Batalin–Vilkovisky formalism beyond perturbation theory via derived geometry

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This is hard, and problematic if there is a gauge symmetry.

Idea of BV-theory

Look at its **derived critical locus** $\mathbb{R}\mathrm{Crit}(S)$, a derived enhancement of $\mathrm{Crit}(S)$.

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Informally speaking,

$$\mathbb{R}\mathrm{Crit}(S) \;=\; \left\{ \phi \in \begin{matrix}\text{configuration}\\ \text{space}\end{matrix},\ \phi^{+} \in \text{antifields} \;\middle|\; \delta S(\phi) \xrightarrow{\phi^{+}} 0 \right\}$$

E.o.m. not imposed on the nose, but up to something, a 1-simplex.

1.1 BV-theory

Main approaches to make classical (and quantum) BV-theory precise in the literature:

1. *NQP*-**manifolds/**L_∞**-algebroid approach.** [Jurčo, Raspollini, Sämann, Wolf, ...]
   Algebra of classical observables is given by a Poisson dg-Lie algebra of functions on an *NQP*-manifold, i.e. a differential-graded manifold (dg-manifold) equipped with a (-1)-shifted symplectic form. (Equivalently, a symplectic L_∞-algebroid.)

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3. **Factorisation Algebras approach.** [Costello, Gwilliam, Williams, ...]
   Algebra of classical observables is given by the differential-graded $\mathbb{P}_0$-algebra of functions on a (-1)-shifted symplectic formal moduli problem, which is sheaved on spacetime.

Approaches (1) & (3) very close. Approaches (2) & (3) related by [Schenkel, Benini, ...].

1.2 Familiar recipe for BV-theory

- Ingredients:
  1. an L_∞-algebra $\mathfrak{L}$ (*BRST-algebra*),
  2. an element $S \in \mathrm{CE}(\mathfrak{L})$ (*action functional*),

  where:

$$\mathrm{CE}(\mathfrak{L}) \;=\; \big(\mathrm{Sym}\,\mathfrak{L}^\vee[-1],\; \mathrm{d}_{\mathrm{CE}(\mathfrak{L})}\big).$$

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$$T^\vee[-1]\mathfrak{L}[1] \;=\; (\mathfrak{L} \oplus \mathfrak{L}^\vee[-3])[1].$$

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- Since $\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} = 0$, we have a new L_∞-algebra $\mathfrak{Crit}(S)$ given by

$$\mathrm{CE}\big(\mathfrak{Crit}(S)\big) \;\coloneqq\; \big(\mathrm{Sym}(\mathfrak{L}^\vee[-1] \oplus \mathfrak{L}[2]),\;\; Q_{\mathrm{BV}} = \{S_{\mathrm{BV}}, -\}\big).$$

  This is the BV-complex.

Example: Yang-Mills theory

- **BRST algebra**

$$\mathfrak{L}[1] \;=\; \Big(\; \Omega^0(M,\mathfrak{g}) \xrightarrow{\;\mathrm{d}\;} \Omega^1(M,\mathfrak{g}) \;\Big)$$

$$\mathrm{deg} = \qquad -1 \qquad\qquad\qquad\qquad 0$$

$$\begin{aligned} \ell_1(c) &= \mathrm{d}c, \\ \ell_2(c_1,c_2) &= [c_1,c_2]_{\mathfrak{g}}, \\ \ell_2(c,A) &= [c,A]_{\mathfrak{g}}, \end{aligned}$$

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$$\mathfrak{Crit}(S)[1] \;=\; \Big(\; \Omega^0(M,\mathfrak{g}) \xrightarrow{\;\mathrm{d}\;} \Omega^1(M,\mathfrak{g}) \xrightarrow{\;\mathrm{d}\star\mathrm{d}\;} \Omega^{d-1}(M,\mathfrak{g}) \xrightarrow{\;\mathrm{d}\;} \Omega^{d}(M,\mathfrak{g}) \;\Big)$$

$$\mathrm{deg} = \qquad -1 \qquad\qquad\qquad 0 \qquad\qquad\qquad 1 \qquad\qquad\qquad 2$$

$$S_{\mathrm{BV}}(\mathsf{c},\mathsf{A},\mathsf{A}^+,\mathsf{c}^+) \;=\; \int_M \Big(\underbrace{\frac{1}{2}\langle F_{\mathsf{A}}, \star F_{\mathsf{A}}\rangle_{\mathfrak{g}}}_{S} - \underbrace{\langle \mathsf{A}^+, \nabla_{\mathsf{A}}\mathsf{c}\rangle_{\mathfrak{g}} + \frac{1}{2}\langle \mathsf{c}^+, [\mathsf{c},\mathsf{c}]_{\mathfrak{g}}\rangle_{\mathfrak{g}}}_{S_{\mathrm{BRST}}} \Big).$$

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$$\begin{aligned} \ell_1(c) &= \mathrm{d}c\,, \\ \ell_2(c_1,c_2) &= [c_1,c_2]_{\mathfrak{g}}\,, \\ \ell_2(c,A) &= [c,A]_{\mathfrak{g}}\,, \end{aligned}$$

