Higher Lie integration I

L_∞-algebras and the Chevalley-Eilenberg construction: The stuff we integrate

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The Maurer-Cartan form

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and for any $g \in G$ let $L_g : G \to G$ denote the left translation by g:

$$L_g(h) = gh.$$

Observe that:

- L_g is a diffeomorphism for each g
- $TL_{g^{-1}} : T_gG \to T_eG$ is an isomorphism for each g
- $T_eG \cong \mathfrak{g}$

Definition

Denote by $\omega_G \in \Omega^1(G, \mathfrak{g})$ the *(left-invariant) Maurer-Cartan form* on G. Given $g \in G$ and $v \in T_gG$, we define

$$(\omega_G)_g(v) = (TL_{g^{-1}})v.$$

ω_G is the unique left-invariant 1-form on G.

The Darboux derivative

Definition

Let $f : M \to G$ be a smooth map valued in a Lie group G. The *(left) Darboux derivative* of f is the $\mathfrak{g}$-valued 1-form $\omega_f = f^*\omega_G$.

Moral:

- For $f : M \to N$,

$$Tf : TM \to TN$$

  is usually referred to as "the derivative of f", but Tf still contains information about f

- In the case that $N = G$ a Lie group, the composition:

$$TM \xrightarrow{Tf} TG \xrightarrow{\omega_G} \mathfrak{g}$$

  has the effect of forgetting the data of f and keeping only the information about the "honest derivative of f"

The Darboux derivative cont'd

Example:

- Take $f : \mathbb{R} \to \mathbb{R}$
- Then $Tf : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ with

$$Tf(x, v) = (f(x), f'(x)v)$$

- Recall that $\mathbb{R}$ is a Lie group with Lie algebra $\mathbb{R}$, and Maurer-Cartan form given by $dt : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where $t : \mathbb{R} \to \mathbb{R}$ is the identity function and thereby

$$dt(x, v) = v$$

- The Darboux derivative ω_f is then given by

$$(\omega_f)_x(v) = f'(x)v$$

The nonabelian fund. theorem of calculus

Observation: Since ω_G satisfies:

$$d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0$$

and d is natural, we have

$$d\omega_f + \frac{1}{2}[\omega_f, \omega_f] = 0$$

Theorem

Let G be a Lie group with Lie algebra $\mathfrak{g}$, M a manifold, and let $\omega \in \Omega^1(M; \mathfrak{g})$ such that $d\omega + \frac{1}{2}[\omega, \omega] = 0$. Then, for each $p \in M$, there exists an open neighborhood $p \in U$ and smooth function $f : U \to G$ such that $\omega|_U = \omega_f$. f is unique up to translation by an element in G.

The Chevalley-Eilenberg algebra of a Lie algebra

Definition

Let $\mathfrak{g}$ be a (finite-dimensional) Lie algebra, denote by $\mathrm{CE}(\mathfrak{g})$ the differential graded algebra whose underlying graded algebra is given by the Grassmann algebra on the dual of $\mathfrak{g}$:

$$\mathrm{CE}(\mathfrak{g}) = \wedge^{\bullet}\mathfrak{g}^*,$$

and whose differential is given on generators by the dual of the Lie bracket on $\mathfrak{g}$ considered as a linear map, with an added sign:

$$-[-,-]^* : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*,$$

and extended by the graded Leibniz rule.

Remark:

- For $\mathfrak{g}$ a finite-dimensional vector space, dg-structures on $\wedge^{\bullet}\mathfrak{g}^*$ are in bijection with Lie algebra structures on $\mathfrak{g}$

Definition

Given a short exact sequence of Lie algebras:

$$0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0$$

one says that $\mathfrak{e}$ is an extension of $\mathfrak{g}$ by $\mathfrak{h}$. When $[h, -]_{\mathfrak{e}}$ vanishes for any $h \in \mathfrak{h}$, the extension is said to be central. Two extensions are equivalent when we have a diagram:

$$\begin{array}{ccccc}
\mathfrak{h} & \xrightarrow{i'} & \mathfrak{e}' & \xrightarrow{s'} & \mathfrak{g} \\
\uparrow{\scriptstyle id_{\mathfrak{h}}} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle id_{\mathfrak{g}}} \\
\mathfrak{h} & \xrightarrow{i} & \mathfrak{e} & \xrightarrow{s} & \mathfrak{g}
\end{array}$$

in which f is an isomorphism.

