# Higher Lie theory

**Def** A *Lie group* is a group object in the category of smooth manifolds, i.e., a group $G$, $G \in Man$, $G \times G \xrightarrow{\mu} G$, $G \xrightarrow{(-)^{-1}} G$ smooth

**Examples** $GL(n, \mathbb{R}) = \{ \mathbb{R}^n \xrightarrow{T} \mathbb{R}^n \mid \exists T^{-1} \}$

$O(n) \subset GL(n, \mathbb{R})$
$\quad = \{ T^{-1} = T^t \}$

$U(n) \subset GL(n, \mathbb{C})$
$\quad = \{ T^{-1} = T^* \}$

$SU(n) \subset U(n)$
$\quad = \{ T \mid \det T = 1 \}$

**Def** A *Lie algebra* is a vector space $V$,
$[-,-] : V \otimes V \to V \qquad [x, y] = -[x, y]$
Jacobi: $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$

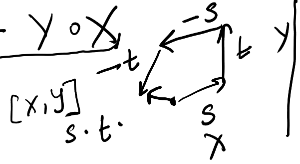
How to study a Lie group $G$?

$G \curvearrowright G \quad g \cdot x = gx$

One tool: Study left-invariant differential geometry on $G$.

| differential geometry | left-invariant differential geometry $\simeq$ differential geometry of $\mathbb{B}G$ |
|---|---|
| smooth manifold $M$ | Lie group $G$ |
| vector field $X$ on $M$; $X \in \mathfrak{X}M$ <br> $D(fg) = D(f)\cdot g + f \cdot D(g)$ <br> $Der(C^\infty M, C^\infty M)$ | left-invariant vector field on $G$ <br> $X \in \mathfrak{g}$ = the Lie algebra of $G$ |
| $[-,-] : \mathfrak{X}M \otimes \mathfrak{X}M \to \mathfrak{X}M$ (Lie derivative of a vector field) | $\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{[-,-]} \mathfrak{g}$ bracket |

$\mathcal{L}_X Y = [X, Y] = X \circ Y - Y \circ X$ (implies Jacobi)
$\quad = -\mathcal{L}_Y X$



| | |
|---|---|
| Smooth function $M \xrightarrow{f} \mathbb{R}$ | Constant $f \in \mathbb{R}$ |
| Lie derivative of a smooth function<br>$X \in \mathfrak{X}M$, $f \in C^\infty M$<br>$\mathcal{L}_X f = X(f)$ | $\mathcal{L}_X f = 0$ |
| CDGA of differential forms $\Omega M$ on $M$ | CDGA Chevalley-Eilenberg algebra of $\mathfrak{g}$ $CE(\mathfrak{g})$ |
| $\Omega^n M = \Gamma(\Lambda^n T^* M)$<br>$= \Lambda^n_{C^\infty M} \Gamma(TM)^*_{C^\infty M}$ | $CE^n(\mathfrak{g}) = \Lambda^n \mathfrak{g}^*$ |
| de Rham differential $d: \Omega^n M \to \Omega^{n+1} M$<br>$n=1$: $\omega \in \Omega^1 M$; $d\omega \in \Omega^2 M$<br>$X, Y \in \mathfrak{X}M$<br>$(d\omega)(X,Y) = \mathcal{L}_X(\underbrace{\omega(Y)}_{\in C^\infty M}) - \mathcal{L}_Y \omega(X) - \omega([X,Y])$<br>$= X(\iota_Y \omega) - Y(\iota_X \omega) - \iota_{\mathcal{L}_X Y} \omega$ | Chevalley-Eilenberg differential $d: CE^n_\mathfrak{g} \to CE^{n+1}_\mathfrak{g}$<br>$n=1$ $\omega \in \mathfrak{g}^*$, $d\omega \in \Lambda^2 \mathfrak{g}^*$<br>$d\omega(X,Y) = -\omega([X,Y])$<br>$\mathfrak{g}^* \xrightarrow{-[-,-]^*} \Lambda^2 \mathfrak{g}^*$<br>$\mathfrak{g} \xleftarrow{[-,-]} \Lambda^2 \mathfrak{g}$ |
| Contraction $\mathfrak{X}M \otimes \Omega^{n+1} M \to \Omega^n M$<br>$X, \omega \mapsto \iota_X \omega$<br>$X \lrcorner \omega$<br>$(\iota_X \omega)(Y_1, \ldots, Y_n) = \omega(X, Y_1, \ldots Y_n)$ | contraction $\mathfrak{g} \otimes \Lambda^{n+1} \mathfrak{g}^* \to \Lambda^n \mathfrak{g}^*$<br>$\iota_X \omega$ |
| Lie derivative of differential forms<br>$\mathfrak{X}M \otimes \Omega^n M \to \Omega^n M$<br>$X, \omega \longmapsto \mathcal{L}_X \omega = X(\omega)$<br>$(\mathcal{L}_X \omega)(Y_1, \ldots, Y_n) = \pm\, \omega([X,Y_1], Y_2, \ldots Y_n)$<br>$\pm\, \omega(Y_1, [X,Y_2], Y_3, \ldots Y_n)$ | $\mathfrak{g} \otimes \Lambda^n \mathfrak{g}^* \to \Lambda^n \mathfrak{g}^*$ |