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$$\begin{aligned} \ell_1(c) &= \mathrm{d}c\,, & & \\ \ell_1(A) &= \mathrm{d}\star\mathrm{d}A\,, & \ell_1(A^+) &= \mathrm{d}A^+\,, \\ \ell_2(c_1,c_2) &= [c_1,c_2]_{\mathfrak{g}}\,, & \ell_2(c,c^+) &= [c,c^+]_{\mathfrak{g}}\,, \\ \ell_2(c,A) &= [c,A]_{\mathfrak{g}}\,, & \ell_2(c,A^+) &= [c,A^+]_{\mathfrak{g}}\,, \\ & & \ell_2(A,A^+) &= [A \overset{\wedge}{,} A^+]_{\mathfrak{g}}\,, \end{aligned}$$

$$\begin{aligned} \ell_2(A_1,A_2) &= \mathrm{d}\star[A_1 \overset{\wedge}{,} A_2]_{\mathfrak{g}} + [A_1 \overset{\wedge}{,} \star\mathrm{d}A_2]_{\mathfrak{g}} + [A_2 \overset{\wedge}{,} \star\mathrm{d}A_1]_{\mathfrak{g}}\,, \\ \ell_3(A_1,A_2,A_3) &= \big[A_1 \overset{\wedge}{,} \star[A_2 \overset{\wedge}{,} A_3]_{\mathfrak{g}}\big]_{\mathfrak{g}} + \big[A_2 \overset{\wedge}{,} \star[A_3 \overset{\wedge}{,} A_1]_{\mathfrak{g}}\big]_{\mathfrak{g}} + \big[A_3 \overset{\wedge}{,} \star[A_1 \overset{\wedge}{,} A_2]_{\mathfrak{g}}\big]_{\mathfrak{g}}\,, \end{aligned}$$

1.3 BV-theory as deformation theory

- **Artinian dg-algebra**: finite-dimensional, non-positively graded, dg-commutative algebra $\mathcal{R}$ s.t. it has a unique maximal differential ideal $\mathfrak{m}_{\mathcal{R}}$ which is nilpotent and $\mathcal{R}/\mathfrak{m}_{\mathcal{R}} \cong \mathbb{R}$.

- **Formal Moduli Problem**: (algebraic) stack on Artinian dg-algebras, i.e.

$$F : \mathsf{dgArt}^{\leq 0} \longrightarrow \mathsf{sSet}.$$

- Any formal moduli problem is equivalent to $F \simeq \mathbf{MC}(\mathfrak{g})$, for some L_∞-algebra $\mathfrak{g}$, where

$$\mathbf{MC}(\mathfrak{g}) : \mathcal{R} \longmapsto \mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}_{\mathcal{R}}).$$

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In BV-theory:

$\mathbf{MC}(\mathfrak{Crit}(S))$ is the derived critical locus of the action functional S on $\mathbf{MC}(\mathfrak{L})$

Example: Yang-Mills theory

How does the story go for Yang-Mills theory?

$$\mathbf{MC}\big(\mathfrak{Crit}(S)\big) \;:\; \mathcal{R} \;\longmapsto\; \mathrm{MC}(\mathfrak{Crit}(S) \otimes \mathfrak{m}_{\mathcal{R}})$$

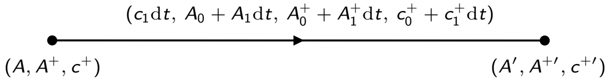
Example: Yang-Mills theory

How does the story go for Yang-Mills theory?

$$\mathbf{MC}(\mathfrak{Crit}(S)) \;:\; \mathcal{R} \longmapsto \mathrm{MC}(\mathfrak{Crit}(S)\otimes \mathfrak{m}_{\mathcal{R}})$$

$$\mathrm{MC}(\mathfrak{Crit}(S)\otimes \mathfrak{m}_{\mathcal{R}})_0 = \left\{ \begin{array}{ll} A & \in \Omega^1(M,\mathfrak{g})\otimes \mathfrak{m}_{\mathcal{R},0} \\ A^+ & \in \Omega^{d-1}(M,\mathfrak{g})\otimes \mathfrak{m}_{\mathcal{R},-1} \\ c^+ & \in \Omega^{d}(M,\mathfrak{g})\otimes \mathfrak{m}_{\mathcal{R},-2} \end{array} \;\middle|\; \begin{array}{rl} \nabla_A \star F_A & = \mathrm{d}_{\mathcal{R}} A^+ \\ \nabla_A A^+ & = \mathrm{d}_{\mathcal{R}} c^+ \end{array} \right\},$$