Interlude: Lie algebra cohomology and extensions cont'd

Proposition

Let $\mathfrak{g}$ be a Lie algebra and M a left $\mathfrak{g}$-module. The second degree cohomology of $\mathfrak{g}$ with values in M is in bijective correspondence with equivalence classes of central extensions:

$$M \to \mathfrak{e} \to \mathfrak{g}$$

Of particular interest to us is the case when $M = \mathbb{R}$.

Proof sketch in one direction:

- The cohomology of $\mathrm{CE}(\mathfrak{g})$ is precisely the real valued Lie algebra cohomology of $\mathfrak{g}$
- From the definition of $\mathrm{CE}(\mathfrak{g})$ we see that a 2-cocycle is a linear map

$$\mu : \mathfrak{g} \wedge \mathfrak{g} \to \mathbb{R}$$

  such that

$$\mu(g_1, [g_2, g_3]) + \mu(g_2, [g_3, g_1]) + \mu(g_3, [g_1, g_2]) = 0$$

Proof sketch cont'd:

- Take the following exact sequence of vector spaces:

$$\mathbb{R} \to \mathfrak{g} \oplus \mathbb{R} \to \mathfrak{g}$$

- Endow $\mathfrak{g} \oplus \mathbb{R}$ with the bracket

$$[(g_1, t_1), (g_2, t_2)]_{\mathfrak{g}\oplus\mathbb{R}} = ([g_1, g_2], \mu(g_1, g_2))$$

- $[-,-]_{\mathfrak{g}\oplus\mathbb{R}}$ satisfies the Jacobi identity as a result of the fact that μ is a cocycle

Flat forms as morphisms of dga's

Proposition

Let $X \in \mathsf{Man}$ and $\mathfrak{g}$ be a Lie algebra, there is a bijection (of sets):

$$\mathsf{DGCA}(\mathrm{CE}(\mathfrak{g}), \Omega(X)) \cong \Omega^1_\flat(X; \mathfrak{g}),$$

where on the right we have the set of flat $\mathfrak{g}$-valued one-forms on X.

Proof sketch in one direction:

- Given $\omega \in \Omega^1_\flat(X; \mathfrak{g})$ define a map $\psi : \mathrm{CE}(\mathfrak{g}) \to \Omega(X)$ by defining it on a generator $\alpha \in \mathfrak{g}^*$:

  $$\psi(\alpha) = \alpha \circ \omega$$

  we verify that ψ respects differentials
- $\psi(d_{\mathrm{CE}}\alpha) = (d_{\mathrm{CE}}\alpha) \circ (\omega \wedge \omega) = -\alpha([\omega \wedge \omega])$
- $d_{\mathrm{dR}}(\psi(\alpha)) = d_{\mathrm{dR}}(\alpha \circ \omega) = \alpha \circ d_{\mathrm{dR}}\omega$
- $d_{\mathrm{dR}}(\psi(\alpha)) - \psi(d_{\mathrm{CE}}\alpha) = \alpha(d_{\mathrm{dR}}\omega + [\omega \wedge \omega]) = \alpha(0) = 0$

“Classical” Lie integration

Let G be a Lie group with Lie algebra $\mathfrak{g}$.

- The non-abelian fund. theorem of calculus tells us that flat $\mathfrak{g}$-valued 1-forms are locally equivalent to smooth G-valued functions modulo translation
- The proposition on the previous slide then gives way to an isomorphism of sheaves on Man:

$$\mathsf{DGCA}(\mathrm{CE}(\mathfrak{g}), \Omega(-)) \cong y(G)/G$$

"Classical" Lie integration cont'd

- Consider the smooth singular complex of $y(G)/G$, which we will denote by $\mathrm{Sing}(y(G)/G) \in \mathsf{sSet}$. This is the simplicial set given by:

$$[n] \mapsto \mathsf{Sh}(\mathsf{Man})(y(\boldsymbol{\Delta}^n), y(G)/G)$$

  where $\boldsymbol{\Delta}^n$ is a model for the smooth n-simplex (we will define this in detail later, for now just think of $\mathbb{R}^n$)
- We have a weak equivalence of simplicial sets

$$\mathbf{cosk}_2(\mathrm{Sing}(y(G)/G)) \simeq \mathbf{B}G$$

We'll revisit this example with a proof when we introduce the integration machinery proper, this is just meant to be a preview!

