Characterizing Paths and Surfaces via (Higher) Holonomy

joint work with H. Oberhauser (Oxford)

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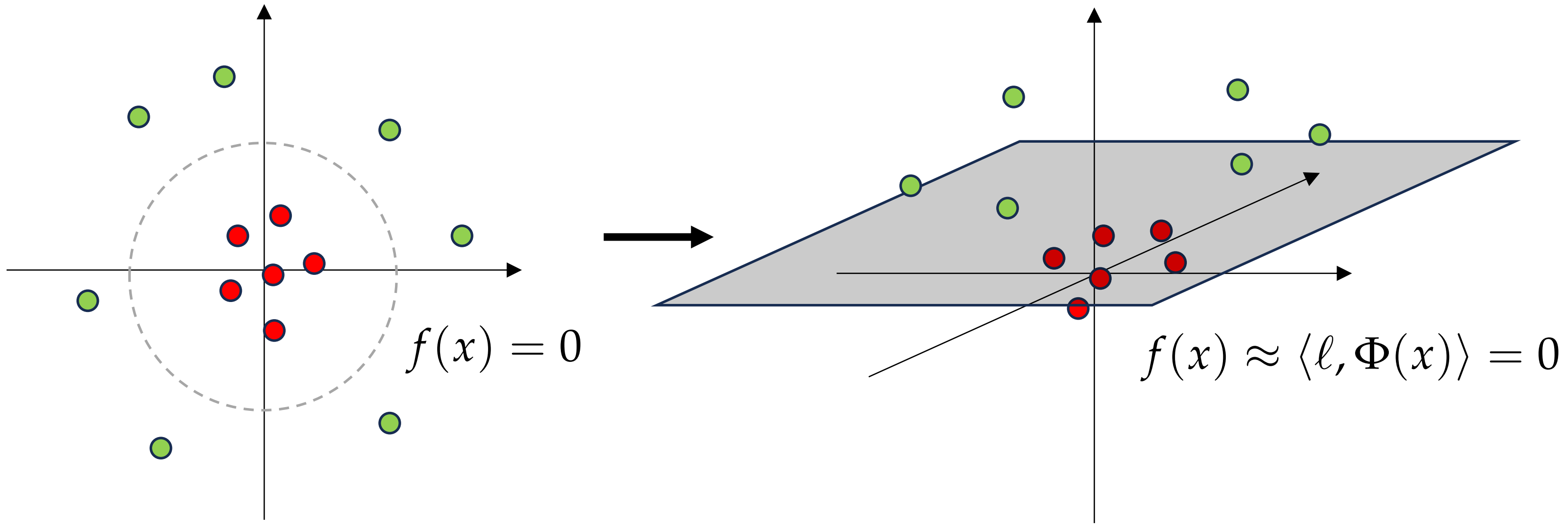
Motivation: Classification in Machine Learning

Input Space $\mathcal{X}$
(Topological Space)

Feature Space $\mathcal{H}$
(Hilbert Space)



Feature Map $\Phi : \mathcal{X} \to \mathcal{H}$

Classification

Find a function
$f : \mathcal{X} \to \mathbb{R}$
such that

$f(x) < 0$	x is red
$f(x) > 0$	x is green

What is a *good* feature map?

A feature map is **universal** if linear functionals $\langle \ell, \Phi(\cdot) \rangle : \mathcal{X} \to \mathbb{R}$ with $\ell \in \mathcal{H}$ are dense in $C(\mathcal{X}, \mathbb{R})$.

Example (The Monomial Map): $\mathcal{X} \subset \mathbb{R}^d$ compact

$$\Phi : \mathcal{X} \to \prod_{m=0}^{\infty} (\mathbb{R}^d)^{\otimes m} =: T((\mathbb{R}^d)) \qquad \Phi(x) = \left(\frac{x^{\otimes m}}{m!} \right)_{m \geq 0}$$

Specify coordinate in $(\mathbb{R}^d)^{\otimes m}$

$$I = (i_1, \ldots, i_m) \qquad \Phi^I(x) = \frac{x^{i_1} \cdots x^{i_m}}{m!}$$

Motivation: Characterizing Probability Measures

How can we characterize vector-valued random variables?

Random Variable

$$X = (X^1, \dots, X^d) : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}^d$$

corresponding probability measure

$$\Longrightarrow$$

Law of X

$$\mu = X_* \mathbb{P} \in \mathcal{P}(\mathbb{R}^d)$$

Can we use feature maps to characterize measures?

$$\Phi : \mathcal{X} \to \mathcal{H}$$

A feature map is **characteristic** if the expectation $\mathbb{E}[\Phi] : \mathcal{P}(\mathcal{X}) \to \mathcal{H}$ is injective.

Example (Moments): $\mathcal{X} \subset \mathbb{R}^d$ compact

$$\Phi : \mathcal{X} \to \prod_{m=0}^{\infty} (\mathbb{R}^d)^{\otimes m} =: T((\mathbb{R}^d)) \qquad \Phi(x) = \left(\frac{x^{\otimes m}}{m!} \right)_{m \geq 0}$$

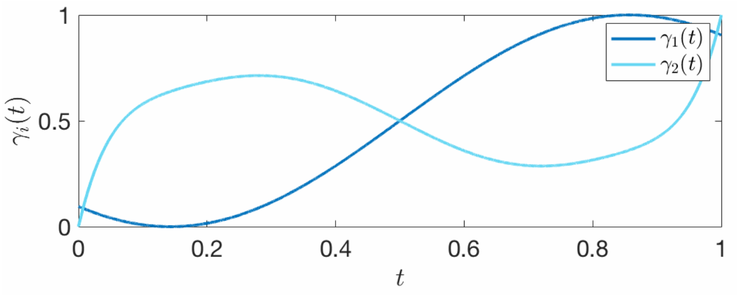
Specify coordinate in $(\mathbb{R}^d)^{\otimes m}$

$$I = (i_1, \dots, i_m)$$

$$\mathbb{E}[\Phi^I(X)] = \frac{\mathbb{E}[X^{i_1} \cdots X^{i_m}]}{m!}$$

Paths and Surfaces

Time Series



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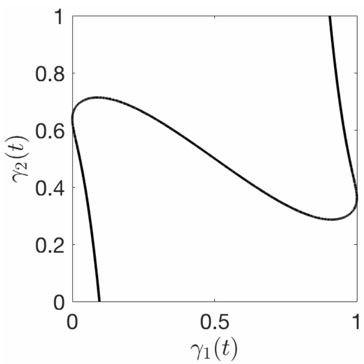


Paths

$$x : [0,1] \to \mathbb{R}^d$$

Images

Surfaces

$$X : [0,1]^2 \to \mathbb{R}^d$$

This talk:
How can we build feature maps for *functional data*?

Motivation: Fourier Transforms and Characteristic Functions

The Characteristic Function (Fourier Transform of Probability Measures)

$$\left(\mathcal{F}^\alpha(\mu) = \mathbb{E}_{X\sim\mu}[\exp(i\alpha X)]\right)_{\alpha\in\mathbb{R}}$$

The Fourier transform is injective (it characterizes measures).

Extension to Nonabelian Groups (Fourier-Stieltjes Transform)

G	compact, Hausdorff topological group
$\Phi^\alpha : G \to U^\alpha$	irreducible finite-dimensional unitary representation
$\alpha \in \mathcal{M}$	set of (equivalence classes) of such representations
U^α	unitary group of finite dimensional Hilbert space V^α

The **Fourier-Stieltjes transform** $\left(\mathcal{F}^\alpha(\mu) = \mathbb{E}_{X\sim\mu}[\Phi^\alpha(X)]\right)_{\alpha\in\mathcal{M}}$ characterizes measures.

Can we use representations of paths and surfaces to achieve a similar result?

Representations of Paths

Thin Fundamental Groupoid

How do we encode paths into a groupoid?