$\mathcal{L}_X W = (t \mapsto \underbrace{Flow(X,t)}_{\text{local diffeomorphism}}{}^* W)'(0)$

$H^n_{dR}(M) = H^n(\Omega M)$
de Rham cohomology

the graded Lie algebra of natural (in M) graded derivations of $\Omega M$
is generated by

$-1: \quad \iota_X : \Omega^{n+1} M \to \Omega^n M$

$0: \quad \mathcal{L}_X : \Omega^n M \to \Omega^n M$

$1: \quad d : \Omega^n M \to \Omega^{n+1} M$

subject to the relations
(É. Cartan's calculus)

$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$

$[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$

$[d, \iota_X] = \mathcal{L}_X$ Cartan's magic formula

$[\iota_X, \iota_Y] = 0$
$[d, \mathcal{L}_X] = 0$
$d^2 = 0 \iff [d, d] = 0$

naturality in M:
$M_1 \overset{f}{\hookrightarrow} M_2$

$f^*(\mathcal{L}_X \omega) = \mathcal{L}_{f^* X} f^* \omega$

differential operators $\mathrm{Diff}^{\leq n}(M)$ filtered algebra
$\cong \Gamma(\mathrm{Hom}(J^{\leq n} M, \mathbb{R}))$
total symbol map
nth jet bundle of M

$H^n(CE(\mathfrak{g})) = H^n(\mathfrak{g}, \mathbb{R})$
Chevalley–Eilenberg cohomology

graded derivations of $CE(\mathfrak{g})$

**H. Cartan** 1950

universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ (filtered)

Clairaut's theorem $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \cdots$

$$\mathrm{Diff}^{\le(n-1)} M \to \mathrm{Diff}^{\le n} M \to \mathrm{Sym}^n \Gamma(TM)$$

principal symbol map

not naturally split unless $m \in M$: $j^{\le(n-1)} f = 0$

$$\mathrm{Sym}^n T^* M \to J^{\le n} M \to J^{n-1} M$$

Poincaré – Birkhoff – Witt

$$\mathrm{gr}\, U\mathfrak{g} = U_n\mathfrak{g}/U_{n-1}\mathfrak{g} \cong \mathrm{Sym}^n \mathfrak{g}$$

## Topics to present:

Talk Dmitri

1) Lie's second and third theorem

simply connected Lie groups $\xrightarrow{G \mapsto \mathfrak{g}}$ finite-dim Lie algebras

$\xleftarrow{\text{Lie integration}}$

Talk Jacek

2) What is a higher Lie group?