Definition

An L_∞-algebra is given by that data of $\mathfrak{g}$ a graded vector space, together with linear maps

$$l_k : \wedge^k \mathfrak{g} \to \mathfrak{g}$$

of degree $k-2$ for $k \geq 0$, subject to the generalized Jacobi identity $\mathcal{J}_n = 0$ for all $n \geq 0$, where $\mathcal{J}_n$ is the following sum:

$$\sum_{p=1}^{n} (-1)^{p(n-p)} \sum_{\sigma \in \text{unshuff}(p,n-p)} (-1)^{\sigma} \varepsilon(v_\sigma) l_{n-p+1}\big(l_p(v_{\sigma(1)}, \ldots, v_{\sigma(p)}), v_{\sigma(p+1)}, \ldots, v_{\sigma(n)}\big)$$

in which $(-1)^\sigma$ denotes the sign of σ as a permutation, and $\varepsilon(v_\sigma)$ denotes the total Koszul sign imparted by permuting the v_i by σ.

Generalized Jacobi identity

Let's have a look at the first few generalized Jacobi identities:

- $\mathcal{J}_1 = l_1 \circ l_1 = 0$ says that l_1 is a chain differential
- $\mathcal{J}_2 = -l_2(l_1(v_1), v_2) + (-1)^{|v_1||v_2|} l_2(l_1(v_2), v_1) + l_1(l_2(v_1, v_2)) = 0$
  from this one can derive the equality:

  $$l_1(l_2(v_1, v_2)) = l_2(l_1(v_2), v_1) + (-1)^{|v_1|} l_2(v_1, l_1(v_2))$$

  which says l_1 is a graded derivation of l_2
- $\mathcal{J}_3 = 0$ is quite a long expression, but its contents are the following ...

Generalized Jacobi identity cont'd

Let $f : \wedge^3\mathfrak{g} \to \mathfrak{g}$ denote the following linear map:

$$f = \sum_{\sigma \in \text{shuff}(2,1)} \varepsilon(\sigma, -, -) l_2(l_2(-, -), -),$$

where $\varepsilon(\sigma, v_1, v_2)$ is the total sign contribution from σ and the Koszul rule. Consider the following diagram:

$$\begin{array}{ccccc}
(\wedge^3\mathfrak{g})_{n-1} & \xleftarrow{\;l_1\;} & (\wedge^3\mathfrak{g})_n & \xleftarrow{\;l_1\;} & (\wedge^3\mathfrak{g})_{n+1} \\
\downarrow f & \searrow^{l_3} & \downarrow f & \searrow^{l_3} & \downarrow f \\
\mathfrak{g}_{n-1} & \xleftarrow[\;l_1\;]{} & \mathfrak{g}_n & \xleftarrow[\;l_1\;]{} & \mathfrak{g}_{n+1}
\end{array}$$

The equality $\mathcal{J}_3 = 0$ is equivalent to the statement that l_3 is a chain homotopy between f and 0. $\mathcal{J}_n = 0$ for $n > 3$ encode higher homotopy coherences: *homotopies between homotopies between ...*

Examples of L_∞-algebras

- Ordinary Lie algebras
  - Take $\mathfrak{g}$ concentrated in degree 0 with $l_k = 0$ for all $k \neq 2$
  - l_2 defines a Lie bracket on $\mathfrak{g}_0$
- Differential graded Lie algebras
  - Take $\mathfrak{g}$ to be an arbitrary graded vector space with $l_k = 0$ for all $k > 2$
  - $\mathcal{J}_1$ gives that l_1 is a differential
  - l_2 is a bracket and $\mathcal{J}_2$ gives that l_1 is a graded derivation of l_2 as we have seen
- The line Lie n-algebra $\mathbf{b}^{n-1}\mathbb{R}$
  - Take $\mathfrak{g}$ concentrated in degree n with $\mathfrak{g}_n = \mathbb{R}$
  - Take all brackets to be trivial

