Quantum Homotopy Seminar

February 12, 2025

Differential forms, Lie groups, Lie algebras

Three points of view on differential forms:

(1) Traditional/classical: a differential k-form on $M \in \mathrm{Man}$ is a smooth section of $\Lambda^k T^* M$.

This means:

$$\begin{tikzcd} M \arrow[r, "\omega"] \arrow[dr, bend right, "\mathrm{id}_M"'] & \Lambda^k T^* M \arrow[d, "\pi"] \\ & M \end{tikzcd} \qquad \omega \in \Omega^k M = \underbrace{\Gamma(\Lambda^k T^* M)}_{C^\infty M\text{-module}}$$

That is, for every $m \in M$ we have

$$\omega_m : \overbrace{T_m M \times \cdots \times T_m M}^{k \text{ factors}} \longrightarrow \mathbb{R}$$

antisymmetric multilinear map

Alternatively: a differential k-form is

$$\omega : \overbrace{\mathfrak{X} M \times \cdots \times \mathfrak{X} M}^{k} \longrightarrow C^\infty M$$

antisymmetric $C^\infty M$-multilinear map

Operations on differential forms: wedge product, de Rham differential

$$\Omega^k M \times \Omega^\ell M \xrightarrow{\wedge} \Omega^{k+\ell} M$$

$C^\infty M$-bilinear

$$\omega_1, \omega_2 \longmapsto \omega_1 \wedge \omega_2$$

(☆) $$(\omega_1 \wedge \omega_2)(X_1, \ldots, X_{k+\ell}) = \sum_{\sigma \in S_{k+\ell}} \frac{(-1)^\sigma}{k!\,\ell!} \omega_1(X_{\sigma(1)} \ldots X_{\sigma(k)}) \cdot \omega_2(X_{\sigma(k+1)}, \ldots X_{\sigma(k+\ell)})$$

$\sigma_1 < \sigma_2 < \cdots < \sigma_\ell$
$\sigma_{\ell+1} < \cdots < \sigma_{k+\ell}$

$$= \sum_{\sigma \in Sh(k,\ell)} \omega_1(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\, \omega_2(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+\ell)})$$

Observation: For $k=0$: $\omega_1 \wedge \omega_2 = \omega_1 \cdot \omega_2$

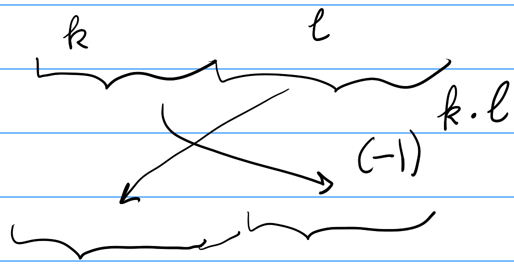
$\omega_1 \in \Omega^0 M \cong C^\infty M$ $\qquad C^\infty M \times \Omega^\ell M \to \Omega^\ell M$

Properties $\wedge$ is $C^\infty M$-bilinear

$(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$

graded commutative: $\omega_2 \wedge \omega_1 = \underbrace{(-1)^{|\omega_1| \cdot |\omega_2|}}_{\text{sign of permutation}} \omega_1 \wedge \omega_2$

Koszul sign



De Rham differential $\Omega^k M \xrightarrow{d} \Omega^{k+1} M$

$\omega \longmapsto d\omega$

$(d\omega)(X_0, \ldots, X_k)$

(★★) $$= \sum_i (-1)^i \mathcal{L}_{X_i} \left(\omega(X_0, \ldots, \widehat{X_i}, \ldots, X_k)\right)$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots)$$

Example $k=0$: $\Omega^0 M = C^\infty M \xrightarrow{d} \Omega^1 M$

$\omega \in C^\infty M$

$(d\omega)(X) = \mathcal{L}_X \omega$ (multivariable calculus).