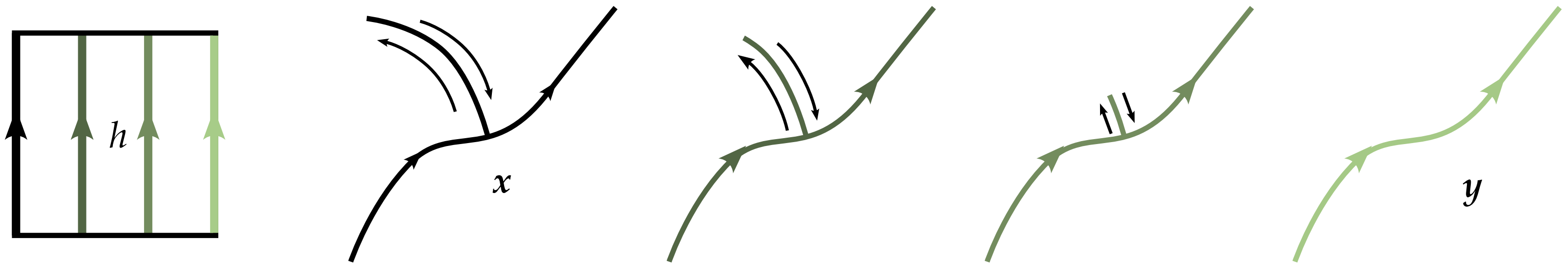
The **thin fundamental groupoid** Π is a groupoid

Objects: $\Pi_0 = \mathbb{R}^d$ **Edges:** $\Pi_1 = C^\infty([0,1], \mathbb{R}^d) / \sim_{\text{thin}}$

We need to quotient out paths by **thin homotopies**.

A smooth homotopy $h : [0,1]^2 \to \mathbb{R}^n$ between paths x and y is **thin** if $\operatorname{rank}(dh) \leq 1$

- The homotopy sweeps out zero area



Representation of Paths

Representation of Paths as a Functor

($\mathrm{B}G$ is the group G as a groupoid with one object)

$$F : \Pi \to \mathrm{B}G$$

1. Parallel Transport

$$F^{\alpha} : \Pi \to \mathrm{BGL}^{n}$$

is a functor

2. Path Signature

$$S : \Pi \to \mathrm{B}T_{\mathrm{gl}}((\mathbb{R}^{d}))$$

is a functor

$T_{\mathrm{gl}}((\mathbb{R}^{d}))$ is the group-like elements of $T((\mathbb{R}^{d})) := \prod_{m=0}^{\infty} (\mathbb{R}^{d})^{\otimes m}$

Example 1: Parallel Transport on Principal Bundles

Consider parallel transport in a trivial G-bundle over $\mathbb{R}^d$.

Path

$x : [0,1] \to \mathbb{R}^d$

Group

GL^n

(Translation-Invariant) Connection

$\alpha \in L(\mathbb{R}^d, \mathfrak{gl}^n)$

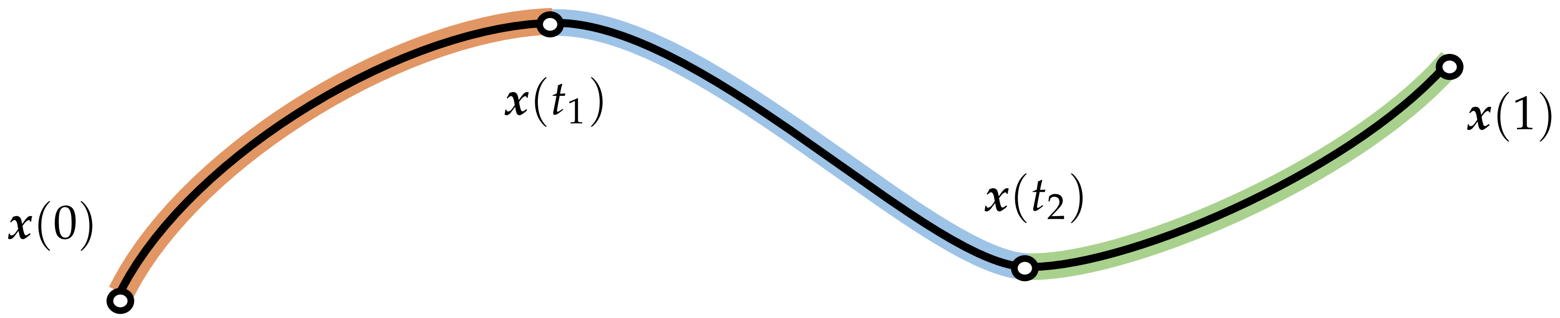
$\alpha = \sum_{i=1}^{d} \alpha^i dx^i \in \Omega^1(\mathbb{R}^d, \mathfrak{gl}^n) \qquad \alpha^i \in \mathfrak{gl}^n$

Parallel Transport

$$\frac{dF_t^\alpha(x)}{dt} = F^\alpha(x) \cdot \alpha\left(\frac{dx_t}{dt}\right)$$

$$F_0^\alpha(x) = I$$

1. Partition Path



2. Approximate and multiply

$\exp\left(\alpha\left(\frac{dx(0)}{dt}\right)\Delta t\right)$ $\quad \exp\left(\alpha\left(\frac{dx(t_1)}{dt}\right)\Delta t\right)$ $\quad \exp\left(\alpha\left(\frac{dx(t_2)}{dt}\right)\Delta t\right)$

3. Take the limit as partition gets finer:

$$F^\alpha(x) = \lim_{n\to\infty} \prod_{i=1}^{n} \exp\left(\alpha\left(\frac{dx(t_{i-1})}{dt}\right)\Delta t\right)$$

Universal and Characteristic

Compact $\mathcal{K} \subset \Pi_1 = C^\infty([0,1], \mathbb{R}^d) / \sim_{\text{th}}$

Universal: The span of linear functionals

$$\left\{ \langle \ell, F^\alpha(\cdot) \rangle : \mathcal{K} \to \mathbb{R} \; : \; n > 0, \, \alpha \in L(\mathbb{R}^d, \mathfrak{gl}^n), \, \ell \in \mathfrak{gl}^n \right\}$$

is dense in $C(\mathcal{K}, \mathbb{R})$

Characteristic: For probability measures $\mu, \nu \in \mathcal{P}(\mathcal{K})$ such that $\mu \neq \nu$ there exists some $n > 0$, $\alpha \in L(\mathbb{R}^d, \mathfrak{gl}^n)$, and $\ell \in \mathfrak{gl}^n$ such that

$$\mathbb{E}_{\boldsymbol{x} \sim \mu}[\langle \ell, F^\alpha(\boldsymbol{x}) \rangle] \neq \mathbb{E}_{\boldsymbol{y} \sim \nu}[\langle \ell, F^\alpha(\boldsymbol{y}) \rangle]$$

Applications:

Time Series Classification

- Lou, Li, Hao, *Path development network with finite-dimensional Lie group representations,* preprint, 2022.

Time Series Generation

- Lou, Li, Hao, *PCF-GAN: generating sequential data via the characteristic function on path space,* preprint, 2023

Example 2: The Path Signature

The path signature is the universal translation-invariant parallel transport map.

Lie Algebra

$\mathfrak{f} := \mathrm{FL}(\mathbb{R}^d)$ (Completion) of Free Lie Algebra

Universal Enveloping Algebra

$$U(\mathfrak{f}) := T((\mathbb{R}^d)) := \prod_{m \geq 0} (\mathbb{R}^d)^{\otimes m}$$

Lie Group

$G := T_{\mathrm{gl}}((\mathbb{R}^d)) \subset T((\mathbb{R}^d))$ Group-Like Elements

Connection

$Z_1, \ldots, Z_d$ Lie algebra generators

$$\zeta = \sum_{i=1}^{d} Z_i \, dx_i \in \Omega^1(\mathbb{R}^d, \mathfrak{f})$$

Path Signature

$$\frac{dS_t(\boldsymbol{x})}{dt} = S_t(\boldsymbol{x}) \otimes \frac{d\boldsymbol{x}}{dt} \qquad S_0(\boldsymbol{x}) = 1$$

Universal Property

$$\begin{array}{ccc} \Pi & \xrightarrow{F^{\alpha}} & \mathrm{BGL}^n \\ {\scriptstyle S}\downarrow & \nearrow_{\exists!\, \widetilde{\alpha}} & \\ \mathrm{B}T_{\mathrm{gl}}((\mathbb{R}^d)) & & \end{array}$$