$$\mathrm{MC}(\mathfrak{Crit}(S)\otimes \mathfrak{m}_{\mathcal{R}})_1 = \left\{ \begin{array}{ll} c_1\mathrm{d}t & \in \Omega^0(M,\mathfrak{g})\otimes \mathfrak{m}_{\mathcal{R},0}\otimes \Omega^1([0,1]) \\ A_0 & \in \Omega^1(M,\mathfrak{g})\otimes \mathfrak{m}_{\mathcal{R},0}\otimes \Omega^0([0,1]) \\ A^1\mathrm{d}t & \in \Omega^1(M,\mathfrak{g})\otimes \mathfrak{m}_{\mathcal{R},-1}\otimes \Omega^1([0,1]) \\ A_0^+ & \in \Omega^{d-1}(M,\mathfrak{g})\otimes \mathfrak{m}_{\mathcal{R},-1}\otimes \Omega^0([0,1]) \\ A_1^+\mathrm{d}t & \in \Omega^{d-1}(M,\mathfrak{g})\otimes \mathfrak{m}_{\mathcal{R},-2}\otimes \Omega^1([0,1]) \\ c_0^+ & \in \Omega^{d}(M,\mathfrak{g})\otimes \mathfrak{m}_{\mathcal{R},-2}\otimes \Omega^0([0,1]) \\ c_1^+\mathrm{d}t & \in \Omega^{d}(M,\mathfrak{g})\otimes \mathfrak{m}_{\mathcal{R},-3}\otimes \Omega^1([0,1]) \end{array} \;\middle|\; \begin{array}{rl} \nabla_{A_0}\star F_{A_0} & = \mathrm{d}_{\mathcal{R}} A_0^+ \\ \nabla_{A_0} A_0^+ & = \mathrm{d}_{\mathcal{R}} c_0^+ \\ \frac{\mathrm{d}}{\mathrm{d}t}A_0 + \nabla_{A_0} c_1 & = \mathrm{d}_{\mathcal{R}} A_1 \\ \frac{\mathrm{d}}{\mathrm{d}t}A_0^+ + \nabla_{A_0}\star F_{A_1} + & \\ + [c_1, A_0^+] & = \mathrm{d}_{\mathcal{R}} A_1^+ \\ \frac{\mathrm{d}}{\mathrm{d}t}c_0^+ + \nabla_{A_0} A_1^+ + & \\ + [c_1, c_0^+] & = \mathrm{d}_{\mathcal{R}} c_1^+ \end{array} \right\},$$



1.4 Global BV-theory?

Formal Moduli Problem: (algebraic) derived stack on Artinian dg-algebras, i.e.

$$F : \mathsf{dgArt}^{\leq 0} \longrightarrow \mathsf{sSet}$$

Artinian dg-algebras $\simeq$ algebras of function on "derived thickened points".

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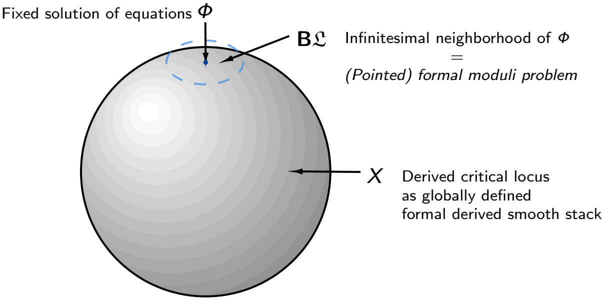
A (-1)-symplectic Formal Moduli Problem can be seen as the formal completion of a fully-fledged (-1)-symplectic derived stack at some given point.



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We have the following picture:

$$\text{Formal Moduli Problem} \longleftrightarrow \text{Perturbative physics}$$
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In fact, in terms of **configuration spaces**:

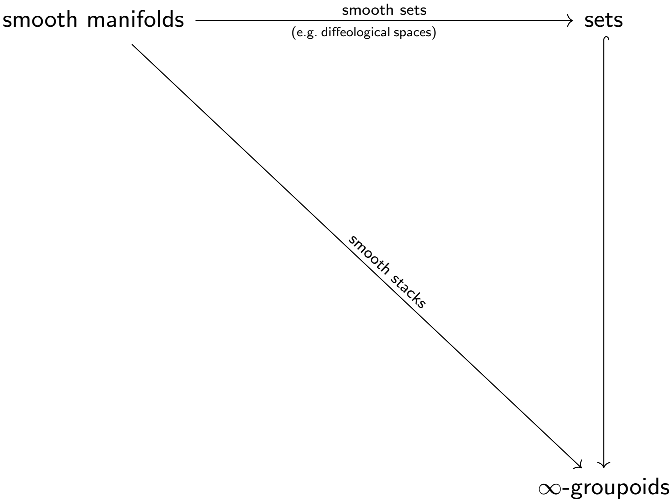
Example (Stack of G-bundles with connection)

$$\underbrace{[\Omega^1(M,\mathfrak{g})/\mathcal{C}^\infty(M,\mathfrak{g})]}_{L_\infty\text{-algebroid}} \quad \neq \quad \underbrace{\mathbf{Bun}_G^\nabla(M)}_{\text{stack of } G\text{-bundles}} := [M, \mathbf{B}G_{\mathrm{conn}}]$$

Physics includes (higher) gauge theories

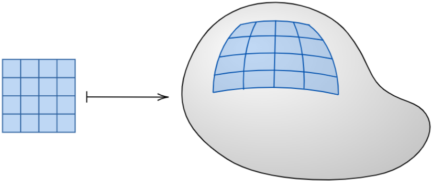
- Quantisation requires BV-theory, i.e. **derived geometry**
- Finite (higher) gauge transformations and global properties require stacks, i.e. **higher geometry** (e.g. Aharonov-Bohm phase and magnetic charge for electromagnetic field)

1.5 Smooth stacks



1.6 Global BRST formalism

- An ordinary geometric space can be encoded by its **functor of points**, which is an ordinary sheaf.



