Twisted Equivariant Thom classes in Topology & Physics

Texas Tech, 11/28/23

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I. Review of Thom classes

Given data:

1. h a multiplicative cohomology theory
2. $V \to X$ a real vector bundle, $\dim(V) = n$.

A class $\sigma_V \in h^n(V)_c$ is a Thom class if the map

$$- \boxtimes \sigma_V : h^k(X) \longrightarrow h^{k+n}(V)_c \quad \text{is an isomorphism.}$$

[$\boxtimes$ = external product
$h(-)_c$ = cpt support.]

Examples:

0. For any multiplicative cohomology theory h, $\underline{\mathbb{R}}^n = \mathbb{R}^n \times X$ has a Thom class implementing the suspension isomorphism $h^k(X) \xrightarrow[- \boxtimes \sigma_{\underline{\mathbb{R}}^n}]{\sim} h^{k+n}(X \times \mathbb{R}^n)_c \simeq \tilde{h}^{k+n}(\Sigma^n X)$.

1. For $h = H(-;\mathbb{R})$ de Rham cohomology,
   An orientation on $V \to X$ specifies a Thom class, $u_V \in H(V;\mathbb{R})_c$.

2. $h = KO$ real K-theory

A spin structure on $V \to X$ specifies a Thom class $\mathcal{S}_V \in KO(V)_c$

③ $h = TMF$ topological modular forms.

A string structure on $V \to X$ specifies a Thom class $\sigma_V \in TMF(V)_c$.

II Universal Thom classes & equivariant refinements

In examples above, there are universal Thom classes coming from twisted equivariant cohomology of $O(n) \circlearrowright \mathbb{R}^n$.

① $\xi_n \to BSO(n)$ universal oriented n-dim'l vector bundle

$$\begin{array}{ccccc} H(\xi_n;\mathbb{R})_c & \cong & H_{SO(n)}(\mathbb{R}^n)_c & \leftarrow & H^{w_1}_{O(n)}(\mathbb{R}^n)_c \\ \Psi & & & & \Psi \\ u_{\xi_n} & \longleftarrow & & \longmapsto & u_n \end{array}$$

[Mathai-Quillen]

w_1-twist: $O(n) \xrightarrow{\det} \{\pm 1\}$

② $\xi_n \to BSpin(n)$ universal spin n-dim'l vector bundle

$$\begin{array}{ccccc} KO(\xi_n)_c & \longleftarrow & KO_{Spin(n)}(\mathbb{R}^n)_c & \leftarrow & KO^{w_1+w_2}_{O(n)}(\mathbb{R}^n) \\ \Psi & & & & \Psi \\ \mathcal{S}_{\xi_n} & \longleftarrow & & \longmapsto & \mathcal{S}_n \end{array}$$

w_2-twist: $\mathbb{Z}/2 \to Pin(n)$
$\downarrow$
$O(n)$

ξ_n [Karoubi, Freed-Hopkins-Teleman]

Punchlines ① The "most universal" Thom class comes from the twisted $O(n)$-equivariant cohomology / K-theory of $\mathbb{R}^n$ w/ standard $O(n)$-action.

② In these examples, the universal Thom class is determined by representation-theoretic data.

③ $\xi_n \to BString_n$ universal string n-dim'l vector bundle

$$\begin{tikzcd}
\sigma_{\xi_n} \in TMF(BString_n)_c & & \text{“}TMF_{String(n)}(\mathbb{R}^n)_c\text{”} \arrow[ll] \\
\sigma_n \in TMF^{\lambda}_{O(n)}(\mathbb{R}^n)_c \arrow[u, mapsto] \arrow[u, dashed] \arrow[rr] & & TMF^{p_1/2}_{Spin(n)}(\mathbb{R}^n)_c \arrow[u, phantom, "\shortparallel"]
\end{tikzcd}$$

$\frac{p_1}{2}$-twist

$BU(1) \to String(n)$
$\downarrow$
$Spin(n)$

$\lambda = w_1 + w_2 + \frac{p_1}{2}$

(mostly unexplored...)

Hope: The "most universal" TMF Thom class comes from the twisted equivariant TMF of $\mathbb{R}^n \circlearrowleft O(n)$.

Optimism comes from Stolz-Teichner program relating TMF & field theories.

II. Field theories, free fermions & Thom classes in $H(-;\mathbb{R})$

Def: (rough) A classical field theory is

① A space (of fields) $\mathcal{F}$

② An (action) function $S : \mathcal{F} \to \mathbb{C}$

(+ additional locality properties...)

