Functorial Field Theory: Homework 1

Notation.

- Cart: the cartesian site. Objects: $\mathbf{R}^n$, morphisms: smooth maps, covering families: good open covers.
- $\mathbf{L}$: the site of smooth loci. Objects: finitely generated C^∞-rings, i.e., $\mathsf{C}^\infty(\mathbf{R}^n)/I$. Morphisms: C^∞-homomorphisms in the opposite direction. Covering families are induced by open covers of $\mathbf{R}^n$.
- Ω^n, $\Omega^n_{\mathsf{closed}}$: the presheaves of sets of Cart that send an object S to the set of differential n-forms (respectively closed n-forms) on S and send a map $f\colon S \to S'$ to the pullback map on forms.
- $\mathsf{B}G$ (G is a Lie group): the presheaf of groupoids on Cart that sends an object S to the groupoid with one object whose automorphism group is $\mathsf{C}^\infty(S, G)$; a map $f\colon S \to S'$ is sent to the induced functor of groupoids given by precomposition with f on morphisms.
- $\mathsf{B}_\nabla G$: same as $\mathsf{B}G$, but S is sent to the groupoid of trivial G-bundles with connection on S.
- $\mathbf{T}$: an object of $\mathbf{L}$ corresponding to the C^∞-ring $\mathbf{R}[x]/(x^2)$.
- $\mathsf{Hom}(-,-)$: internal hom in sheaves of sets or sheaves of groupoids.
- $\mathsf{RHom}(-,-)$: derived internal hom in simplicial presheaves.
- Facts about RHom: it preserves homotopy limits in the second argument and maps homotopy colimits in the first argument to homotopy limits. If the target is a sheaf of sets, RHom can be computed as the ordinary Hom. If the target is a

Problems. (Questions about the statements are welcome.) (Partial solutions and attempts at partial solutions are welcome.)

1. Compute $\mathbf{L}(\mathbf{T}, M)$ for a smooth manifold M and show that it is isomorphic to the set of tangent vectors of M. Working in sheaves of sets on $\mathbf{L}$, compute $\mathsf{Hom}(\mathbf{T}, M)$ and show that it is isomorphic to the total space of the tangent bundle of M as a smooth manifold. Explain how to recover the algebraic operations (e.g., addition and multiplication by a scalar) on the tangent bundle. Reference: Nestruev, Smooth manifolds and observables, Second Edition, Chapter 9.
2. Working in sheaves of sets on $\mathbf{L}$, compute $\mathsf{Hom}(\mathbf{T}, \mathsf{Hom}(M, N))$, where M and N are smooth manifolds. Express your answer in terms of vector fields along smooth maps.
3. Working in sheaves of groupoids on the site $\mathbf{L}$, compute $\mathsf{Hom}(\mathbf{T}, \mathsf{B}G)$, where G is a Lie group.
4. Using the algebraic description of smooth vector fields on a smooth manifold as derivations, explain how to define a sheaf of sets Ω^n (respectively $\Omega^n_{\mathsf{closed}}$) on the site $\mathbf{L}$. (Look up the notion of a C^∞-derivation.)
5. Working in sheaves of sets on $\mathbf{L}$, compute $\mathsf{Hom}(\mathbf{T}, \Omega^n)$ and $\mathsf{Hom}(\mathbf{T}, \Omega^n_{\mathsf{closed}})$.
6. Look up homotopy pullbacks of chain complexes. Working in presheaves of chain complexes on Cart, use de Rham's theorem to show that $\mathsf{B}^{n-1}_\nabla \mathrm{U}(1)$ is the homotopy pullback of $\Omega^n_{\mathsf{closed}}[0] \to \mathbf{R}[n] \leftarrow \mathbf{Z}[n]$. (You can concentrate on the case $n = 2$, where $\mathsf{B}^{n-1}_\nabla \mathrm{U}(1) = \mathsf{B}_\nabla \mathrm{U}(1)$ is the chain complex $\Omega^1 \leftarrow \mathsf{C}^\infty(-, \mathrm{U}(1))$. More generally, for $n > 2$, we have $\Omega^{n-1} \leftarrow \cdots \leftarrow \Omega^1 \leftarrow \mathsf{C}^\infty(-, \mathrm{U}(1))$.)
7. Look up Palais's theorem on natural operations on differential forms. Working in sheaves of groupoids on Cart, use the previous problem to compute $\mathsf{RHom}(\Omega^1, \mathsf{B}G)$ and $\mathsf{RHom}(\Omega^1, \mathsf{B}_\nabla G)$, assuming $G = \mathrm{U}(1)$ for simplicity. What if Ω^1 is replaced by $\Omega^1_{\mathsf{closed}}$? How about Ω^n or $\Omega^n_{\mathsf{closed}}$? Bonus points for figuring out the case of nonabelian G.
8. Look up action groupoids (alias homotopy quotients). Working in sheaves of groupoids on Cart, show that $\mathsf{B}_\nabla G$ is the action groupoid of G (i.e., its representable presheaf) on the presheaf $\Omega^1 \otimes \mathbf{g}$, where $\mathbf{g}$ is the Lie algebra of G and $\otimes$ denotes the objectwise tensor product over real numbers. (Look up the description of connections on trivial principal G-bundles, e.g., in the book of Kobayashi–Nomizu.)
9. Working in sheaves of groupoids on Cart, use the previous problem and Palais's theorem to compute $\mathsf{RHom}(\mathsf{B}_\nabla G, \Omega^n)$.