Simplicial (pre)sheaves and higher Lie groups

Simplicial presheaves: $C^{op} \to sSet$, $C = Cart$

Classical model structure on sSet:

$f_n : X_n \to Y_n$

- cofibrations: monomorphisms $f: X \to Y$ (levelwise injections)
- w.e.: weak homotopy equivalences (morphisms whose geom. realization is w.h.e. of top spaces)
- fibrations: Kan fibrations

maps f s.t.

$$\begin{array}{ccc} \Lambda^n_k & \to & X \\ \downarrow & \nearrow \exists & \downarrow f \\ \Delta^n & \to & Y \end{array}$$

Fibrant objects are Kan complexes. All objects are cofibrant.

$[Cart^{op}, sSet_{Quillen}]_{proj}$ - simplicial presheaves with global projective model structure

W.e. and fibrations are objectwise those of simplicial sets.

A morphism $f: A \to B$ is w.e. wrt. a global model str. precisely if for all $U \in Obj(Cart)$ the morphism $f(U): A(U) \to B(U)$ is a w.e. of simp. sets

$[Cart^{op}, sSet_{Quillen}]_{inj}$ $\rightsquigarrow$ w.e. and cofibrations are objectwise those of simplicial sets.

$[\mathrm{Cart}^{op}, \mathrm{sSet}]_{proj}$ presents the ∞-cat. of ∞-presheaves.

Cart is a site with coverage = differentially good open covers.

an open cover $\mathcal{U} = \{U_i \to X\}$ s.t. for all $n \in \mathbb{N}$ $U_{i_1} \cap \dots \cap U_{i_n} = U_{i_1 \dots i_n}$ is either empty or diffeomorphic to $\mathbb{R}^{\dim X}$.

A sheaf on a site is a presheaf $A: C^{op} \to \mathrm{Set}$ s.t. for each covering sieve $S(\mathcal{U}) \to \underset{\overset{\parallel}{X}}{U}$ the morphism

$$A(U) \simeq [C^{op}, \mathrm{Set}](\underset{\overset{\parallel}{X}}{U}, A) \to [C^{op}, \mathrm{Set}](S(\underset{\uparrow}{\mathcal{U}}), \underset{\uparrow}{A})$$ is an isomorphism.

$\mathcal{U}$ $\quad U \in \mathrm{Cart}$ $\quad \mathcal{U} = \{U_i \to U\}$

Čech nerve:

$\mathcal{U}$ - diff. good open cover of U

$C(\mathcal{U})$ - simplicial object in Cart $\quad \coprod_{ij} U_{ij}$

$U_{ijk} := U_i \cap U_j \cap U_k$

$$C(\mathcal{U}) := (\ \overset{\to}{\underset{\to}{\equiv}} \underline{\{U_{ijk}\}} \overset{\to}{\underset{\to}{\to}} \{U_{ij}\} \rightrightarrows \{U_i\}\ )$$ $\longleftarrow$ Kan complex

in degree k is the disjoint union of the k-fold intersections of open subsets

$\{U_{ijk}\} = \coprod_{ijk} U_{ijk}$

$C(\mathcal{U}) \to U$

Čech nerve

Homotopy function complex:

$[\mathrm{Cart}^{op}, \mathrm{sSet}](X, A) \in \mathrm{sSet}$

$[\mathrm{Cart}^{op}, \mathrm{sSet}](X, A)_n := \mathrm{Hom}_{\mathrm{sSet}}(X \times \Delta^n, A)$

cofibrant repl. fibr. repl.

$\mathbb{R}\mathrm{Map}(X,A) := [\mathrm{Cart}^{op}, sSet](Q(X), P(A))$ – (right) derived mapping space

well-defined up to equivalence

Left Bousfield localization:

M - model str. with a class of maps S.

$L_S M$:

1. W.e. is enlarged to include S-local equivalences.
2. Cofibs. remain the same.
3. The fibrant objects are now S-local objects. + fibrant in M

This is a proposition!

$W \subset W_S$

- An obj. X in M is S-local if, for every $f: A \to B$ in S, the induced map on derived mapping spaces (homotopy function complexes) $\mathbb{R}\mathrm{Map}(f,X): \mathbb{R}\mathrm{Map}(B,X) \to \mathbb{R}\mathrm{Map}(A,X)$ is a weak equiv. in the simp. cat.

- A map $g: Y \to Z$ is an S-local equiv. if it induces a we. on all S-local objects X, $\mathbb{R}\mathrm{Map}(g,X): \mathbb{R}\mathrm{Map}(Z,X) \to \mathbb{R}\mathrm{Map}(Y,X)$ is a w.e.

Čech model structure on $[\mathrm{Cart}^{op}, sSet]_{proj}$ is the left Bousfield localization at the set of Čech cover nerves morphisms.