Example 2: The Path Signature

Path

$$x = (x^1, \dots, x^d) \in C^\infty([0,1], \mathbb{R}^d)$$

Path Signature

$$S : C^\infty([0,1], \mathbb{R}^d) \to \prod_{m=0}^{\infty} (\mathbb{R}^d)^{\otimes m} =: T((\mathbb{R}^d))$$

(represents paths/time series as an infinite sequence of tensors)

$$S(x) = \left(S^{(m)}(x)\right)_{m \geq 0}$$

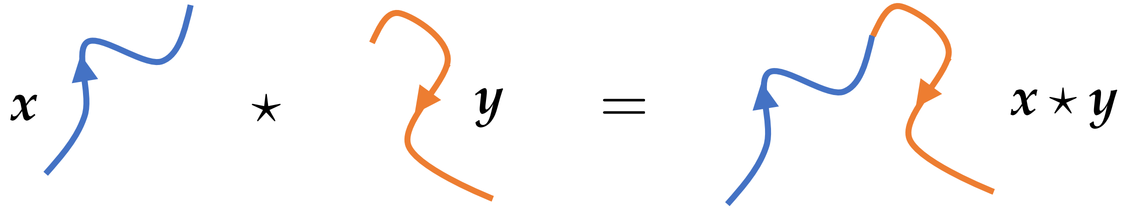
$$S^{(m)}(x) = \int_{\Delta^m} x'_{t_1} \otimes \dots \otimes x'_{t_m} dt_1 \dots dt_m$$

$$\Delta^m = \{0 \leq t_1 < \dots < t_m \leq 1\}$$

Example (m=1):

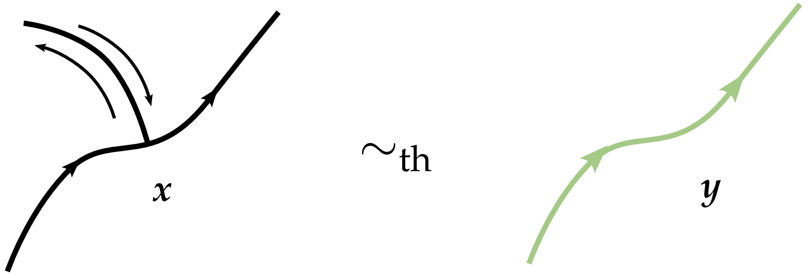
$$S^{(1)}(x) = x_1 - x_0$$

Chen's Identity
(preserves algebraic structure of paths)



$$S(x \star y) = S(x) \otimes S(y)$$

Characterization of Thin Homotopy Classes



$S(x) = S(y)$ **if and only if** $x \sim_{\text{th}} y$

K. T. Chen, *Integration of Paths – A Faithful Representation of Paths by Noncommutative Power Series*, Transactions of the AMS, 1958
B. Hambly, T. Lyons, *Uniqueness for the Signature of a Path of Bounded Variation and the Reduced Path Group*, Annals of Math., 2010
H. Boedihardjo, X. Geng, T. Lyons, D. Yang, *The Signature of a Rough Path: Uniqueness*, Advances in Math., 2016

Universal and Characteristic

The path signature is universal and characteristic.

Compact $\mathcal{X} \subset \Pi_1 = C^\infty([0,1], \mathbb{R}^d) / \sim_{\text{th}}$

Universal

$$\langle \ell, S(\cdot) \rangle : \mathcal{X} \to \mathbb{R}$$

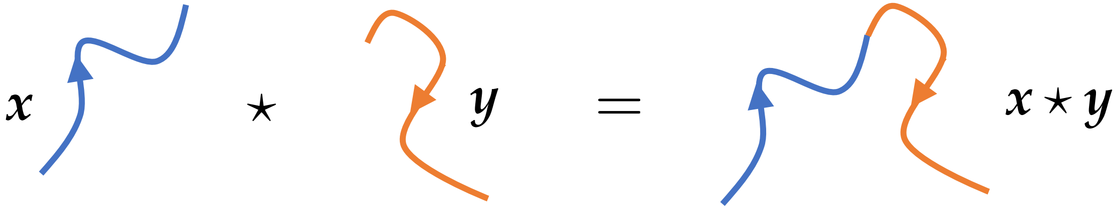
Linear functionals can approximate continuous functions on $\mathcal{X}$.

Characteristic

$$\mathbb{E}[S] : \mathcal{P}(\mathcal{X}) \to T((\mathbb{R}^d))$$

Expected features characterize probability measures on $\mathcal{X}$.

Why do we need Chen's Identity?



$$S(\boldsymbol{x} \star \boldsymbol{y}) = S(\boldsymbol{x}) \otimes S(\boldsymbol{y})$$

Parallelism: Allows Efficient GPU Implementations

Signature Computation: Split path into small pieces, compute signature in parallel, then multiply.

- Kidger and Lyons, *Signatory: differentiable computations of the signature and logsignature transforms, on both CPU and GPU*, ICLR 2021.

Signature Kernel Computations: Recursive algorithm for computing inner products of signatures.

- Kiraly, Oberhauser, *Kernels for sequentially ordered data,* Journal of Machine Learning Research, 2019.

Low Rank Approximations: Recursive algorithm for low rank approximations to signatures.

- Toth, Bonnier, Oberhauser, *Seq2Tens: An efficient representation of sequences by low-rank tensor projections*, ICLR 2021.

Graph Neural Networks: Recursive algorithm for signature-based graph neural network layer.

- Toth, **L**., Hacker, Oberhauser, *Capturing graphs with hypo-elliptic diffusions,* NeurIPS, 2022.

Generalizations of Universality / Characteristicness

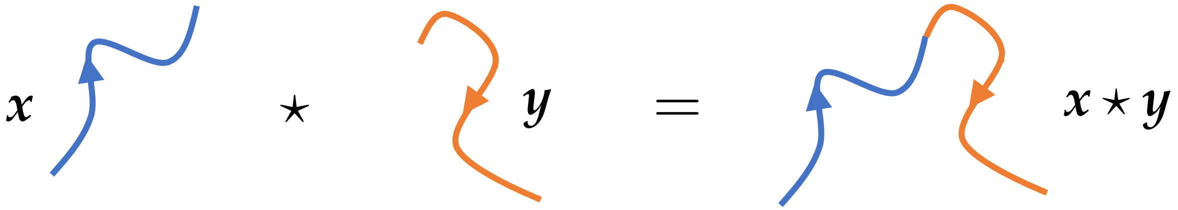
Universal/Characteristic properties can be extended to rough paths.

- Chevyrev, Lyons, *Characteristic functions of measures on geometric rough paths,* Annals of Probability, 2016.
- Chevyrev, Oberhauser, *Signature moments to characterize laws of stochastic processes,* Journal of Machine Learning Research, 2022.
- Cuchiero, Schmocker, Teichmann, *Global universal approximation of functional input maps on weighted spaces,* preprint 2023.

Representation of Surfaces

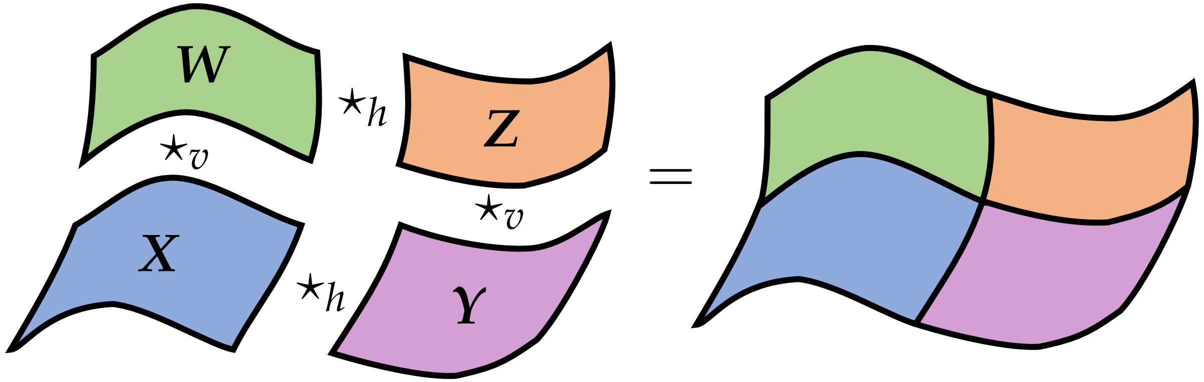
Structure Preserving Maps

Paths



$$\Phi(x \star y) = \Phi(x) \cdot \Phi(y)$$

Surfaces



$$\Phi\begin{pmatrix} \mathbf{W} & \star_h & \mathbf{Z} \\ \star_v & & \star_v \\ \mathbf{X} & \star_h & \mathbf{Y} \end{pmatrix} = \begin{matrix} \Phi(W) & \odot_h & \Phi(Z) \\ \odot_v & & \odot_v \\ \Phi(X) & \odot_h & \Phi(Y) \end{matrix}$$