Examples of L_∞-algebras cont'd: The string Lie 2-algebra

Recall the definition of the Killing form:

Definition

Let $\mathfrak{g}$ be a Lie algebra and consider the linear map defined for each $X \in \mathfrak{g}$

$$\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$$

given by

$$\mathrm{ad}(X)(Y) = [X, Y].$$

The Killing form

$$\langle -, - \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

is the symmetric non-degenerate bilinear form given by:

$$\langle X, Y \rangle = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)).$$

Examples of L_∞-algebras cont'd: The string Lie 2-algebra

The finite-dimensional or *skeletal* model of the string Lie 2-algebra is defined as follows (does the underlying vector space look familiar?):

Definition

Let $\mathfrak{g}$ be a Lie algebra of compact type, denote by $\mathfrak{str}_{\mathfrak{g}}$ or simply $\mathfrak{str}$ the L_∞-algebra comprised of the following data:

$$\mathfrak{str} = \mathfrak{g} \oplus \mathbb{R}$$

as a graded vector space. The brackets are given by:

- $l_1 : \mathbb{R} \to \mathfrak{g}$ is trivial
- $l_2 : (\wedge^2 \mathfrak{g}) \oplus (\mathfrak{g} \wedge \mathbb{R}) \oplus (\wedge^2 \mathbb{R}) \to \mathfrak{g} \oplus \mathbb{R}$ is trivial except for in degree 0 where $l_2(g_1, g_2) = [g_1, g_2]_{\mathfrak{g}}$
- l_3 is trivial except for in degree 0 where $l_3|_{\wedge^3 \mathfrak{g}} \colon \wedge^3 \mathfrak{g} \to \mathbb{R}$ is given by $l_3(g_1, g_2, g_3) = \langle [g_1, g_2]_{\mathfrak{g}}, g_3 \rangle$

The Chevalley-Eilenberg algebra of an L_∞-algebra

Definition

Let $\mathfrak{g}$ be a degree-wise finite-dimensional L_∞-algebra. The Chevalley-Eilenberg algebra of $\mathfrak{g}$ has as its underlying graded algebra the shifted symmetric algebra of $\mathfrak{g}^*$:

$$\mathrm{CE}(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}[1]^*)$$

and whose differential is given on generators by the sum of the duals of the brackets:

$$d_{\mathrm{CE}(\mathfrak{g})} = \sum_k l_k^*$$

and extended by linearity + the graded Leibniz rule.

Remark:

- For a degree-wise finite-dimensional graded vector space $\mathfrak{g}$, dg-structures on $\mathrm{Sym}(\mathfrak{g}[1]^*)$ are in bijection with L_∞-structures on $\mathfrak{g}$

The Chevalley-Eilenberg algebra of an L_∞-algebra

Let's immediately unpack this definition, let $\mathfrak{g}$ be an L_∞-algebra:

- The shifted symmetric algebra $\mathrm{Sym}(\mathfrak{g}[1]^*)$ looks like the following graded vector space:

$$\mathrm{Sym}(\mathfrak{g}[1]) \cong \mathrm{Sym}(\bigoplus_p \mathfrak{g}^*_{2p+1}) \otimes \bigwedge(\bigoplus_q \mathfrak{g}^*_{2q})$$

  So we have
  - $\mathrm{CE}(\mathfrak{g})^1 \cong \wedge^1 \mathfrak{g}^*_0$
  - $\mathrm{CE}(\mathfrak{g})^2 \cong \mathrm{Sym}^1(\mathfrak{g}^*_1) \oplus \wedge^2 \mathfrak{g}^*_0$
  - $\mathrm{CE}(\mathfrak{g})^3 \cong \wedge^3 \mathfrak{g}^*_0 \oplus \wedge^1 \mathfrak{g}^*_2 \oplus (\mathrm{Sym}^1(\mathfrak{g}^*_1) \otimes \wedge^1 \mathfrak{g}^*_0)$
  - ...

To see how the differential works it's best to look at a concrete example ...

