Manifold Theory

- **Def.** (preliminary)

A smooth n-manifold M is an n-dimensional surface inside $\mathbb{R}^m$. $(M \subset \mathbb{R}^m)$

This means: for every point $m \in M$ there is $\varepsilon > 0$ and a diffeomorphism

$\varphi : B(m, \varepsilon) = x \in \mathbb{R}^m / \|x - m\| < \varepsilon \} \longrightarrow \mathbb{R}^n$

s.t. $\varphi(M)$ is a vector subspace of $\mathbb{R}^m$ $\circledast$

There is $U_{\ni m} \subset \mathbb{R}^m$ open and $V \subset \mathbb{R}^m$ open diffeo $\varphi : U \to V$ such that $\varphi(U \cap M) = V \cap W$, $W \subset \mathbb{R}^m$ vector subspace.

- **Def.** Suppose $\mathcal{U} \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ are open subsets

A map $\varphi : U \to V$ is a diff if f is a bijection and φ and φ^{-1} are smooth

- **Def.** A subset $\mathcal{U} \subset \mathbb{R}^m$ is open for every point $u \in \mathcal{U}$ there is $\varepsilon > 0$ such that $B(u, \varepsilon) \subset \mathcal{U}$.

- **Ex.** a.) $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ is not a diffeomorphism

$x \mapsto x^3$. However, φ^{-1} is not smooth

$x \to x^{1/3}$

b.) $(0, \infty) \xrightarrow{\varphi} (0, \infty)$ is a diffeomorphism

$x \mapsto x^2$ ✓

$x \mapsto \sqrt{x}$.

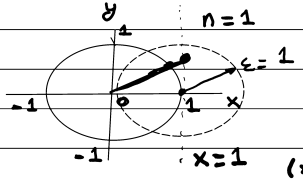
c.) $\mathbb{R}^m \xrightarrow{A} \mathbb{R}^n$ where A is a linear map

A is a diffeomorphism $\Leftrightarrow$ $m = n$ and the matrix of A is inv. (i.e. $\det A \neq 0$).

d.) The inverse function theorem: if $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth bijective map, then φ^{-1} is smooth

$\Leftrightarrow \forall x \in \mathbb{R}^m : (D\varphi)(x)$ is an inv. Linear map (matrix)

• <u>Exa.</u> $S^1 = \{(x,y) \mid x^2+y^2=1\}$



$\{(x,y) \mid x>0\}$

$\longrightarrow \{(x,y) \mid x>0\}$

$(x,y) \longmapsto t\cdot(x,y),\ t = \sqrt{\tan^2(\frac{y}{x})+1}$

Implicit function Theorem!

Suppose $f: \mathbb{R}^m \to \mathbb{R}^p$ is a smooth map and $Df(0^{\psi})$ has rank p.

Then, $\exists \varepsilon > 0 : f^{-1}(f(0)) \cap B(0,\varepsilon)$ which satisfies ✪

0^{ψ}, $0 \in \mathbb{R}^m$ s.t. $f(0)=0$.

• <u>Exa.</u> $m=2, p=1 \quad f: \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \longmapsto x^2+y^2=1$

Let $a=(x,y)$.

$$Df = \left(\frac{\partial f}{\partial x}(x,y) \quad \frac{\partial f}{\partial y}(x,y)\right)$$

$$= (2x \quad 2y)$$

$$\underset{\uparrow}{\text{rank}(Df)(x,y)} = \begin{cases} 0, & \text{if } x=y=0 \\ 1, & \text{otherwise} \end{cases}.$$

rank $=1=p$, IFT applies.

• <u>Exa.</u> $S^n = \{(x_0,\dots,x_n) \mid x_0^2+\dots+x_n^2=1\}$

is also a smooth manifold.

- **Exa.** $T^n = \{ (x_1, \cdots, x_{2n}) \mid x_1^2 + x_2^2 = 1,$
$x_3^2 + x_4^2 = 1 \dots \}$

$f : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n$

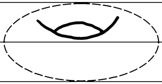
$(x_1, \dots, x_{2n}) \xmapsto{f} (x_1^2 + x_2^2 - 1, \dots, x_{2n-1}^2 + x_{2n}^2 - 1)$

$$Df(x_1, \dots, x_{2n}) = \begin{pmatrix} 2x_1 & 2x_2 & 0 & \dots & \dots & 0 \\ 0 & 0 & 2x_3 & 2x_4 & 0 \dots & 0 \\ 0 & 0 & \dots & \dots & 2x_{2n-1} & 2x_{2n} \end{pmatrix}$$

rank n iff

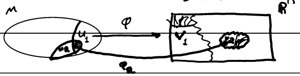
$(x_1, x_2) \neq 0$. Basically no zeroes, we have LID so rank $= n$, number of columns.

- **Exa.** $T^2 = \{ (x, y, z) \mid \left(x - \frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(y - \frac{y}{\sqrt{x^2+y^2}}\right)^2 + z^2 = \left(\frac{1}{2}\right)^2 \}$.



- **Def.** Suppose M is a set

a.) A chart on M is a bijection $\varphi : U \longrightarrow V$ where $U \subset M$ and $V \subset \mathbb{R}^n$ open.
subset



Two charts are compatible

$\varphi_1 : u_1 \longrightarrow v_1 \subset \mathbb{R}^{n_1} \wedge \varphi_2 : u_2 \longrightarrow v_2 \subset \mathbb{R}^{n_2}$

if $\varphi_1(u_1 \cap u_2) \subset \mathbb{R}^{n_1}$ is open

$\varphi_2(u_1 \cap u_2) \subset \mathbb{R}^{n_2}$ is open

and $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(u_1 \cap u_2) \longrightarrow \varphi_2(u_1 \cap u_2)$ is smooth

and $\varphi_1 \circ \varphi_2^{-1} : \varphi_2(u_1 \cap u_2) \longrightarrow \varphi_1(u_1 \cap u_2)$ is smooth.

c) An atlas is a compatible family of charts $(\varphi_i : U_i \longrightarrow V_i)_{i \in I}$ s.t. $\bigcup_{i \in I} U_i = M$.

d.) A smooth manifold is a set (M) w/ an atlas $\mathcal{A}$ $(M, \mathcal{A})$.

Notes (08/29/24)

- **Recall:**

(a) A chart on M is $\varphi : U \longrightarrow V$ (bijection)
$V \subset \mathbb{R}^n$ open, $U \subset M$

(b) $\varphi_1 \sim \varphi_2$ (compatible)
$\varphi_1 : U_1 \longrightarrow V_1$
$\varphi_2 : U_2 \longrightarrow V_2$
if $\varphi_1(U_1 \cap U_2) \subset \mathbb{R}^n$ open
if $\varphi_2(U_1 \cap U_2) \subset \mathbb{R}^n$ open
$\alpha = \varphi_2 \circ \varphi_1^{-1}$ is smooth, $\beta = \varphi_1 \circ \varphi_2^{-1}$ is smooth

(c) An atlas on M is a compatible family
$\{\varphi_i : U_i \rightarrow V_i\}_{i \in I}$

such that $\bigcup_{i \in I} U_i = M$

(d) **def.** A smooth manifold is a set M equipped with an atlas.

- **Exa.** If $U \subset \mathbb{R}^n$, then U is a smooth manifold w/ a single chart $\varphi : U \xrightarrow[\text{id}]{} U$. In particular, $\emptyset, \mathbb{R}^n, B(c, \varepsilon)$
identity are all smooth manifolds.

(b) $S^n = \{ x \in \mathbb{R}^{n+1} \mid ||x|| = 1 \ (x_0^2 + \dots + x_n^2 = 1) \}$

North pole, $N = \{1, 0, \dots, 0\}$

South Pole, $S = \{-1, 0, \dots, 0\}$

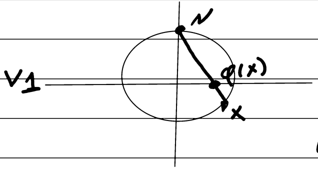
$\varphi_1 : U_1 \to V_1$

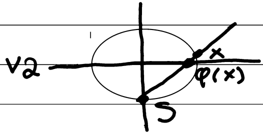
$S^n \setminus \{N\} \longrightarrow \mathbb{R}^n$

Stereographic projection

$\varphi_2 : U_2 \to V2$

$S^n \setminus \{S\} \longrightarrow \mathbb{R}^n$





Kind of like ungluing a point and unravelling into a line.

$\varphi_1(x_0, \dots, x_n) = \gamma \cdot (x_1, \dots, x_n)$

$t \mapsto N + t(x - N) = (1 + t \cdot (x_0 - 1), t \cdot x_1, \dots, t \cdot x_n)$

where $t = \frac{1}{1 - x_0}$

$\varphi_2(x_0, \dots, x_n) = \frac{1}{1 + x_0} \cdot (x_1, \dots, x_n)$

cont. on next page

$\varphi_1^{-1}(y_1,\dots,y_n) =$

$t \mapsto N + t\cdot\big((0,y_1,\dots,y_n) - N\big) = (1-t\ ,ty_1,\dots)$

$(1-t)^2 + t^2(y_1+\dots+y_n^2) = 1 \qquad = N - t\cdot y$

$t^2\cdot(1+y_1^2+\dots+y_n^2) - 2t = 0$

Assuming, $t \neq 0$

$\Rightarrow \quad t = \dfrac{2}{1+y_1^2+\dots+y_n^2}$

$\varphi^{-1}_1(\dots) = N \ + \dfrac{2}{1+y_1^2+\dots+y_m^2}(-1,y_1,\dots,y_n)$

$\varphi_2^{-1}(y_1,\dots,y_n) = S + \dfrac{2}{1+y_1^2+\dots+y_n^2}(1,y_1,\dots,y_n)$

$\varphi_2\circ\varphi_1^{-1}$

$= \varphi_2\left(N + \dfrac{2}{1+y_1^2+\dots+y_m^2}(-1,y_1,\dots,y_n\right) \qquad \dfrac{2}{\|y\|^2}\cdot\dfrac{1}{2-\dfrac{2}{\|y\|^2}}$

$= \dfrac{1}{1+1-\dfrac{2}{1+y_1^2+\dots+y_n^2}}\cdot\dfrac{2}{1+y_1^2+\dots+y_n^2}(y_1,\dots,y_n) \to$ smooth

$1-x_0 = \dfrac{1}{t}, \qquad x_0 = 1-\dfrac{1}{t},\ t = \dfrac{2}{1+y_1^2+\dots+y_n^2}$

This guy is smooth because $1+y_1^2+\dots > 1$.

$= \dfrac{y}{\|y_2\|}$, $y = (y_1,\dots,y_n)$ w.L.O.G on $\varphi_1\circ\varphi_2^{-1}$.

$\square$

$\left(\dfrac{y_1}{\|y\|^2}, \dfrac{y_2}{\|y_2\|^2},\right.$

HW 1: Prove $\mathbb{T}^n$ is a smooth manifold

Hint: $T^n = \mathbb{R}^n/\sim$

$(x_1,\dots,x_n) \sim (y_1,\dots,y_n)$

if $(x_1-y_1,\dots,y_1-y_n) \in \mathbb{Z}^n$

- Recall equivalence relations:

(a) A relation on a set S is a subset $R \subset S\times S$ we write xRy or $x\sim y$ instead of $(x,y)\in R$.

† (b) A relation is reflexive if $\forall x: xRx$

(c) " " " symmetric if $\forall x,y$ $xRY \iff yRx$

(d) " " " transitive if $\forall x,y,z$:

$(xRy \wedge yRz) \Rightarrow xRz$

if † hold, then R is an equivalence relation.

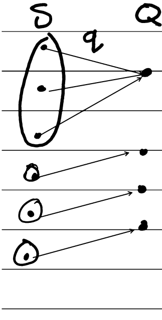
"x is equivalent to y"

(e) • **Thm.** R is an equivalence relation iff ∃ a surjective function $q: S \longrightarrow Q$ such that $\forall x,y$:

$xRy \iff q(x) = q(y)$

$q: S \to Q$

$Q := S/R$



These groups in S are called equivalence classes...

$q: S \to S/R$

- **Thm.** R is an equivalence relation $\iff \exists P \subset 2^S$, $2^S = \{u \mid u \subset S\}$. "powerset"

Quotients in essence induce the establishment of equivalence classes.

2^S

- $\forall p \in P : p \neq \emptyset$
- $\forall p, q \in P : (p \neq q) \Rightarrow (p \cap q = \emptyset)$
- $S = \bigcup_{p \in P} p$.

Also $\forall x, y : xRy \Longleftrightarrow \exists p \in P :$
$(x \in p) \wedge (y \in p)$.

• Rem. P is unique (i)

elements of P are eq. classes (ii)

q is called a quotient map (iii) $q : S \to Q$

Q is called a quotient (iv) $Q = S/R$

of S by R. $Q = S/R$ (v) $q : S \to S/R$

we also might write $[x]$ instead $q(x)$

$[x]_R$

• Rem. we can take $Q = P$

• $q(x) = \{ y \in S \mid yRx \}$. $\longrightarrow$ causes a lot of suffering!

• Exa. $S = \mathbb{R}$, $\sim = R \subset S \times S$

$x \sim y$ if $x - y \in \mathbb{Z}$

easy to check equivalence relation.

$P = \{ \{ -, \dots, -2, -1, 0, 1, 2, \dots \}, \{ \dots, -3/2, 1/2, 1/2, 3/2 \},$

$\{ \dots, x-2, x-1, x, x+1, x+2, \dots \} \}$.

$P \overset{iso}{\cong} [0,1)$. Take $x \in [0,1)$

- **Prop.** $\mathbb{R}/\sim$ is a smooth manifold
- ~~Proof~~ → Two charts

$$\varphi_1 : \underset{(0,1)}{U_1} \longrightarrow V_1 \subset \mathbb{R} \quad \text{open}$$

$U_1 = (0,1)$
$U_2 = [0,1/2) \cup (\frac{1}{2},1)$
$U_1 \cup U_2 = [0,1)$.

$$\varphi_2 : \underset{[0,1/2) \cup (\frac{1}{2},1)}{\overset{U_2}{\shortparallel}} \to (-\tfrac{1}{2},\tfrac{1}{2}) \text{ open} \subset \mathbb{R}$$

$$\varphi_2(x) = \begin{cases} x, & x < 1/2 \\ x-1, & x > 1/2 \end{cases}.$$

check compatibility and if this forms
an Atlas HW (Hint)

Next time: more on diffeomorphisms, etc.

FIN!

Show that $\mathbb{R}/\sim$ torii smooth
where $x \sim y$ by $(x-y) \in \mathbb{Z}$. manifold!

$P \cong [0,1)$

$\varphi_1 : \underset{(0,1)}{U_1} \longrightarrow V_1 \subset \mathbb{R}$

$\varphi_2 : \underset{\shortparallel}{U_2}$

$[0,1/2) \cup (\frac{1}{2},1) \to (-\frac{1}{2},\frac{1}{2})$

$$\varphi_2(x) = \begin{cases} x, & x < 1/2 \\ x-1, & x > 1/2 \end{cases}.$$

x when $x < \frac{1}{2}$, $x - 1$ $x > \frac{1}{2}$

Thus, $x < \frac{1}{2}$ $x = \varphi_2^{-1}(x)$

$$x = \begin{cases} \varphi_2^{-1}(x), & x < 1/2 \\ \varphi_2^{-1}(x-1), & x > 1/2 \end{cases}.$$

$\varphi_2^{-1}(x) = \varphi_2(x)$, $\varphi_2(x)$ id.

$\varphi_1(x)$

Thisequivalence relation is the set that contains setsof units

$\{\ldots -1, 0, 1, \ldots\}$

$\{\ldots -1/2, 0, 1/2\}$.

$\{\ldots x-1, x, x+1\}$.

How is the power set $P \cong [0,1)$?

well, all of the sets repeat themselves.

we can essentially just copy $\{\ldots, x-1, x, x+1, \ldots\}$ a bunch of times in a row...

Why in particular does this help define a Torus?

$\mathbb{R} \setminus \sim$. This quotient sends all of the real numbers to these equivalence classes. Sends

$$\{\ldots, -1, 0, 1, \ldots\}$$

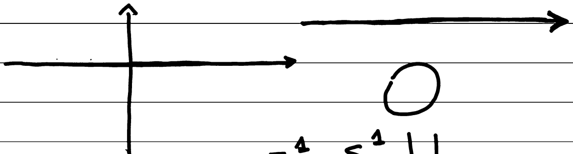
$p \overset{\sim}{=} [0,1)$.

$$\varphi_1 : \underset{(0,1)}{U} \longrightarrow V \subset \mathbb{R} \text{ open}$$

$$\varphi_2 : [0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1) \longrightarrow \underset{\underset{\mathbb{R} \text{ open}}{\cap}}{(-1/2, 1/2)}$$

$$[0, 1/2) \cup (\tfrac{1}{2}, 1) \rightarrow (-\tfrac{1}{2}, \tfrac{1}{2})$$

$$\varphi_2(x) = \begin{cases} x, & x < 1/2 \\ x-1, & x > 1/2 \end{cases}.$$



$T^1 = S^1$!!

$(0,1) \to \mathbb{R}$

uc

$\varphi_1^{-1} : \mathbb{R} \to (0,1)$

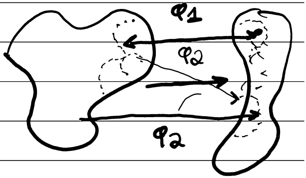
$\frac{1}{2} \to 0,\ 0 \to -n,\ 1 \to n$

$\varphi_2 : \begin{cases} x & \text{if } x < 1/2 \\ x-1 & \text{if } x > 1/2 \end{cases}$. cont too! smooth!

$[0,1/2) \cup (\frac{1}{2}, 1)$.

Thus, $\to (-1/2, 1/2)$

$(\varphi_2 \circ \varphi_1^{-1}) = \varphi_2$



$\varphi_2 = \begin{cases} x & \text{if } x < 0 \\ x-n & \text{if } x > 0 \end{cases}$

$x \in [-n,0) \cup (0,n)$.

Take $x = -n$ $\quad -n$. Let $x = n-1$

$\varphi_2 = -1$? $\quad n \to -n$?

no. $n \to -n$. $\qquad \varphi_2' = \begin{cases} 1 \\ 1 \end{cases}$

Smooth ?

$\varphi_1 : (0,1) \to \mathbb{R}$

$\varphi_1 \circ \varphi_2^{-1}$ $\qquad$ Smooth

$\varphi_1[(-1/2, 1/2)] \to \mathbb{R}$

$\varphi_1 : -1/2 \to -n$; $\varphi_2 : 1/2 \to n$.//

Last time

A smooth$^{\wedge}$manifold is

- M : set
- $\varphi_i : \underset{\underset{M}{\cap}}{U_i} \longleftrightarrow \underset{\underset{\mathbb{R}^n}{\cap \text{opn}}}{V_i}$

Atlas!

$i \in I$ - a set $\qquad \forall\, i,j$

- φ_i is compabible w/ φ_j : $\varphi_j \circ \varphi_i^{-1}$ is smooth
- $\bigcup_{i\in I} u_i = M$.

Today

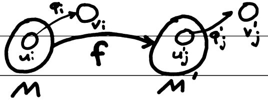
- **Def.** A smooth map

dim = m $\qquad$ dim m'

$f : (M, (\varphi_i : U_i \to V_i)_{i\in I}) \longrightarrow (M', (\varphi'_j : u'_j \to v'_j))_{j\in J}$

is a map of sets $f : M \to M'$ such that ...

$u_i \subset \mathbb{R}^m$, $V_j \subset \mathbb{R}^{m'}$.



,

... such that for every $i \in I, j \in J$ the map

$\varphi'_j \circ f \circ \varphi_i^{-1} : \underset{\underset{\mathbb{R}^n}{\cap \text{opn}}}{\hat{V}_i} \longrightarrow \hat{V}'_j \subset \mathbb{R}^m$ open.

is smooth. $\qquad \hat{V}_i \subset V_i$

$\hat{V}_i = \varphi_i(f^{-1} u'_j)$ is open in $\mathbb{R}^m$.

• **Exa.**

(a) pick some $q \in M'$. set $f(p) = q$ $\forall p \in M$. Then f is smooth.

Indeed $f^{-1} U'_j = \begin{cases} M, & \text{if } q \in U'_j \\ \emptyset, & \text{if } q \notin U'_j \end{cases}$

$$\hat{V}_i = \begin{cases} V_i, & \text{if } q \in U'_j \\ \emptyset, & \text{if } q \notin U'_j \end{cases}$$

$$\varphi'_j \circ f \circ \varphi_i^{-1} = \begin{cases} V_i \longrightarrow V'_j,\ v \mapsto \varphi'_j(q) & \text{if } q \in U_j \\ \emptyset \longrightarrow V'_j & \text{if } q \notin U'_j . \end{cases}$$

(b) $(M, (\varphi_i : U_i \longrightarrow V_i)_{i \in I})$. $\mathbb{R}^m$

Pick some $k \in I$. Then $\varphi_k^{-1} : V_k \longrightarrow U_k \subset M$ is a smooth map. $f =$

$$\begin{tikzcd} V_k \arrow[r] \arrow[rr, bend right] & U_k \subset & M \end{tikzcd}$$

• **Proof.** Have to check that $\forall j \in I$ the map

$$\varphi_j \circ \varphi_k^{-1} \circ (\mathrm{id}_{V_i})^{-1} = \varphi_j \circ \varphi_k^{-1} \text{ is smooth.}$$

This follows simply by definition of M being a smooth manifold.

• Riemann Def. 18.., Veblen, Whitehead 1931.

• **Def.** Given a n-manifold $(M, (\varphi_i : U_i \to V_i)_{i \in I})$

A subset $W \subset M$ is open if for every $i \in I$:

$\varphi_i(W \cap U_i)$ is an open subset of $\mathbb{R}^n$.

• **Recall** A topological space is a pair $(X, \mathcal{U})$

X is a set

$\mathcal{U} \subset 2^X = \{S \mid S \subset X\}$. Elements of $\mathcal{U}$ are known as open subsets of X...

• $\mathcal{U}$ is closed under finite intersections:

$\forall A, B \in \mathcal{U} : A \cap B \in \mathcal{U}$, <u>and</u> $X \in \mathcal{U}$.

• $\mathcal{U}$ is closed under arbitrary union: If W is a subset of $\mathcal{U}$. $W \subset \mathcal{U}$, then $\bigcup W = \{x \in X \mid \exists w \in W : x \in w\} \in \mathcal{U}$.

• <u>Exa.</u> $W = \emptyset$, $\bigcup W = \emptyset \in \mathcal{U}$.