$$= \sum_\ell X_\ell \frac{\partial \omega}{\partial x_\ell} \quad \rightsquigarrow \text{differential}$$

$$d\omega = \sum_\ell \frac{\partial \omega}{\partial x_\ell} dx_\ell$$

$\underline{Ex}$ $\underline{k=1}$

$$\Omega^1 M \xrightarrow{d} \Omega^2 M$$
$$\omega \longmapsto d\omega$$

$$(d\omega)(X_0, X_1) = \underbrace{\mathcal{L}_{X_0}}\underbrace{(\omega(X_1))}_{\in C^\infty M} - \mathcal{L}_{X_1}\underbrace{(\omega(X_0))}_{\in C^\infty M} - \omega(\underbrace{[X_0, X_1]}_{\in \mathfrak{X} M})$$

② The algebraic point of view on differential forms

~ Dubuc, Kock, 1982

Theorem The $\overset{\text{graded}}{\text{commutative}}$ $\overset{d}{\text{differential}}$ $\overset{\Omega^k M}{\text{graded}}$ C^∞-algebra ΩM is the free CDGC$^\infty$A on the C^∞-algebra $C^\infty M$. (placed in degree 0)

Corollary Suppose M is a "smooth space", (not necessarily a manifold) A is the C^∞-algebra of smooth functions on M. Then we can define the algebra ΩM of differential forms on M as $\mathrm{Free}(A)$, where Free is the functor defined in the theorem.

Remark: In a C^∞ CDGA A $d: A_0 \xrightarrow{\in \mathrm{CAlg}_{\mathbb{R}}} A_1$ is a C^∞-derivation:
for every smooth function $f \in C^\infty \mathbb{R}^n$
and every n-tuple $a_1, \ldots, a_n \in A_0$
we have

$$d(\underbrace{f}(a_1, \ldots, a_n)) = \sum_\ell \frac{\partial f}{\partial x_\ell}(a_1, \ldots, a_n) \cdot da_\ell$$

makes sense
if f is a polynomial
Replacing polynomials with smooth functions
yields C^∞-rings / C^∞-algebras

In our case: $A_0 = C^\infty M$

__Remark__ Unfolding the free functor in the theorem, a differential k-form on M is a finite sum of terms of the form

$$f \wedge dg_1 \wedge dg_2 \wedge \cdots \wedge dg_k,$$

where $f, g_1, \ldots, g_k \in C^\infty M \cong \Omega^0 M$.

(subject to the relations of a C^∞DGA: CDGA + chain rule: $d(\overline{h(g_1,\ldots,g_k)}) = \sum_i \frac{\partial h}{\partial x_i}(g_1,\ldots,g_k)\cdot dg_i$. $h \in C^\infty(\mathbb{R}^k)$)

① Dubuc — Kock On 1-form classifiers, 1984

C^∞-rings: ② Moerdijk — Reyes Models for smooth infinitesimal analysis.

③ Carchedi — Roytenberg: On theories of superalgebras of differentiable functions.

__Poincaré lemma__ $M = \mathbb{R}^n$

__Tyler__: Fermions, Dirac, etc. (3 weeks +)

The canonical homomorphism $\mathbb{R}_0 \longrightarrow \Omega M$

__Jacek__ Higgs (after Tyler)

is a quasi-isomorphism.

__Proof__: $\left\{ \begin{array}{l} \Omega M = \mathrm{Free}(C^\infty M) \\ C^\infty M = \underbrace{C^\infty(\mathbb{R}^n)}\ \text{is a free } C^\infty\text{-ring on } n \text{ generators.} \end{array} \right.$

$\Rightarrow \underbrace{\Omega M}$ is the free C^∞CDGA on n elements.

$\Omega\mathbb{R}^n$
$= \mathrm{Free}(\mathbb{R}^n \xleftarrow{id} \mathbb{R}^n)$
$\simeq \mathrm{Free}(0)$
$\cong \mathbb{R}[0]$

$$\begin{tikzcd}
\Omega\mathbb{R}^n \in C^\infty\mathrm{CDGA} \arrow[d, shift right] \arrow[rr, "\perp", "\text{Forget } A \mapsto A_0"'] & & C^\infty\mathrm{Ring} \ni C^\infty((\mathbb{R}^n)^*) \arrow[ll, bend right, "\mathrm{Free}"'] \arrow[d, shift right] \\
\mathrm{Ch}_\mathbb{R} \ni \underbrace{(\mathbb{R}^n \xleftarrow{id} \mathbb{R}^n)} \arrow[u, shift right, "\mathrm{Free}"'] \arrow[u, phantom, "\dashv"] \arrow[rr, "\perp", "C \mapsto C_0"'] & & \mathrm{Vect}_\mathbb{R} \ni \mathbb{R}^n \arrow[ll, bend left, "(V \xleftarrow{id} V) \leftarrow\!\shortmid V"'] \arrow[u, shift right, "V \mapsto C^\infty(V^*)"'] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

contractible!