Def: A symmetry is an automorphism $\varphi: \mathcal{F} \to \mathcal{F}$ satisfying $\varphi^* S = S$.

Symmetries form a group K under composition and $K \circlearrowright \mathcal{F}$.

Def: (Path integral) quantization is a map

(*) $\int (\ ) e^{-S} d\phi : \{\text{functions on } \mathcal{F}\}^K \longrightarrow \mathbb{C}$

invariant under symmetries (+ locality properties)

Remarks:

① Don't take the integral sign too seriously... When the symmetry group is large, there are often very few maps (*).

② Supersymmetry turns out to be particularly restrictive, and $\int$ is (almost) algebraic.

③ When $\int$ is only invariant under a subgroup $H \subsetneq K$ there is an anomaly.

Mathai & Quillen's insight

For $\omega \in \mathrm{Lie}(O(n))$ a skew matrix, define the classical theory

Fields: $\mathcal{F} = \Pi\mathbb{R}^n$ ← $\mathbb{R}^n$ as an "odd" vector space

Action: $S_\omega(\psi) = \frac{1}{2}\langle \psi, \omega\psi\rangle \in \Lambda^2 \mathbb{R}^n \subseteq \Lambda^\bullet \mathbb{R}^n =: C^\infty(\Pi\mathbb{R}^n)$

$O(n) \circlearrowright \mathcal{F}$ acts by symmetries.

This is the (0-dimensional) n free fermions.

Define the path integral via Berezin integration:

$$C^\infty(\Pi V) = \Lambda^\bullet \mathbb{R}^n \xrightarrow{\int(\ )d\psi} \mathbb{C}$$

$$\text{proj} \quad \dashrightarrow \quad \Lambda^{top}\mathbb{R}^n \quad (\wr \text{ choice of orientation})$$

Prop: [M-Q] $\int e^{\frac{1}{2}\langle\psi,\omega\psi\rangle}d\psi = \mathrm{Pf}(\omega)$ the Pfaffian.

This has an anomaly: $\mathrm{Pf}(A\omega A^{-1}) = \underbrace{\det(A)}_{\in\{\pm 1\}}\mathrm{Pf}(\omega)$ for $A \in O(n)$.

Sketch: $\langle\psi,\omega\psi\rangle \in \Lambda^2\mathbb{R}^n \subseteq \Lambda^\bullet\mathbb{R}^n$, $e^x = \sum_k \frac{x^k}{k!}$. Compute! ∎

Corollary: $\int e^{\frac{1}{2}\langle\psi,\omega\psi\rangle}d\psi$ represents the universal twisted Euler class, $\mathrm{Euler}_n \in H^{w_1}_{O(n)}(pt)$.

Sketch: $\omega \mapsto \int e^{\frac{1}{2}\langle\psi,\omega\psi\rangle}d\psi$ is a polynomial function on the Lie algebra of $O(n)$ with an equivariance property under the adjoint $O(n)$-action.

Note: $H^{w_1}_{O(n)}(pt) \cong \left(\mathrm{Sym}(\mathrm{Lie}(O(n))^\vee)\otimes\det\right)^{O(n)}$

$$\mathrm{Eu}_n \longleftrightarrow \mathrm{Pf}$$

∎

The n free fermions have an $\mathbb{R}^n$-family of deformations:

$$S_{\omega,v}(\psi) = \frac{1}{2}\langle \psi, \omega\psi\rangle + \langle v, \psi\rangle \qquad v \in \mathbb{R}^n$$

Theorem [M-Q]: $\int e^{\frac{1}{2}\langle\psi,\omega\psi\rangle + \langle v,\psi\rangle} d\psi$ is a cocycle representing the universal twisted Thom class:

$$\left[\int e^{\frac{1}{2}\langle\psi,\omega\psi\rangle + \langle v,\psi\rangle} d\psi\right] = \mathrm{Um} \in H^{\omega_1}_{O(n)}(\mathbb{R}^n)_c .$$

Idea: generalize Mathai-Quillen forms using other QFTs.

III. Free fermions in dimensions 1 & 2:

Fields:
$$\mathcal{F} = \begin{cases} \mathrm{Map}(S^1, \Pi\mathbb{R}^n) & S^1 = \mathbb{R}/\mathbb{Z} \\ \mathrm{Map}(T^2, \Pi\mathbb{R}^n) & T^2 = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \end{cases}$$

Action: $\omega \in \mathrm{Lie}(O(n))$

$$S_\omega = \begin{cases} \int_{S^1} \langle \psi, (\partial_t + \omega)\psi\rangle + \langle v, \psi\rangle \\ \int_{T^2} \langle \psi, (\partial_{\bar z} + \omega)\psi\rangle + \langle v,\psi\rangle \end{cases}$$

$v \in \mathbb{R}^n$

Theorem [B-E] Using zeta regularization to define the paths integral, the function

$$\int e^{S_{WIV}}\, d\psi$$

gives a cocycle representative of the Chern character of the universal twisted equivariant Thom class in KO-theory when $d=1$ & TMF when $d=2$.