• <u>Exa.</u> $\mathcal{U} = \{ S \subset \mathbb{R}^n \mid S = \bigcup_{B(s,\varepsilon) \subset S} B(s,\varepsilon) \}$.

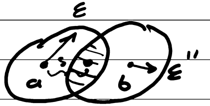
$(\mathbb{R}^n, \mathcal{U})$ is a topological space.

Unions are given basically all by definition.

$A, B \in \mathcal{U}$, $A = \bigcup_{B(\alpha,\varepsilon)} B(\alpha, \varepsilon)$

$B = \bigcup_{B(b,\varepsilon') \subset B} B(b, \varepsilon')$ $\nearrow$ can I use demorgans law here?

$$\bigcup_{\substack{B(a,\varepsilon) \subset A \\ B(b,\varepsilon') \subset B}} \left(B(a,\varepsilon) \cap B(b,\varepsilon') \right)$$



Suppose $s \in B(a,\varepsilon) \cap B(b,\varepsilon)$,

Set $\varepsilon'' = \min(\varepsilon - \|s-a\|, \varepsilon' - \|s-b\|)$

Then $B(s,\varepsilon'') \subset B(a,\varepsilon) \cap B(b,\varepsilon')$

Pick some $x \in B(s, \varepsilon'')$.

Then $\|x-a\| \leq \|x-s\| + \|s-a\| < \varepsilon'' + \|s-a\|$
$\leq \varepsilon - \|s-a\| + \|s-a\| = \varepsilon$.

- **Prop.** $(M, \{W \subset M \mid W \text{ is open}\})$.
is a topological space
- **Proof.** a) M is open:

$$\forall_{i\in I}\ \varphi_i(M\cap U_i) = \varphi_i(U_i) = V_i \subset \mathbb{R}^m$$

by def.

b) $W_1, W_2 \subset M$ open

$$\varphi_i(W_1\cap W_2\cap U_i)$$

$$\varphi_i((W_1\cap U_i)\cap(W_2\cap U_i))$$

$$\underbrace{\varphi_i(W_1\cap U_i)}_{\text{open.}} \subset \mathbb{R}^m \cap \underbrace{\varphi_i(W_2\cap U_i))}_{\text{open.}} \subset \mathbb{R}^m$$

Thus, open.

$W_k \subset M$ open

c) unions $\bigcup_{k\in I} W_k$

$$\forall i\in I: \varphi_i\left(\left(\bigcup_{k\in I} W_k\right)\cap U_i\right) = \varphi_i\left(\bigcup_{k\in K}(W_k\cap U_i)\right)$$

$$\bigcup_{k\in K}\underbrace{\varphi_i(W_k\cap U_i)}_{\text{open.}} \subset \mathbb{R}^m$$

This verifies smooth manifolds have an underlying topological space

- **Def.** A continuous map $f:(X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$
is a map of sets $f: X \to Y$ s.t.
$\forall v\in \mathcal{V}: f^{-1}(v)\in \mathcal{U}$.

Prop. Every smooth map of smooth manifolds of smooth manifolds is continuous

proof. (next time...)

Def. A topological space is $(X, \mathcal{U})$
X set, $\mathcal{U} \subset 2^X$ closed under finite $\cap$, arbitrary $\cup$.

Def. A continuous map $(X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ is a map of sets $f : X \to Y$ s.t. $\forall v \in \mathcal{V}$: $f^{-1} v \in \mathcal{U}$.

Rem. $U \subset \mathbb{R}^m \xrightarrow{f} V \subset \mathbb{R}^n$ is continuous $\iff \forall u \in U \ \forall \varepsilon > 0, \exists \delta > 0 \ \forall x \in U$:

$$\|x - u\| < \delta \implies \|f(x) - f(u)\| < \varepsilon.$$

Proof.
$\Rightarrow$ assume f is continuous, $u \in U$, $\varepsilon > 0$.
Also, $B(f(u), \varepsilon) \subset \mathbb{R}^n$ (open). Also $\underset{u\in}{f^{-1}}(B(f(u), \varepsilon)) \overset{\text{open}}{\subset} U$.
For some $\delta > 0$ |

$$B(u, \delta) \cap U, \ \forall x \in U : \|x - u\| < \delta$$
$$\iff x \in B(u, \delta) \implies \|f(x) - f(u)\| < \varepsilon$$
$$\iff f(x) \in B(f(u), \varepsilon) \iff x \in f^{-1}(B(f(u)), \varepsilon).$$

$\Leftarrow$ try other side at home.

Prop. $(M, (\varphi_i : U_i \to V_i)_{i \in I}) \xrightarrow{f} (N, (\psi_j : W_j \to X_j)_{j \in J})$.
The underlying map of topological spaces is continuous

• **Proof.** Suppose $S \underset{open}{\subset} N$. w.t.s.

$f^{-1}(S) \subset M \iff \forall i \in I \quad \varphi_i(u_i \cap f^{-1}(S)) \overset{open}{\subset} \mathbb{R}^n$

$S = S \cap N = S \cap \bigcup_{j\in J} w_j = \bigcup_{j\in J} (S \cap w_j)$

$\varphi_i (u_i \cap f^{-1} \left(\bigcup_{j\in J} (S \cap w_j) \right)$

$= \varphi_i (u_i \cap \bigcup_{j\in J} (S \cap w_j) = \varphi_i \left(\bigcup_{j\in J} u_i \cap f^{-1}(S \cap w_j) \right)$

$= \bigcup_{j\in J} \underbrace{\varphi_i (u_i \cap f^{-1}(S \cap w_j))}$

$= \cup_{j\in J} \varphi_i (u_i \cap f^{-1} S \cap f^{-1} w_j)$

$= \cup_{j\in J} \underbrace{(\psi_j \circ f \circ \varphi^{-1})}^{-1} (S \cap w_j)$

$\varphi_i \circ f^{-1} \circ \psi_j^{-1} \circ (\psi_j (S \cap w_j))$

smooth!

Now, but why is $\psi_j(S \cap w_j)$ open?

$\psi_j(\underset{\underset{\text{by ass}}{\uparrow}}{S} \cap \underset{\underset{\text{open.}}{\uparrow}}{w_j})$ is open $\quad S \cap w_j$ is open.

$= (\psi^{-1})^{-1}(S \cap w_j)$

w_j is open $\iff \forall k : \varphi_k(w_j \cap w_k) \subset X_k$.

• **Prop.** $(X, U) \xrightarrow{f} (Y, V) \xrightarrow{g} (Z, W)$

If f and g are continuous, then $f \circ g$ is too.

• **Proof.** suppose $w \in W$, then $(g \circ f)^{-1} w = f^{-1} \underbrace{g^{-1} w}_{open} . \subset X$ open.

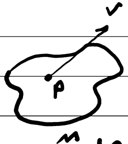
What is a tangent vector?

Three different points of view:

- Transfer tangent vectors in $\mathbb{R}^n \to M$ using charts
- Kinematic tangent vector: an equivalenc class of trajectories with the same velocity.
  - $\mathbb{R} \xrightarrow[\text{smooth}]{p} M$ $\quad \mathbb{R} \xrightarrow{p} M$ $p \sim q$ if $\underset{\text{(in a chart)}}{p'(0) = q'(0)}$
- Algebraic/observational:

$$\begin{tikzcd} C^{\infty}(M,\mathbb{R}) \arrow[d, "\text{directional derivative}"] & = \{f: M \to \mathbb{R} \mid f \text{ smooth}\}. \\ \mathbb{R} & \end{tikzcd}$$

- If $M \underset{\text{open}}{\subset} \mathbb{R}^n$, $p \in M$, $v \in \mathbb{R}^n$



$f: M \longrightarrow \mathbb{R}$

$$\nabla_v f(p) = v_p(f) = \nabla_{(p,v)} f = \nabla_v f = \frac{d f(p + t \cdot v)}{dt}$$

Leibniz rule: $v_p(fg) = v_p \cdot g(p) + f(p) \cdot v_p(g)$

Banach manifolds... charts

$(0,1) \to \mathbb{R}$

uc

$\varphi_1^{-1} : \mathbb{R} \to (0,1)$

$\frac{1}{2} \to 0, \quad 0 \to -n, \quad 1 \to n$

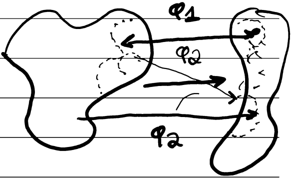
$$\varphi_2 : \begin{cases} x & \text{if } x < 1/2 \\ x-1 & \text{if } x > 1/2 . \end{cases}$$

cont too! smooth!

$[0, 1/2) \cup (\frac{1}{2}, 1).$

Thus, $\longrightarrow (-1/2, 1/2)$

$(\varphi_2 \circ \varphi_1^{-1}) = \varphi_2$



$$\varphi_2 = \begin{cases} x & \text{if } x < 0 \\ x-n & \text{if } x > 0 \end{cases}$$

$x \in [-n, 0) \cup (0, n).$

$(x_1, x_2, \ldots, x_n)$

$\varphi_1(x_1, x_2, \ldots, x_n) = (\varphi_1(x_1), \varphi_1(x_2), \ldots \varphi_1(x_n))$

$\varphi_2 \circ \varphi_1^{-1}(x_1, x_2, \ldots, x_n)$

$\varphi_2(\varphi_1^{-1}(x_1), \varphi_1^{-1}(x_2), \varphi_1^{-1}(x_3), \ldots, \varphi_1^{-1}(x_n))$

$\varphi_1^{-1}(x_1)$ smooth

$\varphi_2(\varphi_1^{-1}(x_1))$ smooth.

Hint: $T^n = \mathbb{R}^n/\sim$

$(x_1, \dots, x_n) \sim (y_1, \dots, y_n)$

if $(x_1 - y, \dots, y_1 - y_n) \in \mathbb{Z}^n$

$\mathbb{R}/\sim$ is a smooth manifold.

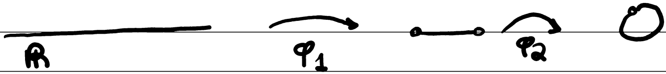
i.e. $T^1 = S^1 = \mathbb{R}^1/\sim$.

xRY if $x - y \in \mathbb{Z}$.

Thus,

$\varphi_1 \circ \varphi_2^{-1}(x_1)$ is smooth

$\varphi_1(x_1, \dots, x_n) = \gamma \cdot (x_1, \dots, x_n)$.



$\mathbb{R}\setminus\sim = T^1 = S^1$. shown to be a smooth manifold

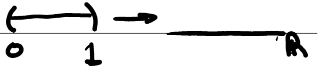
$\varphi_1 \circ \varphi_2^{-1}(x_1, x_2, \dots, x_n)$

$\varphi_1 : (0,1) \longrightarrow \underset{\text{open}}{V_1} \subset \mathbb{R}$

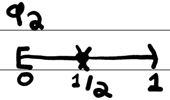
$\varphi_2 : [0, 1/2) \cup (\tfrac{1}{2}, 1) \longrightarrow (-1/2, 1/2)$

$$\varphi_2(x) = \begin{cases} x , & x < 1/2 \\ x-1, & x > 1/2 . \end{cases}$$

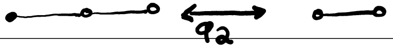
$\underline{\varphi_1 \circ \varphi_2^{-1}}$



$$x = \begin{cases} \varphi_2^{-1}(x), & x < 1/2 \\ \varphi_2^{-1}(x-1), & x > 1/2 \end{cases}$$



$$\varphi_2(x) = \begin{cases} x & x < 1/2 \\ (x-1) & x > 1/2 \end{cases}$$



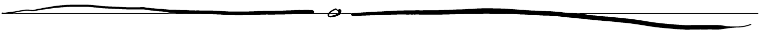
$\varphi_1 \circ \varphi_2^{-1} : (-1/2, 1/2) \longrightarrow V_1 \subset \mathbb{R}$

$$\varphi_1 \circ \varphi_2^{-1} = \begin{cases} x & \text{if } x < 0 \\ x - \sup(V) & \text{if } x > 0 \end{cases} \quad \text{smooth}$$

$\varphi_2 \circ \varphi_1^{-1} : V_1 \subset \mathbb{R} \longrightarrow (-1/2, 1/2)$

$\varphi_1^{-1} : V_1 \subset \mathbb{R} \longrightarrow (0,1) \longrightarrow (-1/2, 1/2)$

$$\varphi_2 = \begin{cases} x , & x < 1/2 \\ x-1, & x > 1/2 \end{cases} \quad \text{Smooth}$$



Last time

A tangent vector in M is

1) A tangent vector in a chart

2) An equivalene class of curves

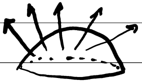
3) A derivation of the ring of smooth functions on M.

TM denotes the tangent bundle $TM \to M$ is a vector bundle

$\downarrow$ Projection map

M base space

The tangent bundle of an open subset $U \subset \mathbb{R}^n$



finite dimensional real vector-space.

$$\begin{tikzcd} TU = U \times \mathbb{R}^n \arrow[d, "p"'] \\ U \end{tikzcd} \quad p(u,v) = u$$

$(u,v_1) + (u,v_2)$, $t \cdot (u,v) = (u, t \cdot v)$

$(u,0)$

Then $u \in U$, then $\{(u,v) \mid v \in \mathbb{R}^n\} \subset TU$ is a vector space.

- Products of manifolds

what do we want?

Given $M, N \in$ Man, want $M \times N \in$ an,

$\pi_1 : M \times N \to M$, $\pi_2 : M \times N \to N$ (smooth maps)

Given $L, M, N \in$ Man, $f_1 : L \to M$, $f_2 : L \to N$

want $(f_1, f_2) : L \to M \times N$

(For sets: $(f_1, f_2)(x) = (f_1(x), f_2(x))$.

$\pi_1 \circ (f_1, f_2) = f_1$, $\pi_2 \circ (f_1, f_2) = f_2$

- Given some $g : L \to M \times N$, we have

  $g = (\pi_1 \circ g, \pi_2 \circ g)$.

Forsets

$M \times N = \{ (m,n) \mid m \in M, n \in N \}$

$\pi_1(m,n) = m$, $\pi_2(m,n) = n$

$(f_1, f_2)(l) = (f_1(l), f_2(l))$

$\pi_1((f_1,f_2)(l)) = \pi_1(f_1(l), f_2(l)) = f_1(l)$

$(\pi_1 \circ g, \pi_2 \circ g)(l) = (\pi_1(g(l)), \pi_2(g(l)))$

$g(l) = (m,n)$ for unique $m \in M$, $n \in N$

$= (\pi_1(m,n), \pi_2(m,n)) = (m,n) = g(l)$.

- **Def.** $M = (M, (\varphi_i : \mathcal{U}_i \to V_i)_{i \in I})$

  $N = (N, (\psi_j : \mathcal{U}'_j \to V'_j)_{j \in J})$

$M \times N = (M \times N, (\varphi_i \times \psi_j : U_i \times U'_j \times V_i \times V'_j)_{(i,j) \in I \in J})$

$(\varphi_i \circ \pi_1, \psi_j \circ \pi_2) = (\varphi_i(u), \psi_j(u')$

$\bigcup_{i,j} U_i \times U'_j$

$\bigcup_i u_i \times \left(\bigcup_j u'_j \right) = \bigcup_i (u_i \times N) = \left(\bigcup_i u_i \right) \times N$

$= M \times N$

$(i,j),(k,l) \in I \times J$

$\varphi_i \times \psi_j : \mathcal{U}_i \times \mathcal{U}_j' \longrightarrow V_i \times V_j'$

$\varphi_k \times \psi_l : U_k \times \mathcal{U}_l' \longrightarrow V_k \times V_l'$

$(\varphi_k^{-1} \times \psi_l^{-1}) \circ (\varphi_i \times \psi_j)$

$= \underbrace{(\varphi_k^{-1} \circ \varphi_i)}_{\text{smooth}} \times \underbrace{(\psi_l^{-1}, \psi_j)}_{\text{smooth}}$

$D(f \times g)$

$$\begin{pmatrix} Df & 0 \\ 0 & Dg \end{pmatrix}.$$

- **Prop.** $\pi_1 : M \times N \longrightarrow M$ is a smooth map.
- **proof.** pick $i \in I, j \in J$

passing to the corresponding chart (i,j) in $M \times N$, j in N, π_1 becomes $V_i \times V_j' \longrightarrow V_i$

restriction of a linear map to opensubset, smooth

- **Proof.** $L = (L, (\chi_k : W_k \to X_k)_{k \in K})$

pick $k \in K$, $(i,j) \in I \times J$

(f_1, f_2) becomes $X_k \xrightarrow[(f_1,f_2)]{} V_i \times V_j'$

$(\underbrace{\varphi_i \circ f_1 \circ \chi_k^{-1}}_{\text{smooth}}, \underbrace{\psi_j \circ f_2 \circ \chi_k^{-1}}_{\text{smooth}})$ $D(f,g) = \begin{pmatrix} Df \\ Dg \end{pmatrix}$

* * *

- **Def.** $M = (M, (\varphi_i : U_i \longrightarrow V_i))$

$TM = (\ldots, (\psi_i : \ldots . \longrightarrow V_i \times \mathbb{R}^n)_{i \in I}).$

The underlying set $TM = (\ldots, (\psi_i : U_i \times \mathbb{R}^n \to V_i \times \mathbb{R}^n)_{i \in I}).$

$$\left(\bigsqcup_{i\in I} U_i \times \mathbb{R}^n\right) / \sim$$

- **Def.** I : set, $i \in I$: A_i : set

$$\bigsqcup_{i\in I} = \bigcup_{i\in I} \{i\} \times A_i$$

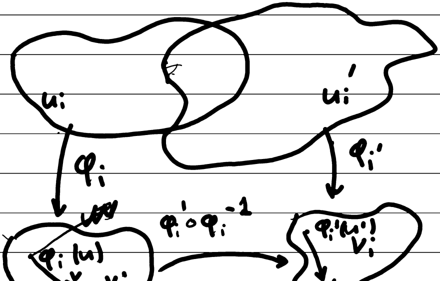
$\uparrow$ coproduct of sets.

$\sim$: $i, i' \in I$ $\quad u \in U_i$ $\quad u' \in U_{i'}$

$\quad v \in \mathbb{R}^n$ $\quad v' \in \mathbb{R}^n$

$(i, (u, v)) \sim (i, (u', v'))$

if $u = u'$ and $D(\varphi_i' \circ \varphi_i^{-1})(v| = v'$



$\varphi_1 : (0,1) \longrightarrow V_1 \subseteq \mathbb{R}$

$\varphi_2 : [0, 1/2) \cup (1/2, 1) \longrightarrow \overset{V_2}{(-1/2, 1/2)}$

$\varphi_1 \circ \varphi_2^{-1} : \varphi_2(u_1 \cap u_2) \longrightarrow \varphi_1(u_1 \cap u_2)$

$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(u_1 \cap u_2) \longrightarrow \varphi_2(u_1 \cap u_2).$

$\varphi_1(u_1 \cap u_2)$

$u_1 = (0,1) \cap u_2 = [0, 1/2) \cup (1/2, 1)$

$u_1 \cap u_2 = (0, 1/2) \cup (1/2, 1)$

$\varphi_1(u_1 \cap u_2)$

maps $0 \to \inf(V_1)$

$1 \to \sup(V_1)$

$(\inf(v_1), 0) \cup (0, \sup(V_1))$

$\longrightarrow (\inf(v_1), 0) \cup (0, \sup(v_1))$

$$\begin{cases} x & x < 0 \\ (x + \inf(v_1)) & x > 0 \end{cases}$$

$$Df(x) = \begin{cases} 1 & x < 0 \\ 1 & x > 0 \end{cases}$$

smooth for domain.

$\varphi_1 \circ \varphi_2^{-1} : \varphi_2(u_1 \cap u_2) \longrightarrow \varphi_1(u_1 \cap u_2)$

$\varphi_2 : [0, 1/2) \cup (1/2, 1) \longrightarrow (-1/2, 1/2)$

$$\varphi_2 = \begin{cases} x & \text{if } x < 0 \\ x-1 & \text{if } x > 0 \end{cases}$$

$= x \quad \text{if } x < 0$

$= x-1 \text{ if } x > 0$

$x = \varphi_2^{-1}(x) \text{ if } x < 0$

$x = \varphi_2^{-1}(x-1) \text{ if } x > 0$

$\varphi_2^{-1} = \text{id if } x < 0$

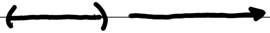
$\varphi_2^{-1} =$

$x-1 \overset{\varphi_2^{-1}}{\longmapsto} x$, thus $x \overset{\varphi_2^{-1}}{\longmapsto} x+1$

$\circ$

$\varphi_1 : (0,1) \longrightarrow V \subset \mathbb{R}$

$(0,1)$



$f(x) = 1/x, \quad y = 1/x, \; x = 1/f(x)$

$\varphi_1(x) = -1/x - 1/2 \quad = -\frac{1}{x}$

$\tan(\pi x - \pi/2)$ is a suitable map.

$\varphi_1 = \tan(\pi x - \pi/2)$

Last time: $\underset{\cup}{\mathbb{R}^n}$ need to write a homeomorphism

$M = (M, (\varphi_i : U_i \to V_i)_{i\in I})$ transition map

$TM = \left(\bigsqcup_{i\in I} U_i \times \mathbb{R}^n \right) \Big/ (u_i, v_i) \sim (u_j, v_j)$ φ

$\iff (u_i = u_j) \wedge (D(\varphi_j \circ \varphi_i^{-1})(u_i) = V_j$

$(\varphi_i \times id_{\mathbb{R}^n} : U_i \times \mathbb{R}^n \to V_i \times \mathbb{R}^n)_{i\in I}.$

$\bigcup_i U_i \times \mathbb{R}^n = TM$ / compatibility of charts

$(\varphi_j \times id\mathbb{R}_n \circ (\varphi_i \times id\mathbb{R})^{-1} = (\varphi_j \times \varphi_i)^{-1} \times id\mathbb{R}^n.$

coddom) =

- Exa. a) $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$

$TV = V \times V$ $\quad V \in Vect_{\mathbb{R}}^{f.d.}$

$U \underset{open}{\subset} V : \quad TU = U \times V$

b) S^n: start with $V \in Vect_{\mathbb{R}, \langle -,- \rangle}^{f.d.}$

$S^V = (V \cup \{\infty\}, \quad \varphi_1 : V \xrightarrow{id} V$

$\varphi_2 : V \cup \{\infty\} \setminus \{0\} \to V$

$v \mapsto \frac{1}{\|v\|} \cdot v \quad)$

$\varphi_2 = \varphi_2 : v \setminus \{0\} \to v \setminus \{0\}$

$v \mapsto \frac{1}{\|v\|^2} \cdot v$

$\varphi_1 \circ \varphi_2^{-1} : \varphi_2^{-1} : V \setminus \{0\} \to V \setminus \{0\}$

$$\frac{1}{\|v\|^2} \cdot \|v\| = \|w\|$$

$$\|v\| = \frac{1}{\|w\|}$$

$$v = w \cdot \|v\|^2 = \frac{w}{\|w\|^2}.$$

$$T(S^V) = \left(TV \sqcup V,\ \varphi \times \mathrm{id}_V,\ \varphi_2 \times \mathrm{id}_V \right)$$

In general $TS^V \cong V \times S^V$

However $TS^1 \cong \mathbb{R} \times S$, $TS^2 \cong \mathbb{R}^2 \times S^2$?