$(\mathbb{R}^n \xleftarrow{id} \mathbb{R}^n)$
$\simeq 0$

dmitripavlov.org/homotopy (Notes)

2025-2-19: Cartan calculus, Lie groups

Graded derivations on ΩM:

a) $d : \Omega^k M \longrightarrow \Omega^{k+1} M$ degree 1

b) $\iota_X : \Omega^{k+1} M \longrightarrow \Omega^k M$ degree -1 $X \in \mathfrak{X} M$

c) $\mathcal{L}_X : \Omega^k M \longrightarrow \Omega^k M$ degree 0 $X \in \mathfrak{X} M$

Theorem (Élie Cartan, ...) The graded Lie algebra of natural graded derivations of Ω, admits the following system of generators and relations
generators: d, ι_X, $\mathcal{L}_X$ $(X \in \mathfrak{X} M)$
relations: Cartan calculus (6 relations)

naturality: with respect to restrictions to open subset
$U \underset{\text{open}}{\subset} M$

$$\begin{tikzcd} \Omega M \arrow[r, "D"] \arrow[d, "\text{restrict}"'] \arrow[dr, phantom, "\#"] & \Omega M \arrow[d, "\text{restrict}"] \\ \Omega U \arrow[r, "D"'] & \Omega U \end{tikzcd}$$

Notation: $D_1 \in \mathrm{Der}^k(\Omega M, \Omega M)$, $D_2 \in \mathrm{Der}^\ell(\Omega M, \Omega M)$
graded commutator

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{k \cdot \ell} D_2 \circ D_1$$

$[D_1, D_2] \in \mathrm{Der}^{k+\ell}(\Omega M, \Omega M)$.

If k and ℓ are odd: $[D_1, D_2] = D_1 \circ D_2 + D_2 \circ D_1$.

Cartan calculus: $[\iota_X, \iota_Y] \overset{①}{=} 0$ $[d, d] \overset{②}{=} 0$ $[d, \mathcal{L}_X] \overset{③}{=} 0$

(differential forms are antisymmetric) "$2d^2$" (d commutes with f^*: $f^* d\omega = d f^* \omega$)

$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$ (by def) ④

$[\mathcal{L}_X, \iota_Y] \overset{⑤}{=} \iota_{[X,Y]}$ (follows from the formula for $\mathcal{L}_X$)

$[d, \iota_X] \overset{⑥}{=} \mathcal{L}_X$ Cartan's magic formula

$$(\mathcal{L}_X \omega)(Y_1, \dots, Y_k) = \underbrace{\mathcal{L}_X\big(\omega(Y_1, \dots, Y_k)\big)} - \sum_i \omega(Y_1, \dots, [X, Y_i], \dots, Y_k)$$

A quick way to prove ① – ⑥:
a graded ~~derivation~~ $\mathcal{D}$ of degree p satisfies $p \cdot k$

$$\mathcal{D}(\omega_1 \wedge \omega_2) = \mathcal{D}\omega_1 \wedge \omega_2 + (-1)^{|\mathcal{D}| \cdot |\omega_1|} \omega_1 \wedge \mathcal{D}\omega_2 .$$

Thus, it suffices to prove ① - ⑥ for $\omega \in \Omega^0 M = C^\infty M$ and $\omega = df$, $f \in \Omega^0 M = C^\infty M$.

Lie groups and Lie algebras

Idea: study Lie groups G by studying left-invariant differential geometric on G

Left-invariant: If $g \in G$, $\ell_g : G \to G$, $h \mapsto g \cdot h$ left action.