$C(\mathcal{U}) \to V$, $\mathcal{U}$ is a a covering of V.

Section Schreiber et al „Čech cocycles ..."

Fiorenze Stasheff

$[\mathrm{Cart}^{op}, sSet]_{proj,loc}$

- Every w.e. in $[..]_{proj}$ is a w.e. in $[...]_{Čech}$.
- Cofibs. are the same.

$$X \xrightarrow{\sim_{loc}} Q(X) \longrightarrow *$$

$$\mathcal{U} \longrightarrow X$$

Prop.

In $[\mathrm{Cart}^{op}, \mathrm{sSet}]_{\check{C}ech}$, the Čech nerve $C(\mathcal{U}) \to X$ is a cofibrant resolution of X.

Corollary:

The fibrant objects of the Čech str. are precisely the objects that are fibrant in the global structure and in addition satisfy the descent condition:

for all $U \in \mathrm{Cart}$

$$A(U) \simeq [\mathrm{Cart}^{op}, \mathrm{sSet}](U, A) \longrightarrow [\mathrm{Cart}^{op}, \mathrm{sSet}](C(\mathcal{U}), A)$$

is a weak equivalence of Kan complexes (in $\mathrm{sSet}_{Quillen}$).

We can consider $[\mathrm{Cart}^{op}, \mathrm{sSet}]_{\check{C}ech}$ as a model for ∞-sheaves.

$$\| \qquad\qquad \mathrm{Sh}(\mathcal{C}) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathrm{PSh}(\mathcal{C})$$

$$\mathrm{Sh}_\infty(\mathrm{Cart})$$

- The localization ~~right~~ left Quillen functor

$$[\mathrm{Cart}^{op}, \mathrm{sSet}]_{proj} \underset{\longleftarrow}{\overset{\mathrm{Id}}{\underset{\perp}{\longrightarrow}}} [\mathrm{Cart}^{op}, \mathrm{sSet}]_{\check{C}ech}$$

presents ∞-sheafification, which is a left adjoint left exact functor.
This means that all homotopy colimits and all finite homotopy limits in Čech m. str. can be computed in the global m. str.

An object in $\mathbb{R}\mathrm{Map}(X, A)$ is represented by a diagram:

$$\begin{array}{ccc} C(\mathcal{U}) & \xrightarrow{g} & A \\ {\scriptstyle s}\downarrow\wr & & \\ X & & \end{array}$$

Smooth ∞-groups

In classical homotopy th.:

Monoid = top. space equipped with A_∞-structure.

∞-group is a groupal A_∞-space where the homotopy-associative product is invertible, up to homotopy.

May's recognition theorem: such ∞-groups are, up to equivalence, loop spaces.

$X_0 = \{*\}$

Thus, $\infty Grp \underset{B}{\overset{\Omega}{\rightleftarrows}} \infty Grpd_* \simeq sGrp \underset{B}{\overset{\Omega}{\rightleftarrows}} sSet_*$.

This can be generalized to the smooth case:

delooping

$$[Cart^{op}, sGrp] \underset{\Omega}{\overset{\mathbb{B}}{\rightleftarrows}} [Cart^{op}, sSet_*].$$

$G \simeq \Omega \mathbb{B} G$

We can think of smooth ∞-groups $\underset{\sim}{G}$ in terms of their deloopings $\mathbb{B}G$ which carries exactly the same information.

Def.
G - Lie group. $\mathbb{B}G$ is defined to associate $U \in Cart$ a nerve of the action groupoid $\underbrace{* /\!/ C^\infty(U,G)}$.

one object groupoid with $C^\infty(U,G)$ as its set of morphisms

$\mathbb{B}G$ is not fibrant over Man but is fibrant in $[Cart^{op}, sSet]_{Čech}$.

$\mathbb{R}Map(X, \mathbb{B}G) = GBund(X)$ - groupoid of G-bundles on X.

$\mathbb{R}Map(X, \mathbb{B}G) = BG(X)$

$$\mathbb{R}\mathrm{Map}(X, \mathbf{B}G) = \overbrace{\mathrm{Hom}_{[\mathrm{CartSp}^{op},\, \mathrm{sSet}]}}^{\mathrm{Sh}_\infty}(C(\mathcal{U}), \mathbf{B}G)$$

$$C(\mathcal{U}) \overset{\sim}{=} \mathrm{hocolim}\Big(\cdots \coprod_{ijk} U_{ijk} \substack{\to\\ \to\\ \to} \coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i \Big) \simeq \int^{[k]\in\Delta} \Delta[k] \cdot \underbrace{\coprod_{i_0 \dots i_k} U_{i_0 \dots i_k}}_{C(\mathcal{U})_k}$$

tensoring of simp presheaves over simp sets.