HW 2: write a smooth map that

assigns $S^2 \xrightarrow{s} TS^2$

s.t.

$$\begin{tikzcd} S^2 \arrow[r, "s"] \arrow[rr, bend right, "\mathrm{id}"'] & TS^2 \arrow[r, "p"] & S^2 \end{tikzcd}$$

section at most at one point, i.e

$\forall v \in S^2 \setminus \{\infty\}$ $s(v) \neq 0$, $0 \in TS^2$.

HW 4: $T(T^n) = T^n \times \mathbb{R}^n$?

HW 3: $V \in \mathrm{Vect}_{\mathbb{R}}^{f.d.}$ ★

a lattice $L \subset V$: $L = \varphi(\mathbb{Z}^n)$

$L = \mathbb{Z}^n \subset \mathbb{R}^n = V$ $\quad \varphi : \mathbb{R}^n \xrightarrow[\text{iso}]{\cong} V$

$V/L = V/\sim$, $v_1 \sim v_2$ if $v_1 - v_2 \in L$.

quotient group. abelian L is normal

a.) equip the quotient V/L with a structure of smooth manifold

b.) Prove $\overset{\text{elliptic curve}}{V/L \times V/L} \xrightarrow{+} V/L$ is a smooth map.

c.) Prove that $\underset{\varphi}{V/L} \xrightarrow{\cong} V'/L'$ $\dim V = \dim V'$

ellipticcurve vector bundle.

- **Proposition** $M \in \mathrm{man}$

$$\begin{tikzcd} TM \in \mathrm{Man} \arrow[d, "p\ \mathrm{smooth}"] \\ M \end{tikzcd}$$

$V \in \mathrm{Vect}_{\mathbb{R}}$ $\quad 0 \in V$

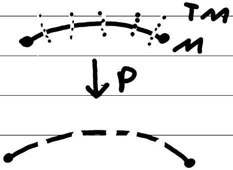
$0 : 1 \longrightarrow V$

$\cdot : \mathbb{R} \times V \longrightarrow V$

$+ : V \times V \longrightarrow V$

$|A \times B| = |A| \cdot |B|$

$|\prod_{\emptyset}| = 1 \ , |A| = |A|$



In a lecture, submersion will come in handy.

$0 : M \longrightarrow TM$

$\cdot : \mathbb{R} \times TM \longrightarrow TM$

$+ : TM \times TM \not\longrightarrow TM$

$: \underbrace{TM \times TM}_{M} \longrightarrow TM$

$\shortparallel$

$\{(v_1, v_2) \in TM \times TM \mid p(v_1) = p(v_2)\}.$

- **Proof.**

— Prop. — Suppose we have two smooth manifolds $M = (M, (\varphi_i : U_i \longrightarrow V_i)_{i \in J})$
$N = (N, (\psi_i : U_i' \longrightarrow V_i')_{i \in I})$

If $f : M \longrightarrow N$ is a map of sets such that $\forall i : f(U_i) \subset U_i'$.
Then f is smooth iff $\forall i \quad \psi_i \circ f \circ \varphi_i^{-1} : V_i \longrightarrow V_i'$ is a smooth map.

Proof $\psi_j \circ f \circ \varphi_i^{-1} = (\psi_j \circ \psi_i^{-1}) \circ (\psi_i \circ f \circ \varphi_i^{-1})$

a) $0: M \longrightarrow TM$ is smooth

$M = (M, (\varphi_i : U_i \longrightarrow V_i))$

$TM = (TM, (\varphi_i \times id_{\mathbb{R}^n} : U_i \times \mathbb{R}^n \longrightarrow V_i \times \mathbb{R}^n)_{i \in I})$

$0: M \rightarrow TM$ is smooth $\Leftrightarrow \forall i$

$m \mapsto (m, 0)$ $((\varphi_i \times id) \circ 0 \circ \varphi_i^{-1})(v)$

$= (\varphi_i \times id_{\mathbb{R}^n})(\varphi_i^{-1}(v), 0)$

$= (\varphi_i(\varphi_i(v)), id\, 0) = (v, 0).$

exercise for b) and c)

a) $V_i \longrightarrow V_i \times \mathbb{R}^n$ in the Ith chart.

$v \longmapsto (v, 0)$

b) $\mathbb{R} \times V_i \times \mathbb{R}^n \longrightarrow V_i \times \mathbb{R}^n$

$(t, v, w) \mapsto (v, t \cdot w)$

c) $V_i \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow V_i \times \mathbb{R}^n$

$(v, w_1, w_2) \rightarrow (v, w_1 + w_2).$

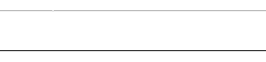
<u>Construction 2:</u>

$TM = C^\infty(\mathbb{R}, M)/\sim$

$f \sim g \Leftrightarrow f(0) = g(0)$ <u>and</u> $f'(0) = g'(0)$

(in some chart)

$C^\infty(\mathbb{R}, S)/\sim$



• Proposition

$$C^\infty(\mathbb{R}, M)/\sim \longrightarrow TM$$

is a bijection

• Recall $X/R \xrightarrow{f} Y$ are in bijection with maps of sets $X \xrightarrow[g]{} Y$ such that g respects R.

$$x_1 R x_2 \Rightarrow g(x_1) = g(x_2).$$

"f is well-defined"

$$f([x]) = g(x).$$

• Proof of Prop.

$$\begin{tikzcd} \underset{f\in}{C^\infty}(\mathbb{R}, M) \arrow[r] & TM \arrow[d, equal] \\ f(0) & \coprod_{i\in I} U_i \times \mathbb{R}^n/\sim \end{tikzcd}$$

Pick some i s.t. $f(0) \in U_i$.

$(u_i, w_i) \sim (u_j, w_j)$ $\iff u_i = u_j$ and

Set $r(f) = [(f(0), D(\varphi_i \circ f)(0)]$ $D(\varphi_j \circ \varphi_i^{-1})(w_i) = w_j$.

If

$$[(f(0), D(\varphi_i \circ f)(0)] = [(f(0), D(\varphi_j \circ f)(0)]$$

and $D(\varphi_j \circ \varphi_i^{-1})(w_i) = w_j$.

Tensors

$$D(\varphi_j \circ \varphi_i^{-1})\left(D(\varphi_i \circ f)(0)\right)$$

$$= D(\varphi_j \circ \underbrace{\varphi_i^{-1} \circ \varphi_i}_{I} \circ f)(0)$$

$$= D(\varphi_j \circ f)(0).$$

$D(\varphi_i \circ f)(\quad 0 \quad) = D(\varphi_i \circ g)(0)$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "f"] \arrow[rrd] & M \supset U_i \arrow[r, "\varphi_i"] & V_i \\
& & \mathbb{R}^n \arrow[u, phantom, "\subset" rotate=90]
\end{tikzcd}$$

2) suppose $f, g \in C^\infty(\mathbb{R}, M)$, $f \sim g$

$D(\varphi_i \circ f)(\quad 0 \quad) = D(\varphi_i \circ g)(\quad 0 \quad)$

$f(0) = g(0)$

want: $r(f) = r(g)$

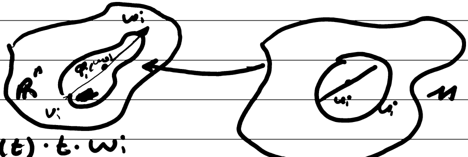
$[(f(0), D(\varphi_i \circ f)(0)] = [(g(0), D(\varphi_i \circ g)(0))]$

$'' \; D(\varphi_i)(f(0)) \cdot (Df(0)) = f'(0).$

3) $s: \left(\bigsqcup_{i \in I} U_i \times \mathbb{R}_n \right) / \sim \; \longrightarrow C^\infty(\mathbb{R}, M)/\sim$

pick some $i \in I$: $U_i \times \mathbb{R}^n \longrightarrow C^\infty(\mathbb{R}, M)/\sim$

$(u_i, w_i) \longmapsto \varphi_i^{-1} \circ h$

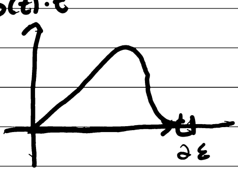


$h: \mathbb{R} \longrightarrow V_i$

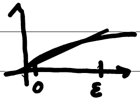
$h(t) = \varphi_i(u_i) + b(t) \cdot t \cdot w_i$

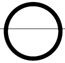
$b(t) = \begin{cases} 1, & t \in (-\varepsilon, \varepsilon) \\ 0, & |t| \geq 2\varepsilon \end{cases}$

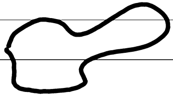
$b(t) \in (0,1)$, $\varepsilon \leq |t| < 2\varepsilon.$

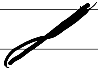


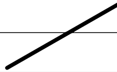
$h(t) = \varphi_i(U_i) + c(t) \cdot w_i$











$(u_i, w_i) \sim (u_j, w_j)$

$u_i = u_j$ and $D(\varphi_j \circ \varphi_i^{-1})(\varphi_i(u_i))(w_i) = w_j$

want: $\underset{\overset{\|}{[\varphi_i^{-1} \circ h_i]}}{S(u_i, w_i)} = \underset{\overset{\|}{[\varphi_j^{-1} \circ h_j]}}{S(u_j, w_j)}$

$\Longleftrightarrow \varphi_i^{-1} \circ h_i \sim \varphi_j^{-1} \circ h_j$

$$\varphi_j^{-1}(h_j(0)) = \varphi_j^{-1}(\varphi_j(u_j)) = u_j = u_i$$

In the J-th chart

$$D(\varphi_i \circ \varphi_i^{-1} \circ h_i)(0) = \underset{\overset{\|}{D(\varphi_i \circ \varphi_j^{-1})(\underset{\overset{\|}{\varphi_j(u_j)}}{h_j(0)})(\underset{\overset{\|}{w_j}}{h_j'(0)})}}{D(\varphi_i \circ \varphi_j^{-1} \circ h_j)(0)}$$

$$= \underset{\overset{\|}{w_i}}{h_i'(0)}$$

5) $r \circ s = id_{TM}$

6) $s \circ r = id_{C^\infty(\mathbb{R}, M)/\sim}$

prove 5 and 6.

Last time:

(1) TM (charts) $\cong_{(2)} TM$ kinematic

$\wr\wr^{2}$ TM (3)

(derivations)

The algebra of smooth manifolds

what is $C^\infty(M,\mathbb{R})$ in physics " observables

Answer: a commutative real algebra

Better answer a C^∞-ring.

Def. A commutative real algebra is

$(A, 0, +, \cdot, \iota)$

A: underlying set

$0 \in A$

$+ : A\times A \longrightarrow A$

$\cdot : A\times A \longrightarrow A$

$\iota : \mathbb{R} \to A$

$0 := \iota(0)$, $1 := \iota(1)$

$\forall r,s \in \mathbb{R}$

Such that $\forall a,b,c \in A$

$a+(b+c) = (a+b)+c$

$0 + a = a$

$a\cdot(b\cdot c) = (a\cdot b)\cdot c$

$a\cdot(b+c) = ab+ac$

$\iota(r+s) = \iota(r) + \iota(s)$

$\iota(rs) = \iota(r)\cdot\iota(s)$

Alternatively a homomorphism of commutative rings $\iota: \mathbb{R} \longrightarrow A$

Exa. $C^\infty(M)$ All operations are defined pointwise $(f+g)(m) = f(m)+g(m)$

$\iota(r)(m) = r$

$\iota(r+s)(m) = r+s = \iota(r)(m) + \iota(s)(m)$

$= (\iota(r)+\iota(s))(m)$

$M = \emptyset : C^{\infty}(M, \mathbb{R})$ $f: A \to B$

$G \subset A \times B$, $\forall a \in A \; \exists! \, B \in b : (a,b) \in G$.

$A = \emptyset$: $A \times B = \emptyset \times B = \emptyset$

$G = \emptyset$.

$\forall a \in \emptyset \; \exists! \, b \in B : (a,b) \in \emptyset$

Fact: A smooth manifold M can be reconstructed from $C^{\infty}(M, \mathbb{R})$

suppose A is a commutative real algebra such that $\exists \varphi : A \xrightarrow{\cong} C^{\infty}(M)$.

Then M can be reconstructed from A as follows:

The underlying set of M is the set of homomorphisms of algebras $A \to \mathbb{R}$

Reference: Jet Nestruev

Smooth manifolds and observables.

Second Edition

- Proposition If M is a hausdorff, second countable smooth manifold, then, the canonical map of sets $M \xrightarrow{ev} \operatorname{Hom}(C^{\infty}M, \mathbb{R})$ is a bijection

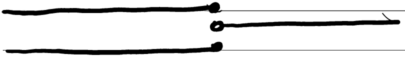
ev(m)

- Def. Topological space (X, U) is hausdorff if for any points $x, y \in X$

$(x \neq y) \Rightarrow \exists V, W \in X$ s.t. $x \in V$, $y \in W$ and

$V \cap W = \emptyset$.

• Exa: 1) $\mathbb{R}$ 3) $\mathbb{R}^n$
2) $\mathbb{Q}$ 4) any embedded



$\mathcal{M} = (-\infty,0]_1 \sqcup (-\infty,0]_2 \sqcup (0,\infty)$

Two charts

$\varphi_1 : (-\infty,0]_1 \sqcup (0,\infty) \longrightarrow \mathbb{R}$

$\varphi_2 : (-\infty,0]_2 \sqcup (0,\infty) \longrightarrow \mathbb{R}$

compatibility:

$\varphi_2 \circ \varphi_1^{-1} : (0,\infty) \xrightarrow[id]{} (0,\infty)$

$\varphi_1 \circ \varphi_2^{-1} : (0,\infty) \xrightarrow[id]{} (0,\infty)$

• **Def.** A topological space $(X,\mathcal{U})$ is second countable if $\exists$ countable $\mathcal{B} \subset \mathcal{U}$ such that $\forall V \in \mathcal{U} : V = \bigcup_{B \in \mathcal{B}, b \subset V} B$

• Exa. a) $\mathbb{R}$ is second countable

$B = \{ (a,b) \mid a,b \in \mathbb{Q} \}$

$(r,s) = \bigcup_{(a,b) \subset (r,s) \to a,b \in \mathbb{Q}} (a,b)$

b) $\mathbb{R}^n$ is also second countable

$B = \{ B(x,\varepsilon) \mid \varepsilon \in \mathbb{Q}_{>0}, x \in \mathbb{Q}^n \}$

c) embedded smooth manifolds

- Non-example

The long line

- Proof of proposition

a) ev is injective

if $m_1 \neq m_2$, then $ev(m_1) \neq ev(m_2)$.

That is, $\exists f \in C^\infty(M)$:

$$\underset{f(m_1)}{\overset{}{ev(m_1)(f)}} \neq \underset{f(m_2)}{ev(m_2)(f)}$$

$$ev(m_1)(f) \neq ev(m_2)(f)$$
$$\shortparallel \qquad\qquad \shortparallel$$
$$f(m_1) \neq f(m_2)$$

Since M is Hausdorff, $\exists$ open subsets V_1, V_2 s.t. $m_1 \in V_1$, $m_2 \in V_2$, $V_1 \cap V_2 = \emptyset$

must show that

can assume V_1, V_2 are domains of charts

$\varphi_1 : V_1 \longrightarrow W_1$ (compatible w/ M)

$\varphi_2 : V_2 \longrightarrow W_2$

pick $\varepsilon > 0$: $B(\varphi_1(m_1), 2\varepsilon) \subset W_1$

Take $f : M \longrightarrow \mathbb{R}$

$$f = \begin{cases} 0 & \text{if } m \notin V_1 \\ b\left(\|\varphi_1(m) - \varphi_1(m_1)\|^2 \cdot 1/\varepsilon^2\right), & \text{if } m \in V_1 \end{cases}$$

$b : \mathbb{R} \to \mathbb{R}$ smooth s.t.

$$b(0) = 1 \qquad b\big|_{[1,\infty)} = 0$$

b) Use Whitney's embedding theorem (requires M to be second countable)

Assume M to be embedded $M \subset \mathbb{R}^n$.

Given a homomorphism $\varphi : C^\infty(M) \to \mathbb{R}$,

•

• **Thm.** $M \xrightarrow{\cong} \mathrm{Hom}(C^\infty(M), \mathbb{R})$

$m \longmapsto ev_m = (f \mapsto f(m))$

• **Proof.** Injectivity ✓

Surjectivity pick a smooth embedding w/ a closed image $M \subset \mathbb{R}^m$, i.e. $\mathbb{R}^m / M$ is open.

$C^\infty(\mathbb{R}^m) \longrightarrow C^\infty(M)$

• Restriction preserves all operations

If $\varphi : C^\infty M \longrightarrow \mathbb{R}$ is a homomorphism

$$\begin{tikzcd} C^\infty \mathbb{R}^m \arrow[r, "r"] \arrow[rr, bend right, "\varphi \circ r"'] & C^\infty M \arrow[r, "\varphi"] & \mathbb{R} \end{tikzcd}$$

$\exists\, p \in \mathbb{R}^m$ s.t. $\varphi \circ r = ev_p$.

claim: $p \in M$. (Thus, $\varphi = ev_p$

Suppose $p \notin M$. There $\exists\, g \in C^\infty(\mathbb{R})$ s.t. $g|_M = 0$ and $g(p) \neq 0$.

(Need: $\mathbb{R}^m \setminus M$ is open).

(★) evaluate at g:

$\varphi(r(g)) = g(p) \neq 0$

$\varphi(r(0))$

$\varphi(0) = 0$

Therefore, $p \in M$.

- *Prop.* $\mathbb{R}^m \xrightarrow{\cong} \hom(C^\infty \mathbb{R}^m, \mathbb{R})$
- Proof: Injec ✓

Surjectivity: Assume $\varphi : C^\infty \mathbb{R}^m \longrightarrow \mathbb{R}$.

Hadamard's Lemma:

If $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is smooth, then $\exists\, \underset{\text{smooth}}{g_i : \mathbb{R}^m \to \mathbb{R}}$ $(1 \le i \le m)$

$$f = f(0) + \sum_i x_i \cdot g_i$$

- Example $m = 1$ $\qquad f'(0) = g(0)$

$$f(x) - f(0) = x \cdot g(x) \qquad m > 1 \text{ not unique}$$

$$g(x) = \frac{f(x) - f(0)}{x - 0}$$

Set $p_i = \varphi(x_i)$ $\qquad x_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ ith coordinate

Now, $p \in \mathbb{R}^m$ and we claim that $\varphi = ev_p$.

Pick $f \in C^\infty M$ $\quad f = f(p) = \sum_i (x_i - p_i) \cdot g_i$

$\exists g_i$: Hadamard's lemma

we have $\varphi(f) = \varphi(f(p) + \sum (x_i - p_i) \cdot g_i)$

$= f(p) + \sum(\varphi(x_i) - p_i) \cdot \varphi(g_i))) = f(p)$.

Sar's thm? Reference:

·Transformations of deformation

HW6 Prove M, N are smooth manifolds then $C^\infty(M, N) \xrightarrow{\cong} \mathrm{Hom}(C^\infty N, C^\infty M)$

$$\overset{\psi}{f} \longmapsto \quad (g^{\in} \longmapsto g \cdot f).$$ ✯

Show bijection.

- **Remark** $M = \mathbb{R}^0 \cong$

$$N \cong C^\infty(\mathbb{R}^0, N) \xrightarrow{\cong} \hom(C^\infty N, \mathbb{R}).$$

$$N \cong C^\infty(\mathbb{R}^0, N),\ M \cong C^\infty(\mathbb{R}^0, M) \Rightarrow \hom(C^\infty N, \mathbb{R}) \times \hom(C^\infty M, A)$$

- **Remark** The functor

$$C^\infty \cdot \mathrm{Man}^{op} \longrightarrow \mathrm{Comm\,Alg}_{\mathbb{R}} \overset{F}{\supset}$$

is fully faithful. In particular,

$$C^\infty : \mathrm{Man}^{op} \longrightarrow F$$

is an equivalence of categories

$\mathrm{Ob}\,F = \{A \in (\mathrm{Alg}_{\mathbb{R}}) \mid \exists M : A \cong C^\infty(M, \mathbb{R})$

- **Remark** constructions of differential geometry continue to make sense for commutative real algebras

$$A \in \mathrm{CAlg}_{\mathbb{R}} \quad , A \notin F.$$

- **Exa.**

a) Take $G \subset \mathbb{R}^m$ closed (i.e. $\mathbb{R}^m \setminus G$ is open.)

Take $I = \{ f \in C^\infty \mathbb{R}^m \mid f|_G = 0 \}$

Then $A = C^\infty(\mathbb{R}^m) / I$ is the algebra of smooth functions on G.