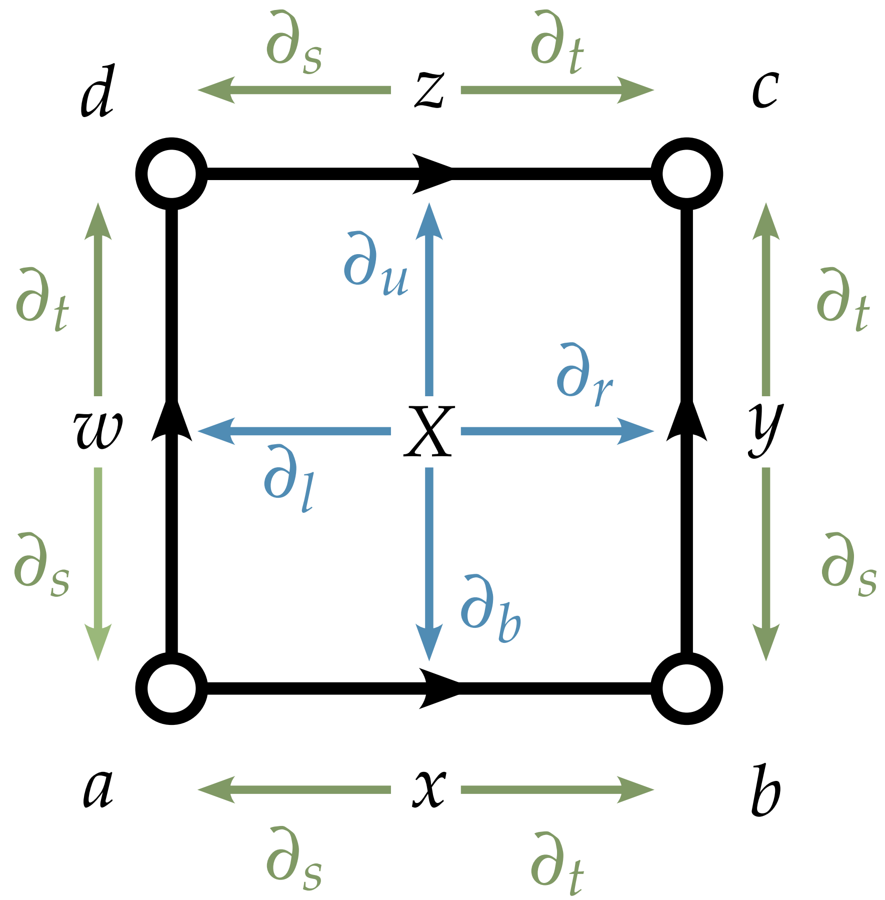
Other approaches to higher dimensional signatures do not preserve this algebraic structure.

C. Giusti, D. Lee, V. Nanda, H. Oberhauser, *A topological approach to mapping space signatures,* preprint (2022)
J. Diehl, L. Schmitz, *Two-parameter sums signatures and corresponding quasisymmetric functions* , preprint (2022)
J. Diehl, K. Ebrahimi-Fard, F. Harang, S. Tindel, *On the signature of an image,* preprint (2024)

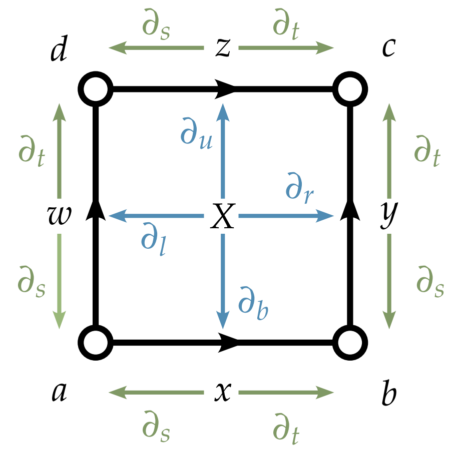
Double Groupoids

Def: A *(edge symmetric) double groupoid* $\mathbf{G}$ is:

- A set of of *objects* $\mathbf{G}_0$ and a set of *edges* $\mathbf{G}_1$ which form a groupoid
- A set of *squares* $\mathbf{G}_2$ with (left, right, up, bottom) *boundary maps* $\partial_l, \partial_r, \partial_u, \partial_b : \mathbf{G}_2 \to \mathbf{G}_1$



Double Groupoids



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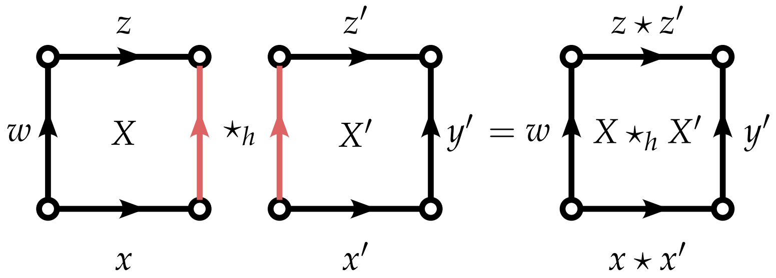
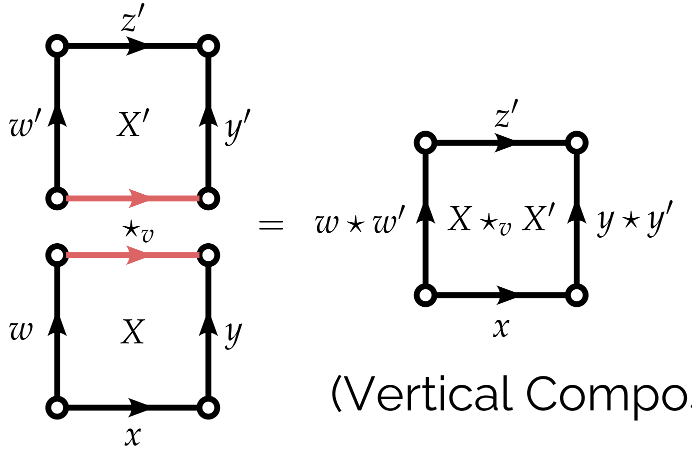
- A set of *squares* $\mathbf{G}_2$ with (left, right, up, bottom) *boundary maps* $\partial_l, \partial_r, \partial_u, \partial_b : \mathbf{G}_2 \to \mathbf{G}_1$
    - **Horizontal/Vertical Composition** (associative):

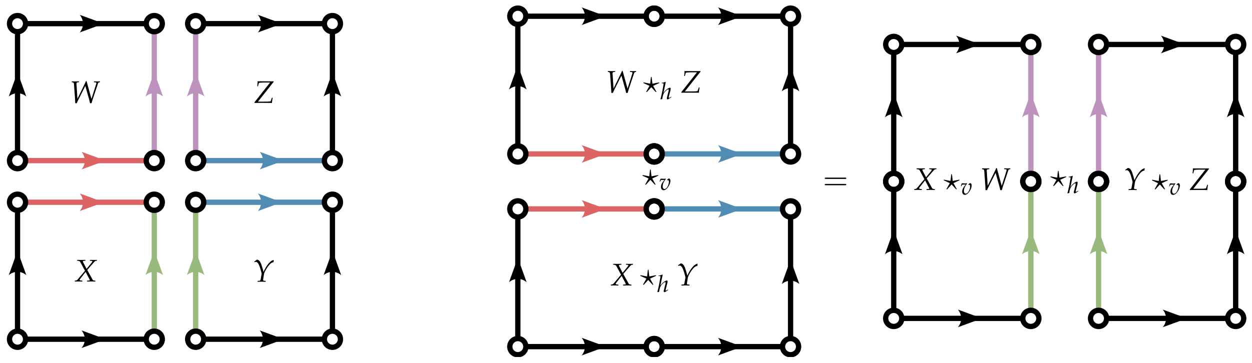


(Horizontal Composition)