Simplicial presheaves;

$\mathrm{PSh}(\mathrm{Man}, \mathrm{sGrp}) \xrightarrow{\mathbb{B}} \mathrm{sPSh}(\mathrm{Man})_*$ — $\mathbb{B}$ delooping/classifying stack

$\xleftarrow{\Omega}$ loop group

Examples: 1) ordinary Lie groups; $\mathbb{B}U(1)$

3) String Lie 2-group. $\mathrm{String}(n) \to \mathrm{Spin}(n)$

(nonlinear $\sigma$-model in dim 2, Wess-Zumino-Witten) WZW

2) $\mathbb{B}^n U(1)$

principal bundles = bundle $n$-gerbes

($n=1$: B-field; $n=2$ C-field in SUGRA).

Talk 3 Integration of higher Lie algebras

Talk 4

Emilio

dmitripavlov.org/homotopy

FSS, RW } + suggest other papers

# The Cartan — Lie theorem
## ("Lie's third theorem")

Question: to what extent does $\mathfrak{g}$ determine $G$?

<u>Notation</u> $G$ Lie group

$G_0$ : the connected component of the identity element in $G$

$\tilde{G}_0$ : the universal covering of $(G_0, e)$

$\tilde{G}_0 = P_e G_0 / \sim$ $\quad P_e G_0 = \{\gamma : [0,1] \xrightarrow[C^\infty]{} G_0 \mid \gamma(0) = e\}$

group operations are defined pointwise

$\gamma_0 \sim \gamma_1$ if $\exists h : [0,1] \times [0,1] \xrightarrow[C^\infty]{} G_0$

$h(-,0) = \gamma_0$

$h(-,1) = \gamma_1$

$h(0,-) = e \in G_0$

$h(1,-)$ is constant

(in particular, $\gamma_0(1) = \gamma_1(1)$)

$\cong P_e G / H_e G$ $\quad H_e G \subset P_e G$

$H_e G = \{\gamma \in P_e G \mid \gamma \sim e\}$.

smooth manifold

charts: take a chart $(\mathcal{U}, \varphi)$ in $G_0$

$\mathcal{U} \subset G_0$, $\varphi : \mathcal{U} \xrightarrow{\cong} \mathbb{R}^n$; take some element $\delta \in P_e G$, $\underline{\delta(1) \in \mathcal{U}}$

$\rightsquigarrow$ construct a chart in $\tilde{G}_0 = P_e G / H_e G$

$$V_{\delta, \mathcal{U}} = \{[\gamma] \in P_e G / H_e G \mid [\gamma] = [\varepsilon * \delta]\}$$

($*$: concatenation of paths)

$\varepsilon : [0,1] \xrightarrow[C^\infty]{} \mathcal{U}$ $\quad \varepsilon(0) = \delta(1)$


<u>Observation</u> $\tilde{G}_0 \longrightarrow G_0 \longrightarrow G$ (homomorphisms of Lie groups)

induces a sequence of isomorphisms of Lie algebras

$$\mathfrak{g} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}$$

$\mathfrak{g} \cong T_e G$

$\cong T_e G_0$

$\cong T_e \tilde{G}_0$

# Theorem (Lie, É. Cartan)

The functor

"Lie differentiation"
$G \longmapsto \mathfrak{g}$

simply connected Lie groups $\longrightarrow$ finite-dimensional Lie algebras
$\longleftarrow$
"Lie integration"

is an equivalence of categories.

**Terminology** Lie's second theorem = fully faithful ((relatively) easy)
Lie's third theorem = essential surjectivity for germs of Lie groups = local Lie groups

É. Cartan: essential surjectivity
~ 1920 "Cartan—Lie theorem"

**Proof** (essential surjectivity)

Suppose $\mathfrak{g}$ is a Lie algebra.
Want to construct $G$: a simply connected Lie group with Lie algebra $\mathfrak{g}$

Since $G$ must be simply connected,

which category???

**Proof** $G \cong \tilde{G}_0 \cong P_e G/\sim \xrightarrow[D]{FTC \cong} \Omega^1([0,1],\mathfrak{g}) \,/\, \Omega^2([0,1]\times[0,1],\mathfrak{g})$

$d\omega + [\omega\wedge\omega] = 0$

$P_e G/\sim$ can be defined using $\mathfrak{g}$ only!