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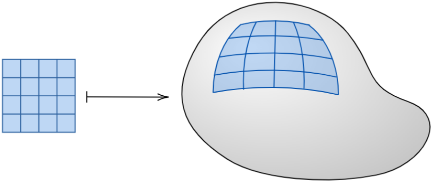

- A higher geometric space can be defined as a **stack**, which is a functor

$$X : \mathsf{Mfd}^{\mathrm{op}} \longrightarrow \mathsf{sSet}$$

satisfying a higher sheaf condition, i.e. it is an element of

$$\mathbf{SmoothStack} := [\mathsf{Mfd}^{\mathrm{op}}, \mathsf{sSet}]^{\circ}_{\mathsf{proj,loc}}.$$

$\Longrightarrow$ geometry encompassing **gauge principle** from physics.

1.5 Global BRST formalism

Now, let us go back to smooth stacks.

Moduli stack of principal G-bundles:

$$\mathbf{B}G = [*/G]$$

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Let M be a smooth manifold and $\coprod_{\alpha \in I} V_\alpha \twoheadrightarrow M$ be a good open cover of it.

A map $M \xrightarrow{g_{\alpha\beta}} \mathbf{B}G$ is given by a G-bundle $\{g_{\alpha\beta} \in \mathcal{C}^\infty(V_\alpha \cap V_\beta, G) \,|\, g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}\}$.

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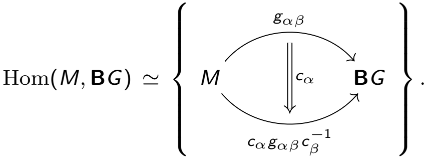

Smooth stack of principal G-bundles on the manifold M:

$$\mathbf{Bun}_G(M) := [M, \mathbf{B}G].$$

1.6 Global BRST formalism

Smooth stack $\mathbf{Bun}_G^{\nabla}(M)$ of principal G-bundles with connection:

$$\mathrm{Hom}\big(U,\, \mathbf{Bun}_G^{\nabla}(M)\big) \;\simeq\; \mathrm{cosk}_2\left(Z_2 \; \overset{\left(c_\alpha,\, {g_{\alpha\beta},A_\alpha \atop g'_{\alpha\beta},A'_\alpha}\right)}{\longrightarrow} \; \overset{\left(c'_\alpha,\, {g'_{\alpha\beta},A'_\alpha \atop g''_{\alpha\beta},A''_\alpha}\right)}{\longrightarrow} \; \underset{\left(c'_\alpha\cdot c_\alpha,\, {g_{\alpha\beta},A_\alpha \atop g''_{\alpha\beta},A''_\alpha}\right)}{\longrightarrow} \; Z_1 \; \overset{(g_{\alpha\beta},A_\alpha)}{\longrightarrow} \; \underset{(g'_{\alpha\beta},A'_\alpha)}{\longrightarrow} \; Z_0 \right),$$

where:

$$Z_0 \;=\; \left\{ \begin{array}{ll} g_{\alpha\beta} & \in \mathcal{C}^\infty(V_\alpha \cap V_\beta \times U, G) \\ A_\alpha & \in \Omega^1_{\mathrm{ver}}(V_\alpha \times U, \mathfrak{g}) \end{array} \;\middle|\; \begin{array}{l} g_{\alpha\beta}\cdot g_{\beta\gamma}\cdot g_{\gamma\alpha} = 1 \\ A_\alpha = g_{\beta\alpha}^{-1}(A_\beta + \mathrm{d})g_{\beta\alpha} \end{array} \right\},$$

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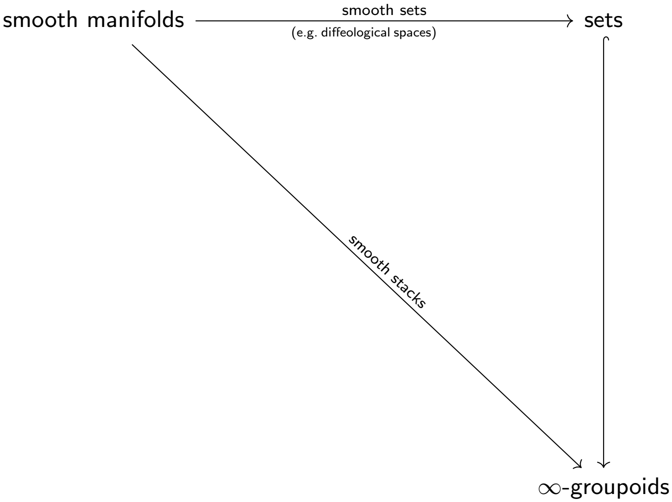
$Z_2 \;=\; \{$compositions of gauge transformations$\}$,

2.1 Family tree of smooth stacks



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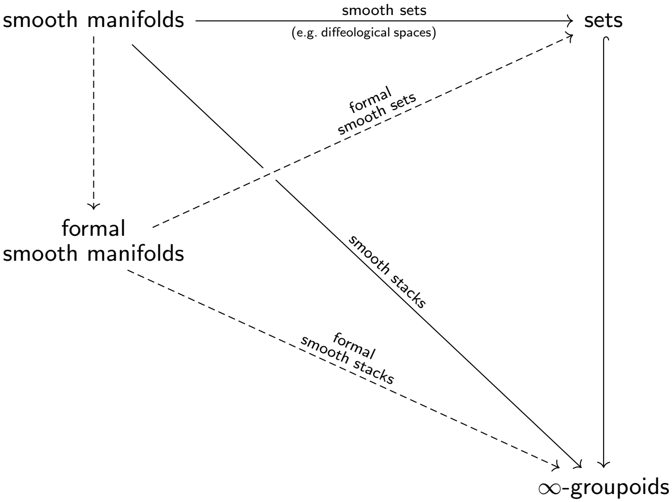