(Proof is computational.)

<u>Extending down:</u> Classical field theory w/ boundaries:

$$\mathcal{F} = \begin{cases} \mathrm{Map}(\bullet\!\!-\!\!\!-\!\!\bullet,\ \Pi\mathbb{R}^n) & \text{"world line"} \\ \mathrm{Map}(\quad,\ \Pi\mathbb{R}^n) & \text{"world sheet"} \end{cases}$$

Get an (odd) symplectic mfld of classical solutions, (canonical) quantization constructs a Clifford module.

<u>$d=1$:</u> (Extension of Stolz–Teichner '04)

- n^{th} Clifford algebra $C\ell_n$ is $\otimes$-invertible $\Rightarrow$ fully dualizable, 2-d TFT restricts

$$\begin{tikzcd} 1\text{-Bord}(pt/\!\!/O(n)) \arrow[r, bend left=50, "C\ell_n"] \arrow[r, bend right=50, "\mathbb{1}"'] \arrow[r, phantom, "\Uparrow C\ell_n"] & \mathrm{Alg} \end{tikzcd} \qquad \underset{\circlearrowleft O(n)}{\bullet} \longmapsto \underset{C\ell_n}{}\ \underset{\circlearrowleft O(n)}{C\ell_n}$$

$C\ell_n$ as a $C\ell_n$-module produces a relative/twisted field thy.

Everything has a (projective) $O(n)$-action.

Produces the K-theory Euler class: $[{}_{Cl_n}Cl_n] \in KO^{\text{Witte}}_{O(n)}(pt)$
(Atiyah-Bott-Shapiro)

- This 1-TFT admits an $\mathbb{R}^n$-family of non-topological (supersymmetric) deformations: Hamiltonian at $v \in \mathbb{R}^n$ given by $|v|^2$.

Value on intervals: $\underset{t,\theta}{\bullet\!\!-\!\!-\!\!\bullet} \longmapsto e^{-t|v|^2 + \theta v}$

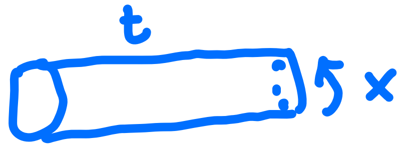
$\Rightarrow$ KO Thom class in $KO^{\text{Witte}}_{O(n)}(\mathbb{R}^n)_c$

$\underline{d=2}$: (joint work in progress w/ Meng Guo & Kiran Luecke)

- Get a conformal field theory determined by by $Cl(L\mathbb{R}^n) \circlearrowleft L\widehat{\mathrm{Spin}}(n)$, "spinor rep."

$\Rightarrow$ Euler class in ${}^{\ell}K^{Tate}_{Spin(n)}(pt)$, equivariant elliptic cohomology at the Tate curve

Value on cylinders:



$S(L\mathbb{R}^n) \xrightarrow{q^E} S(L\mathbb{R}^n)$

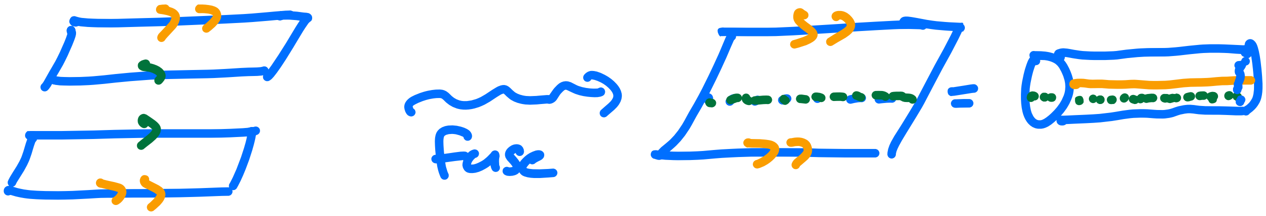
E = energy

$q = e^{2\pi i(x+it)}$

- This CFT has an $\mathbb{R}^n$-family of non-conformal (supersymmetric) deformations

$\Rightarrow$ Thom class in ${}^{\ell}K^{Tate}_{Spin(n)}(\mathbb{R}^n)_c$.