- **Exa.** $f'(a)$ $\quad f(a+\varepsilon) = f(a) + \varepsilon \cdot f'(a) + \varepsilon^2 \cdot g(a+\varepsilon)$

Take $\varepsilon^2 = 0$ $\quad = f(a) + \varepsilon \cdot f'(a)$

$$f'(a) : f(a+\varepsilon) - f(a) = \varepsilon \cdot f'(a)$$

$A \in CAlg_{\mathbb{R}}$

$A = \mathbb{R}[\varepsilon]/\varepsilon^2 = \{a + b\cdot\varepsilon \mid a, b \in \mathbb{R}\}$

$$(a_1 + b_1\cdot\varepsilon) + (a_2 + b_2\cdot\varepsilon) = (a_1 + a_2) + (b_1 + b_2)\cdot\varepsilon$$

$$(a_1 + b_1\cdot\varepsilon)(a_2 + b_2\cdot\varepsilon| = a_1 a_2 + (a_1 b_2 + a_2 b_1)\cdot\varepsilon$$

(·· · • · ··) infinitesimal neighborhood/fuzz

geometry $\longleftrightarrow$	algebra
M: Smooth manifold	$C^\infty M$: comm real algebra
$p \in M$ $\xrightarrow{p \mapsto ev_p}$	$C^\infty M \xrightarrow{p} \mathbb{R}$: homomorphism
$M \to N$ smooth map	$C^\infty N \to C^\infty M$: homomorphism
$v \in TM$	$C^\infty M \to \mathbb{R}$: derivation
$C^\infty M \xrightarrow{d} \Omega^1 M$	$C^\infty M \to \Omega^1 C^\infty M$ kahler C^∞-diff
$M \to TM$ vectorfield	$C^\infty M \to C^\infty M$: derivation

Recall A module over a commutative real algebra
A is a real vector space M together with
a homomorphism of commutative real algebras
$A \to End(M)$. where $End M$ is the $\ell(a)m = a\cdot m$
algebra of endomorphisms are R-linear maps
$M \to M$ $+, -, 0$ defined pointwise. $1 = id_M$
$f\cdot g = f \circ g$

Unfold: M is a real vector space with addit

$A \times M \longrightarrow M$: scalar multiplication

$(a_1+a_2)m = a_1 m + a_2 m$; $0 \cdot m = 0$

$a \cdot (m_1 + m_2) = a \cdot m_1 + a \cdot m_2$; $a \cdot 0 = 0$

$(a_1 \cdot a_2) \cdot m = a_1 \cdot (a_2 \cdot m)$; $1 \cdot m = m$

$(r \cdot a) \cdot m = a \cdot (r \cdot m) = r \cdot (a \cdot m)$ $r \in \mathbb{R}$

$V \cong \mathbb{R}^n$

- **Exa.** $A = C^\infty M$, $V = \mathbb{R}$

An A-module structure on V is a homomorphism of $\mathbb{R}$-algebras $C^\infty(M) \longrightarrow \mathrm{End}(M) \cong \mathbb{R}$

i.e., a point $p \in M$

$f \in C^\infty M$, $r \in V = \mathbb{R}$, $f(r) = f(p) \cdot r$

- **Exa.** $A = \mathbb{R}$, $V \in \mathrm{Vect}_{\mathbb{R}}$

$$l(r) = l(r \cdot 1) = r \cdot l(1) = r \cdot \mathrm{id}_V$$

An $\mathbb{R}$ module $\equiv$ $\mathbb{R}$-vector space

Recall $C^\infty(M)$ - module structures on $\mathbb{R}$

$\equiv$ homomorphisms $C^\infty M \longrightarrow \mathbb{R}$

$\equiv$ points in M.

$p \in M$: $\underset{C^\infty M}{f} \cdot \underset{\in \mathbb{R}}{r} = f(p) r$

- **Def.** Suppose A is a commutative real algebra, M is an A-module. A derivation of A w/ values in M is an $\mathbb{R}$ linear map $A \xrightarrow{d} M$ s.t. Leibniz rule holds

$$d(a_1,a_2) = \underbrace{d(a_1)}_{m}\cdot\underbrace{a_2}_{a} + \underbrace{a_1}_{a}\cdot\underbrace{d(a_2)}_{m}$$

• **Exa.** $A = C^\infty S \quad M = C^\infty S$

$f \mapsto f'$ is a derivation

• **Exa.** $f \mapsto D_v f(s)$

$$\begin{array}{c} D_v f(s) \\ \| \\ t \mapsto f(s+t\cdot v)'_{t=0} \end{array}$$

$$\begin{tikzcd} C^\infty S \arrow[r, "\text{derivation}"'] \arrow[d, phantom, "\|"] & \mathbb{R} \arrow[d, phantom, "\|"] \\ A & M \end{tikzcd}$$
$M \leftarrow$ A-module structure by ev_s.

Leibniz rule: $d(fg) = d(f)g + fd(g)$

$D_v(f\cdot g)(s) = (D_v f(s))\cdot g(s) + f(s)\cdot D_v g(s)$.

$\underset{\overset{\cap}{A}}{f} \bullet \underset{\in \mathbb{R}}{m} = f(s)\cdot m$

arbitrary module structure ↙

• **Prop.** The map $S \times \mathbb{R}^n \longrightarrow Der(C^\infty S, \mathbb{R})$
$(s,v) \longmapsto (f \mapsto D_v f(s))$.

is a bijection

• **Proof.**

Injectivity: Suppose we have (s_1,v_1) and $(s_2,v_2) \in S \times \mathbb{R}^n$ map to the same derivation $C^\infty S \to \mathbb{R}$. The module structures on $\mathbb{R}$ must coincide $\Rightarrow s_1 = s_2$. Take $f \in C^\infty S$:

$f(s) = \langle w, s - s_1\rangle$ where $w \in \mathbb{R}^n$.

$D_{v} f(s_1) = \langle w, v_1\rangle \quad \forall\, w \in \mathbb{R}^n$
$\qquad\qquad\qquad\ \|$
$D_{v_2} f(s_2) = \langle w, v_2\rangle \qquad \langle w, v_1 - v_2\rangle = 0$

$w = v_1 - v_2 \quad \Rightarrow \quad v_1 - v_2 = 0 \Rightarrow v_1 = v_2$

Surjectivity: Suppose $d: C^\infty S \longrightarrow \mathbb{R}$ is a derivation.

The $C^\infty S$-module structure on $\mathbb{R} \equiv$ a unique $s \in S$

$f \cdot m := f(s)\cdot m$. Now $d(H_{s,w}) \in \mathbb{R}\ \forall\ w \in \mathbb{R}^n$

we get a $\mathbb{R}$-linear map $\underset{\mathbb{R}^n}{\underset{\cap}{w}} \longmapsto d(H_{s,w})$

$$\mathbb{R}^n \longmapsto \mathbb{R}.$$

$d(H_{s,w}) = \langle w, v\rangle$; w.t.s. that

$D_v f(s) = d(f)\ \forall\ f \in C^\infty S$.

Recall Hadamard Lemma

$$f(t) = f(s) + \sum_{1\le i\le n} (t_i - s_i)\cdot g_i(t)$$

$$d(f(s)) + \sum_i \left[d(t_i - s_i)\cdot g_i + (t_i - s_i)\cdot d(g_i)\right]$$

prove derivation vanishes constants

$$H_{s,e_i}(t) = \langle e_i, t-s\rangle = t_i - s_i$$

$$= \sum dH_{s,e_i}\cdot g_i + \underbrace{(t_i - s_i)\cdot dg_i}_{\text{vanishes if } t=s}$$

$$= \sum \langle e_i, v\rangle \cdot g_i + (t_i - s_i)\cdot dg_i = \sum_i \langle e_i, v\rangle\, g_i(s)$$

$$= \sum v_i \cdot \partial_i f(s) = D_v f(s).$$

$$\partial_i f = \frac{\partial f}{\partial x}(s) = g_i(s).$$

Corollary: Every derivation is also a C^∞ derivation

- Def. A map of sets $C^\infty S \longrightarrow M$ is a C^∞ derivation if the chainrule holds

$g_i \in C^\infty S$ $$d(f(g_1,\dots,g_i)) = \sum_i \frac{\partial f}{\partial x_i}(g_1,\dots,g_m)\cdot d(g_i)$$

$f \in C^\infty \mathbb{R}^m$

Remark Every C^∞ derivation is a derivation.

Take $f_1(x_1, x_2) = x_1 + x_2$ (additivity)

$f_2(x_1, x_2) = x_1 \cdot x_2$ (Leibniz)

$f_3(x_1) = r \cdot x_1$

$f_1 : d(g_1 + g_2) = dg_1 + dg_2$

$f_2 : d(g_1 \cdot g_2) = g_2 \cdot dg_1 + g_1 \cdot dg_2$

Def. $S \in Man, w \in TS \xrightarrow{D} Der(C^\infty S, \mathbb{R})$

(i) $W = [(\overset{\in S}{s}, \overset{\in \mathbb{R}^n}{v}, \overset{\in I}{i})]$ $\quad S = (S, (\varphi_i : U_i \to \overset{\subset \text{open } \mathbb{R}^n}{V_i})_{i \in I})$

$s \in \mathcal{U}_i$. $D_W f = D_v(f \circ \varphi_i^{-1})(\varphi_i(s))$

HW + hint it's chain rule.

HW 7: Show well-defined.

(ii) $W = [\gamma]$ $\quad \gamma : \mathbb{R} \to S$

$D_W f = (f \circ \gamma)'(0)$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"'] \arrow[rr, bend right, "f \cdot \gamma"'] & S \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

Proposition D is a bijection HW8

Def. A tangent vector in $S \in man$ is a point derivation $C^\infty S \to \mathbb{R}$.

Today
$$\begin{tikzcd} M \arrow[r] \arrow[d, "p_M"'] \arrow[dr, phantom, "\#"] & N \arrow[d, "p_N"] \\ TM \arrow[r, "Tf"'] & TN \end{tikzcd}$$
$p_N \circ Tf = f \circ p_M$

• Example $m \overset{open}{\subset} \mathbb{R}^n$

$TM = M \times \mathbb{R}^n$

$Tf(m, v) = Df(m)(v) = D_v f(m)$

• Def. ① $f : M \to N$

$M = (M, (\varphi_i : U_i \to V_i)_{i \in I})$

$N = (N, (\psi_j : W_j \to X_j)_{j \in J})$

$m \in M, v \in \mathbb{R}^m, i \in I$

$Tf([(m, v, i)]) = [(f(m), D_v(\psi_j \circ f \circ \varphi_i^{-1})(\varphi_i(m)),$
$j)]$ pick $j \in J$ s.t. $f(m) \in W_j$

→ Independence of the choice of j

$[(f(m), D_v(\psi_j \circ f \circ \varphi_i^{-1})(\varphi_i(m))] = ☆$
$\overset{?}{=} [(f(m), D_v(\psi_{j'} \circ f \circ \varphi_i^{-1})(\varphi_i(m)), j')]$

$☆ = [(f(m), D_v(\psi_{j'} \circ \psi_j^{-1} \circ \psi_j \circ f \circ \varphi_i^{-1})(\varphi_i(m), j'))]$

Independence of i: similar.

• Def. ② $f([\gamma]) = [f \circ \gamma]$, $\gamma : \mathbb{R} \to M$

independent of γ : chart $\}$ chain rule

• Def. ③ $f : M \to N$

$C^\infty M \xrightarrow{v} \mathbb{R}$ ☆ derivation $\qquad C^\infty f : C^\infty N - C^\infty M$

$Tf(v) \qquad v \circ C^\infty f \qquad\qquad g \longmapsto (g \circ f) \cdot r$ ☆

$v \circ C^\infty f : C^\infty N \longrightarrow \mathbb{R} \qquad g \cdot r = (g \circ f) \cdot r$

Claim: $v \circ C^\infty f$ is a derivation

$\mathbb{R}$-linear ✓

$(v \circ c^{\infty} f)(h_1 \cdot h_2) = v(c^{\infty} f(h_1 \cdot h_2))$

$v((c^{\infty} f)(h_1) \cdot_{\star} (c^{\infty} f)(h_2)) = v((c^{\infty} f)(h_1))$

$\cdot\, c^{\infty} f(h_2) + (c^{\infty} f)(h_1) \cdot_{\star} v((c^{\infty} f)(h_2))$

$= (v \circ c^{\infty} f)(h_1) \cdot_{\square} h_2 + h_1 \cdot_{\square} (v \circ c^{\infty} f)(h_2)$

Q: why are these def. equivalent?

$Tf① = Tf② = Tf③$

Pass to a single chart in M and N.

①=②

$$\begin{tikzcd}
{[f \circ \gamma]} \arrow[r, mapsto] \arrow[d, mapsto] & (f \circ \gamma)'(0) \arrow[d, equal] \\
Tf(\gamma'(0)) & D_{\gamma'(0)} f(\gamma(0))
\end{tikzcd}$$

①=③

$M \subset \mathbb{R}^m \longrightarrow N \subset \mathbb{R}^n$

$$\begin{tikzcd}
(m, v) \arrow[r, mapsto] \arrow[d, mapsto] & (f(m), D_v f(m)) \arrow[dd, mapsto, bend left] \\
(g \mapsto D_v g(m)) \arrow[r, mapsto] & (h \mapsto D_v(h \circ f)(m)) \arrow[d, equal, "\text{chain rule}"] \\
 & D_{D_v f(m)} h(f(m))
\end{tikzcd}$$

Example. $M = N = \mathbb{C}$ $\quad TM = M \times \mathbb{R}^2 = M \times \mathbb{C}$

HW 9: $\quad TN = N \times \mathbb{R}^2 = N \times \mathbb{C}$

$f : M \longrightarrow N$

$z \longmapsto z^2$

$\Vert$

$x + iy \longmapsto x^2 - y^2 + 2xyi$

$$Tf(z, \underbrace{a \cdot \partial_x + b \cdot \partial_y}_{\begin{pmatrix} a \\ b \end{pmatrix}}) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

$= 2 \cdot (xa - yb, ya + xb)$

$= 2 \cdot z \cdot v$

$v = a + ib$

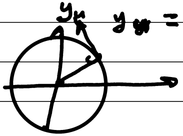
Example $S^1 \subset \mathbb{R}^2$ $\quad S^1 = \{(x,y) \mid x^2 + y^2 = 1\}$

$S^1 \xrightarrow{f} S^1$

$f(x,y) = (x^2 - y^2, 2xy)$

sidenote $M \subset \mathbb{R}^a$ $\quad$ HW9: ④ $\Longleftrightarrow$ TM, Tf

$TM = (m, u)$ $\; m \in M, u \in \mathbb{R}^a$ $\quad$ ①∨②∨③



$y = (u_x, u_y)$

$\langle (x,y), (u_x, u_y) \rangle = 0$

$= x u_x + y u_y = 0 \Longleftrightarrow Re(z \bar{u}) = 0$

$(u_x, u_y) = \lambda (y - x)$

$z = x + iy$

$Tf(z, u) = (f(z), 2z \cdot u)$

$= (\underset{\overset{\uparrow}{S^1}}{z^2}, 2zu)$

$Re(z^2 \cdot 2z\bar{u})$

$= Re(z \cdot z \cdot \bar{z} \cdot \bar{u} \cdot 2)$

$= Re(z \cdot \bar{u} \cdot 2) = 0$

•Example : V real vectorspace : V_1, V_2

$f: V_1 \longrightarrow V_2$

$TV_i = V_i \times V_i$

$$Tf : \underset{\substack{\shortparallel \\ V_1 \times V_1}}{TV_1} \longrightarrow \underset{\substack{\shortparallel \\ V_2 \times V_2}}{TV_2}$$

$Tf(p,v) = (f(p), f(v)) = f(p,v)$

•Exa.

$S^n = \{ v \in \mathbb{R}^{n+1} \mid \|v\| = 1$

$f: \mathbb{R}^{n+1}/0 \longrightarrow S^n$

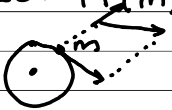
$v \longmapsto \frac{1}{\|v\|} \cdot v$

$T\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$

$TS^n = \{ (m, u) \mid m \in S^n, u \in \mathbb{R}^{n+1}$

$\langle m, u \rangle = 0 \}$

HW 10: $Tf(m,v) = (f(m), v - \frac{\langle m, v \rangle \cdot v}{\|v\|})$



Last time

tangent vector	tangent map	vector field
$v \in TM$	$Tf : TM \to TN$	$v : M \to TM$
$v = (u,w) \in U_i \times \mathbb{R}^n$ in a chart	$Tf(u,w) = D_w f(u)$	$X_i : U_i \to \mathbb{R}^n$
$v = [\gamma : \mathbb{R} \to M]$	$Tf(v) = Tf([\gamma])$	$X = [\Gamma]_i$
$= [\gamma : U \to M]$	$= f \circ \gamma$	$\Gamma : W \to M$
$v : C^\infty M \to \mathbb{R}$		$W \underset{\text{open}}{\subset} M \times \mathbb{R}$
endomorphisms ?	$Tf(v) = v \circ C^\infty f$	$X : C^\infty M \xrightarrow[\text{deriv.}]{} C^\infty M$
	$C^\infty f : C^\infty N \to C^\infty M$	
$v = [(u,w,i)]$	$g \to g \circ f$	

- **Def.** A vector field X on a smooth manifold M is a smooth map $X : M \to TM$

s.t.

$$\begin{tikzcd} M \arrow[r, "X"] \arrow[dr, "id_M"'] \arrow[dr, phantom, "\#", near start] & TM \arrow[d, "\phi"] \\ & M \end{tikzcd}$$

- **Example** $M \underset{\text{open}}{\subset} \mathbb{R}^n$ $\quad TM \cong M \times \mathbb{R}^n \xrightarrow{p} M$

$$\begin{tikzcd} M \arrow[r, "X"] \arrow[drr, "id_M"'] & TM = M \times \mathbb{R}^n \arrow[d, phantom, "\#", near start] \arrow[dr] & \\ & & M \end{tikzcd}$$

$X(m) = (m, \bar{x}(m))$

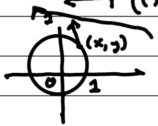
$\bar{x} : M \to \mathbb{R}^n$

- **Example** $M = S^1$ $\quad TS^1 \cong S^1 \times \mathbb{R}$

$S^1 \subset \mathbb{R}^2$

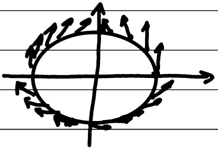
$\{(x,y) \mid x^2 + y^2 = 1\}$

$\longleftarrow ((x,y), t)$



$((x,y), (-ty, tx))$

$X(x,y) = \big((x,y), x, (-xy, x^2)\big)$



• Def. $M = (M, (\varphi_i : \underset{\underset{M}{\cap}}{U_i} \longrightarrow \underset{\underset{\mathbb{R}^n}{\cap}}{V_i})_{i\in I})$

A vector field X on M is

$$(X_i : U_i \longrightarrow \mathbb{R}^n)_{i\in I}$$

such that $\forall\, i,j \in I$

$$D(\varphi_j \circ \varphi_i^{-1})(X_i\big|_{U_i\cap U_j}) = X_j\big|_{U_i \cap U_j}$$

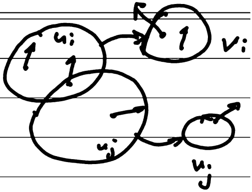
$$D_{X_i(u)}(\varphi_j \circ \varphi_i^{-1})(\varphi_i(u)) = X_j(u) \text{ or}$$

$$D(\varphi_i \circ \varphi_j^{-1})\big|_{\varphi(u)} \cdot X_i(u) = X_j(u)$$

$$X : M \to TM \longmapsto (X_i : U_i \longrightarrow \mathbb{R}^n)_{i\in I}$$

$$\begin{tikzcd} X|_{U_i} : U_i \arrow[r] \arrow[dr] \arrow[drr, bend right=40, "(\varphi_i, X_i)"'] & TM & \\ & TU_i \arrow[u, "T\iota_i"'] \arrow[r, "\cong", "T\varphi_i"'] & TV_i \cong V_i \times \mathbb{R}^n \end{tikzcd}$$

$$U_i \xrightarrow{\iota_i} M$$



$D(\varphi_j \circ \varphi_i^{-1})(\varphi_i(u))(x_i(u))$

$D(\varphi_j \circ \varphi_i^{-1})(\varphi_i(u))(T\varphi_i(x(u))$

$= D(\varphi_j)(u)(T(x(u))$

• Prop. (chain rule)

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[dr, "g\circ f"'] & M \arrow[d, "g"] \\ & N \end{tikzcd}$$

$$\begin{tikzcd} TL \arrow[r, "Tf"] \arrow[dr, "T(g\circ f) = Tg\circ Tf"'] & TM \arrow[d, "Tg"] \\ & TN \end{tikzcd}$$

"T is a functor"

$T: Man \to Man$

• Proof. ③ $T(g\circ f)([\gamma: \mathbb{R} \to L])$

$= [g\circ f\circ \gamma: \mathbb{R} \to N)$

$= Tg ([f\circ\gamma])$

$= Tg(Tf([\gamma]))$.

precom-position ④ $T(g\circ f)(v) = v\circ C^\infty(g\circ f) = v\circ(C^\infty f \circ C^\infty g)$

$= (Tf(v))\circ C^\infty g = Tg(Tf(v))$

↳ $T(\varphi_j \circ \varphi_i^{-1})(T\varphi_i(X_i(u))$

$T(\varphi_j \circ \varphi_i^{-1} \circ \varphi_i)(X_i(u)$

$T(\varphi_j)(X_i(u)) = X_j(u)$.

Last time: vectorfield on M.

① $M \xrightarrow{v} TM$ $\quad p\circ v = id_M$

② $M = (M, (\varphi_i : U_i \longrightarrow V_i)_{i\in I})$

$X_i : V_i \longrightarrow \mathbb{R}^n$

$T(\varphi_j \circ \varphi_i^{-1})(x, X_i(x)) = (\varphi_j(\varphi_i^{-1}(x)), X_j(\varphi_j(\varphi_i^{-1}x)))$

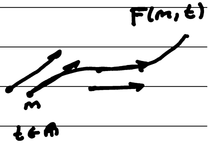
③ $W \subset M \times \mathbb{R}$ $\quad , W \supset M \times \{0\}$

$F : W \longrightarrow M \quad [F]$

$F(F(x,s),t) = F(x, s+t)$

$F_1 \sim F_2$ if $\exists\ W_3 \subset W_1 \cap W_2$

$F_1|_{W_3} = F_2|_{W_3}$



"Local semigroup, Local action of Lie group."