The Chevalley-Eilenberg algebra of $\mathfrak{str}$

Recalling that $\mathfrak{str}$ has underlying graded vector space given by $\mathfrak{str}_0 = \mathfrak{g}$ and $\mathfrak{str}_1 = \mathbb{R}$, we have that:

- $\mathrm{CE}(\mathfrak{str})^1 \cong \mathfrak{g}^*$
- $\mathrm{CE}(\mathfrak{str})^2 \cong \wedge^2\mathfrak{g}^* \oplus \mathbb{R}$
- $\mathrm{CE}(\mathfrak{str})^3 \cong \wedge^3\mathfrak{g}^* \oplus \mathfrak{g}^*$
- $\mathrm{CE}(\mathfrak{str})^4 \cong \wedge^4\mathfrak{g}^* \oplus \wedge^2\mathfrak{g}^* \oplus \mathbb{R}$

The only non-trivial bracket data are the following maps:

$$l_2|_{\wedge^2\mathfrak{g}} = [-,-] : \wedge^2\mathfrak{g} \to \mathfrak{g}$$

and

$$l_3|_{\wedge^3\mathfrak{g}} = \langle [-,-],-\rangle : \wedge^3\mathfrak{g} \to \mathbb{R}$$

The Chevalley-Eilenberg algebra of $\mathfrak{str}$ cont'd

Thus the differential is computed as follows:

- In degree 1
$$d_{\mathrm{CE}(\mathfrak{str})} : \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^* \oplus \mathbb{R}$$
is given by $[-,-]^* + 0$
- In degree 2
$$d_{\mathrm{CE}(\mathfrak{str})} : \wedge^2 \mathfrak{g}^* \oplus \mathbb{R} \to \wedge^3 \mathfrak{g}^* \oplus \mathfrak{g}^*$$
is given on $\wedge^2 \mathfrak{g}^*$ by extending $[-,-]^*$ by the graded Leibniz rule to produce values in $\wedge^3 \mathfrak{g}^*$, and on $\mathbb{R}$ by $\langle [-,-],-\rangle^* + 0$
- In degrees ≥ 3 everything follows from the extensions of what happens in degrees 1 and 2

$\mathfrak{str}$ as a central extension

The underlying graded vector space of $\mathfrak{str}$,

$$\mathfrak{str} = \mathfrak{g} \oplus \mathbb{R}$$

is the same as we previously saw in the context of the central extension associated to a 2-cocycle, this is no coincidence.

Definition

Let $\mathfrak{g}$ be an L_∞-algebra, a $\mathfrak{g}$-cocycle of degree n is a morphism of L_∞-algebras:

$$\mu : \mathfrak{g} \to \mathbf{b}^{n-1}\mathbb{R},$$

or equivalently, a morphism of their Chevalley-Eilenberg algebras:

$$\mathrm{CE}(\mathbf{b}^{n-1}\mathbb{R}) \to \mathrm{CE}(\mathfrak{g})$$

$\mathfrak{str}$ as a central extension cont'd

The definition is unpackaged as follows:

- A morphism $\mathrm{CE}(\mathbf{b}^{n-1}\mathbb{R}) \to \mathrm{CE}(\mathfrak{g})$ picks out an element $\mu \in CE^n(\mathfrak{g})$, this is the image of the single generator of $\mathrm{CE}(\mathbf{b}^{n-1}\mathbb{R})$ in degree n
- That this morphism respects differentials implies that $d_{\mathrm{CE}(\mathfrak{g})}\mu = 0$

Observe the following:

- Given any n-cocycle $\mu : \mathfrak{g} \to \mathbf{b}^{n-1}\mathbb{R}$ we can form the homotopy pullback (or *homotopy fiber*):

$$\begin{array}{ccc} \mathfrak{g}_\mu & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathfrak{g} & \xrightarrow[\mu]{} & \mathbf{b}^{n-1}\mathbb{R} \end{array}$$

Observation cont'd:

- It follows that
$$\mathbf{b}^{n-2}\mathbb{R} \to \mathfrak{g}_\mu \to \mathfrak{g}$$
is a central extension of L_∞-algebras

We now apply this argument to the dual of the Killing form:

- $\langle[-,-],-\rangle^* \in \mathrm{CE}(\mathfrak{g})$ is a closed element of degree 3, thereby it is a Lie 3-cocycle
- The central extension it classifies is precisely $\mathfrak{str}$

This is the infinitesimal version of gerbes being classified by higher cohomology classes!