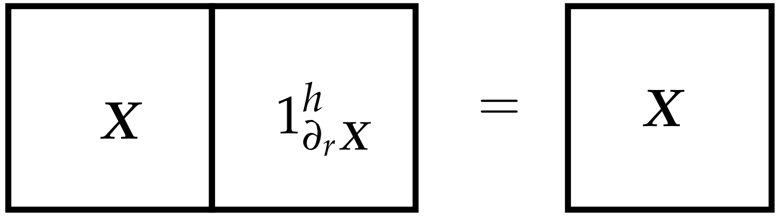
(Vertical Composition)

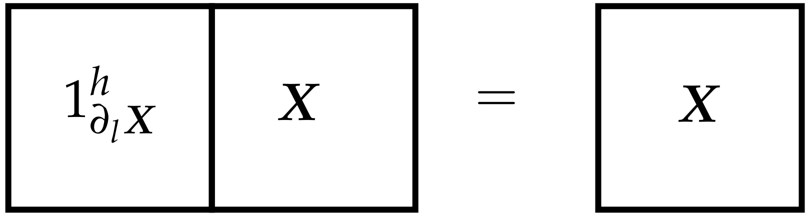
- **Interchange Law**



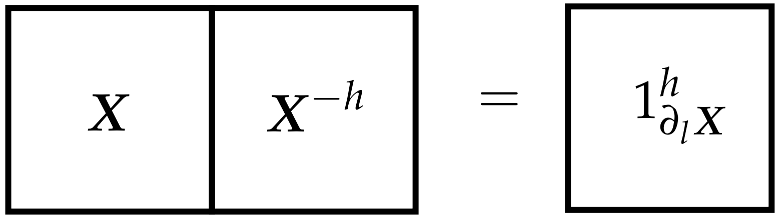
Double Groupoids and Functors

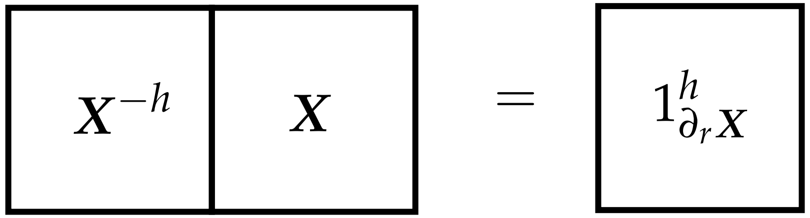
- **Identity Squares:** For every $x \in \mathbf{G}_1$, there exist *horizontal and vertical identity squares* $1^h_x, 1^v_y \in \mathbf{G}_2$





- **Inverse Squares:** For every $X \in \mathbf{G}_2$ there exist *horizontal and vertical inverse squares* $X^{-h}, X^{-v} \in \mathbf{G}_2$





Def: A *functor* $F : \mathbf{G} \to \mathbf{H}$ between double groupoids consist of maps

$$F_0 : \mathbf{G}_0 \to \mathbf{H}_0 \qquad F_1 : \mathbf{G}_1 \to \mathbf{H}_1 \qquad F_2 : \mathbf{G}_2 \to \mathbf{H}_2$$

such that $F_1(x \star y) = F_1(x) \star F_1(y)$ and identities / inverses are preserved.

$$F_2(X \star_h Y) = F_2(X) \star_h F_2(Y)$$
$$F_2(X \star_v Y) = F_2(X) \star_v F_2(Y)$$

Thin Fundamental Double Groupoids

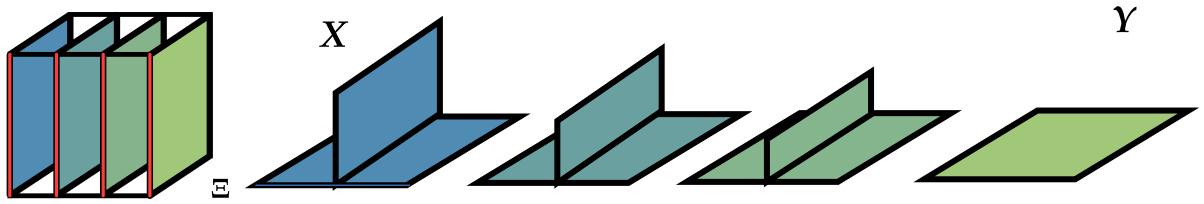
How do we encode surfaces as a double groupoid?

The **thin fundamental double groupoid** Π is a double groupoid

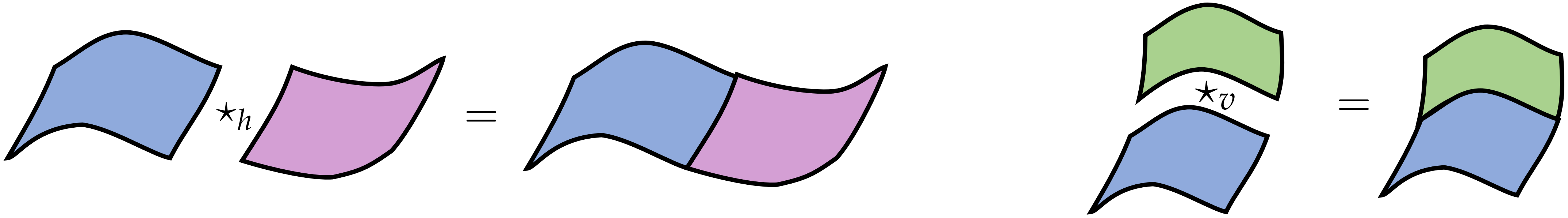
Objects: $\Pi_0 = \mathbb{R}^n$ **Morphisms:** $\Pi_1 = C^\infty([0,1], \mathbb{R}^n)/\sim_{th}$ **Squares:** $\Pi_2 = C^\infty([0,1]^2, \mathbb{R}^n)/\sim_{th}$

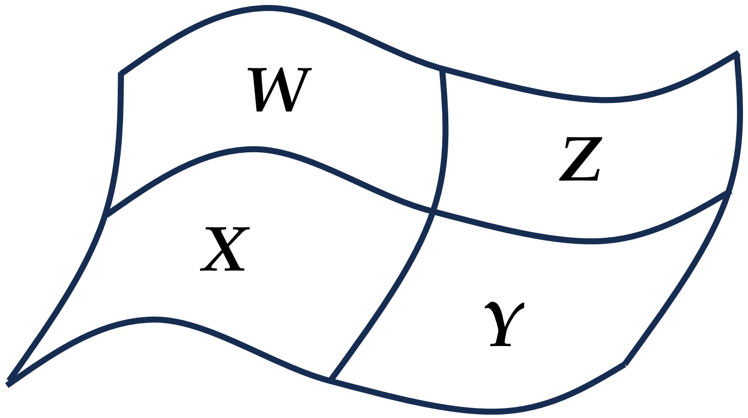
A smooth homotopy $\Xi : [0,1]^3 \to \mathbb{R}^d$ between surfaces X and Y is **thin** if $\operatorname{rank}(d\Xi) \leq 2$

- The homotopy sweeps out zero volume



Squares are equipped with horizontal and vertical compositions (associativity, identity, inverse):

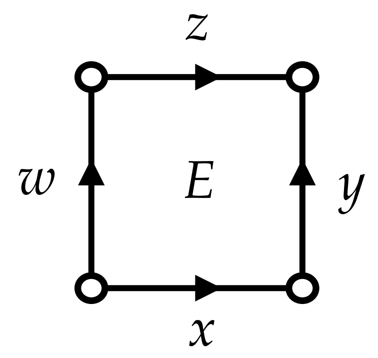




$$\Phi\begin{pmatrix} W \star_h Z \\ \star_v \quad\quad \star_v \\ X \star_h Y \end{pmatrix} = \begin{matrix} \Phi(W) \odot_h \Phi(Z) \\ \odot_v \quad\quad\quad \odot_v \\ \Phi(X) \odot_h \Phi(Y) \end{matrix}$$

What is the generalization of a group which allows for two multiplication operators?