④ Derivation

② → ①

$p \in TM$, $p = [(m, w, i)]$. Fix $i \in I$

$U_i \xrightarrow{Y_i} TM$

$\overset{\cup}{m} \longrightarrow [(m, \ {}_i(\varphi_i(m)), i]$

$Y_i|_{U_i \cap U_j} = Y_j|_{U_i \cap U_j}$

$[(m, X_i(\varphi_i(m)), i)] = [(m, X_j(\varphi_j(m)), j)]$

$Y : \bigcup_i U_i \longrightarrow TM$

$$\begin{tikzcd}
\mathbb{R}^m & \mathbb{R}^n \\
\cup & \cup \\
U \arrow[r, "f"] & V \\
TU = U \times \mathbb{R}^n \arrow[u] \arrow[r, bend right, "Tf"] & TV = V \times \mathbb{R}^n \arrow[u]
\end{tikzcd}$$

$Tf(u,w) = (f(u), D_w f(u))$

• Example

$M = \mathbb{R}$

$X : M \to \mathbb{R} \quad X(t) = t^2$

$\frac{\partial F}{\partial t}(m,t) = X(F(m,t))$

$\frac{\partial F}{\partial t}(m,t) = F(m,t)^2$

Assume $F(m,t) \neq 0$ $\quad F(m,t)^{-2} \cdot \frac{\partial f}{\partial t}(m,t) = 1$

$\frac{\partial}{\partial t}\left(\frac{F(m,t)^{-1}}{-2} \right) = \frac{\partial}{\partial t}(t) \quad \longrightarrow 0$

$-F(m,t)^{-1} = t + c \qquad F(m,t)^{-1} = -t + c$

$F(m,t) = -\frac{1}{t+c} = -\frac{1}{t-1/m}$

$F(m,0) = -\frac{1}{c}, \quad c = -1/m$

$F(m,t) = \begin{cases} 0, & m = 0 \\ \frac{1}{\frac{1}{m} - t}, & m \neq 0 \end{cases}$

HW 11

$T(M \times N) \cong TM \times TN$

③ → ① $[F]$ $\quad F : \underset{M\times\mathbb{R}}{\underset{\cap}{W}} \longrightarrow M$

$TF : TW \xrightarrow{TF} TM$

$\cap$

$T(M \times \mathbb{R})$

$\cong$

$TM \times T\mathbb{R}$

$TM \times \underbrace{\mathbb{R}}_{t} \times \underbrace{\mathbb{R}}_{\frac{\partial}{\partial t}}$

① $M \xrightarrow{\nu} TM$

$\nu(m) = TF\big((m,0), 0, 1\big)$

$\nu(m) = \frac{\partial F}{\partial t}(m,0) \in TM$

①, ② ⟶ ③

Theorem $M \in \mathrm{Man}$, $v \in \mathfrak{X}M$

$$\mathfrak{X}M = \{ v : M \to TM \mid p \circ v = \mathrm{id}_M \}$$

$\exists!$ ③ $\quad v = \frac{\partial F}{\partial t}(m,0)$

$F : W \to M$ (existencc and uniqueness to ODEs)

If $F_1 : W_1 \to M$ ③, then $W_1 \subset W$

$\forall m \in M$: $W \cap (\{m\} \times \mathbb{R})$ is aninterval of the maximal integral curveofM. MC

Proof. Immediate reduction to $\mathbb{R}^n$ using charts, apply the Banach Fixed Point theorem.

$$\frac{\partial F}{\partial t}(m,t) = G(F(m,t),(m,t))$$

jets

$$x_i : M \times \underset{t}{\mathbb{R}} \longrightarrow V$$

$$\frac{dx_i}{dt} = G_i(x_1(t), x_2(t), \dots, x_n(t), t)$$

$$\frac{\partial F}{\partial t}(m,t)$$ $\quad X : M \times \mathbb{R} \to TM$

$$= X(F(m,t),t) \quad N = M \times \mathbb{R}$$

$$Y : N \longrightarrow TN$$

$$Y(m,t) = (x(m,t),t,1$$

$$\frac{\partial G}{\partial t}(m,t) = Y(G(m,t))$$

Citw manifold thm.

.

④ · **Def.** A vector field on M is a derivation $X : C^\infty M \longrightarrow C^\infty M$, i.e., X is an $\mathbb{R}$ - linear map s.t.

$$X(fg) = X(f)\cdot g + f \cdot X(g)$$

· ex. think of how to get a derivation using a flow or curve

· **Remark** The value of a vector field X at some point $p \in M$ $\quad C^\infty M \xrightarrow[ev_m]{} \mathbb{R}$ is the point derivation

$$\begin{tikzcd} C^\infty M \arrow[rr, "X_m"] \arrow[dr, "X"'] & & \mathbb{R} \\ & C^\infty M \arrow[ur, "ev_m"'] & \end{tikzcd}$$

$$(ev_m \circ X)(f \cdot g) = ev_m(X(fg)) = ev_m(X(f)g + fX(g))$$
$$= ev_m(Xf)\cdot g(m) + f(m) \cdot ev_m(X(g))$$

④ Given a derivation $X : C^\infty M \xrightarrow{deriv} C^\infty M$

Given a point $m \in M$: $ev_m \circ X$ is the point derivation corresponding to the tangent vector X_m.

Given point derivations $X_m : C^\infty M \longrightarrow \mathbb{R}$ $(m \in M)$, we have $X : C^\infty M \longrightarrow \mathbb{R}^M$ (a derivation)

$$f \longmapsto m \longmapsto X_m(f)$$

X factors as an actual derivation to smooth functions if X_m depends smoothly on $m \in M$.

③ $F : W \longrightarrow M$

$W \subset M \times \mathbb{R} \qquad W \supset W \times \{0\}$

$$F(F(x,t),s) = F(F(m,s),t)$$

$$F(m,0) = m.$$

$$X : C^\infty M \longrightarrow C^\infty M \qquad X = [F]$$

$$\overset{\in}{f} \longrightarrow \mathcal{L}_{[F]} f$$

$$= \lim_{t \to 0} \frac{1}{t} \big(f(F(m,t)) - f(m) \big)$$

$$= \big(t \longmapsto f(F(m,t)) \big)'(0).$$

- Prop. if $[F]$ ③, then X is a derivation.

- Claim: X is a derivation

$\mathbb{R}$-Linear

Leibniz

$$\mathcal{L}X(fg)^{(m)} = \big(t \longmapsto (f \cdot g)(F(m,t)) \big)'(0)$$

$$= \big(t \longmapsto f(F(m,t)) \big)'(0) \cdot g\big(\underbrace{F(m,0)}_{m}\big)$$

$$\big(t \mapsto g(F(m,t)) \big)'(0) \cdot f\big(\underbrace{F(m,0)}_{m}\big)$$

$$= \mathcal{L}_{[F]}(f)(m) \cdot g(m) + f(m) \cdot \mathcal{L}_{[F]}(g)(m)$$

$$\mathcal{L}_{[F]}(fg) = \mathcal{L}_{[F]} f \cdot g + f \cdot \mathcal{L}_{[F]} g$$

④ → ①

- Prop. If $X : C^\infty M \xrightarrow{\text{deriv}} C^\infty M$, then

$$s : M \longrightarrow TM \qquad s(m) = ev_m \circ X$$

is a smooth map and a section $p \circ s = id\,M$.

<u>• Proof.</u> ① The base point of $ev_m \circ X$ is ev_m. Thus $p \circ s = id\,M$.

② s is smooth. Working in a chart, assume $M \underset{open}{\subset} \mathbb{R}^n$.

$s(m)_i = (ev_m \circ X)(x_i)$

$x_i \in C^\infty M$, $x_i : \underset{p}{\mathbb{R}^n} \longrightarrow \underset{p_i}{\mathbb{R}}$

$s(m)_i = X(x_i)(m)$.

$s_i \in C^\infty M$ smooth function

Thus s is smooth

□

• Proof ④ $\xrightarrow[\checkmark]{}$ ① $\xrightarrow[\checkmark]{}$ ③

Lie derivatives of vector fields

= Lie brackets of vector fields

$X, Y \in \mathfrak{X}(M)$

$\mathcal{L}_X Y = [X,Y] \in \mathfrak{X}(M)$

• Def. $X, Y : C^\infty M \xrightarrow{deriv} C^\infty M$

Then, : $[X,Y] : C^\infty M \to C^\infty M$

$[XY](f) = X(Y(f)) - Y(X(f))$

Claim: $[X,Y]$ is a derivation. $\mathbb{R}$ linear ✓

$[X,Y](fg) = X(Y(fg)) - Y(X(fg))$

$= X(Y(f)\cdot g + f\cdot Y(g)) - Y(X(f)\cdot g + f\cdot X(g))$

$= X(Y(f))\cdot g + Y(f)\cdot X(g) + X(f)\cdot Y(g) + f\cdot X(Y(g))$

$- Y(X(f))\cdot g - X(f)\cdot Y(g) - Y(f)X(g)$

$- f\cdot Y(X(g))$

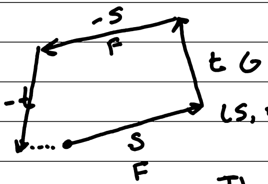
$= [X,Y](f)\cdot g + f\cdot [X,Y](g)$

• **Def ②** [F], [G]: local flows on M.

[F, G]: vector field, Assume $M \subset \mathbb{R}^n$. In a chart,

$$[F,G](\underset{\hat{m}}{m}) = \lim_{(s,t)\to 0} \frac{1}{s\cdot t}\left(G(F(G(F(m,s),t),-s),-t) - m\right)$$

In general: $\overset{W\subset}{\mathbb{R}^2} \xmapsto{h} M$



$(s,t) \longmapsto G(F(G(F(m,s),t),-s,-t)$

Th: $W \times \mathbb{R}^2 \to TM$

$(s,t),\ (s,0) \to M$

$W \xrightarrow{\frac{\partial h}{\partial s}} M$

$\frac{\partial^2 h}{\partial s \partial t} : W \to TM$ evaluated at $(0,0)$

[F, G]

$\frac{\partial^2 h}{\partial s \partial t}(0,0)$.

[F, G] product of two infinitesimal.

Commuting vector fields, coordinate vector fields.

Incoordinates $M = \mathbb{R}^n$

$$x_i : \mathbb{R}^n \xrightarrow{ith} \mathbb{R}$$

$$\frac{\partial}{\partial x_i} : C^\infty M \longrightarrow C^\infty M$$
$$f \longmapsto \frac{\partial f}{\partial x_i}$$

$$M \longrightarrow TM = M \times \mathbb{R}^n$$
$$m \longmapsto (m, e_i)$$

$$e_i = (0,0,\dots,0,\overset{ith}{1},0,\dots,0)$$

• Remark. $\mathfrak{X}(M) = \{ X : M \to TM \mid p \circ X = id M \}$ is a module over $C^\infty M$

$$f \in C^\infty M,\ X \in \mathfrak{X}(M) : f \cdot X \in \mathfrak{X}(M)$$

$$(f \cdot X)(m) = \underset{\mathbb{R}}{\underset{\rotatebox{90}{$\in$}}{f(m)}} \cdot \underset{T_m M}{\underset{\rotatebox{90}{$\in$}}{X(m)}}$$

$$X : C^\infty M \xrightarrow{deriv} C^\infty M f \cdot X : C^\infty M \to C^\infty_M (f X)(g)$$
$$= f \cdot X(g).$$

• Prop. If $M \underset{open}{\subset} \mathbb{R}^n$, then $\mathfrak{X}(M)$ is a free $C^\infty M$ module of rank n with a basis $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \in \mathfrak{X}$

• Proof. Given some $X \in \mathfrak{X}(M)$ set $f_i = X(x_i)$ then $X = \sum_i f_i \frac{\partial}{\partial x_i}$

∎

• Proof. $X(g) = \sum_i f_i \cdot \frac{\partial g}{\partial x_i}$. Pick some $m \in M$

$$X(g(m)) = \sum f_i(m) \frac{\partial g}{\partial x_i}(m)$$

$$g(p) = g(m) + \sum h_i(p) \cdot (p_i - m_i)$$

Hadamard's Lemma : $\exists h_i$

★

$$X\Big(g(m) + \sum_i (p_i - m_i) h_i\Big)$$

$$= \Big(\sum_i X(p_i - m_i) \cdot h_i + \underset{p=m}{\overset{0}{\cancel{(p_i - m_i) h}}}\Big)(m)$$

$$= \Big[\sum f_i \cdot h_i\Big](m) = \sum_i f_i(m) \cdot h_i(m)$$

$$= \sum_i f_i(m) \cdot \frac{\partial g}{\partial x_i}(m) \quad ★ \ ✓$$

• Prop. The Lie bracket

1. $[-,-] : \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{X}$ is $\mathbb{R}$ - bilinear

   $[f \cdot X, g \cdot Y] = f \cdot g [X, Y]$

4. $[X, f \cdot Y] - f[X, Y] = X(f) \cdot Y$

2. $[X, [Y, Z]] - [Y, [X, Z]] + [Z, [X, Y]] = 0$

3.) $[X, Y] = -[Y, X]$ ↑ Jacobi Identity

1 - 3 Lie algebra

• Prop. Derivations of $A \in CAlg_{\mathbb{R}}$ form a Lie algebra over $\mathbb{R}$ $[D_1, D_2] = D_1 D_2 - D_2 D_1$

• Proof. 0) $[D_1, D_2]$ is a derivation

1) $\mathbb{R}$ bilinear ✓

2) ✓

3) ✓

- Prop. 4 is true
- <u>Prop.</u> Evaluate on some $g \in C^\infty M$

$$X(fY(g)) - \cancel{f \cdot Y(X(g))} - f \cdot X(Y(g))$$

$$\cancel{+ f\, Y(X(g))}$$

$$X(f)\cdot Y(g) + fX(y(g)) - f\cdot X(y(g))$$

$$X\ (f)\cdot Y)\,(g)$$

- Example $M = \mathbb{R}^n$

$\mathfrak{X}M$: free $C^\infty M$-module

w/basis $\frac{\partial}{\partial x_i}$, $1 \le i \le n$

$\forall X \in \mathfrak{X}(M) : \exists!\ (f_i)_{i \le 1 < n}$

$$X = \sum_i f_i \frac{\partial}{\partial x_i}$$

$$\left[\sum_i f_i \frac{\partial}{\partial x_i}, \sum_j g_j \cdot \frac{\partial}{\partial x_j}\right] = \sum_{i,j} \left[f_i \frac{\partial}{\partial x_i}, g_j \cdot \frac{\partial}{\partial x_j}\right]$$

$$\mathcal{L}_X(f\cdot y) = (\mathcal{L}_X f)\cdot Y + f\cdot \mathcal{L}_X Y$$

$$= \sum_{ij} \left(f_i \frac{\partial}{\partial x_i}\right)(g)_j \cdot \frac{\partial}{\partial x_j} + g_j \left[f_i \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right].$$

noncommutative

$$= \sum_{ij} f_i \cdot \frac{\partial g_j}{\partial x_i} \cdot \frac{\partial}{\partial x_j} + \overset{0}{\cancel{g_j f_i \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]}} - g_j \frac{\partial f}{\partial x_j} \cdot \frac{\partial}{\partial x_i}$$

$$\sum_k \left(\sum_i f_i \frac{\partial g_k}{\partial x_i} - g_i \frac{\partial f_k}{\partial x_i} \right) \cdot \frac{\partial}{\partial x_k}$$

Multi-linear Algebra of Real vector spaces
(and modules)

• Def. $V_1, \dots, V_n, W \in \mathrm{Vect}_{\mathbb{R}}$

A multilinear map $V_1, \dots, V_n \longrightarrow M$

is a map of sets $V_1 \times \dots \times V_n \xrightarrow{t} W$

such that $\forall i \ \forall v_j \in V_j$: $V_i \longrightarrow W$ is an $\mathbb{R}$ linear map

$j \neq i$ $\quad v_i \longmapsto T(v_1, \dots, v_i)$

• Def. $V, W \in \mathrm{Vect}_{\mathbb{R}}$ $\quad \mathrm{Hom}(V, W) \in \mathrm{Vect}_{\mathbb{R}}$

$f \quad f : V \to W \in \mathrm{Vect}_{\mathbb{R}}$

operations are nom pointwise $\qquad$ convolution.

$$(f_1 + f_2)(v) = f_1(v) + f_2(v)$$

$$\mathbb{R} \to (r \cdot f)(v) = r \cdot f(v)$$

• Def. $V_1, \dots, V_n, W \in \mathrm{Vect}_{\mathbb{R}}$

$\mathrm{Hom}(V_1, \dots, V_n ; W)$

• Prop. $\overset{T\in}{\mathrm{Hom}(V_1, V_2 ; W)} \cong \overset{Q\ \in}{\mathrm{Hom}(V_1, \mathrm{hom}(V_2, W))}$

$\cong \mathrm{Hom}\Big(V_2, \mathrm{Hom}(V_1, W)\Big)$

Given T, $Q(v_1)(v_2) = T(v_1, v_2)$

Given Q, $T(v_1, v_2) = Q(v_1)(v_2)$

• Example $\qquad$ Terminology $W = \mathbb{R}$

T is a multilinear form

• Example $\mathrm{Hom}(V, \mathbb{R}) = V^*$ the dual vector space.

$$\mathrm{Hom}(\mathbb{R}, V) \cong V \qquad \mathrm{Hom}(V, \mathbb{R}^n) \cong V^* \oplus \dots \oplus V^*$$

$$f \longmapsto f(1)$$

$\text{Hom}(\mathbb{R}^n \to V) \cong V \oplus \ldots \oplus V$

- Def. $V_1, V_2 \in \text{Vect}\,\mathbb{R}$

$V_1 \otimes V_2 \in \text{Vect}_{\mathbb{R}}$

tensor product of V_1 and V_2

$\mathbb{R}$-bilinear $\quad V_1, V_2 \xrightarrow{\otimes} V_1 \otimes_{\mathbb{R}} V_2$

$$v_1, v_2 \overset{\otimes}{\longmapsto} v_1 \otimes_{\mathbb{R}} v_2$$

Essentially, we define tensor product of elements and spaces in the same package.

• Thm. The universal property:

if $V_1, V_2 \xrightarrow[T]{\text{bilinear}} W$, then, there is a unique $\mathbb{R}$ linear map $S: V_1 \otimes V_2 \longrightarrow W$ such that $\forall v_1, v_2: T(v_1+v_2) = S(v_1 \otimes v_2)$

S is only linear, whereas T is multi-linear (bi in this case)

• Prop. $\otimes$ exists and is unique up to a unique isomorphism

• Proof. U_1: real vector space with basis $V_1 \times V_2$

$U_2 \subset U_1$: vector subspace linear, span of $(v_1+v_1', v_2)-(v_1, v_2)-(v_1', v_2)$

$$(v_1, v_2+v_2') - (v_1, v_2) - (v_1, v_2')$$

$$(r \cdot v_1, v_2) - r \cdot (v_1, v_2)$$

$$(v_1, r \cdot v_2) - r \cdot (v_1, v_2)$$

set $V_1 \otimes V_2 = U_1/U_2$

$= U_1$

- **Recall**

$U_1 = \text{Free}(V_1 \times V_2) = \bigoplus_{V_1 \times V_2} \mathbb{R}$

$U_2 = \text{Span} \left\langle (u_1+u_1', u_2) - (u_1,u_2) - (u_1',u_2), \dots \right\rangle$

$$\begin{tikzcd} V_1, V_2 \arrow[r] \arrow[dr, "\text{bilinear}"'] & \underline{V_1 \otimes V_2} \arrow[d, dashed, "\exists ! S"] \\ & W \end{tikzcd}$$

$(v_1,v_2) \longmapsto [(v_1,v_2)] = v_1 \otimes v_2$

$$\begin{tikzcd} \langle 1,2 \rangle \\ \mathbb{R} \arrow[u] \\ 4 \end{tikzcd}$$

$T(v_1,v_2) = S(v_1 \otimes v_2)$

- existence of S

$$\begin{tikzcd} V_1 \otimes V_2 = U_1/U_2 \arrow[r, "S"] & W \\ U_1 \arrow[r, "s_1"] \arrow[d, dashed, leftrightarrow, bend right] & W \\ V_1 \times V_2 \arrow[r, "s_2"] & W \end{tikzcd}$$

vanish on U_2

map of sets

$s_1(\langle (u_1+u_1', u_2) - (u_1,u_2) - (u_1',u_2), \dots \rangle$

$T(u_1+u_1', u_2) - T(u_1,u_2) - T(u_1',u_2) = 0$

$$\begin{tikzcd} V_1 \otimes V_2 = U_1/U_2 \arrow[r, shift left, "S"] \arrow[r, shift right, "S'"'] & W \\ V_1, V_2 \arrow[u, "\otimes"] \arrow[ur, "T"'] & \end{tikzcd}$$

$S - S' = 0 \Rightarrow S = S'$.