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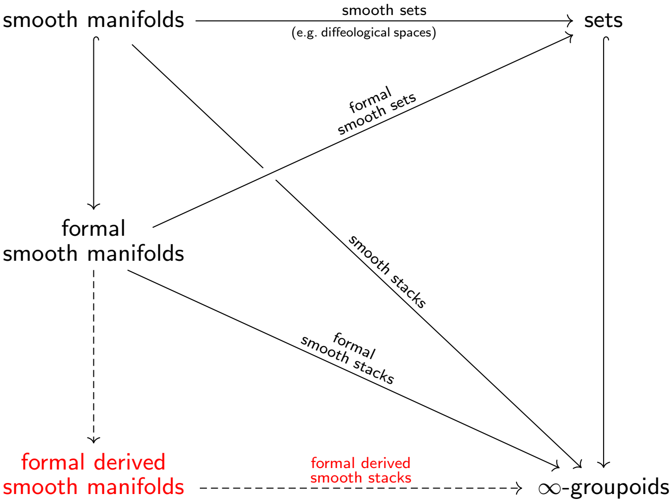

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2.3 Derived smooth manifolds

- **Observation**: given manifolds $M, N \hookrightarrow B$, the intersection $M \cap N := M \times_B N$ is not generally well-defined in Mfd.

- **Solution**: derived smooth manifolds [Spivak, Joyce, Carchedi, Steffens, ...].

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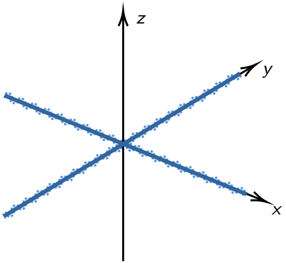
- **Observation**: given manifolds $M, N \hookrightarrow B$, the intersection $M \cap N \coloneqq M \times_B N$ is not generally well-defined in Mfd.
- **Solution**: derived smooth manifolds [Spivak, Joyce, Carchedi, Steffens, ...].

Use the natural embedding:

$$i \ : \ \mathbf{N}(\mathrm{Mfd}) \longrightarrow \mathbf{dMfd}$$

The derived intersection always exists in the $(\infty, 1)$-category **dMfd**:

$$\begin{array}{ccc} i(M) \times^h_{i(B)} i(N) & \longrightarrow & i(M) \\ \downarrow & & \downarrow{\scriptstyle i(f)} \\ i(N) & \xrightarrow{i(g)} & i(B). \end{array}$$



For example, in $B = \mathbb{R}^3$,
$M = \{(x, y, x^2y^2) \in \mathbb{R}^3 \,|\, (x, y) \in \mathbb{R}^2\}$
and $N = x, y$-plane.

2.2 $\mathcal{C}^\infty$-algebras

Let CartSp be the category of Cartesian spaces $\{\mathbb{R}^n\}_{n\in\mathbb{N}}$ and smooth maps between them.

This is a Lawvere theory, as any object is such that $\mathbb{R}^n \cong \mathbb{R} \times \cdots \times \mathbb{R}$.

Definition

A $\mathcal{C}^\infty$-**algebra** is a product-preserving functor

$$A \,:\, \mathsf{CartSp} \longrightarrow \mathsf{Set}.$$

Example

Let $M \in \mathsf{Mfd}$ be a smooth manifold.

$$\mathcal{C}^\infty(M) \,:\, \mathbb{R}^n \longmapsto \mathcal{C}^\infty(M, \mathbb{R}^n)$$

There is a natural embedding:

$$\mathsf{Mfd} \hookrightarrow \mathsf{C}^\infty\mathsf{Alg}^{\mathrm{op}}$$

2.3 Derived smooth manifolds

Homotopy $\mathcal{C}^\infty$-algebras: simplicial $\mathcal{C}^\infty$-algebras with projective model structure, i.e.

$$\mathbf{hC^\infty Alg} \;\coloneqq\; \mathbf{N}_{hc}([\Delta^{\mathrm{op}}, \mathsf{C}^\infty\mathsf{Alg}]^\circ_{\mathrm{proj}}),$$

where Δ is the simplex category.

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The following will be our effective definition of formal derived manifolds.

Theorem [Carchedi, Steffens 2019]

There is a canonical equivalence of $(\infty, 1)$-categories

$$\mathbf{dMfd} \;\simeq\; \mathbf{hC^\infty Alg}^{\mathrm{op}}_{\mathrm{fp}}$$

between the $(\infty, 1)$-category of derived manifolds, and the opposite of the $(\infty, 1)$-category of homotopically finitely presented homotopy $\mathcal{C}^\infty$-algebras.

At an intuitive level, $U \in \mathbf{dMfd}$ is a geometric object whose algebra of smooth function is a homotopically finitely presented homotopy $\mathcal{C}^\infty$-algebra modelled as

$$\mathcal{O}(U) \;=\; \Big(\;\cdots\; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{O}(U)_3 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathcal{O}(U)_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathcal{O}(U)_1 \;\substack{\longrightarrow\\\longrightarrow}\; \mathcal{O}(U)_0 \Big)$$

where each $\mathcal{O}(U)_i$ is an ordinary $\mathcal{C}^\infty$-algebra.