Sharpe: Differential geometry

Fiorenza—Schreiber—Stasheff: Čech... Duistermaat—Kolk

Bad answer: Banach manifolds

Lie groups answer: category of sheaves of sets on the site of smooth manifolds.

**Def** (Darboux derivative)

Suppose $f\colon M \xrightarrow{C^\infty} G$, $M \in$ Man, $G \in$ Lie group

$g \in G$: $\ell_g : G \to G$, $x \mapsto g \cdot x$ $\quad T(\ell_g) : TG \to TG$, $T_c G \to T_{g\cdot c} G$ $\quad T(\ell_{g^{-1}}) : T_g G \to T_e G \cong \mathfrak{g}$

$v \longmapsto (p(v), T(\ell_{(p(v))^{-1}})(v))$

$$Tf : TM \longrightarrow TG \cong G \times T_e G \cong G \times \mathfrak{g}$$

$v \longmapsto (p(v), \omega_G(v))$

$$Df = \pi_{\mathfrak{g}}(Tf) \qquad Df : TM \longrightarrow \mathfrak{g}$$

**Example** $G = (\mathbb{R}, +)$ $\quad T\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ $\quad \mathfrak{g} = (\mathbb{R}, [-,-] = 0)$

$f : M \longrightarrow \mathbb{R}$ $\quad Df = f'$ (Calculus I)

**Proposition** a) $Df = 0 \iff f$ is a locally constant function (constant on every connected component)

assume $M$ is connected

b) $Df_1 = Df_2 \iff \exists g \in G : g \cdot f_1 = f_2$

$\iff f_1^{-1} f_2$ is constant

$\iff f_2^{-1} f_1$ is constant

Which functions $M \to \mathfrak{g}$ admit Darboux antiderivatives?

**Observation** If $f : M \to G$, then $Df : TM \longrightarrow \mathfrak{g}$ (linear on every fiber)

the "Darboux differential of $f$" $Df \in \Omega^1(M, \mathfrak{g})$

$\| \qquad \uparrow f^*$

$f^* \omega_G \qquad \Omega^1(G, \mathfrak{g})$

**Def** $id_G : G \to G$

$D(id_G) : TG \to \mathfrak{g}$

$\omega_G \in \Omega^1(G, \mathfrak{g})$

the Maurer — Cartan form of $G$

$\omega_G \in \Omega^1(G, \mathfrak{g})$

$f^* \omega_G \in \Omega^1(M, \mathfrak{g})$

$$(f^* \omega_G)(v) = \omega_G(Tf(v)) = (Df)(v), \quad v \in TM$$

**Example** $G = \mathbb{R}$

$\omega_G = dx$ $\quad x : G = \mathbb{R} \xrightarrow{id} \mathbb{R}$

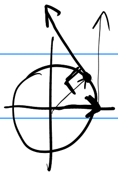
$f : G \to \mathbb{R}$ $\quad dx \in \Omega^1(\mathbb{R})$

$$Df = f^* \omega_G = f^* dx = d(f(x)) = f'(x) \cdot dx = f' \cdot dx$$

$f' \cdot dx$ — the differential of $f$

## Example $G = S^1 \cong U(1) = \{x+iy \mid x^2+y^2=1\}$

$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$



$v \in TS^1$ $\quad x \cdot v_x + y \cdot v_y = 0$ $\qquad \mathfrak{g} \cong i\mathbb{R}$ "purely imaginary numbers" $\cong \mathbb{R}$

$v = ((x,y),(v_x,v_y))$

$$\omega_G((x,y),(v_x,v_y)) = (x - iy)\cdot(v_x + i v_y)$$

$$= (\underbrace{x v_x + y v_y}_{0},\ x v_y - y v_x)$$

$(dx)((x,y),(v_x,v_y)) = v_x \qquad = (0,\ \underline{x v_y - y v_x})$

$(dy)((x,y),(v_x,v_y)) = v_y \qquad \omega_{S^1} = \underline{x\,dy - y\,dx}$ Maurer-Cartan form of $S^1$

$$= i(x\,dy - y\,dx)$$

$f: M \to S^1 \subset \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

$\parallel$

$(f_x, f_y : M \to \mathbb{R}) \qquad (f^* \omega_{S^1})(v) = \omega_{S^1}(Tf(v))$