Trivial Double Group

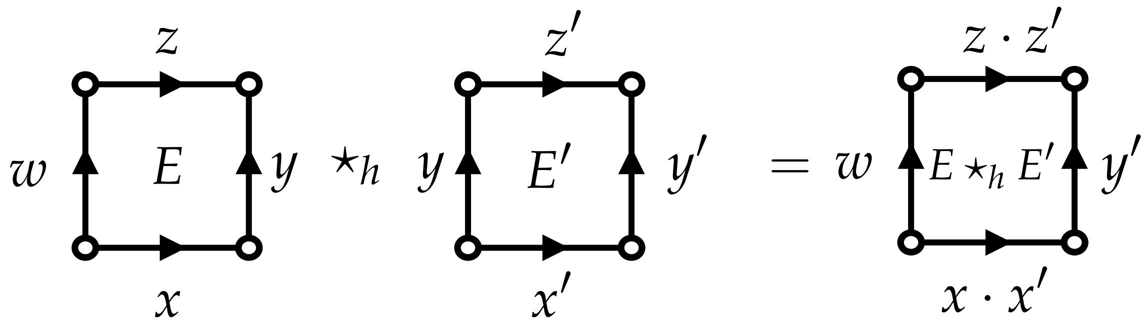


Def: A *double group* is a double groupoid with one object.

Given a group G, we define the *trivial double group of* G, denoted $\mathbf{D}(G)$, by

$$\mathbf{D}_1(G) = G \qquad \mathbf{D}_2(G) = \{(x, y, z, w; E) \in G^5 \,:\, E = xyz^{-1}w^{-1}\}$$

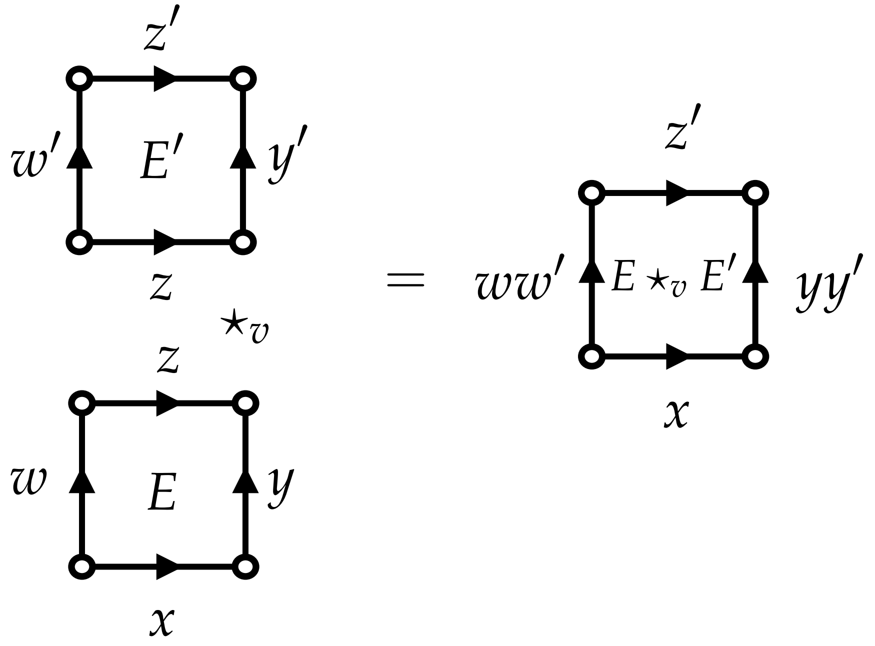
Horizontal Composition



$$\begin{aligned} E \star_h E' &= xx'y'(z')^{-1}z^{-1}w^{-1} \\ &= x(x'y'(z')^{-1}y^{-1})x^{-1}(xyz^{-1}w^{-1}) \\ &= (x \triangleright E') \cdot E \end{aligned}$$

$$x \triangleright y := xyx^{-1}$$

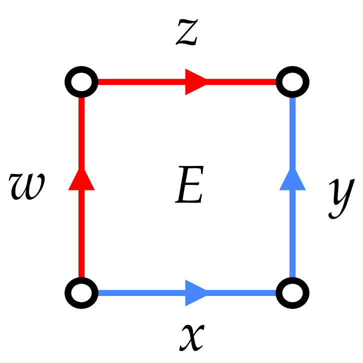
Vertical Composition



$$E \star_v E' = E \cdot (w \triangleright E')$$

Double Groups

We interpret E as a relationship between the two paths wz and xy .



To obtain more general double groups, we the interior element E should be valued in another group H

Crossed Modules

Def: A *crossed module* $\mathbf{G} = (\delta : \mathbf{G}_2 \to \mathbf{G}_1, \triangleright)$ consists of:

- **Groups:** $(\mathbf{G}_1, \cdot), (\mathbf{G}_2, *)$
- **Boundary Map:** $\delta : \mathbf{G}_2 \to \mathbf{G}_1$ (group homomorphism)
- **Action:** $\triangleright : \mathbf{G}_1 \to \mathrm{Aut}(\mathbf{G}_2)$ (g acting on h is written $g \triangleright h$) such that

CM1. $\delta(g \triangleright h) = g \cdot \delta(h) \cdot g^{-1}$ for all $g \in \mathbf{G}_1, h \in \mathbf{G}_2$

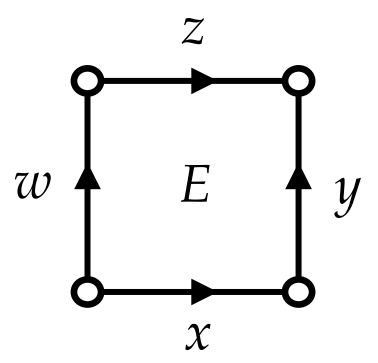
CM2. $\delta(h_1) \triangleright h_2 = h_1 * h_2 * h_1^{-1}$ for all $h_1, h_2 \in \mathbf{G}_2$

Ex: Trivial crossed module

$$\mathbf{G} = (\mathrm{id} : G \to G, \triangleright)$$
$$x \triangleright y = xyx^{-1}$$

Given a crossed module $\mathbf{G} = (\delta : \mathbf{G}_2 \to \mathbf{G}_1, \triangleright)$, we construct a double group $\mathbf{D}(\mathbf{G})$

$\mathbf{D}_1(\mathbf{G}) = \mathbf{G}_1$ $\quad\quad$ $\mathbf{D}_2(\mathbf{G}) = \{(x, y, z, w; E) \in \mathbf{G}_1^4 \times \mathbf{G}_2 \ : \ \delta(E) = xyz^{-1}w^{-1}\}$



Horizontal Composition

$E \star_h E' = (x \triangleright E') * E$

Vertical Composition

$E \star_v E' = E * (w \triangleright E')$

The boundary formulas hold (CM1) and the interchange law holds (CM2).

General Linear Crossed Module

The *general linear crossed module* is defined by automorphisms of **(Baez-Crans) 2-vector spaces**:

$$\mathcal{V}^{n,m,p} := \mathbb{R}^{n+m} \xrightarrow{\phi} \mathbb{R}^{n+p} \qquad\qquad \phi = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$$

Def: The crossed module $\mathbf{GL}^{n,m,p} = (\delta : \mathbf{GL}_2^{n,m,p} \to \mathbf{GL}_1^{n,m,p}, \triangleright)$ is defined by

$$\mathbf{GL}_1^{n,m,p} = \left\{ F, G = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}, \begin{pmatrix} A & D \\ 0 & E \end{pmatrix} \in \mathrm{GL}^{n+m} \times \mathrm{GL}^{n+p} \right\}$$

$$\begin{array}{ccc} \mathbb{R}^{n+m} & \xrightarrow{\phi} & \mathbb{R}^{n+p} \\ F \downarrow & & \downarrow G \\ \mathbb{R}^{n+m} & \xrightarrow{\phi} & \mathbb{R}^{n+p} \end{array}$$

$$\mathbf{GL}_2^{n,m,p} := \left\{ H = \begin{pmatrix} A - I & D \\ B & U \end{pmatrix} \in \mathrm{Mat}_{n+m,n+p} \ : \ A \in \mathrm{GL}^n \right\}$$

$$\begin{array}{ccc} \mathbb{R}^{n+m} & \xrightarrow{\phi} & \mathbb{R}^{n+p} \\ & \swarrow H & \\ \mathbb{R}^{n+m} & \xrightarrow{\phi} & \mathbb{R}^{n+p} \end{array}$$