2.4 Formal derived smooth manifolds

Derived smooth manifold do not include objects like

$$\mathrm{Spec}\left(\frac{\mathcal{C}^\infty(\mathbb{R})}{(x^2)}\right)$$

Definition

A homotopy $\mathcal{C}^\infty$-algebra A is finitely generated if $\pi_0 A$ is finitely generated as an ordinary $\mathcal{C}^\infty$-algebra, i.e. such that $\pi_0 A \cong \mathcal{C}^\infty(\mathbb{R}^n)/\mathcal{I}$.

Let $\mathbf{sC^\infty Alg}_{\mathrm{fg}}$ be the $(\infty, 1)$-category of finitely generated homotopy $\mathcal{C}^\infty$-algebras.

Definition

We define the $(\infty, 1)$-category of formal derived smooth manifolds by

$$\mathbf{dFMfd} \;:=\; \mathbf{sC^\infty Alg}_{\mathrm{fg}}^{\mathrm{op}}.$$

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3.1 Formal derived smooth stacks

- We can define étale maps of formal derived smooth manifolds so that they truncate to ordinary étale maps (they generalise local diffeomorphisms of ordinary manifolds).
- By using étale maps, we can make dFMfd into a $(\infty, 1)$-site.
- By [Toen, Vezzosi 2006], we can define formal derived smooth stacks by

$$\textbf{dFSmoothStack} \;\coloneqq\; [\mathsf{dFMfd}^{\mathrm{op}}, \mathsf{sSet}]^{\circ}_{\mathsf{proj,loc}}.$$

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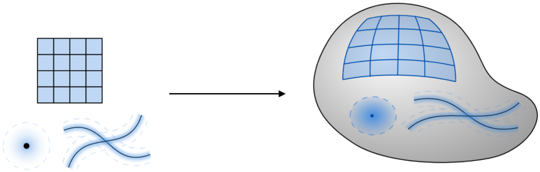

One has a natural (coreflective) embedding

$$\mathbf{dFSmoothStack} \underset{t_0}{\overset{\longleftarrow}{\longrightarrow}} \mathbf{SmoothStack}.$$

[See Carchedi's and Steffens' current foundational work on derived differential geometry.]

3.2 Derived differential geometry

dFSmoothStack comes with differential structure, as defined in [Schreiber 2013]

de Rham stack

Given a formal derived smooth stack X, define

$$\mathfrak{I}(X) : R \longmapsto X(R^{\mathrm{red}}) \quad \text{with} \quad R^{\mathrm{red}} := \pi_0 R/\mathfrak{m}_{\pi_0 R}.$$

There is a natural map

$$\mathfrak{i}_X : X \longrightarrow \mathfrak{I}(X)$$

Similarly to [Khavkine, Schreiber], the differential structure can be used to deal with infinitesimal geometry.

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dFSmoothStack comes with differential structure, as defined in [Schreiber 2013]

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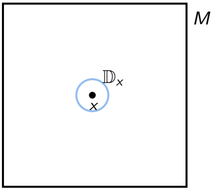
There is a natural map

$$\mathfrak{i}_X : X \longrightarrow \Im(X)$$

Similarly to [Khavkine, Schreiber], the differential structure can be used to deal with infinitesimal geometry.

A derived infinitesimal disks at $x \in X$ is defined by the pullback

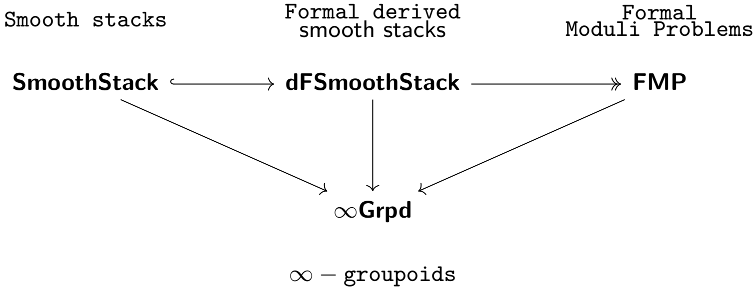
$$\begin{array}{ccc} \mathbb{D}_x & \xhookrightarrow{\iota_x} & M \\ \downarrow & & \downarrow \mathfrak{i}_X \\ * & \xhookrightarrow{x} & \Im(X) \end{array}$$



where $\Im(X)$ is the de Rham stack of X.

3.3 Formal moduli problems as infinitesimal cohesion

Let **FMP** be the $(\infty, 1)$-category of Formal Moduli Problems, which can be seen as formal derived stacks on derived infinitesimal disks.