$f_x^2 + f_y^2 = 1 \qquad v \in TM$

$$\big((f_x(p(v)), f_y(p(v))), (df_x(v), df_y(v))\big)$$

$$= f_x(p(v)) \cdot df_y(v) - f_y(p(v))\, df_x(v)$$

$$= f_x\, df_y - f_y\, df_x$$

## Theorem (Fundamental theorem of calculus)

$M$ simply connected

$$C^\infty(M, G) \xrightarrow{D} \Omega^1(M, \mathfrak{g})$$

$\ker D = C^\infty_{\text{lconst}}(M,G) \qquad \operatorname{im} D = \{\omega \in \Omega^1(M,\mathfrak{g}) \mid d\omega + [\omega \wedge \omega] = 0\}$

(Maurer-Cartan identity)

Existence: follows from existence of solutions for smooth ODE: $\widetilde{D}f = \omega$

$\alpha, \beta \in \Omega^2(M, \mathfrak{g})$ $\quad \alpha \wedge \beta \in \Omega^2(M, \mathfrak{g} \wedge \mathfrak{g}) \xrightarrow[{[-]}]{} \Omega^2(M, \mathfrak{g})$

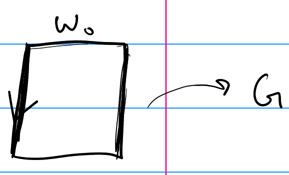
<u>Proof</u> $\quad d\omega_G = -[\omega_G \wedge \omega_G] \in \Omega^2(G, \mathfrak{g})$
using the <u>formula for d</u>

$f: M \to G \qquad Df = f^* \omega_G \in \Omega^1(M, \mathfrak{g})$

$$\begin{aligned} & d(Df) + [Df \wedge Df] \\ &= d(f^*\omega_G) + [f^*\omega_G \wedge f^*\omega_G] \\ &= f^* d\omega_G + [f^*(\omega_G \wedge \omega_G)] \\ &= f^* d\omega_G + f^*[\omega_G \wedge \omega_G] \\ &= f^*(d\omega_G + [\omega_G \wedge \omega_G]) = f^* 0 = 0 \end{aligned}$$

Encoding $P_e G/\sim$ as a construction in smooth sets
$\shortparallel$

$Q = \Omega^1([0,1], \mathfrak{g}) / \sim \qquad \omega_0 \sim \omega_1$ if $\exists \psi \in \Omega^1([0,1] \times [0,1], \mathfrak{g})$
$d\psi + [\psi \wedge \psi] = 0$




$A/B$

Know: $G, \mathfrak{g} \Rightarrow Q \cong G$
(discrete groups)

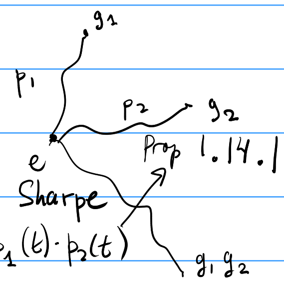
$\psi|_{[0,1] \times \{i\}} = \omega_i$

$\psi|_{\{i\} \times [0,1]} = 0$

$A/B \in Sh(\text{Cart}, \text{Group})$
$\cong$
$y(G)$
$A(S)/B(S) \in \text{Group}$

$S \in \text{Cart} \qquad \overbrace{A(S)}^{\text{magma}} = \Omega^1_{fw}([0,1] \times S, \mathfrak{g})$
$\downarrow$
$S$