$$\mathrm{Sh}_\infty(C(\mathcal{U}), A) \simeq \mathrm{Sh}_\infty\Big(\int^{[k]} \Delta[k] \cdot C(\mathcal{U})_k, A \Big) \simeq$$

$$\simeq \int_{[k]} \mathrm{Sh}_\infty(\Delta[k] \cdot C(\mathcal{U})_k, A) \simeq \int_{[k]} \mathrm{sSet}(\Delta[k], \mathrm{Sh}_\infty(C(\mathcal{U})_k, A)) \simeq$$

$$\simeq \int_{[k]} \mathrm{sSet}\Big(\Delta[k], \prod_{i_0 \dots i_k} A(U_{i_0 \dots i_k})\Big) = \int_{[k]} \mathrm{Hom}\Big(\Delta[k], \prod_{i_0 \dots i_k} \mathbf{B}G(U_{i_0 \dots i_k})\Big)$$

a collection of morphisms in sSet of the form

$$\left(\begin{array}{ccc} \vdots & & \vdots \\ \Delta[2] & \xrightarrow{g^{(2)}} & \prod_{ijk} \mathbf{B}G(U_{ijk}) \\ \Delta[1] & \xrightarrow{g^{(1)}} & \prod_{ij} \mathbf{B}G(U_{ij}) \\ \uparrow\uparrow & & {\scriptstyle d_0}\uparrow\uparrow{\scriptstyle d_1} \downarrow {\scriptstyle s_0} \\ \Delta[0] & \xrightarrow{g^{(0)}} & \prod_i \mathbf{B}G(U_i) \end{array} \right)$$

This is a collection $(\{g_i\}, \{g_{ij}\}, \{g_{ijk}\}, \dots)$

a vertex in $\mathbf{B}G(U_i)$

an edge in $\mathbf{B}G(U_{ij})$

a 2-simplex in $\mathbf{B}G(U_{ijk})$

etc.

$$\left(\begin{tikzcd} & g_j|_{U_{ijk}} \arrow[dr, "g_{jk}|_{U_{ijk}}"] \arrow[d, phantom, "\Downarrow g_{ijk}"] & \\ g_i|_{U_{ijk}} \arrow[ur, "g_{ij}|_{U_{ijk}}"] \arrow[rr, "g_{ik}|_{U_{ijk}}"'] & {} & g_k|_{U_{ijk}} \end{tikzcd} \right) \in \mathbf{B}G(U_{ijk})$$

This is a Čech-cocycle on X with values in $\mathbf{B}G$ relative to $\mathcal{U}$.

A (gauge) transformation between to cocycles g and g':

- 1-morphisms $h_i : g_i \to g_i'$ in $\mathbb{B}G(U_i)$ $\quad h_i : U_i \to G$
- 2-morphisms

$$\begin{tikzcd}
g_i|_{U_{ij}} \arrow[r, "g_{ij}"] \arrow[d, "h_i"'] & g_j|_{U_{ij}} \arrow[d, "h_j"] \\
g_i'|_{U_{ij}} \arrow[r, "g_{ij}'"] & g_j'|_{U_{ij}}
\end{tikzcd} \quad \text{on } U_{ij}$$

- ...

By definition of $\mathbb{B}G$, the above reduces to:

$\mathbb{B}G$-cocycle:
- a collection of smooth maps $g_{ij} : U_{ij} \to G$
- a collection of identities $g_{ij}|_{U_{ijk}} \cdot g_{jk}|_{U_{ijk}} = g_{ik}|_{U_{ijk}}$

$\mathbb{B}G$-coboundary / Gauge transformation:
- a collection of smooth maps $h_i : U_i \to G$
- a collection of identities $g_{ij} \cdot h_j|_{U_{ij}} = h_i|_{U_{ij}} \cdot g'_{ik}$
-

<u>Theorem</u>

classifying stack of princ G-bundles

groupoid of G-bundles on X

$$\mathbb{R}\mathrm{Map}(C(\mathcal{U}), \mathbb{B}G) \simeq G\mathrm{Bund}(X)$$

<u>Prop.</u>

$$\mathrm{Ch}_{\geq 0} \underset{N_\bullet}{\overset{DK}{\rightleftarrows}} s\mathrm{Ab}$$

non-neg. graded chain complexes

simplicial abelian groups

forms normalized chain complexes

$N_\bullet(G)$ = Moore complex

$$DK : [\mathrm{Cart}^{op}, \mathrm{Ch}_{\geq 0}] \xrightarrow{m} [\mathrm{Cart}^{op}, s\mathrm{Ab}] \xrightarrow{F} [\mathrm{Cart}^{op}, s\mathrm{Set}]$$

takes levelwise surjections to fibrations and levelwise quasi-isomorphisms to w.e.