Group Multiplication	Inverse	Crossed Module Boundary	Crossed Module Action
$H * H' = H + H' + H\phi H'$	$H^{-*} = -H(I + \phi H)^{-1}$	$\delta(H) = (H\phi + I, \phi H + I)$	$(F, G) \triangleright H = FHG^{-1}$

M. Forrester-Barker, *Representations of crossed modules and cat[1]-groups,* PhD Thesis, University of Wales, Bangor, 2003

J. Baez and A. Crans, *Higher dimensional algebra VI: Lie 2-algebras,* Theory and Applications of Categories, 2004.

General Linear Differential Crossed Module

Def: A *differential crossed module* $\mathfrak{g} = (\delta : \mathfrak{g}_2 \to \mathfrak{g}_1, \triangleright)$ has:

- **Lie Algebras:** $(\mathfrak{g}_1, [\cdot,\cdot]), (\mathfrak{g}_2, [\cdot,\cdot]_*)$
- **Boundary Map:** $\delta : \mathfrak{g}_2 \to \mathfrak{g}_1$ (Lie algebra homomorphism)
- **Action:** $\triangleright : \mathfrak{g}_1 \to \mathrm{Der}(\mathfrak{g}_2)$

DCM1. $\delta(x \triangleright E) = [x, \delta(E)]$ for all $x \in \mathfrak{g}_1, E \in \mathfrak{g}_2$

DCM2. $\delta(E) \triangleright \delta(F) = [E,F]_*$ for all $E, F \in \mathfrak{g}_2$

Def: The *general linear differential* crossed module $\mathfrak{gl}^{n,m,p} = (\delta : \mathfrak{gl}_2^{n,m,p} \to \mathfrak{gl}_1^{n,m,p}, \triangleright)$ is defined by

$$\mathfrak{gl}_1^{n,m,p} = \left\{ X, Y = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}, \begin{pmatrix} A & D \\ 0 & E \end{pmatrix} \in \mathfrak{gl}^{n+m} \oplus \mathfrak{gl}^{n+p} \right\}$$

$$\mathfrak{gl}_2^{n,m,p} := \left\{ Z = \begin{pmatrix} R & S \\ T & U \end{pmatrix} \in \mathrm{Mat}_{n+m,n+p} \right\}, \qquad [Z, Z']_* = Z\phi Z' - Z'\phi Z$$

Crossed Module Boundary

$$\delta(Z) = (Z\phi, \phi Z)$$

Lie Algebra Action

$$(X, Y) \triangleright Z = XZ - ZY$$

Induced Lie Group Action

$$\triangleright : \mathbf{GL}_1^{n,m,p} \to \mathfrak{gl}_2^{n,m,p}$$

$$(F, G) \triangleright X = FXG^{-1}$$

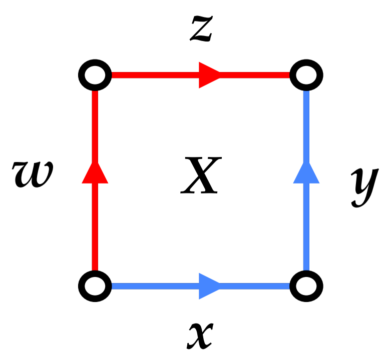
Towards Surface Holonomy

Mathematical physicists (Baez, Schreiber, Waldorf + others 2000's) developed **surface holonomy** to study higher gauge theories.

Martins, Picken, *Surface holonomy for non-abelian 2-bundles via double groupoids,* Advances in Mathematics, 2011

Surface	Lie Group	Lie Algebra	Translation-Invariant Connection
$\mathbf{X} : [0,1]^2 \to \mathbb{R}^d$	$\mathbf{GL}_1^{n,m,p} \subset \mathrm{GL}^{n+m} \times \mathrm{GL}^{n+p}$	$\mathfrak{gl}_1^{n,m,p} \subset \mathfrak{gl}^{n+m} \oplus \mathfrak{gl}^{n+p}$	$(\alpha, \beta) \in L(\mathbb{R}^d, \mathfrak{gl}_1^{n,m,p})$

If the connection is **not flat**, then $F(\boldsymbol{w} \star \boldsymbol{z}) \neq F(\boldsymbol{x} \star \boldsymbol{y})$.



Surface holonomy of $\boldsymbol{X}$ with respect to a **fake-flat 2-connection** provides a chain homotopy between $F(\boldsymbol{w} \star \boldsymbol{z})$ and $F(\boldsymbol{x} \star \boldsymbol{y})$.

2-Connection

$$\gamma \in L(\Lambda^2\mathbb{R}^d, \mathfrak{gl}_2^{n,m,p})$$

Fake Flatness Condition

$$\delta(\gamma) = [(\alpha, \beta), (\alpha, \beta)]$$

Surface Holonomy

Surface Holonomy Functor

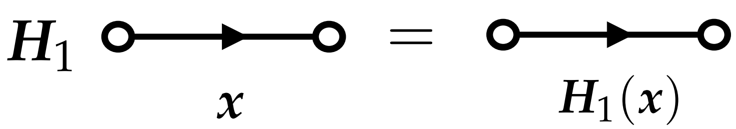
$$\boldsymbol{H} : \Pi \to \mathbf{D}(\mathbf{GL}^{n,m,p})$$

2-Connection

$$\omega = (\alpha, \beta, \gamma)$$

Edges

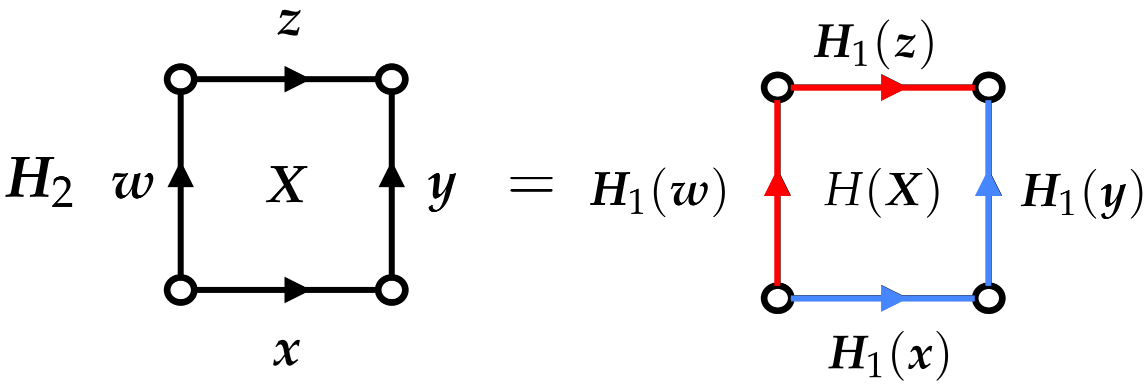
$$\boldsymbol{H}_1 : \Pi_1 \to \mathbf{D}_1(\mathbf{GL}^{n,m,p}) \subset \mathrm{GL}^{n+m} \times \mathrm{GL}^{n+p}$$



This is a classical parallel transport functor.

Squares

$$\boldsymbol{H}_2 : \Pi_2 \to \mathbf{D}_2(\mathbf{GL}^{n,m,p})$$



Surface Holonomy Map

$$H : \Pi_2 \to \mathbf{GL}_2^{n,m,p}$$

This provides a homotopy between the path holonomy along paths wz and xy .