$(S-S')([(v_1,v_2)]) = T(v_1,v_2) - T(v_1,v_2) = 0$

- Proposition

$$U_1 \otimes (V_1 \oplus V_2)$$

$$\cong u \otimes v_1 \oplus u \otimes v_2$$

a.)

$$\begin{tikzcd}
u \otimes (v_1 \oplus v_2) \arrow[r, "S"] & u \otimes v_1 \oplus u \otimes v_2 \\
u, v_1 \oplus v_2 \arrow[u, "\otimes"] \arrow[ur, "S \circ \otimes"] & (u \otimes v_1, u \otimes v_2) \\
(u, (v_1, v_2)) \arrow[ur, mapsto] &
\end{tikzcd}$$

$$\underbrace{(u+u', (v_1, v_2))}_{\text{Bilinearity property}} \xmapsto{S} \left((u+u' \otimes v_1, (u+u' \otimes v_2 \right) = \left(u \otimes v_1 + u' \otimes v_1, + u \otimes v_2 + u' \otimes v_2 \right)$$

$$(u \otimes v_1, u \otimes v_2) + (u' \otimes v_1, u' \otimes v_2)$$

$$= S(u, (v_1, v_2) + (v_1', v_2'))$$

$$= S(u, (v_1, v_2)) + S(u, (v_1', v_2'))$$

b.) $T : u\otimes v_1 \oplus u\otimes v_2 \longrightarrow u\otimes(v_1\oplus v_2)$

$T_1 : u\otimes v_1 \longrightarrow u\otimes(v_1\oplus v_2)$

$T_2 \quad u\otimes v_2 \longrightarrow u\otimes(v_1\oplus v_2)$

$T(p_1,p_2) = T_1(p_1) + T_2(p_2)$

$p_1 \in u\otimes V_1 , \quad p_2 \in u\otimes V_2$

$$\begin{tikzcd}
u\otimes v_1 \arrow[rr, "T_1"] & & u\otimes(v_1\oplus v_2) \\
u,v_1 \arrow[u] \arrow[urr] & & u\otimes(v_1,0) \\
(u,v_1) \arrow[urr, mapsto, "\text{bilinear}"'] & &
\end{tikzcd}$$

$T(u\otimes v_1, u\otimes v_2) = u_1\otimes(v_1,0) + u_2\otimes(0,v_2)$

$T_2(u\otimes v_2) = u\otimes(0,v_2)$

c.) $S(T(u_1\otimes v_1, u_2\otimes v_2)) =$

$S(u_1\otimes(v_1,0) + u_2\otimes(0,v_2))$

$= (u_1\otimes v_1, u_1\otimes 0) + (u_2\otimes 0, u_2\otimes v_2) =$

$(u_1\otimes v_1, u_2\otimes v_2)$

d.) $T(S(u\otimes(v_1,v_2)))$

$= T(u\otimes v_1, u\otimes v_2)$

$= u\otimes(v_1,0) + u\otimes(0,v_2)$

$= u\otimes(v_1,v_2)$ .//

tombstone halmos

- Corollary

a)
$$\begin{tikzcd} \mathbb{R}^m \otimes \mathbb{R}^n \arrow[r, phantom, "\cong"] & \mathbb{R}^{m\cdot n} & g_{(i,j)} \\ & \mathbb{R}^m, \mathbb{R}^n \arrow[ul, "\otimes"] \arrow[u] & \\ & (e_i, f_j) \arrow[uur, bend right] & \end{tikzcd}$$

$$\left(\bigoplus_m \mathbb{R}\right) \otimes \left(\bigoplus_n \mathbb{R}\right) = \bigoplus_m \left(\mathbb{R} \times \bigoplus_n \mathbb{R}\right)$$

$$S\left(=\sum_i x^i e_i, \sum y^j f_j\right)$$

$$= \bigoplus_m \bigoplus_n \mathbb{R}\otimes\mathbb{R} \cong \bigoplus_{m\cdot n} \mathbb{R} \cong \mathbb{R}^{m\cdot n}. \qquad = \sum_{i,j} x^i y^j g_{(i,j)}$$

- Lemma $\mathbb{R}\otimes\mathbb{R} \xrightarrow{\cong} \mathbb{R}$

$$a\otimes b \longmapsto a\cdot b$$

$$c\cdot 1\otimes 1 = c\otimes 1 \longleftarrow\!\shortmid\ c$$

$$a\otimes b \longmapsto a\cdot b$$

$$(a\cdot 1)\otimes(b\cdot 1) \longleftarrow a\cdot b$$

$$ab\cdot(1\otimes 1)$$

$$\begin{tikzcd} c\cdot(1\otimes 1) \arrow[dr] & c \arrow[l, maps to] \arrow[d, phantom, "\shortparallel"] \\ & c\cdot 1\cdot 1 \end{tikzcd}$$

b.) $\dim(V_1 \otimes V_2) = \dim V_1 \cdot \dim V_2$

c.) if $(e_i)_{i\in I}$ is a basis for V_1,
$(f_j)_{j\in I}$ is a basis for V_2,
$(e_i \otimes f_j)_{(i,j)\in I\times J}$ is a basis for $V_1 \otimes V_2$
$V_1 \oplus V_2$
for $(e_i \oplus f_j)$ has
$(e_i, 0) + (0, f_j)$

• Def. $$\begin{tikzcd} V_1^* \otimes V_2 \arrow[r, "T"] & Hom(V_1, V_2) \\ V_1^* , V_2 \arrow[u, "\otimes"] \arrow[ur, "bilinear"'] & \end{tikzcd}$$

$f \ , v_2 \in V_2 \mapsto (v_1 \mapsto f(v_1) \cdot v_2)$

$f : V_1 \to \mathbb{R}$

imT = finite rank linear maps $V_1 \to V_2$

• Def. $V_1, V_2 \in$ vect

$s \in V_1 \otimes V_2$

s is a tensor, which is a member of the tensor space, which is also a vector space

the rank of s is ★

$$\text{rank} s = \min k \quad s = \sum_{i=1}^{k} \underset{V_1}{\underset{\uparrow}{v_i}} \otimes \underset{V_2}{\underset{\uparrow}{v_i'}}$$

• Exa. rank s = 0 $\Longleftrightarrow$ s = 0

rank s = 1 $\Longleftrightarrow$ $v \in V_1, v' \in V_2$

$s = v \otimes v'$

tensor decomposable tensor

• Prop. a) $\text{im}T = \{ f : V_1 \to V_2 \mid \dim \text{im} f < \infty \}$

$$\underset{★}{\overset{\|}{\text{rank} f}} = \underset{\dim \text{im} f}{\overset{\|}{\underbrace{\text{rank} f}}}$$

b) $\forall g \in V_1^* \otimes V_2$

$\text{rank} g = \dim \text{im} T(g)$

• **Thm.** $V^* \otimes W \longrightarrow \mathrm{Hom}(V,W)$

$f \otimes w \longmapsto (v \mapsto f(\underset{\mathbb{R}}{\underset{\uparrow}{v}}) \cdot w)$

$\mathrm{im}\, T = \{ f : V \to W \mid \dim \mathrm{im} f < \infty)$ finite rank operators

$\dim T(t) = \mathrm{rank}\, t = \min k$

$t = \sum_{i=1}^{k} f_i \otimes w_i$

ask for a little more clarification over this guy...

• Corollary $\dim V < \infty$

$V^* \otimes W \xrightarrow{\cong} \mathrm{Hom}(V,W)$

$V \otimes W \xrightarrow{\cong} \mathrm{Hom}(V^*, W)$

• Corollary $\dim V, W < \infty$

$V \otimes W \cong \mathrm{Bilm}(V^*, W^*; \mathbb{R})$

i) $t \in V^* \otimes W$

elements of tensor product have finite rank

$t = \sum_{i=1}^{k} f_i \otimes w_i$, k min

$T(t)(v) = \sum_{i=1}^{k} f_i(v) \cdot w_i \in \mathrm{Span}(w_i)_i \; \forall i$

$\dim \leq k$

$\dim = k.$

ii.) $f : V \to W$ $\dim \mathrm{im} f < \infty.$

$\mathrm{im}(f) = \mathrm{Span}(w_1, \ldots, w_k)$ $k = \dim \mathrm{im} f$

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[dr, "f"] \arrow[ddr, bend right, "g_i"'] & W \\
& \mathrm{im} f \arrow[u] \arrow[d, "w_i^*"] \\
& \mathbb{R}
\end{tikzcd}$$

$$f(v) = \sum_{i=1}^{k} w_i^*(f(v)) \cdot w_i$$

$$= T\left(\sum_{i=1}^{k} g_i \otimes w_i\right)(v)$$

Tensors

$V \in \mathrm{Vect}_{\mathbb{R}}$

k times contravariant
and l times covariant is an element of

$$\underbrace{V\otimes V\otimes V\otimes\ldots\otimes V}_{k \text{ times}}\otimes\underbrace{V^*\otimes V^*\otimes\ldots\otimes V^*}_{l \text{ times}}$$

$$\dim(\ldots.) = (\dim V)^{k+l}$$

suppose $\mathbb{R}^n \xrightarrow[\cong]{e} V$ is a basis

$$V \xrightarrow[\cong]{x=e^{-1}} \mathbb{R}^n \quad v \in V = \sum_i x_i(v) e_i$$

Suppose $t = v_1 \otimes \ldots. \otimes v_k \otimes f_1 \otimes \ldots \otimes f_l$

Then $t = \left(\sum x^i(v_1 \cdot e_i)\right) \otimes \ldots \otimes \left(\sum e_i^*(f_i)\cdot x_j^*\right)$
$\otimes \ldots.$

$$\sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l}} \left(x^{i_1}(v_1) \cdots e^*_{j_1}(f_1)\, x^*_{j_1}\right) \cdot \underbrace{(e_{i_1} \otimes \ldots \otimes e_{i_k} \otimes f_{j_1} \ldots \otimes f_{j_l})}_{\text{basis}}$$

$$\begin{tikzcd}
 & V & \\
\mathbb{R} \arrow[ur, "e"] \arrow[rr, "T"'] & & \mathbb{R}^n \arrow[ul, "d"']
\end{tikzcd}$$

$$e_i = \sum_j T_i^j d_j$$

$$\begin{tikzcd} & & \mathbb{R}^n \\ \mathbb{R}^n & & \\ & V \arrow[ul, "x=c^{-1}"] \arrow[uur, "y=d^{-1}"'] & \end{tikzcd}$$

$e = d \circ T$

$x_i = \sum T^{-1}{}_i^j\, y_j$

$e^{-1} = T^{-1} \circ d^{-1}$

$x = T^{-1} \circ y$

- Recall multilinear maps

  $\mathrm{Hom}(V, V, \dots, V\,;\, W)$

  $\cong \mathrm{Hom}(V \otimes V \otimes \dots \otimes V, W)$

  $= \mathrm{Hom}(V^{\otimes n}, W)$

  if $\dim V < \infty$ $\cong V^{*\otimes n} \otimes W$

- **Def.** A multi-linear map $V, \dots, V \xrightarrow{T} W$ is symmetric if $T(v_1, \dots, v_{i-1}, v_j, v_{i+1}, \dots v_{j-1}, v_i\, v_{j+1}) = T(v_1, \dots, v_n)$.

  $T(\dots) = -T(v_1, \dots, v_n)$ pairity of transpositions.

- **Proposition** $V \in \mathrm{Vect}$, $n \geq 0$

  $\exists\ \mathrm{Sym}^n V \in \mathrm{Vect}$, symmetric multilinear map $V, \dots, V \longrightarrow \mathrm{Sym}^n V$

  such that $\forall\, W \in \mathrm{vect}$ symmetric multilinear map $V, \dots, V \xrightarrow{T} W$

  $\exists!\ \mathrm{Sym}^n V \underset{\text{linear}}{\xrightarrow{S}} W$ s.t. $T = S \circ \vee$

  $T(v_1, \dots, v_n) = S(v_1 \vee v_2 \vee \dots \vee v_n$

6.) $\exists \Lambda^n V$ evect, antisymmetric multi-linear map $V, \dots, V \xrightarrow{\wedge} \Lambda^n V$

$$v_1, \dots, v_n \overset{\wedge}{\longmapsto} \underbrace{v_1 \wedge v_2 \wedge \dots \wedge v_n}_{\text{multivector}}$$

Corollary $\mathrm{Hom}_{sym}(V, \dots, V; W)$

$\cong \mathrm{Hom}(\mathrm{Sym}^n V, W)$

$\mathrm{Hom}_{antisym}(V, \dots, V; W) \cong \mathrm{Hom}(\Lambda^n V, W)$.

elements ingrassman algebra

Prop. Σ_n = permutation of n elements

$\overset{\shortparallel}{S_n}$

$$\sigma \in S^n : V^{\otimes n} \xrightarrow{\sigma} V^{\otimes n}$$

$$v_1, \dots, v_n \longmapsto V_{\sigma(1)} \otimes V_{\sigma(2)} \otimes \dots \otimes V_{\sigma(n)}$$

$$V_1 \otimes \dots \otimes V_n \longmapsto$$

Symmetric Tensor construction.

Set $\mathrm{Sym}^n V = V^{\otimes n} / \mathrm{span}(V_1 \otimes \dots \otimes V_n - V_{\sigma(1)}^{(n)} \otimes \dots \otimes V_{\sigma(n)})$

$$v_1 \vee \dots \vee v_n = [V_1 \otimes \dots \otimes V_n]$$

$$\Lambda^n V = V^{\otimes n} / \mathrm{Span}(v_1 \otimes \dots \otimes v_n) - (-1)^{\sigma} v_{\sigma_1} \otimes \dots \otimes v_{\sigma_n}$$

$v_1 \wedge \dots \wedge v_n = [v_1 \otimes \dots \otimes v_n]$. HW 12

use universal property

• Def A tensor $t \in V^{\otimes n}$ is • Symmetric

$$\sigma \cdot t = t$$

for all $\sigma \in S^n$ only for fields char $= 0$

• antisymmetric

$$\sigma \cdot t = (-1)^{\sigma} \cdot t$$

• Example

$V = \langle e_1, e_2 \rangle$

$$\left.\begin{array}{l} t = e_1 \otimes e_2 + e_2 \otimes e_1 \\ t = e_1 \otimes e_1 \\ t = e_2 \otimes e_2 \end{array}\right\} \text{Symmetric}$$

$t = e_1 \otimes e_2 - e_2 \otimes e_1 \}$ antisymmetric.

$t = 0$ symmetric and anti-symmetric

$$V^{\otimes n}_{sym} \qquad V^{\otimes n}_{antisym}$$

• Proposition. working over a field of char 0 (e.g., $\mathbb{Q}, \mathbb{R}, \mathbb{C}$)

$$\mathrm{Sym}^n V \xrightarrow{\cong} V^{\otimes n}_{sym}$$

$$\Lambda^n V \xrightarrow{\cong} V^{\otimes n}_{\Lambda}$$

• Proof.

$$\begin{tikzcd} \mathrm{Sym}^n V \arrow[r, shift left, "A"] & V^{\otimes n}_{sym} \arrow[l, shift left, "B"] \end{tikzcd}$$

$$v_1 v \ldots v v_n \qquad v_1 \otimes \ldots \otimes v_n$$

$$v_1, \ldots, v_n \longmapsto \sum_{\sigma \in S_n} v_{\sigma_1} \otimes \ldots \otimes v_{\sigma_n}$$

$(B\circ A)(v_1\vee\dots\vee v_n) = B\left(\sum_{\sigma} v_{\sigma_1}\otimes\dots\otimes v_{\sigma_n}\right)$

$= \sum v_{\sigma_1}\vee\dots\vee v_{\sigma_n} = \sum v_1\vee\dots\vee v_n = n!\,(v_1\vee\dots\vee v_n)$

$B\circ A = n!\cdot id_{Sym^n V}$ $\qquad t\in V^{\otimes n}_{sym} = \sum_{j\in J} v_{j,1}\otimes$

$(A\circ B)(t)$ $\qquad v_{jm}$

$= A\left(\sum_j v_{j1}\vee\dots\vee v_{jn}\right) = \sum_j\sum_{\sigma\in S^{\cdot}} v_{j\sigma_1}\otimes\dots$

$\otimes v_{j\sigma_n}$

Def. $Hom(V_1,W_1)\otimes Hom(V_2,W_2)\to hom(V_1\otimes V_2, W_1\otimes W_2)$

$f_1\otimes f_2 \longmapsto (v_1\otimes v_2\mapsto f_1(v_1)\otimes f_2(v_2))$

Isomorphism iff our vector spaces are finite (V_1, V_2 are finite dimensional or $\dim W_i<\infty$)

$\underbrace{}_{}$ dualizable

modules: iso if V_i are f.g. proj.

Maps out of

$$\begin{tikzcd} M \arrow[r, two heads] & N \\ P \arrow[u, dotted, "\text{linear map}"] \arrow[ur, "\text{proj}"'] & \end{tikzcd}$$

$M\otimes_R M^* \cong Hom_R(M,M)$

$\mathbb{Z}/n\mathbb{Z}$ not projective

R is a commutative ring

M R-module

M is dualizable $\exists M^*$ R-module $\cong Hom_R(M,R)$

$f\otimes m\longmapsto f(m)$

$M^*\otimes_R M\longrightarrow R$

$$\begin{tikzcd} 1 \arrow[r, mapsto] & \sum e_i\otimes e_i^* & id_M \\ R \arrow[u, phantom, "\in"] \arrow[r, "\text{coevaluation}"'] & M\otimes M^* \arrow[r] & Hom_R(M,M) \arrow[u, phantom, "\in"] \end{tikzcd}$$

$M \cong M^{**}$

underlying ring or field $\underset{R}{\otimes}$

$$\begin{tikzcd}
\mathfrak{X}(M) \underset{C^\infty M}{\otimes} \mathfrak{X}(M) \\
\mathfrak{X}(M) \underset{\mathbb{R}}{\otimes} \mathfrak{X}(M) \arrow[u, two heads]
\end{tikzcd}$$

$f\cdot v \otimes v$
$\neq v \otimes f\cdot v$

$C^\infty M \underset{C^\infty M}{\otimes} C^\infty M \cong C^\infty M$

$C^\infty M \underset{\mathbb{R}}{\otimes} C^\infty M \underset{\substack{\uparrow \\ \text{dense}}}{\subset} C^\infty(M \times M)$

$A, B \in CAlg_{\mathbb{R}}$ $\quad A \otimes B$

$(a_1 \otimes b_1)\cdot(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$

$$\begin{tikzcd}
 & A\otimes B \arrow[rr, dashed, "\exists! \ f"] & & C \\
A \arrow[ur, "a \mapsto a\otimes 1"] \arrow[urrr] & & B \arrow[ul, "b \mapsto b\otimes 1"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

$h(a \otimes b) = f(a)\cdot g(a)$

completed tensor $\quad C^\infty$- tensor product

Kolar - Michor - Slovak
natural operations in diff geo
Michor
topics in differential geometry
Moerdijk - Reyes
models for smooth
infinitesimal analysis.

Lulu.com
Nestruev
Ramanan
global
calculus.

• Def. $W_i = \mathbb{R}$ $\quad V_1^* \otimes V_2^* \longrightarrow (V_1 \otimes V_2)^*$

$$V^{*\otimes n} \longrightarrow (V^{\otimes n})^*$$

V dualizable => iso

$$f_1 \otimes \dots \otimes f_n \longmapsto (v_1 \otimes \dots \otimes v_n \longmapsto f_1(v_1) \cdots f_n(v_n)).$$

• Def. $\mathrm{Sym}^n V^* \longrightarrow (\mathrm{Sym}^n V)^* = \hom(\mathrm{Sym}^n V, \mathbb{R})$

$$f_1 \vee \dots \vee f_n \longmapsto \Big(v_1 \vee \dots \vee v_n \longmapsto \sum_\sigma f_1(v_{\sigma_1}) \cdots f_n(v_{\sigma_n})\Big)$$

$$f_1, \dots, f_n \longmapsto \Big(v_1, \dots, v_n \longmapsto \frac{1}{n!} \sum_\sigma \prod_i f_i(v_{\sigma(i)})\Big)$$

$$V^* = \hom(V, \mathbb{R}), \quad V = \hom(V^*, \mathbb{R})$$

$$\hom_{sym}(V, \dots, V; \mathbb{R}) \cong (V^{\otimes n})^*_{sym} \cong (V^*)^{\otimes n}_{sym}$$

• Prop. $\dim V < \infty \iff V \xrightarrow{\cong} V^{**}$

$$v \longmapsto (f \mapsto f(v))$$

$\boxed{\text{Def}}$

$$V^* \otimes V \xrightarrow{ev} \mathbb{R}$$

$$f, v \longmapsto f(v)$$

$$f^i v_i = \sum_i f^i v_i$$

$\boxed{\text{Def.}}$

$$\mathrm{coev} \nearrow V \otimes V^* \xrightarrow{\cong \text{ if } < \infty} \mathrm{Hom}(V, V)$$

$$v, f \longmapsto (w \longmapsto f(w) \cdot v)$$

$$1 \longmapsto id_V \quad e^i e_i$$

- Recall

Lambda abstraction

$$\{U \longrightarrow \mathrm{Hom}(V,W)\} \cong \{U \otimes V \longrightarrow W\}$$

currying

$$\mathrm{Hom}(U, \mathrm{Hom}(V,W)) \cong \mathrm{Hom}(U \otimes V, W).$$

$$f^c (u \longmapsto (v \longmapsto f(u \otimes v)))$$

$$(\mathrm{Sym}^n V^*) \otimes \mathrm{Sym}^n V \longrightarrow \mathbb{R}$$

contravariant in V $\quad V_1 \xrightarrow{g} V_2$

$$\begin{tikzcd}
(u \longmapsto f(u) \circ g) & \mathrm{Hom}(U, \mathrm{hom}(V_1, W)) \\
f^V \arrow[u, mapsto] & \mathrm{Hom}(U, \mathrm{hom}(V_2, W)) \arrow[u]
\end{tikzcd}$$

Recall

$\dim V < \infty$

iff

$$V^* \otimes W \xrightarrow{\cong} \mathrm{Hom}(V,W)$$

$$\underbrace{V^*, \dots, V^*}_{\mathrm{Sym}} ; \underbrace{V, \dots, V}_{\mathrm{Sym}} \longrightarrow \mathbb{R}$$

$$f_1, \dots, f_n ; v_1, \dots, v_n \longmapsto \frac{1}{n!} \sum_\sigma \prod_i f_i(v_{\sigma(i)})$$

$$\frac{(p+q)!}{(p)!\,(q)!}$$

$$v_1, \dots, v_n \longmapsto \frac{1}{n!} \sum_\sigma \prod_i v_{\sigma(i)}$$

$$\begin{tikzcd}
\mathrm{Sym}^n V \arrow[r, shift left] & V^{\otimes n}_{\mathrm{sym}} \arrow[l, shift left]
\end{tikzcd}$$

$v_1 \vee \dots \vee v_n \qquad v_1 \otimes \dots \otimes v_n$

$\Lambda^k V^* \otimes V^{\otimes k}_{\text{antisym}} \longrightarrow \mathbb{R}$

$\Lambda^k V^* \otimes \Lambda^k V$

$f_1, f_2 \in V^* \qquad f_1 \wedge f_2 \in \left(V^{\otimes 2}\right)^*_{\text{antisym}}$

$f_1 \wedge f_2 \in \Lambda^2 V^*$

$$\Lambda^2 V \longrightarrow \mathbb{R}$$
$$v_1, v_2 \longmapsto \frac{1}{2}\left(f_1(v_1) f_2(v_2) - f_1(v_2)\cdot f_2(v_1)\right)$$

$\left(V^{\otimes 2}\right)^*_{\text{antisym}}$

convex hull of points.

symplecics

Symplex ? Cartan - diffforms are cochains

Prop. $\dim\left(V^{\otimes n}\right) = (\dim V)^n$

$$\dim\left(\mathrm{Sym}^n V\right) = \binom{\dim V + n - 1}{n}$$

$$\dim\left(\Lambda^n V\right) = \binom{\dim V}{n}$$

Corollary $\dim \Lambda^{\dim V} V = 1$

proof $v_1 \wedge \dots \wedge v_n \qquad v_i = \sum_j v_i^j e_j$

$$= \left(\sum_{j_1} V_i^{j_1} e_{j_1}\right) \wedge \dots \left(\sum_{j_n} v_i^{j_n} e_{j_n}\right)$$

$$= \sum_{j_1<j_2<j_3<\dots}^{n} a_j (e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_n})$$

Stars and Bars argument

$$e^*_{j_n} \wedge \dots \wedge e^*_{j_n}$$

$$\Lambda^k V = F(V \times V \times \dots V)$$

$$\Lambda^k V = \otimes V / \ker \pi_A \quad \pi_A = \frac{1}{k!} \sum sgn(\sigma) \sigma$$

$$< v^1 \otimes \dots \otimes v^k, v_1 \otimes \dots \otimes v_k > = v^1(v_1) v^2(v_2) \dots v^k(v_k)$$

$$\pi : \otimes^k V \longrightarrow \otimes^k V / \ker \pi_A$$

$$< \pi(v^1 \otimes \dots \otimes v^k), \pi(v_1 \otimes \dots \otimes v_k) > = \frac{1}{k!} det(v^i(v_j))$$

but $< v^1 \wedge \dots \wedge v^k, v_1 \wedge \dots \wedge v_k > = det(v^i(v_j))$ or $\frac{1}{k!} det(v^i(v_j)$

$A_k(V) \cong (\Lambda^k V)^* \cong \Lambda^k V^*$

using (1) $\varphi \wedge \psi(v_1, \dots v_{k+l}) = \frac{1}{k!l!} \sum \operatorname{sgn}\sigma$
$\varphi(v_{\sigma_1} \cdots v_{\sigma k})\, \psi(v_{\sigma(k+1)} \cdots v_{\sigma(k+l)})$

$= \frac{(k+l)!}{k!l!} Alt(\varphi \otimes \psi)$

using ② $(\psi \wedge \varphi)('') := \frac{1}{(k+l)!} \sum ''$

but if $e_1, \dots, e_n$ and $\varepsilon^1, \dots, \varepsilon^n$
$(\varepsilon^{j_1} \wedge \varepsilon^{j_2} \dots \wedge \varepsilon^{j_k})(e_{i_1} \wedge \dots \wedge e_{i_k}) = \frac{1}{k}!$

$(i_v \varphi)(v_1, \dots, v_{k-1}) = \varphi(v_1, v_2, \dots v_{k-1})$

not graded derivation (minor.) Freegroup algebra

$\Lambda^k V^*$ John Lee:

$(\Lambda^k V)^*$ $\Lambda^k V^*, A^k V^* \trianglerighteq$

$\frac{1}{n!}(f_1 \wedge \dots \wedge f_k) \longleftarrow\!\shortmid f_1 \otimes \dots \otimes f_k$

$\Lambda^k V^* \longleftarrow \Lambda^k(V, \mathbb{R})$

$\Longrightarrow \cong (V^*)^k_{Alt}$

$f_1 \wedge \dots \wedge f_k \mapsto \sum (-1)^\sigma f_{\sigma_1} \otimes \dots \otimes f_{\sigma_k}$

Graded derivation Koszul sign dim

Recall: basis $e_1,\dots,e_n$ V n

$e_1^*,\dots,e_n^*$ V^* n

$e_{i_1}\otimes\dots\otimes e_{i_k}$ $\otimes^k V$ n^k

$j_\alpha = i+\alpha$

$e_{i_0}\vee\dots\vee e_{i_{k-1}}$ $\mathrm{Sym}^k V$ $\binom{n+k-1}{k}$

$e_{i_1\leq\dots\leq} e_{i_k}$ $\Lambda^k V$ $\binom{n}{k}$

$i_1<\dots<i_k$

$$\dim V^{\dim V}\ V = 1$$

$\boxed{\text{Def}}$ - An oriented volume elements on V is an element of $\Lambda^{\dim V} V$

- An oriented translation-invariant measure on V is an element of

$$\Lambda^{\dim V} V^* \cong (\Lambda^{\dim V} V)^*$$

signed area

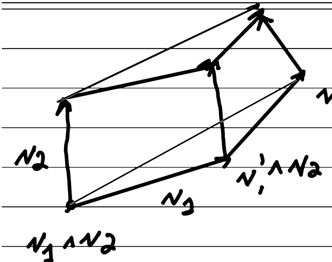
$v_1\wedge v_2 = -v_2\wedge v_1$

" volume as a physical idea."