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4.2 Derived critical locus of Yang-Mills action

- Yang-Mills action functional as morphism of smooth stacks

$$\begin{aligned} S \;:\; \mathbf{Bun}_G^{\nabla}(M) \;&\longrightarrow\; \mathbf{Dens}_M \\ (g_{\alpha\beta},\, A_\alpha) \;&\longmapsto\; \frac{1}{2}\langle F_A \overset{\wedge}{,}\, \star F_A\rangle_{\mathfrak{g}} \end{aligned}$$

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- Yang-Mills e.o.m as morphism of formal derived smooth stacks

$$\begin{aligned} \delta S \;:\; \mathbf{Bun}_G^{\nabla}(M) \;&\longrightarrow\; T^{\vee}_{\mathrm{res}}\mathbf{Bun}_G^{\nabla}(M) \\ (g_{\alpha\beta},\, A_\alpha) \;&\longmapsto\; (g_{\alpha\beta},\, A_\alpha,\, \nabla_{A_\alpha}\star F_{A_\alpha},\, 0), \end{aligned}$$

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- Construct the derived critical locus of Yang-Mills action

$$\begin{array}{ccc} \mathbb{R}\mathrm{Crit}(S)(M) & \longrightarrow & \mathbf{Bun}_G^{\nabla}(M) \\ \big\downarrow & & \big\downarrow{\scriptstyle \delta S} \\ \mathbf{Bun}_G^{\nabla}(M) & \xrightarrow{\;0\;} & T^{\vee}_{\mathrm{res}}\mathbf{Bun}_G^{\nabla}(M), \end{array}$$

4.3 Non-perturbative classical BV-BRST theory

$$\mathbb{R}\mathrm{Hom}\big(U,\, \mathbb{R}\mathrm{Crit}(S)(M)^{\mathrm{pre}}\big) \;\simeq$$

$$\simeq \left(\cdots \;\; \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \;\; Z_2 \;\; \begin{array}{c} \xrightarrow{\left(c_\alpha,\, g_{1,\alpha\beta},\, h_{1,\alpha},\, \substack{g_{\alpha\beta},\, A_\alpha,\, A^+_\alpha,\, c^+_\alpha \\ g'_{\alpha\beta},\, A'_\alpha,\, A^{+\prime}_\alpha,\, c^{+\prime}_\alpha}\right)} \\ -\left(c'_\alpha,\, g'_{1,\alpha\beta},\, h'_{1,\alpha},\, \substack{g'_{\alpha\beta},\, A'_\alpha,\, A^{+\prime}_\alpha,\, c^{+\prime}_\alpha \\ g''_{\alpha\beta},\, A''_\alpha,\, A^{+\prime\prime}_\alpha,\, c^{+\prime\prime}_\alpha}\right)\!\rightarrow \\ \xrightarrow[\left(c''_\alpha,\, g''_{1,\alpha\beta},\, h''_{1,\alpha},\, \substack{g_{\alpha\beta},\, A_\alpha,\, A^+_\alpha,\, c^+_\alpha \\ g''_{\alpha\beta},\, A''_\alpha,\, A^{+\prime\prime}_\alpha,\, c^{+\prime\prime}_\alpha}\right)]{} \end{array} \;\; Z_1 \;\; \begin{array}{c} \xrightarrow{(g_{\alpha\beta},\, A_\alpha,\, A^+_\alpha,\, c^+_\alpha)} \\ \xrightarrow[(g'_{\alpha\beta},\, A'_\alpha,\, A^{+\prime}_\alpha,\, c^{+\prime}_\alpha)]{} \end{array} \;\; Z_0 \right)$$

- 0-simplices:
  - $g_{\alpha\beta}$ transition functions,
  - A_α connection,
  - A^+_α equations of motion,
  - c^+_α Noether identities,
- 1-simplices:
  - c_α gauge transformations,
  - $g_{1,\alpha\beta}$ homotopies of transition functions,
  - $A_{1,\alpha}$ homotopies of connections,
  - $A^+_{1,\alpha}$ homotopies of equations of motions,
  - $c^+_{1,\alpha}$ homotopies of Noether identities,
- $(n \geq 2)$-simplices: compositions of gauge transformations and homotopies of homotopies.

4.4 Recovering usual BV-BRST theory

- Use the de Rham stack to obtain the derived infinitesimal disk of $\mathbb{R}\mathrm{Crit}(S)$ at fixed solution $(P, \nabla_A) \in \mathbb{R}\mathrm{Crit}(S)$ of the e.o.m

$$\begin{array}{ccc} \mathbb{D}_{\mathbb{R}\mathrm{Crit}(S)(M),(P,\nabla_A)} & \longrightarrow & \mathbb{R}\mathrm{Crit}(S)(M) \\ \big\downarrow & & \big\downarrow {\scriptstyle \mathfrak{i}_{\mathbb{R}\mathrm{Crit}(S)(M)}} \\ * & \xrightarrow{(P,\nabla_A)} & \Im\big(\mathbb{R}\mathrm{Crit}(S)(M)\big). \end{array}$$

- Obtain L_∞-algebra with underlying complex

$$\overrightarrow{\mathfrak{Crit}}(S)_{(P,\nabla_A)}[1] \ = \ \Big(\underset{-1}{\Omega^0(M,\mathfrak{g}_P)} \xrightarrow{\nabla_A} \underset{0}{\Omega^1(M,\mathfrak{g}_P)} \xrightarrow{\nabla_A \star \nabla_A} \underset{1}{\Omega^{d-1}(M,\mathfrak{g}_P)} \xrightarrow{\nabla_A} \underset{2}{\Omega^{d}(M,\mathfrak{g}_P)}\Big)$$

deg = −1 0 1 2

with expected bracket structure.