- can try to extend down even further to 2-mflds w/ corners $\longleftrightarrow$ "open strings."



- This theory has higher categorical symmetries that faithfully encode the String group. (ST04)

IV. Power operations in field theories & cohomology theories

For X a manifold with G-action, have $\mathrm{Bord}_d(X/\!\!/G)$ category of bordisms over the quotient stack $X/\!\!/G$.

Construction: [Barthel, B-E, Stapleton] There is a functor:

$$\mathrm{Bord}_d\left(X^{\times k}/\!\!/G \wr \Sigma_k\right) \xrightarrow{\mathbb{P}_k} \mathrm{Bord}_d(X)$$

This induces a k^{th} power operation on twisted field theories.

$d=0$: field thy power ops $\leftrightarrow$ powers of functions/cohomology classes

$$H(X) \xrightarrow{\mathbb{P}_k} H(X^{\times k})^{\Sigma_k}$$

$$f \longmapsto f \boxtimes k \quad \text{(external product)}$$

<u>Observation</u>: Twisted $H\mathbb{R}$-Thom classes are compatible w/ power operations:

$$\begin{array}{ccccc} H^{w_1}_{O(n)}(\mathbb{R}^n) & \xrightarrow{\mathbb{P}_k} & H^{\mathbb{P}_k w_1}_{O(n)\wr\Sigma_k}((\mathbb{R}^n)^{\times k}) & \xleftarrow{res} & H^{w_1}_{O(nk)}(\mathbb{R}^{nk}) \\ \rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} \\ u_n & \longmapsto & \mathbb{P}_k u_n = res\, u_{nk} & \longleftarrow\!\shortmid & u_{nk} \end{array}$$

Furthermore, this refines to Mathai-Quillen Thom forms:

$$\int e^S d\psi \longmapsto \mathbb{P}_k \int e^S d\psi = res \int e^S d\psi \longleftarrow\!\shortmid \int e^S d\psi$$

<u>N.B.</u>: uses the inclusion $O(n)\wr\Sigma_k \hookrightarrow O(nk)$.

The untwisted statement doesn't even make sense: $SO(n)\wr\Sigma_k$ is not a subgroup of $SO(nk)$

(e.g. $n=1$, $k=2$ $SO(1)\wr\Sigma_2 \cong \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle$ not in $SO(2)$.)

<u>d=1</u>: Recall twisted power operations in KO/KU/KR

$$KR(X) \xrightarrow{\mathbb{P}_k} KR_{\Sigma_n}(X^{\times n})$$

$$V \longmapsto V^{\boxtimes n} \circlearrowleft \Sigma_n.$$

<u>Thm</u> [B-E, Meng Guo] Twisted K-Thom classes are compatible w/ power operations:

$$KR^{w_1+w_2}_{O(n)}(\mathbb{R}^n) \xrightarrow{\mathbb{P}_k} KR^{\mathbb{P}_k(w_1+w_2)}_{O(n)\wr\Sigma_k}((\mathbb{R}^n)^{\times k}) \xleftarrow{res} KR^{w_1+w_2}_{O(nk)}(\mathbb{R}^{nk})$$

$$S_n \longmapsto \mathbb{P}_k S_n = res\, S_{nk} \longleftarrow S_{nk}$$

this refines to cocycle representatives

$$Cl_n \overset{\mathbb{P}_k}{\longmapsto} Cl_n^{\otimes k} \simeq Cl_{nk} \overset{res}{\longleftarrow\!\!\shortmid} Cl_{nk}$$

$$\circlearrowleft O(n) \qquad \circlearrowleft O(n)\wr\Sigma_k \qquad \circlearrowleft O(nk)$$

$\underline{d=2}$ In progress! Focus is on ${}^{\ell}K^{Tate}_{Spin}$ version now (not fully extended).

<u>Conjecture</u>: Twisted TMF-Thom classes are compatible with power operations: $(\lambda = w_1 + w_2 + \frac{p_1}{2})$

$$TMF^{\lambda}_{O(n)}(\mathbb{R}^n) \xrightarrow{\mathbb{P}_k} TMF^{\mathbb{P}_k\lambda}_{O(n)\wr\Sigma_k}((\mathbb{R}^n)^{\times k}) \xleftarrow{res} TMF^{\lambda}_{O(nk)}(\mathbb{R}^{nk})$$

This refines to 2d field theories:

$$Fer_n^{\lambda} \longmapsto \mathbb{P}_k Fer_n^{\lambda} \cong res\, Fer_{nk}^{\lambda} \longleftarrow\!\!\shortmid Fer_{nk}^{\lambda}$$