$m\in M$ $t\in T$ $m\cdot t\in M\otimes T$

$$(\lambda v_1)\wedge v_2 = \lambda(v_1\wedge v_2).$$

Shearing $\rightarrow$ $(v_1+v_1')\wedge v_2 = v_1\wedge v_2 + v_1'\wedge v_2$



what about indecomposible elements.

$= v_1 \wedge v_2 \wedge v_3 \wedge \dots \wedge v_n = \lambda(e_1 \wedge \dots \wedge e_n)$ λ is the determinant.

$\left(\sum_{j_1} v_1^{j_1} e_{j_1}\right) \wedge \dots$ $\uparrow$

$v_i = \sum_j v_i^j e_j$

Overlapping terms but antisymmetry kills a ton of terms

$$= \sum_{\sigma} (v_1^{\sigma_1} \cdots v_n^{\sigma_n}) \cdot (e_{\sigma_1} \wedge \dots \wedge e_{\sigma_n})$$

λ " this sum of permuted matrix

Sign of permutation

$$= \left(\sum_{\sigma} (-1)^{\sigma} v_1^{\sigma_1} \cdots v_n^{\sigma_n}\right) \cdot (e_1 \wedge \dots \wedge e_n)$$

$$= \det v \cdot (e_1 \wedge \dots \wedge e_n).$$

Remark V, $n = \dim V$

$\Lambda^n V = F(V\times\dots\times V) /$ additivity

$\shortparallel$

$OrVol(v)$

$(\lambda v_1,\dots,v_n) = \lambda\cdot(v_1,\dots,v_n)$

$(v_1, v_2, v_3, \dots, v_n) =$

$-(v_2, v_1, v_3, \dots, v_n)$

$Vol(v) = F(V\times\dots\times V) /$ additivity

$(\lambda v_1,\dots,v_n) = |\lambda|.(v_1,\dots,v_n)$

$(v_1, v_2, v_3, \dots) = (v_2, v_1, \dots)$

$(v_1, v_2, \dots, v_n)$

$Or(V) = F(V\times\dots\times V) / (v_1+\lambda v_i, v_2,\dots,v_n) =$

$= sgn(\lambda)(v_1,\dots,v_n)$

$(v_1, v_2, v_3, \dots, v_n) = -(v_2, v_1)$

....

$\dim OrVol(v) = \dim vol(v) = \dim Or(v) = 1$

$[e_1,\dots, e_n]$ is a basis

Surrogate of

$v, v, v_3, \dots, v_n$ multi-linearity.

$\boxed{\text{Prop}}$ $\underset{\Lambda^{\dim V} V}{\underset{\parallel}{\mathrm{OrVol}(V)}} \cong \mathrm{Vol}(V) \otimes \mathrm{or}(V)$

$\boxed{\text{Def}}$ A translation invariant measure on V is an element of $\mathrm{Vol}(V)^* \cong \mathrm{Vol}(V^*)$

$\boxed{\text{Prop}}$ $\{$ otim's measures on V $\}$

$\cong \{$ Haar measure

$=$ t-i Radomeasure
locally finite and inner regular

finite dimensional
$V \in \mathrm{Vect}_{\mathbb{R}}$
$\mathrm{Vol}(V) \in \mathrm{Line}_{\mathbb{R}}$ determinant line ↙

$\mathrm{OrVol}(V) = \det(V) \in \mathrm{Line}_{\mathbb{R}}$

$\boxed{\text{Def}}$ $\det V = \Lambda^{\dim V} V$ "fiberwise determinant"
$f: V \longrightarrow W$, $n = \dim V = \dim W$

$\det f : \det V \longrightarrow \det W$
$v_1 \wedge \dots \wedge v_n \longmapsto f(v_1) \wedge \dots \wedge f(v_n)$

Special case : $V = W$; $\det f : \det V \longrightarrow \det W$
$\det f \in \mathrm{Hom}(\det V, \det V) \cong \mathbb{R}$
determinant of an endomorphism is a number.

Properties $\dim V < \infty$

Rem: $\dim V = 1$; $V^* \otimes V \cong \hom(V, V) \cong \mathbb{R}$

$\mathbb{R} \otimes W \cong W$

$\det V \cong \operatorname{Vol} V \otimes \operatorname{Or}(V)$

$\operatorname{vol}(V) \cong \det V \otimes \operatorname{Or}(V)^*$ 1-d vector spaces

$\operatorname{or}(V) \cong \det V \otimes (\operatorname{vol} V)^*$ form a group under tensor product

$(\det V)^* \cong \det(V^*)$

$(\operatorname{vol} V)^*$

"decomposable multilinear form" "two-group"

$$f_1 \otimes \ldots \otimes f_n \longmapsto \frac{1}{n!}(f_1 \wedge \ldots \wedge f_n)$$

$$\sum_\sigma f_{\sigma_1} \otimes \ldots \otimes f_{\sigma_n} \longleftarrow (f_1 \wedge \ldots \wedge f_n)$$

$(\operatorname{Or} V)^* \cong \operatorname{Or}(V^*) \cong \operatorname{Or} V$

Def $\dim W = 1$ $\quad \operatorname{Vol} W$, $\dim = 1$ generated by

a.) $\operatorname{Vol} W = F(W) / \lambda \in \mathbb{R}$: $[\lambda \cdot w] = |\lambda| \cdot [w$

$[w] = [-w]$

Confusion $\quad \operatorname{Or}(\det V) \cong \operatorname{Or} V$

$\operatorname{vol}(\det V) \cong \operatorname{vol} V$

exercise

$[0] = 0$

$$W \xrightarrow{1-1} \mathrm{vol}\, W \qquad \overbrace{|\lambda \cdot w|}^{\mathrm{vol}W} = \overbrace{|\lambda|}^{\mathbb{R}} \cdot \overbrace{|w|}^{\mathrm{vol}W}$$

$$w \longmapsto [w]$$

not linear

Pseudo. forms ? Top-degree pseudo form is a density.

b.) $\mathrm{Or}\,W = F(W) \Big/ \overset{\lambda \in \mathbb{R}}{[\lambda w] = \mathrm{sign}(\lambda) \cdot [w]}$

$$W \xrightarrow{\mathrm{sign}} \mathrm{or}\, W$$

$\mathrm{Sign}(\lambda \cdot w) = \mathrm{sign}(\lambda) \cdot \mathrm{sign}(w)$

$$F\left(W \setminus \{0\} / \mathbb{R}_{>0}\right) \Big/ -[w] = [-w]$$

c.) $\mathrm{or}(\mathrm{vol}\, W) \cong \mathbb{R}$

$\boxed{\text{Recall}}$ (the algebraic construction of ΛV).

$$\Lambda^k V \overset{k\geq 0}{=} (V\otimes \dots \otimes V) \Big/ \overset{+\text{ permutations}}{(v_1 \otimes v_2 \otimes \dots \otimes - v_2 \otimes v_1)}$$

$$= F(V\times \dots \times V) \big/ [v_1 + v_1', v_2, \dots.]$$

$$= [v_1, v_2, \dots] + [v_1', v_2, \dots]$$

$$[\lambda \cdot v_1, v_2, \dots] = \lambda \cdot [v_1, v_2, \dots]$$

+ Permutations

exterior (wedge) product

$$\Lambda^k V \otimes \Lambda^l V \longrightarrow \Lambda^{k+l} V$$

$$(v_1 \wedge \dots \wedge v_k) \otimes (v_k \wedge \dots \wedge v_{k+l}) \longmapsto v_1 \wedge \dots \wedge v_{k+l}$$

koszul sign rule: $\omega_1 \wedge \omega_2 = (\omega_2 \wedge \omega_1) \cdot (-1)^{k\cdot l}$

Sign iff k and l are odd.

$$[v_1, \dots, v_k] \otimes [v_{k+1}, \dots, v_{k+l}] \xmapsto[\wedge]{} [\cdot \dots]$$

$$= [v_1] \wedge \dots \wedge [v_k] \qquad F(v) = [\lambda \cdot v] = \lambda \cdot [v]$$

$$[v] \longleftarrow v \qquad [v + v'] = [v] + [v']$$

$$\Lambda^1 V \cong V$$

This is called a graded commutative algebra.

$\boxed{\text{Def}}$ A $\mathbb{Z}$-graded commutative algebra A is $A_k \in \text{Vect}_{\mathbb{R}}$ $k \in \mathbb{Z}$.

$$A_k \otimes A_l \xrightarrow{\cdot} A_{k+l} \quad , k, l \in \mathbb{Z}$$

$$\begin{tikzcd}
v\otimes w \arrow[d, mapsto] & A_k \otimes A_l \arrow[r, "\cdot"] \arrow[d] & A_{k+l} \\
(w\otimes v) \, -1^{(k\cdot l)} & A_l \otimes A_k \arrow[ur, "\cdot"'] &
\end{tikzcd}$$

$$v \cdot w = (-1)^{k \cdot l} w \cdot v$$

Example: ΛV

$\Lambda^k V$

Example $\deg 2k : \text{sym}^k V$

$\deg 2k+1 : 0$

$\boxed{\text{Rem}}$ $$GCA_{\mathbb{R}} \underset{\Pi}{\overset{\oplus}{\rightrightarrows}} \text{Alg}_{\mathbb{R}}$$

$$\oplus A := \bigoplus_k A_k$$

if $A_k = 0$ for $k < 0$

$$\Pi A := \prod_k A_k$$

$$(a_0, a_1, \ldots) \cdot (b_0, b_1, \ldots,)$$
$$= (a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0, \ldots)$$

The universal property of ΛV

$A \in GCA_{\mathbb{R}}$

$$\{\Lambda V \longrightarrow A\} \cong \{\overset{\Lambda^1 V}{\overset{\cong}{V}} \xrightarrow{g} A_1\}$$

$f \longmapsto f_1$

morphisms of $GCA_{\mathbb{R}}$

linear maps

$\sum_i v_1 \wedge \ldots \wedge v_{i-1} \wedge f(v_i)$

$g \longmapsto f$

$f_k : \Lambda^k V \longrightarrow A_k$

$v_1 \wedge \ldots \wedge v_k \longmapsto f(v_1) \wedge \ldots \wedge f(v_k)$

- **Prop** $V \in Vect_{\mathbb{R}}$, $A \in GCA_{\mathbb{R}}$

$$\{V \xrightarrow[f]{} A_1\} \cong \{\Lambda V \xrightarrow[g]{} A\}$$

$f \longmapsto (v_1 \wedge \ldots \wedge v_k \longmapsto f(v_1) \wedge \ldots \wedge f(v_k))$

$g_1 \longleftarrow\!| \; g_1$

ΛV is the free graded commutative algebra generated by V in degree 1.

ΛV has generators and relations

gen: (W, ω) $W \subset V$ subspace $\omega \in \det W = \Lambda^{\dim W} W$

degree $= \dim W$

rel: $(W_1, \omega_1) \wedge (W_2, \omega_2) = \begin{cases} 0 , & W_1 \cap W_2 \neq \{0\} \\ (W_1 + W_2), & \omega \wedge \omega_2 \end{cases}$

if $W_1 \cap W_2 = \{0\}$

$\dim (W_1 + W_2) = \dim W_1 + \dim W_2$

$\dim W = 1$

In degree 1 : (W, ω) $\omega \in W$

$$(W_1, \omega_1) + (W_2, \omega_2) = \begin{cases} 0, & \omega_1 + \omega_2 = 0 \\ (\operatorname{span}(\omega_1 + \omega_2), \omega_1 + \omega_2) \end{cases}$$

$\boxed{\text{Def}}$ $\Lambda_a V \longrightarrow \Lambda_g V$

is induced by the universal property

$$\begin{tikzcd} V \arrow[r] \arrow[rr, bend right, "\mathrm{id}_V"'] & \Lambda^1_g V & \cong V \end{tikzcd}$$

Essentially,

$$v_1 \wedge \dots \wedge v_k \longmapsto \begin{cases} 0 \text{ if } (v_i)_i \text{, lindep} \\ (W, \omega) \end{cases}$$

$W = \operatorname{span}(v_1, \dots, v_k)$

$\omega = v_1 \wedge \dots \wedge v_k$

Next $\Lambda_g V \longrightarrow \Lambda_a V$

what happens to the generator?

$\omega \in \Lambda^{\dim W} W$

$W \xrightarrow{inc} V$

$\Lambda^{\dim W} W \longrightarrow \Lambda^{\dim W} V$

equiv classes $\longmapsto$ equiv class

$$(W, \omega) \longmapsto \omega \in \Lambda_a^{\dim W} V$$

$$1 \quad \begin{tikzcd} (W_1, \omega_1) \wedge (W_2, \omega_2) \arrow[r, mapsto] \arrow[d, mapsto] & \left\{ \begin{matrix} 0, & W_1 \cap W_2 \neq \{0\} \\ (W_1 + W_2, & \omega_1 \wedge \omega_2 \end{matrix} \right\} \arrow[d, mapsto] \\ \omega_1 \wedge \omega_2 \, , & \left\{ \begin{matrix} 0, & W_1 \cap W_2 \neq \{0\} \\ & \end{matrix} \right. \end{tikzcd}$$

Grassmann calculus

Claim if $W_1 \cap W_2 \neq \{0\}$, then $\omega_1 \wedge \omega_2 = 0$.

$\omega_1 \wedge \omega_2 = u_1 \wedge \dots \wedge u_{k_1} \wedge v_1 \wedge \dots \wedge v_k$
$= 0$ linear dependence.

$\boxed{\text{Prop}}$ $\Lambda_g V^*$ gen & rel $G \subset A_{\mathbb{R}}$

gen(W, ω) $W \subset$ codim k

Substitute V^* in Λ_g
generators (W, ω)
indegree k $\dim V = n$
$\omega \subset V^*$

$$\underset{k}{W} \overset{\text{inclusion}}{\rightarrowtail} \underset{n}{V^*}$$

$$\underset{n-k}{U} \underset{\text{inclusion}}{\overset{\ker}{\rightarrowtail}} \underset{n}{V \cong V^{**}} \overset{i^*}{\twoheadrightarrow} \underset{k}{W^*}$$

$w = \{ f : V \to \mathbb{R} \mid f|_U = 0 \}$

$U = \{ v \in V \mid \forall w \in W : w(v) = 0 \}$

$\omega \in \Lambda^k W$ "the map is the important thing..."

$\boxed{\text{Recall}}$
$$\overset{k}{V_1} \underset{f_1}{\rightarrowtail} \overset{l}{V_2} \underset{f_2}{\twoheadrightarrow} \overset{m}{V_3}$$

$\operatorname{im} f_1 = \ker f_2$ $V_2 = V_1 \oplus V_3$:

$\det V_2 \cong \det V_1 \otimes \det V_3$ $\det(V_1 \oplus V_3)$

$(a_1 \wedge .. \wedge a_k) \otimes (c_1 \wedge \ldots \wedge c_n)$. $\det(V_1) \otimes \det V_3$

$(f_1(a_1) \wedge \ldots \wedge f_1(a_k)) \wedge \tilde{c}_1 \wedge \ldots \wedge \tilde{c}_m)$

$f_2(\tilde{c}_i) = c_i$ choice

$\det V \cong \det U \otimes \det W^*$ from earlier.

$\det W \cong \det U \otimes \det V^*$

$\psi \in \det U \otimes \det V^*$

$\cong (\det(V/U))^*$ Orthogonal compliment ψ is an oriented measure on the quotient V/U

$$(U_1, \psi_1) \wedge (U_2, \psi_2) = \begin{cases} 0 & \operatorname{codim}(U_1 \cap U_2) \neq \operatorname{codim} U_1 + \operatorname{codim} U_2 \\ (U_1 \cap U_2, \psi_1 \wedge \psi_2), & \text{otherwise} \end{cases}$$

"transversality..."

triang

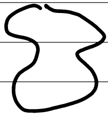
- Cairns
- Prasolov

ball nLab

Demailly

Guillemin - Haine

$U \subset \mathbb{R}^n$ Starshaped



$U \to \mathbb{R}^n$

$p \mapsto f(p) \cdot p$

$f : U \to \mathbb{R}_{>0}$

Sheaves

good cover ?

Redundant!