$\omega_1 \cdot \omega_2 = \omega_1(t) + A_{\omega_1(t)} \omega_2(t)$




Leibniz Sharpe
$p(t) = p_1(t) \cdot p_2(t)$

$s \in S \qquad S$
$\nearrow [0,1] \times S$
$[0,1] \times \{s\} \quad \iota_s$

$T_{fw}\begin{pmatrix} P \\ \downarrow \\ S \end{pmatrix} = \ker \begin{pmatrix} TP \\ \downarrow \\ TS \end{pmatrix}$
$\shortparallel$
$T_{\text{vertical}}$

$p(t) = \begin{cases} p_1(2t), & t \leq \frac{1}{2} \\ g_1 p_2(2t-1), & t \geq \frac{1}{2} \end{cases}$

$\iota_s^* \psi \in \Omega^1([0,1], \mathfrak{g})$

not associative
piecewise smooth $\quad \forall s \in S$

$\downarrow$
$B(S) \to A(S) \times A(S)$
$[0,1] \times \{i\} \times S \hookrightarrow [0,1] \times [0,1] \times S$

$B(S) = \Omega^2_{fw}([0,1] \times [0,1] \times S, \mathfrak{g})$
$\downarrow$
$S$

$(p_1 \cdot p_2) \cdot p_3$
$p_1 \cdot (p_2 \cdot p_3)$

$[0,1] \times [0,1] \times \{s\}$
$p_1(b(2t))$ $\quad [0,1] \times [0,1] \times S$
$p_2(b(2t-1))$

delooping $B$ ↙ Quillen equivalence

$$\underset{\text{loop } \Omega}{s\mathrm{Grp}} \underset{G}{\overset{\overline{W}}{\rightleftarrows}} s\mathrm{Set}_{red} \qquad X_0 = \{*\}$$

$$\Omega X = \mathrm{Hom}(S^1_*, X)$$

strictly associative group operations ↓ Quillen equivalence

$$\mathrm{PSh}(s\mathrm{Grp}) \underset{\Omega}{\overset{\mathbb{B}}{\rightleftarrows}} \mathrm{PSh}(s\mathrm{Set}_{red})$$

$\uparrow G$ Smooth $\infty$-groups ⇢ $\mathbb{B}G$

$\pi_1(\mathbb{B}G) \cong G$ produced by Lie integration

$U(1)$ principal $U(1)$-bundle

$$\begin{array}{ccc} \mathrm{Spin}(n) & \longrightarrow & SO(n) \\ \downarrow & \lrcorner_h & \downarrow \\ E\,U(1) & \longrightarrow & \mathbb{B}U(1) \end{array}$$

manifold; modulating / classifying map; group homomorphism

$\mathbf{B}$

$*$

$\mathbb{B}U(1)$ $\quad \mathbb{B}\,\mathrm{String}$

DCCT Ex 1.2.148

$$\begin{array}{ccc} \mathbb{B}\,\mathrm{Spin}(n) & \longrightarrow & \mathbb{B}\,SO(n) \\ \downarrow & \lrcorner_h & \downarrow W_3 \\ * & \longrightarrow & \mathbb{B}^2 U(1) \end{array}$$

$W_3$: third integral Stiefel–Whitney class

$$\begin{array}{ccc} \mathbb{B}\,\mathrm{String}(n) & \longrightarrow & \mathbb{B}\,\mathrm{Spin}(n) \\ \downarrow & \lrcorner_h & \downarrow \frac{p_1}{2} \\ * & \longrightarrow & \mathbb{B}^3 U(1) \end{array}$$

DCCT

$\frac{p_1}{2}$: first fractional Pontrjagin class

$s\mathrm{PSh}(\mathrm{Cart})_{\check{C}ech}$

looping:

$$\begin{array}{ccc} \mathrm{String}(n) & \longrightarrow & \mathrm{Spin}(n) \\ \downarrow & & \downarrow \\ \mathbf{E}(\mathbb{B}U(1)) & \longrightarrow & \mathbb{B}^2 U(1) \end{array}$$

$\mathrm{PSh}(\mathrm{Cart}, s\mathrm{Grp})$

classifying stack for principal

Lie 2-group ↑ universal principal $\mathbb{B}U(1)$-bundle