M: manifold	$M = G$ Lie group + left-invariance
	$h = e \in G$
① $\underline{f \in C^\infty M}$	$f : G \to \mathbb{R}$ $f(g \cdot h) = f(h)$ $\underline{f \in \mathbb{R}}$
	(Fraktur $\mathfrak{g}$)
② $X \in \mathfrak{X} M$	$X \in \mathfrak{g}$ the Lie algebra of G $(\mathfrak{X} G)^G \xrightarrow[ev_e]{\cong} T_e G$ $(g \mapsto (T\ell_g)(v)) \leftarrow\!\shortmid v$ left invariant vector field
$X, Y \in \mathfrak{X} M$ ③ $[X, Y] \in \mathfrak{X} M$	$X, Y \in (\mathfrak{X} G)^G$ $[X, Y] \in (\mathfrak{X} G)^G$ $\mathfrak{g} \otimes_{\mathbb{R}} \mathfrak{g} \xrightarrow{[-,-]} \mathfrak{g}$ Lie bracket
④ $X \in \mathfrak{X} M$ $f \in C^\infty M$ $\mathcal{L}_X f = Xf \in C^\infty M$	$X \in (\mathfrak{X} G)^G$ $f \in (C^\infty G)^G \cong \mathbb{R}$ $\mathcal{L}_X f = Xf = 0 \in (C^\infty G)^G$

(5) $\omega \in \Omega^k M$

$\omega \in (\Omega^k G)^G \xrightarrow[\mathrm{ev}_e]{\cong} \Lambda^k T_e^* G = \Lambda^k \mathfrak{g}^*$

(6) $\omega_1 \wedge \omega_2$

$\omega_1 \wedge \omega_2 \quad \Lambda^k \mathfrak{g}^* \otimes_{\mathbb{R}} \Lambda^l \mathfrak{g}^* \longrightarrow \Lambda^{k+l} \mathfrak{g}^*$.

(7) $\omega \in \Omega^k M \quad d\omega \in \Omega^{k+1} M$

$$(d\omega)(X_0, \dots, X_k) = \sum_i (-1)^i \mathcal{L}_{X_i} \underbrace{\omega(X_0, \dots, \hat{X}_i, \dots, X_k)}_{\in C^\infty M} + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k)$$

$\omega \in (\Omega^k G)^G \cong \Lambda^k \mathfrak{g}^* \quad d\omega \in (\Omega^{k+1} G)^G \cong \Lambda^{k+1} \mathfrak{g}^*$

$$(d\omega)(X_0, \dots, X_k) = \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k)$$

$(\Lambda \mathfrak{g}^*, \wedge, d) \in CDGA_{\mathbb{R}}$ is the Chevalley – Eilenberg algebra of $\mathfrak{g}$

(8) $H^k M = \Omega^k_{cl} / \Omega^k_{ex}$

$\Omega^k_{cl} = \{\omega \in \Omega^k \mid d\omega = 0\}$, $\quad \Omega^k_{ex} = \{d\psi \mid \psi \in \Omega^{k-1} M\}$

de Rham cohomology of M

$H^k \mathfrak{g} = H^k(\Lambda \mathfrak{g}^*, d)$

(Chevalley-Eilenberg) cohomology of $\mathfrak{g}$

(9) Cartan calculus for manifolds

(9) Cartan calculus for Lie algebras

(10) Differential operators $\mathrm{Diff}^{\le k} M$

$(\mathrm{Diff}^{\le k} G)^G \cong (U^{\le k} \mathfrak{g})$

the universal enveloping algebra

(11) principal symbol isomorphism:

$\mathrm{Diff}^{\le (k-1)} M \to \mathrm{Diff}^{\le k} M \longrightarrow \mathrm{Sym}^k_{C^\infty M} \mathfrak{X} M = \Gamma(\mathrm{Sym}^k TM)$

(11) Poincaré - Birkhoff - Witt theorem

$U^{\le (k-1)} \mathfrak{g} \to U^{\le k} \mathfrak{g} \longrightarrow \mathrm{Sym}^k \mathfrak{g}$

Feb 26: JJ Hao: Principal G-bundles w/ gauge fields ∇