Last time: $W \subset V$

$\underbrace{\Lambda_g V}_{\text{Graded commutative Algebra}} = (W, \omega)$ $\dim W = \text{degree}$

$\omega \in \det W$

① $(W_1, \omega_1) \wedge (W_2, \omega_2) = \begin{cases} 0, & W_1 \cap W_2 \neq \{0\} \\ (W_1 + W_2, \omega_1 \wedge \omega_2) \end{cases}$

degree one: $\dim W = 1$ $\det W = W$

$\omega \in W$, $\omega \neq 0$

$\Rightarrow$ $W = \operatorname{span}\{\omega\}$

$\omega = 0 \Rightarrow (W, \omega) = 0$

$$\omega \in \det W = \Lambda^{\dim W} W \subset \Lambda^{\dim W} V$$

$$W = \{\omega \in V \mid \forall u \in U : u(\omega) = 0\}$$

$$U = \{u \in V^* \mid u \lrcorner \omega = 0\}$$

Contraction

② $V \cong \Lambda^1_g V$

(U, ψ) degree $=$ codim U

$$(U_1, \psi_1) \wedge (U_2, \psi_2) = \begin{cases} 0, \\ (U_1 \cap U_2 \,, \psi_1 \wedge \psi_2) \end{cases}$$

in degree 1: $\psi \in (V/U)^*$

$\psi : V \to \mathbb{R} \,,\ \psi|_U = 0 \Rightarrow U = \ker \psi$

Pairing $\Lambda V \otimes \underset{\substack{\cong \\ \Lambda(V^*)}}{\Lambda V^*} = \mathbb{R}$

Bourbaki
Algebra
chapter III

$$\begin{tikzcd} \Lambda^m V \arrow[r, bend left, leftrightarrow] & \otimes \Lambda^m V^* \end{tikzcd}$$

$$(\underset{m}{W}, \underset{\substack{\in \\ \det W}}{\omega}) \otimes (\underset{n-m}{U}, \underset{\in (\det(V/U))^*}{\psi}) \longmapsto$$

$$\begin{tikzcd} \overset{m}{W} \arrow[r, tail, "\iota"] & \overset{n}{V} \arrow[r, two heads, "q"] & \overset{m}{V/U} \end{tikzcd}$$

$\det(q \circ \iota) : \det W \longrightarrow \det(V/U)$

$\psi\left(\det(q \iota)(\omega) \right)$

Contractions

$$V \otimes \underset{\substack{\wr\| \\ (\Lambda^k V)^*}}{\Lambda^k V^*} \longrightarrow \Lambda^{k-1} V^* \qquad v \lrcorner \omega$$

$$/\!/ \; \iota_v \omega$$

$$v \otimes \underset{\substack{\cap \\ (\Lambda^k V)^*}}{\omega} \longmapsto (v_2, \dots, v_k) \qquad \omega \llcorner v$$

$$\longmapsto \omega(v, v_2, \dots, v_k)$$

$$V^* \otimes \Lambda^k V \relbar\joinrel\relbar\joinrel\relbar \Lambda^{k-1} V$$

$$f \otimes (v_1 \wedge \dots \wedge v_k) \longmapsto \sum_i f(v_i) (-1)^i v_1 \wedge v \dots \wedge \hat{v_i} \wedge \dots \wedge v_k$$

Proposition $A \in GCA_{\mathbb{R}}$, $d \in \mathbb{Z}$

$\overset{\mathbb{R}\text{-Linear}}{\mathcal{D}} : A_k \longrightarrow A_{k+d}$ is a <u>grated derivation</u> of degreed if $\mathcal{D}(a \cdot b) = (\mathcal{D}a) \cdot b + (-1)^{d \cdot |a|} \cdot a \cdot \mathcal{D} b$

$a \in A_{|a|}$ $\quad b \in A_{|b|}$

Proposition $\forall\, v \in V : \iota_v$ g.d. -1 of ΛV^*

$\forall f \in V^* : \iota_f$ g.d. -1 of ΛV

$f : V^*$

$f \lrcorner (v_1 \wedge \dots \wedge v_k \wedge v_{k+1} \wedge \dots \wedge v_{k+\ell})$

$(f \lrcorner (v_1 \wedge \dots \wedge v_k)] \wedge (v_{k+1} \wedge \dots \wedge v_{k+\ell}) + (-1)^k (v_1 \wedge \dots \wedge v_k)$
$\wedge (f \lrcorner (v_{k+1} \wedge \dots \wedge v_{k+\ell}))]$

$= \sum_i^k f(v_i) (-1)^i (v_1 \wedge \dots \wedge \widehat{v_i} \wedge \dots \wedge v_k \wedge v_{k+1} \wedge \dots \wedge v_\ell)$

$+ (-1)^k \sum_{j=1}^{\ell} f(v_{k+j}) \cdot (-1)^j (v_1 \wedge \dots \wedge v_k \wedge v_{k+1} \wedge \dots \wedge \widehat{v_{k+j}} \wedge \dots \wedge v_{k+\ell})$

$= \sum_{i=1}^{k+\ell} f(v_i) (-1)^i (v_1 \wedge \dots \wedge \widehat{v_i} \wedge \dots \wedge v_{k+\ell}).$

$V^* \not\cong \bigoplus_{k \geq 0} \mathbb{R}$ $\quad$ $\det W = \Lambda^{\dim W} W$

$\psi \in \det U \otimes (\det V)^*$

$\det U = \Lambda^{\dim U} U$
$\subset \Lambda^{\dim U} V$

$\Lambda^\ell V \otimes \Lambda^k V^* \xrightarrow{\lrcorner} \Lambda^{k-\ell} V^*$

$(W, \omega) \otimes (U, \psi) \longmapsto (W + U,$

$\ell \ \rotatebox{90}{$\in$} \det W \ n-k \qquad n-k+\ell$

transversally?

if $\dim (W+U) = n-k+\ell$

contractions

$\dim V = m$

$$\Lambda^k V \otimes \Lambda^l V^* \xrightarrow{\lrcorner} \Lambda^{l-k} V^*$$

$$(v_1 \wedge \dots \wedge v_n) \quad \omega \longmapsto (v_{k+1} \wedge \dots \wedge v_{k+l})$$

$$\frac{1}{(l-k)!} \sum_{\sigma} \omega(v_1, \dots, v_k)$$

$$(\det(V/U))^* = \det U \otimes (\det V)^*$$

$$(\overset{k}{W}, \overset{\det W}{\omega}) \lrcorner (\overset{n-l}{U}, \overset{(\det(V/U))^*}{\psi}) = \begin{cases} 0, (\dim W + U) \neq n-l+k \\ (W+U, \omega \otimes \psi \end{cases}$$

$$\Lambda^k V^* \otimes \Lambda^l V \xrightarrow{\lrcorner} \Lambda^{l-k} V$$

$$\underset{k-l}{(U, \psi)} \otimes \underset{l}{(W, \omega)} \longmapsto \begin{cases} 0 \text{ if} \dim(U \cap W) = l-k \\ U \cap W, \ \psi \otimes \omega \end{cases}$$

$$U \cap W \longrightarrow W \longrightarrow W/(U \cap W)$$

$$\cong V/U$$

$$\det(U \cap W) \cong \underset{\omega}{\det W} \otimes \underset{\psi}{\det (V/U)^*}$$

$$(W_1, \omega_1) \wedge (W_1, \omega_2) = \begin{cases} 0, \ \dim(W_1 + W_2) \neq k+l \\ (W_1 + W_2, \ \omega_1 \otimes \omega_2 \end{cases}$$

$$(\overset{n-k}{U_1}, \psi_1) \wedge (\overset{n-l}{U_2}, \psi_2) = \begin{cases} 0, \ \dim(U_1, U_2) \neq n+k-l \\ (U_1 \cap U_2), \ \psi_1 \otimes \psi_2 \end{cases}$$

Important Special case

Hodge star

$\ell = n = \dim V$

Fix $\omega \in \Lambda^m V^*$ (an oriented volume form)

$\Lambda^m V^* \otimes \Lambda^n V \longrightarrow \mathbb{R}$

$\omega \otimes \psi \longmapsto 1$ $\quad V \otimes V \longrightarrow \mathbb{R}$

$* : \Lambda^k V \longrightarrow \Lambda^{\ell-k}(V^*)$ $\quad V \xrightarrow{\cong} \underset{\substack{\cong \\ V^*}}{\mathrm{Hom}(V, \mathbb{R})}$

$\varphi \longmapsto \varphi \lrcorner \omega$

$* : \Lambda^k V^* \longrightarrow \Lambda^{\ell-k} V$

$x \longmapsto x \lrcorner \psi$

$(\overset{\perp}{W}, \varphi \otimes \omega)$

$(\omega, \varphi) \longmapsto (W, \varphi \otimes \omega)$

$\uparrow \qquad\qquad\qquad \Uparrow$

$\in \Lambda^k V \qquad\qquad \Lambda^{\ell-k} V^*$

$(U, x) \longmapsto (U, x \otimes \psi)$

$(U^\perp, x)$

If V is equipped w/ a nondegenerate metric $V/U \cong U^\perp$

$\Lambda^m V^* \cong \Lambda^m V$

Differential Forms

$\omega \in \Omega^n M$

A differential n-form on a smooth manifold M is an anti-symmetric $C^\infty M$-multilinear form

$$\underbrace{\mathfrak{X}M}_{\text{vectorfields}}, \dots, \mathfrak{X}M \longrightarrow C^\infty M.$$

Equivalently,

ω maps $m \in M$

to an element of $\Lambda^n T_m^* M \cong (\Lambda^n T_m M)^*$

has to be a smooth map.

Example: $n=0$: 0-form $\equiv$ smooth function.

Examples

$$\Omega^0 M \cong C^\infty M$$

$M = V \in \mathrm{Vect}_{\mathbb{R}}$ more generally

$T_m M \cong V$ $\quad M \underset{\text{open}}{\subset} V$

$T_m^* M \cong V^*$

An n-form ω is a smooth map

$M \longrightarrow \Lambda^n V^* \cong (\Lambda^n V)^*$

Fix a basis $e_1, \dots, e_m$ of V and

a dual basis $x_1, \dots, x_m$ of $V^* \cong \Lambda^1 V^* \cong (\Lambda^1 V)^*$

$dx_i : M \longrightarrow \Lambda^1 V^*$ if $v \in \mathfrak{X}M$, projection map?

$m \longmapsto x_i$. $dx_i(v) = v_i$

Example

we have the wedge product

$$\begin{aligned} \Omega^m M \otimes \Omega^n M &\longrightarrow \Omega^{m+n} M \\ \omega_1 \otimes \omega_2 &\longmapsto \omega_1 \wedge \omega_2 \end{aligned}$$

Example $M \underset{\text{open}}{\subseteq} V$

$$dx_i \wedge dx_j \in \Omega^2 M$$

$$\begin{aligned} (dx_i \wedge dx_j)(v_1, v_2) &= dx_i(v_1) \cdot dx_j(v_2) \\ &\quad - dx_i(v_2) \cdot dx_j(v_1) \\ &= v_{1,i}\, v_{2,j} - v_{2,i}\, v_{1,j} \end{aligned}$$

$$= \det \begin{pmatrix} v_{1,i} & v_{1,j} \\ v_{2,i} & v_{2,j} \end{pmatrix}$$

a basis for $\Omega^k M$

$$\left(dx_{i_1} \wedge \dots \wedge dx_{i_k}\right)(v_1, \dots, v_k)$$

$$= \det \left(\begin{matrix} v_{1,i_1} & v_{1,i_2} & \dots & v_{1,i_k} \\ & & & \\ & & & \end{matrix} \right.$$

Example $f : M \to \mathbb{R}$

$$\begin{tikzcd} Tf : TM \arrow[r] \arrow[dr, "\text{directional derivative}"'] & T\mathbb{R} = \mathbb{R} \times \mathbb{R} \arrow[d, "\pi_2"] \\ & \mathbb{R} \end{tikzcd}$$

$df \in \Omega^1 M$

$(df)(v) = \partial_v f$

The DGCA of differential forms

$\langle \Omega \rangle$

Def A differential graded algebra A is $A_n \in \mathrm{Vect}_{\mathbb{R}}$ $n \in \mathbb{Z}$

- A is a graded algebra (commutative)
- $d : A_n \longrightarrow A_{n+1}$ $\mathbb{R}$-linear map (cohomological grading convention) $\nearrow$ cochain complex
- $d(a \cdot b) = d(a) \cdot b + a \cdot d(b) \cdot (-1)^{|a| \cdot 1}$
- graded commutative if $a \cdot b = (-1)^{|a||b|} b \cdot a$

$M = \mathbb{R}^n$ df $f = \sum_i x_i \cdot g_i$

<u>Algebraic Def</u> $df = \sum dx_i \cdot g_i + x_i \cdot dg_i$

"Kahler differentials" $\Omega^1 M = C^\infty$-Kahler differenital

Ω is the free C^∞-DGA (A_0 is a C^∞-ring, $d : A_0 \longrightarrow A_1$, C^∞ derivation (

$$d(f(a_1, \dots, a_n)) = \sum_i \frac{\partial f}{\partial x_i}(a_1, \dots, a_n) \cdot da_i \,))$$

on $C^\infty M$ in degree 0. In practice a $\omega \in \Omega^n M$ $\wedge$

C^∞DGCA

$$\left\{ \Omega M \xrightarrow{p} \underset{\uparrow}{B} \right\} \cong \left\{ C^\infty M \xrightarrow{q} \underset{\overset{\cap}{C^\infty \mathrm{Ring}}}{B_0} \right\}$$

$p \longmapsto p_0$ $\qquad p(\sum f \wedge dg_1 \wedge \dots \wedge dg_k)$

$p \longleftarrow q$ $\qquad \sum p(f) \wedge dp(g_1) \wedge \dots \wedge d(p(g_k))$

In practice $\omega \in \Omega^k M$: $\omega = \sum_{\text{finite}} f \wedge dg_1 \wedge \dots \wedge dg_k$

$f, g \in C^\infty M$

What is a concrete example of

$$\mathcal{B} = C^{\infty}(\mathbb{R}^n) \otimes \mathrm{GrSym}(V^*) \qquad V \in \mathrm{GrVect}_{\mathbb{R}}$$

$$\mathrm{GrSym}\, V^* = \bigotimes_{k \geq 1} \mathrm{GrSym}(V_k^*[k])$$

\bigotimes

$$= \bigotimes_{\substack{k \geq 1 \\ k \text{ odd}}} \Lambda V_k^* \otimes \bigotimes_{\substack{k \geq 1 \\ k \text{ even}}} \mathrm{Sym} V_k^*$$

\otimes

Geometric Definition

$$\Omega_g M \in \mathrm{DGCA}_{\mathbb{R}}$$

ΩM = smooth infinitesimal singular cochain complex.

$$\Omega^k M = \Gamma(\Lambda^k T^* M) \cong \Lambda^k_{C^{\infty} M} \Gamma(T^* M)$$

$$(\underset{\underset{M}{\rotatebox{90}{$\in$}}}{m} \longmapsto \omega_m \in \Lambda^k T^*_m M) \qquad \rotatebox{90}{$\cong$}$$

$$\Lambda^k_{C^{\infty} M}(\mathfrak{X} M)^*_{C^{\infty} M}$$

$$\omega_m(v_1, \dots, v_k) \in \mathbb{R}$$

$$v_i \in TM$$

vector-bundles....

$$\omega(X_1, \dots, X_k) \in C^{\infty} M$$

$$\omega(f \cdot X_1, \dots) = f \cdot \omega(X_1, \dots,)$$

$\Omega_a M \longrightarrow \Omega_g M \iff C^\infty M \xrightarrow[id]{a} (\Omega_g^0 M) = C^\infty M$

Def $\Omega^0 M \longrightarrow \Omega^1 M$

$C^\infty M \xrightarrow{d} (\mathfrak{X} M)^*_{C^\infty M}$

$(df)(X) = \mathcal{L}_X f = X(f)$

Thm. Isomorphic!

de Rham time

Proposition $\omega \in \Omega^k M$

$(d\omega)(X_0, \dots, X_k)$

$= \sum_{0 \le i \le k} (-1)^i \mathcal{L}_{X_i} \omega(X_0, \dots, \widehat{X_i}, \dots, X_k)$

$+ \sum_{0 \le i \le j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k)$

Suppose $M \subset_{open} \mathbb{R}^n = V$

$TM \cong M \times V \qquad T^*M \cong M \times V^*$

$\Lambda^k T^* M = M \times \Lambda^k V^*$

$\Omega^k M = C^\infty(M, \Lambda^k V^*)$

take $\omega \in \Omega^k_M = C^\infty(M, \Lambda^k V^*)$

$D\omega$ derivative

c III $\quad D\omega : M \times V \longrightarrow \Lambda^k V^*$

$$(D\omega)_m(v_0, \underbrace{v_1, \dots, v_k}_{\text{antisymetrize thewholething}}) \in \mathbb{R}^n$$

$$d\omega = \mathrm{Alt}_k(D\omega)$$

$$(\mathrm{Alt}(D\omega)(X_0, \dots, X_k)$$
$$= \sum (-1)^i (D\underbrace{\omega)(X_i, X_0, \dots, \widehat{X_i}, \dots, X_k)}_{\text{multi-linear in } \omega \text{ and } x_i}$$

$$D(\omega(X_0, X_1, \dots, \widehat{X_i}, \dots, X_n))$$

in ω and x_i

$$(D_{x_i}\omega)(x_0, \dots, \widehat{x_i}, \dots, x_k) + \sum_j \omega(X_0, \dots, \widehat{X_i}, \dots, D_{x_i} x_j, x_n)$$

$k = 0 : d\omega(X_0) = \mathcal{L}_{X_0}\omega$

$k = 1 : (d\omega)(x_0, x_1) = \mathcal{L}_{x_0}\omega(x_1) - \mathcal{L}_{x_1}\omega(x_0)$

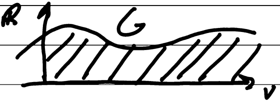
Riemann curvature tensor

$- \omega([x_0, x_1]).$

$\int$ area graph

$f : V \to \mathbb{R}$

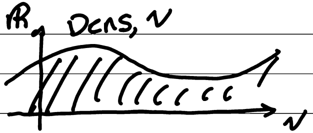
$G \subset \mathbb{R} \times V$



can integrate $f : V \longrightarrow \mathrm{Dens}_1(V)$

$$= \mathrm{Vol}(V^*) = \underset{\dim 1}{\det(V^*)} \otimes \underset{\dim 1}{\mathrm{or}(V^*)}$$

Analogous figure



$f : V \to D$

$V \times \mathrm{Dens}, V$ has a canonical measure!?

Recall: a TIM on $W \in \mathrm{Vect}_{\mathbb{R}}$ is an element of $\mathrm{Vol}(W^*) = \det(W^*) \otimes \mathrm{or}(W^*)$

$$\mathrm{Vol}(W^*) = \mathrm{Vol}\left(V^* \times (\mathrm{Dens}, V)^*\right)$$

$$\cong \mathrm{Vol}(V^*) * \mathrm{Vol}\left((\mathrm{Dens}, V)^*\right)$$

$$= \mathrm{Dens}_1 V \otimes (\mathrm{Dens}_1 V)^* \cong \mathbb{R} \ni 1$$

Suppose

basis

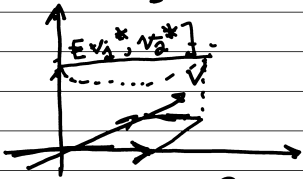
$v_1, \dots, v_n \in V$

$v_1^*, \dots, v_n^* \in V^*$

$[v_1^*, v_2^*, \dots, v^*_n, [v_1, \dots, v_n]]$

$\text{vol}\hat{V} \cong (\text{Dens}, V)^*$

$\text{Dens}_1(V)$



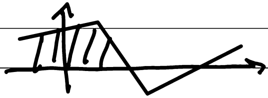
$[v_1, \dots, v_n, [v_1^*, \dots, v_n^*]]$
$\in \text{Vol}(W)$
$\cong (\text{Dens}, W)^*$

$\dim V = 2$

a 1-density

Suppose $f: V \to \text{dens}_1 V$
$= (\text{vol} V)^* = \det V^* \otimes \text{or} V^*$

$\mu \in \text{Vol}(W^*)$
& measure



$$\int f := \mu\left(\{ (v,\omega) \in V \times \text{Dens}_1 V \mid 0 \leq \omega \leq f(v) \right)$$

$$- \mu\left(\{ (v,\omega) \mid f(v) \leq \omega \leq 0 \} \right.$$

Any oriented 1 dim v space has a canonical ordering.

$\mathrm{or}(\mathrm{Dens}_1 V)$

$= \mathrm{or}(\det V^* \otimes \mathrm{or} V^*)$

$= \mathrm{or}(\det V^*) \otimes \mathrm{or}(\mathrm{or}(V^*))$

$= \mathrm{or} V^* \otimes \mathrm{or} V^*$ what is $\mathbb{R}$

$= \mathrm{or} V \otimes \mathrm{or} V^* \cong \mathbb{R}$

If V is oriented,

$\mathrm{Dens}_1 V \cong \det V^*$

$f : V \to \det V^*$ $f \in \Omega^{\dim V} V$

Recall

$\det V^* \xrightarrow{1\text{-}1} \mathrm{Dens}_1 V$

$[v_1, \dots, v_n] \longmapsto [v_1, \dots, v_n]$

non-linear

$[v_2, v_1, v_3 \dots] \longmapsto [v_2, v_1, \dots]$

$-[v_2, \dots, v_n]$ $\quad \| \quad [v_1, \dots, v_n]$

If $\omega \in \Omega^{\dim V} V$

$|\omega| \in \mathrm{dens}_1 V$ or $\in C^\infty(V, \mathrm{dens}_1 V)$

Set $\int \omega = \int |\omega|$

$V = \mathbb{R}^n$

$$\Omega^0 M \xrightarrow{d} \Omega^1 M \to \dots \to \Omega^{\dim M} M$$

$\Lambda^0 T^*_m M \qquad \Lambda^1_{\cdot} T^*_{\cdot} M$

$\Lambda^0 T^* M \otimes \mathrm{or} M \qquad \Lambda^1 T^* M \otimes \mathrm{or} M \qquad \Lambda^{\dim M} T^* M \otimes \mathrm{or} M$

fiber by fiber → Line bundle

$\mathrm{or} M = \mathrm{or}(T^* M)$

$$\begin{tikzcd}
 & & & \mathbb{R} \\
\tilde{\Omega}^0 M \arrow[r] & \tilde{\Omega}^1 M \arrow[r] & \dots \arrow[r] & \tilde{\Omega}^{\dim M} M \arrow[u, "\int"']
\end{tikzcd}$$

$\Gamma(\mathrm{Dens}_1(TM))$

Lower degree are not twisted forms.

$V = \mathbb{R}^n$

$V = \mathbb{R}^n$

$\dim M = k$

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r, "g"'] & \mathbb{R} \end{tikzcd}$$

$$\begin{tikzcd} \Omega^k M & \Omega^k N \arrow[l, "f^*"'] \end{tikzcd}$$

Pullback!

$$\begin{tikzcd} C^\infty N \arrow[r] & C^\infty M \\ g \arrow[r, mapsto] \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & g \circ f \end{tikzcd}$$

remember this

$$\begin{tikzcd} \Omega M & \Omega N \arrow[l] \end{tikzcd}$$

$$\sum (h \circ f) \, d g_1 \circ f \wedge \ldots \wedge d(g_k \circ f) \longmapsfrom \sum h \, d g_1 \wedge \ldots \wedge d g_k$$

$V = \mathbb{R}^n$

$x_i : \mathbb{R}^n \longrightarrow \mathbb{R}$

$dx_i \in \Omega^1 \mathbb{R}^n$

$dx_1 \wedge \ldots \wedge dx_n \in \Omega^n \mathbb{R}^n$

$|dx_1 \wedge \ldots \wedge dx_2| \in |\Omega^n \mathbb{R}^n|$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$

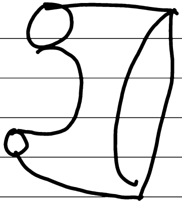
$$\int f \cdot |dx_1 \wedge \ldots \wedge dx_n| \in \mathbb{R}$$

nLab generalized differential form

The Stokes Theorem

$$\int_M d\omega = \int_{\partial M} \iota^* \omega$$

Local $\mathbb{R}^n$ or $\{ x \in \mathbb{R}^n \mid x_1 \geq 0 \}$



$\iota : \partial M \longrightarrow M$

$\omega \in \Omega^{\dim M - 1} M \otimes \text{or} M$

$d\omega \in \Omega^{\dim M} M \otimes \text{or} M$

Example $M = [a, b]$

$\partial M = \underset{-}{\overset{\leftarrow}{a}} \qquad \underset{+}{\overset{b}{\rightarrow}}$

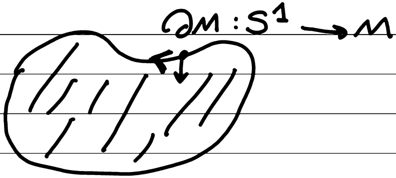
$\omega \in \Omega^0 M \otimes \text{or} M$

$\omega : M \rightarrow \mathbb{R}$

$d\omega = \omega' dx$

$$\int_{[a,b]} \omega' |dx| = \int_{\leftarrow a}^{\rightarrow b} \iota^* \omega = \omega(b) - \omega(a)$$

$\mathbb{R}^2$



$\omega \in \Omega^1 M$

$\omega = f_x dx + f_y dy$

$$d\omega = \frac{\partial f_x}{\partial y}(dy \wedge dx) + \frac{\partial f_y}{\partial x}(dx \wedge dy)$$

$$= \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)(dx \wedge dy)$$

$$\int \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \left| (dx \wedge dy) \right| =$$

$$\int_{S^1} f_x dx + f_y dy$$

Green's Formula

13

bonus HW:

$\boxed{\text{dim 3 Stokes formula}}$