi \rangle = \langle \phi, \psi \rangle^*$
- $\langle \psi, c\phi \rangle = c\langle \psi, \phi \rangle =^* \psi, \phi \rangle$

for all $X \in \mathbb{R}^{s,t}$, $\psi, \phi \in \triangle$, $c \in \mathbb{C}$.
Such a form is called a Dirac form.

For this last section, we play a bit fast and loose. Mostly we are concerned about getting the idea across, and sweep the technical aspects under the rug.

Definition

The Dirac operator $D : \Gamma(S) \to \Gamma(S)$ on the spinor bundle S is defined by:

$$D\psi = \eta^{ab} e_a \cdot \nabla_{e_b} \psi.$$

Choosing a local section $e = (e_1, \ldots, e_n)$ of $SO^+(M)$ (called a vielbein), we can write more explicitly

$$D\psi = i\Gamma^a \left(d\psi(e_a) - \frac{1}{4}\omega_{abc}\Gamma^{bc}\psi \right).$$

As one would expect, this definition is independent of our choice of local vielbein. This can be seen by the composition of maps

$$\Gamma(S) \xrightarrow{\nabla} \Gamma(T^* M \otimes S) \xrightarrow{\eta} \Gamma(TM \otimes S) \xrightarrow{\gamma} \Gamma(S)$$

The Dirac Operator

Theorem

If the dimension n of the manifold M is even, then the Dirac operator D maps

$$D : \Gamma(S_{\pm}) \to \Gamma(S_{\mp}).$$

That is, D takes sections of S_+ to sections of S_- and vice versa.

Theorem

(Dirac Operator is Formally Self-Adjoint) Let M be a manifold without boundary. If the Dirac form satisfies $\delta = -1$, then the Dirac operator $D : \Gamma_0(S) \to \Gamma_0(S)$ is formally self-adjoint, i.e.,

$$\langle D\phi, \psi\rangle_{S,L^2} = \langle \phi, D\psi\rangle_{S,L^2}$$

for all spinors $\phi, \psi \in \Gamma_0(S)$.

Finally, we arrive at the Dirac Lagrangian. Fix the following data:

- ▷ an n-dimensional oriented and time-oriented pseudo-Riemannian spin manifold (M, g) of signature (s, t)
- ▷ a spin structure $Spin^+(M)$ together with a complex spinor bundle $S \to M$
- ▷ a Dirac form $\langle -, - \rangle$ (not necessarily positive definite) on the Dirac spinor space $\triangle = \triangle_n$ with associated Dirac bundle metric $\langle -, - \rangle_S$

Definition

The Dirac Lagrangian for a (free) spinor field $\psi \in \Gamma(S)$ of mass m is defined by

$$L_D[\psi] = \Re\langle\psi, D\psi\rangle_S - m\langle\psi, \psi\rangle_S.$$

The term $\Re\langle\psi, D\psi\rangle_S$ is called the kinetic term and $m\langle\psi, \psi\rangle_S$ is called the Dirac mass term.

The Lagrangian should be real, so taking the real part (in the kinetic term) is necessary.