A

n-th delooping of A $\quad$ K, A - abelian Lie groups

$$\mathbb{B}^n A : U \longmapsto DK(\overset{\text{deg } n}{C^\infty(U,A)} \to 0 \to \dots \to 0)$$

$$K \to A \Rightarrow \mathbb{B}^n(K \to A) : U \longmapsto DK(C^\infty(U,K) \to \overset{\text{deg } n}{C^\infty(U,A)} \to \dots \to 0)$$

$$\mathbb{E}\mathbb{B}^{n-1}A = \mathbb{B}^{n-1}(A \xrightarrow{id} A)$$

FFT : $\mathrm{Bord}^S_d \longrightarrow \mathbb{B}^d U(1)$

Prop. $\mathbb{B}^n A$ is indeed a delooping of $\mathbb{B}^{n-1}A$.

$$\begin{tikzcd} \Omega G \arrow[r] \arrow[d] \arrow[dr, phantom, "(hp)"] & * \arrow[d] \\ * \arrow[r] & G \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbb{B}^{n-1}A \arrow[r] \arrow[d] \arrow[dr, phantom, "hp"] & * \arrow[d] \\ * \arrow[r] & \mathbb{B}^n A \end{tikzcd}$$

$(A,\partial) = \dots \to A_k \to A_{k-1} \to \dots \to A_0$

$A[1] = \dots \to A_{k-1} \to \dots \to A_0 \to 0$

In $[\mathrm{Cart}^{op}, \mathrm{sSet}]$, $\lrcorner$ is a pullback.

$$\begin{tikzcd} \mathbb{B}^{n-1}A \arrow[r] \arrow[d] & \mathbb{E}\mathbb{B}^{n-1}A \arrow[r, "\sim"] \arrow[d] & * \\ * \arrow[r] & \mathbb{B}^n A & \end{tikzcd}$$

replacement of a point inclusion by a fibration

Prop.
For $A = \mathbb{Z}, \mathbb{R}, U(1)$ and all $n \geq 1$ we have that $\mathbb{B}^n A$ satisfies descent over Cart in that it is fibrant in $[\mathrm{Cart}^{op}, \mathrm{sSet}]_{loc}$.

Def.
- For $X \xleftarrow{\sim} QX \xrightarrow{f} \mathbb{B}^n A$ a span in $[\mathrm{Cart}^{op}, \mathrm{sSet}]$, the corresponding $\mathbb{B}^{n-1}A$ - principal bundle is the homotopy pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r, "f"] & \mathbb{B}^n A \end{tikzcd}$$

In $[\mathrm{Cart}^{op}, sSet]$:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \overbrace{\mathbb{E}\mathbf{B}^{n-1}A}^{\mathbb{E}G \simeq *} \arrow[d] \\ QX \arrow[r, "g"] \arrow[d, "s"] & \mathbf{B}^n A \\ X & \end{tikzcd}$$

smooth ∞-group

$\mathbb{R}\mathrm{Map}(X, \mathbf{B}(G))$

• The Kan complex $(\mathbf{B}^{n-1}A)\mathrm{Bund}(X) : [\mathrm{Cart}^{op}, sSet](X, \mathbf{B}^n A)$ is the n-groupoid of smooth $\mathbf{B}^{n-1}A$-principal n-bundles on X.

Ex.

$$\begin{tikzcd} \mathbf{B}\mathrm{String} \arrow[r] \arrow[d] \arrow[dr, phantom, "hp"] & * \simeq \mathbb{E}\mathbf{B}^2 U(1) \arrow[d] \\ \mathbf{B}\mathrm{Spin} \arrow[r] & \mathbf{B}^3 U(1) \end{tikzcd}$$

produced by Lie integration

„first fractional Pontryagin class"

$\frac{1}{2}p_1$

Smooth String 2-group : $\mathrm{String}_{\mathrm{smooth}} := \Omega \mathbf{B}\mathrm{String}$

$$\begin{tikzcd} U(1) \arrow[r] & \mathrm{Spin}^c(n) \arrow[r] \arrow[d] \arrow[dr, phantom, "hp"] & SO(n) \arrow[d] \\ & * \arrow[r] & \mathbf{B}U(1) \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathbf{B}\mathrm{Spin} \arrow[r] \arrow[d] & \mathbf{B}SO(n) \arrow[d, "W_3"] \\ * \arrow[r] & \mathbf{B}^2 U(1) \end{tikzcd}$$

third Stiefel-Whitney class

$$\mathbf{B}G_{\mathrm{conn}} = \Omega^1(U, \mathfrak{g}) // C^\infty(U, G)$$