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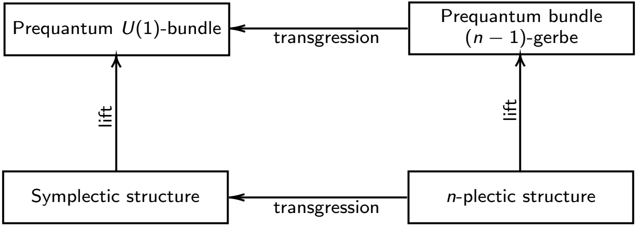
Outlook: the puzzle of quantisation

***n*-plectic geometry** (or higher symplectic geometry) [Rogers, Baez, Saemann, Szabo, Bunk, Fiorenza, Schreiber, Sati, ...] naturally fits in the following picture:



Example (Closed string)

[Waldorf 2009]: transgression of a bundle gerbe on a smooth manifold M to a principal $U(1)$-bundle on the loop space $\mathcal{L}M = [S^1, M]$.

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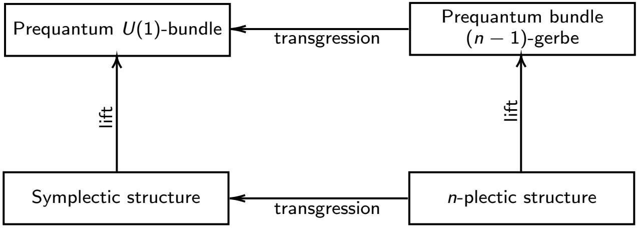

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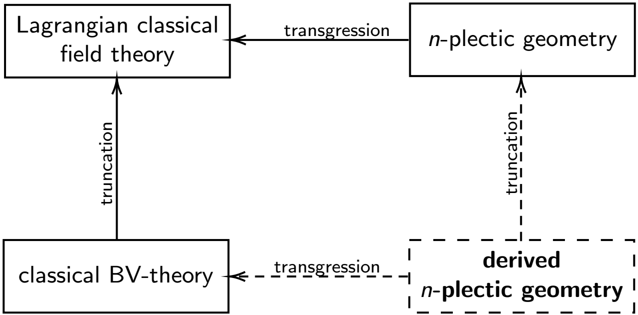
- [Ševera 2000]: Courant 2-algebroid and Vinogradov n-algebroid are higher generalisations of the Poisson 1-algebroid (as symplectic L_∞-algebroids).
- [Rogers 2011], [Sämann, Ritter 2015]: relation between the L_∞-algebras of observables on n-plectic manifolds and Vinogradov n-algebroids.

Outlook: the puzzle of quantisation

Topic of current research: **derived** ***n*****-plectic geometry**.



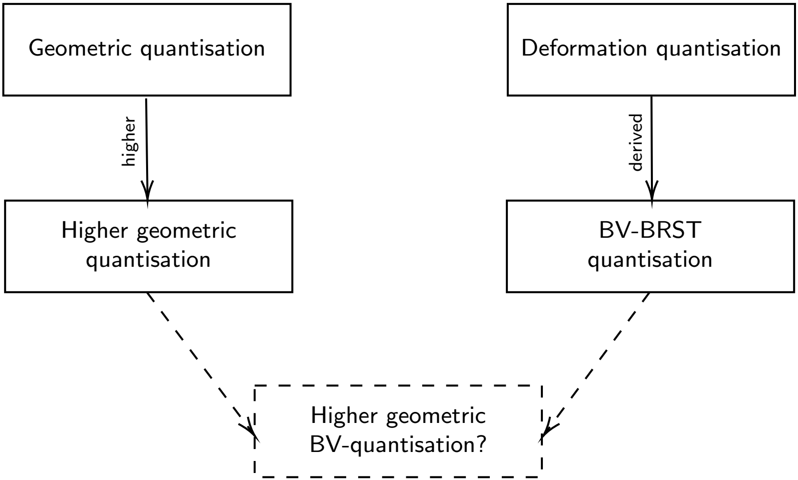
Framework to make contact with BV-BFV theory, possibly unify:

- (-1)-shifted 2-form attached to M (*BV-form*),
- 0-shifted 2-form attached to ∂M (*BFV-form*).

Outlook: the puzzle of quantisation

- Global geometric (non-perturbative)
- Focus on space of states

- Local geometric (perturbative)
- Focus on algebra of observables



Outlook: the puzzle of quantisation

- Setting to go beyond perturbative BV-BRST theory
  - Usually one would consider $\Omega^*(X, \mathfrak{g})$ with L_∞-structure and take shifted cotangent bundle $T^*[-1]\Omega^*(X, \mathfrak{g})$
  - We can consider $\mathbf{Bun}_G^\nabla(X) := [X, \mathbf{B}G_{\mathrm{conn}}]$ (or some concretification of this), and take derived critical locus $\mathbb{R}\mathrm{Crit}(S)(M)$ for a given $S : \mathbf{Bun}_G^\nabla(X) \to \mathbb{R}$

$\Longrightarrow$ Global geometric generalisation of BV-BRST theory

- Setting to go beyond BV-quantisation
  - [Bunk, Sämann, Szabo], [Fiorenza, Sati, Schreiber]: higher geometric prequantisation of n-plectic structures and prequantum bundle n-gerbes
  - [Safronov]: geometric quantisation of derived symplectic structures in derived algebraic geometry via algebraic bundle k-gerbes

$\Longrightarrow$ Beyond BV-quantisation by "higher derived" geometric (pre)quantisation?

Thank you for your attention!