Surface Holonomy

Surface

$$X : [0,1]^2 \to \mathbb{R}^d$$

Double Group

$$\mathbf{GL}^{n,m,p}$$

2-Connection

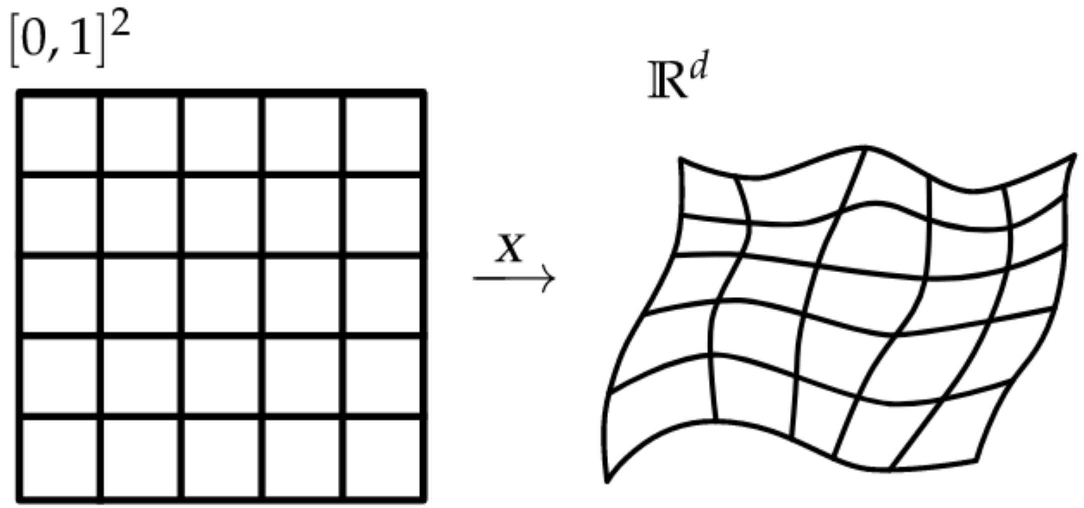
$$\omega = (\alpha, \beta, \gamma)$$
$$(\alpha, \beta) \in L(\mathbb{R}^d, \mathfrak{gl}_1^{n,m,p})$$
$$\gamma \in L(\Lambda^2\mathbb{R}^d, \mathfrak{gl}_2^{n,m,p})$$

$$\widetilde{H}_{i,j} := \exp_* \left(\gamma \left(\frac{\partial \mathbf{X}_{s_i,t_j}}{\partial s}, \frac{\partial \mathbf{X}_{s_i,t_j}}{\partial t} \right) \Delta s \, \Delta t \right)$$

1. Partition surface

2. Approximate on subsquares and multiply

3. Take the limit as partition gets finer



Surface Holonomy

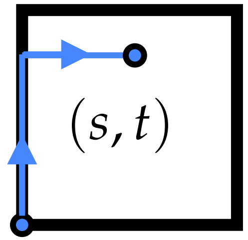
Def: Given a 2-connection $\omega = (\alpha, \beta, \gamma)$ valued in $\mathfrak{gl}^{n,m,p}$ and $\boldsymbol{X} \in C^\infty([0,1]^2, \mathbb{R}^d)$, the *surface holonomy* is

$$H^\omega_{s,t}(\boldsymbol{X}) : [0,1]^2 \to \mathbf{GL}_2^{n,m,p}$$

$$\frac{\partial H^\omega_{s,t}(\boldsymbol{X})}{\partial t} = (I + H^\omega_{s,t}(\boldsymbol{X})\phi) \int_0^s F^\alpha(\boldsymbol{x}^{s',t})\gamma\left(\frac{\partial \boldsymbol{X}_{s',t}}{\partial s}, \frac{\partial \boldsymbol{X}_{s',t}}{\partial t}\right)(F^\beta(\boldsymbol{x}^{s',t}))^{-1} ds' \qquad H^\omega_{s,0}(\boldsymbol{X}) = 0$$

Tail Path

$\boldsymbol{x}^{s,t} : [0,1] \to \mathbb{R}^d$



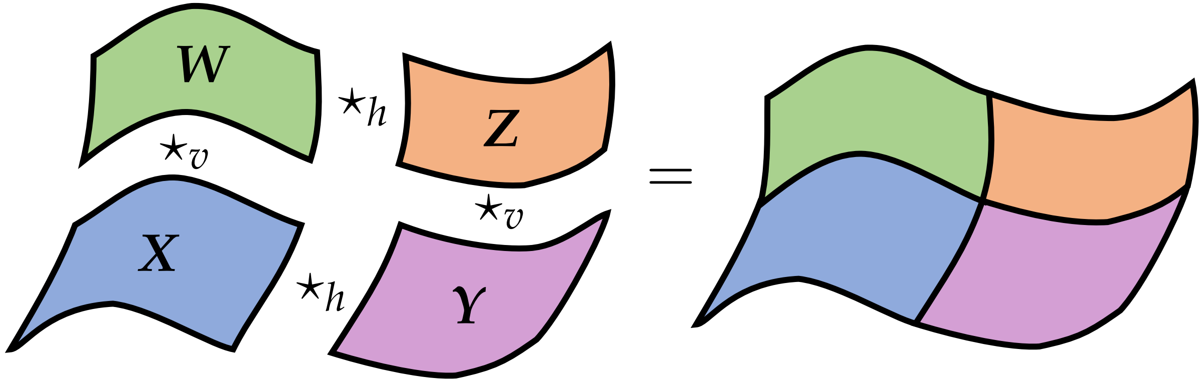
$$H^\omega(\boldsymbol{X}) := H^\omega_{1,1}(\boldsymbol{X})$$

Theorem [Martins, Picken '11]: $\boldsymbol{H} : \boldsymbol{\Pi} \to \mathbf{D}(\mathbf{GL}^{n,m,p})$ is a functor between double groupoids.

1. H is invariant under thin homotopy of surfaces.
2. It preserves horizontal and vertical compositions.

J. Faria Martins & Roger Picken, *Surface holonomy for non-abelian 2-bundles via double groupoids,* Advances in Mathematics, 2011.

Universal and Characteristic Surface Features



$\mathcal{X} \subset C([0,1]^2, \mathbb{R}^d)$

$(\mathcal{G}, \odot_h, \odot_v)$ is a double group

$$\Phi\begin{pmatrix} \mathbf{W} & \star_h & \mathbf{Z} \\ \star_v & & \star_v \\ \mathbf{X} & \star_h & \mathbf{Y} \end{pmatrix} = \begin{matrix} \Phi(\boldsymbol{W}) \odot_h \Phi(\boldsymbol{Z}) \\ \odot_v \qquad\qquad \odot_v \\ \Phi(\boldsymbol{X}) \odot_h \Phi(\boldsymbol{Y}) \end{matrix}$$

$$\overline{\mathbf{X}}_{s,t} = (s, t, \boldsymbol{X}_{s,t}) : [0,1]^2 \to \mathbb{R}^{d+2}$$

Main Results [L., Oberhauser '23]

1. **Nonsmooth:** Generalize surface holonomy to bounded controlled p-variation surfaces (p < 2)
2. **Functorial:** It is compatible with horizontal and vertical concatenation of surfaces.
3. **Computable:** Computational methods for piecewise linear surfaces.
4. **Universal:** The span of the following exponentials of linear functionals is dense in $C_b(C^{p-\mathrm{cvar}}([0,1]^2, \mathbb{R}^d), \mathbb{C})$

$$\{\exp(i\langle \ell, H^\omega(\bar{\cdot})\rangle) : \omega = (\alpha, \beta, \gamma) \text{ 2-connection in } \mathfrak{gl}^{n,m,p}, \ell \in \mathfrak{gl}_2^{n,m,p}\}$$

5. **Characteristic:** If $\mu, \nu \in \mathcal{P}(C^{p-\mathrm{var}}([0,1]^2, \mathbb{R}^d))$ such that $\mu \neq \nu$, there exists a 2-connection ω and ℓ such that

$$\mathrm{Law}_{\mathbf{X}\sim\mu}\langle \ell, H^\omega(\overline{\mathbf{X}})\rangle \neq \mathrm{Law}_{\mathbf{Y}\sim\nu}\langle \ell, H^\omega(\overline{\mathbf{Y}})\rangle$$

Ongoing Work and Conclusion

Surface holonomy provides structure-preserving feature maps for images / surfaces analogous to parallel transport for paths / time series.

Ongoing Work:

- **Machine Learning:** How can we adapt path signature methodology to the setting of images?
- **Universal Surface Holonomy:** Develop the analogue of the path signature for surfaces.
- **Unparametrized Surfaces:** Does surface holonomy characterize thin homotopy classes of surfaces?
- **Rough Surfaces:** How can we go beyond the Young regime?

Thank you!

Preprint: Lee, Oberhauser, *Random surfaces and higher algebra,* arXiv:2311.08366, 2023